DISCUSSION PAPER NO. 113

PROOF THAT PRICES WHICH ARE
PRESENT-DISCOUNTED CERTAINTY EQUIVALENTS
FLUCTUATE RANDOMLY*

by

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November 15, 1974

* Funds for this research were provided by the Center for Advanced Studies in Accounting and Information Systems, Northwestern University, Evanston, Ill., 60201.
1. CONSISTENT BELIEFS

Consider a time sequence ..., \( X_t \), \( Y_t \), ..., of random vectors. As in Samuelson 1965 [4], the price of spot \( \Phi \) wheat in Chicago at time \( t \) might be some component, say, \( X_t \), of the vector \( \bar{X}_t \); another component, say, \( Y_t \), of the vector \( \bar{Y}_t \) might denote, as in Samuelson 1973 [5], the price of common stock of General Motors at time \( t \); and so on. At any time \( t \), the values ..., \( \bar{X}_{t-1} \), \( \bar{Y}_{t-1} \), ..., \( \bar{Z}_{t-1} \), are already history and, so, are fixed forever. But the same cannot be said about the values of \( \bar{X}_{t+1} \), \( \bar{Y}_{t+2} \), ... These values are still in the future, and we may suppose that at time \( t \) they cannot be known with certainty.

Now, fix attention on \( \bar{X}_T \), where \( T \) is some arbitrarily chosen date; and suppose that at any time \( t \leq T \) an individual assigns to \( \bar{X}_T \) a probability distribution

\[
\text{Prob}[X_T \leq X_T \mid Y_T, \bar{Y}_{T-1}, \ldots] = F_T[X_T \mid Y_T, \bar{Y}_{T-1}, \ldots].
\]  

(1)
The probability distribution that the individual assigns to $\tilde{y}_{t+1}$ at time $t+1$ would depend, in general, upon the value that $\tilde{y}_{t+1}$ takes. Suppose that at time $t$ the individual assigns to $\tilde{y}_{t+1}$ the probability distribution
\[
\text{Prob}[\tilde{y}_{t+1} \leq \omega_{t+1} | \omega_t, \omega_{t-1}, \ldots] = P_t(\omega_{t+1} | \omega_t, \omega_{t-1}, \ldots);
\]
and to $\tilde{y}_t$ conditional upon the value that $\tilde{y}_{t+1}$ takes, he assigns the (conditional) probability distribution
\[
\text{Prob}[\tilde{y}_t \leq x_t | \tilde{y}_{t+1} = \omega_{t+1}, \omega_t, \ldots] = P_t(x_t | \omega_{t+1}, \omega_t, \ldots).
\]

**ASSUMPTION 0:** The individual's probability beliefs are **consistent** in the sense that they accord with the fundamental logic of probability calculus.

Then (1), (2) and (3) above necessarily obey the relation
\[
P_t(x_t | \omega_t, \omega_{t-1}, \ldots) = \int \cdots \int P_t(x_t | \omega_{t+1}, \tilde{y}_t, \omega_{t-1}, \ldots) P_t(\omega_{t+1} | \omega_t, \omega_{t-1}, \ldots).
\]

Herein, $\int \cdots \int f(x) \, g(dx)$ denotes a Stieltjes integral; and when $z$ is a vector, it denotes a **multiple Stieltjes integral.**

2. **ECONOMIC BEHAVIOR: A REVIEW**

Suppose, now, that the individual owns a futures contract in #2 wheat for delivery in Chicago at time $T$; and, for this contract, let $\tilde{y}_t$ be his minimum asking price at any time $t \leq T$. Then at any time
t < T, the asking prices ..., \( \tilde{y}_{t-1} \), \( \tilde{y}_t \) are already history and, hence, fixed. But the asking prices \( \tilde{y}_{t+1} \), \( \tilde{y}_{t+2} \), ..., \( \tilde{y}_T \) are still in the future, and we may suppose that they cannot at time \( t \) be stated with certainty. We may, however, presume that when the due date \( T \) for the futures contract arrives arbitrage will ensure that

\[
\tilde{y}_T = y_T \quad \text{iff} \quad y_T = x_T \quad \text{commissions aside.} \quad (5)
\]

For all times \( t < T \), the relation between \( y_t \) and \( \tilde{x}_T \) will depend upon what we posit about how the individual sets his asking price \( y_t \).

For example, it might be posited that at any time \( t < T \) the individual sets \( y_t \) equal to the non-expected level of the terminal spot price \( \tilde{x}_T \). That is,

\[
y_t = \sum_{s=0}^{\infty} x_t \mathbb{E}_t \left( \frac{dx_{t+s}}{G_t, G_{t-1}, \ldots} \right) = \mathbb{E}_t (\tilde{x}_T) \quad (6)
\]

where \( \mathbb{E}_t \) denotes the "expectation operator" with respect to the probability distribution \( \mathbb{P}_t (\tau_t, G_t, G_{t-1}, \ldots) \) which the individual assigns to \( \tilde{x}_T \) conditional upon \( \tilde{y}_t = G_t, \tilde{y}_{t-1} = G_{t-1}, \ldots \). This implicitly assumes that the individual has a linear utility for cash flow or income; further, it ignores the availability (to the individual) of risk-free investments yielding a positive interest.

Accordingly, it might more generally be posited that at any time \( t < T \) the individual sets \( y_t \) equal to the present-discounted expected value of \( \tilde{x}_T \), the discount rate \( (\rho_{t+1}, \rho_{t+2}, \ldots, \rho_T) \) being equal to the risk-free interest rate \( (\rho_{t+1}, \rho_{t+2}, \ldots, \rho_T) \) suitably
adjusted for the individual's risk attitude toward holding out the futures contract for the next period \(((t \text{ to } t+1), (t+1 \text{ to } t+2), \ldots, (T-1 \text{ to } T))\). That is,

\[
Y_t = \frac{\lambda_{t+1}^{-1} \lambda_{t+2}^{-1} \ldots \lambda_T^{-1}}{t} E_t \left( \tilde{X}_T \right),
\]

(7)

where \(\lambda_1 = 1 + r_1\), with \(\lambda_1 > \rho_1\) \(\text{[resp. } \lambda_1 < \rho_1\] \(\text{[resp. risk loving]}\); and \(t \tilde{\Pi}_T\) denotes the "discounting operator" from time \(T\) to \(t\) at the individual's discount rate schedule.

A little reflection shows that both (6) and (7) above are subsumed under the more general behavioral assumption that at any time \(t \leq T\) the individual sets \(Y_t\) equal to the present-discounted certainty equivalent of the terminal spot price \(X_T\), with the discount rate \((r_{t+1}, r_{t+2}, \ldots, r_T)\) being equal to the risk-free interest rate \((\rho_{t+1}, \rho_{t+2}, \ldots, \rho_T)\) for the period \(((t \text{ to } t+1), (t+1 \text{ to } t+2), \ldots, (T-1 \text{ to } T))\).

The question then is what can we say about the sequence \(\ldots, Y_{t-1}, Y_t, \tilde{Y}_{t+1}, \ldots, \tilde{Y}_T\) \((t < T)\)?

Samuelson 1965 [4] -- with a slight reinterpretation -- provides an answer to this question when \(Y_t\) is related to \(\tilde{X}_T\) \((t < T)\) by (6) and (7), respectively. The more general case, when \(Y_t\) is equal to the present-discounted certainty equivalent of \(\tilde{X}_T\), I shall now investigate.
3. CERTAINTY EQUIVALENT: CONSISTENT PREFERENCES WITH TIME DISCOUNTING

Make the following sufficient assumptions to guarantee the existence of cardinal utilities $u_t$ for cash flow or income at time $t$ \((t \in \mathbb{Z})\), where \(\mathbb{Z}\) is the set of natural numbers.

**Assumption 1**: For the purpose of the individual's preferences, a risky alternative is completely characterized by the probability distributions for cash flow or income at time $t$ \((t \in \mathbb{Z})\).

**Assumption 2**: The individual has, over all risky alternatives, preferences which are consistent in the sense that he cannot, so to speak, make book against himself and end up winning -- or losing -- money! In other words, posit the Axiom of Complete Ordering of all risky alternatives, and the Axiom of "Strong Independence" (see, for example, Samuelson 1952 [3]).

This much assumption implies the existence of cardinal utility functions $u_t$ for sure cash flow or income at time $t$ \((t \in \mathbb{Z})\) and, hence, also the expected utility maximization rule for choice among risky alternatives. The following further assumption should be acceptable to all but the mystical few.

**Assumption 3**: All the utility functions $u_t$ \((t \in \mathbb{Z})\) are strictly increasing monotonic in their argument, cash flow or income.
This now allows definition of the individual's "certainty equivalent operator" $C_t$:

$$C_t(X, \gamma) = \gamma_t \text{ iff } u_t(\gamma_t) = \int dx_t \cdot u_t(x_t) \cdot P_t(dx_t|\gamma_t, \gamma_{t+1}, \ldots),$$  

(8)

where $X_t$ ($\tau \in T$) is any random cash flow at time $\tau$ and $P_t(x_t|\gamma_t, \gamma_{t+1}, \ldots)$ is the probability distribution which the individual assigns to $X_t$ conditional upon $\gamma_t = \gamma_t, \gamma_{t+1} = \gamma_{t+1}, \ldots$ ($t \leq \tau$).

Thus, the behavioral axiom that at any time $t \leq T$ the individual sets $\gamma_t$ equal to the present-discounted certainty equivalent of the terminal spot price $X_T$ may be formally stated as

$$\gamma_t = \sum_{t+1}^{T} \frac{\gamma_{t+1}}{\lambda_{t+1}} \ldots \frac{\gamma_{T}}{\lambda_{T}} \cdot C_t(X_T)$$

(9)

where, in this case, $\lambda_t = 1 + \rho_t$ ($4 = t+1, t+2, \ldots, T$).

Lastly, make the following

**ASSUMPTION 4:** The individual's preferences among sure cash flows at different times accord with the usual present-discounted value calculus.

The following Fundamental Consistency Theorem may now be recaptured without proof from Prakash 1974 [1] and [2].

**THEOREM (Prakash 1974):** Grant Assumptions 1 through 4 above.

Then, the family $\{u_t|t \in T\}$ of the individual's cardinal
utility functions is such that, for any \( r \in \mathbb{Z} \), and any \( t \leq r \),

\[
    t^\Pi_t \cdot C_t \cdot C_t = t^\Pi_t .
\]

(10)

Not too roughly, this says that, if an individual has consistent preferences, then it must be that his present-discounted certainty equivalent of any random cash flow \( \bar{X}_r \) is the same as his present certainty equivalent of the random cash flow obtained by discounting \( \bar{X}_r \) to the present. Using (8) above, (10) translates into

\[
    u_t (t^\Pi_t (x)) = \sum_{m=0}^{\infty} u_t (t^\Pi_t (x)) \int_t^\infty \frac{dx}{x} \cdot \phi_t \cdot \phi_{t+1} \cdot \ldots ,
\]

(11)

where \( \phi_t = C_t (\bar{X}_r) \).

4. PRESENT-DISCOUNTED CERTAINTY EQUIVALENTS FLUCTUATE RANDOMLY

Toward enunciating the main theorems, let \( \bar{X}_t \) be a "time" sequence of random vectors \( \bar{X}_t \) of which some component \( X_t \) denotes random cash flow at time \( t \). Fix a date \( T \) arbitrarily. For any \( t < T \), let the probability laws (1) through (4) hold. Further, grant Assumptions 1 through 4, and let \( u_t \) be a cardinal utility function for random cash flows at time \( t \) \( (t < T) \).

**THEOREM:** For \( t < T \), the sequence \( y_t, \bar{Y}_{t+1}, \ldots, \bar{Y}_T \) defined by (2) has the property

\[
    \begin{align*}
        C_t (\bar{Y}_{t+1} \cdot \phi_t) &= \lambda_{t+1} \cdot y_t , \\
        C_t (\bar{Y}_{t+k} \cdot \phi_t) &= \lambda_{t+1} \cdot \lambda_{t+2} \cdot \ldots \cdot \lambda_{t+k} \cdot y_t ,
    \end{align*}
\]

(12)
where $\lambda_1$ is the discount rate for the period $(i-1)$ to $i$

$(I = t+1, t+2, \ldots, t+k)$ and $(t + k) \leq \tau$.

Proof: By definition (9), $(y_{t+1} | \omega_t) = E_{t+1}^{\tau} \cdot c_{t+1}(x_T | \omega_t)$. Then, using (3) and (11),

$$u_{t+1}(y_{t+1} | \omega_t) = \int_0^\infty u_{t+1}(y_{t+1} | \omega_t) \cdot P_t(dx_T | \omega_t, \omega_t, \omega_t) \cdot \ldots$$

Now denote $C_{t+1}(x_T | \omega_t) = C_{t+1}$ for short. Then, by (2) and (8),

$$u_{t+1}(c_{t+1}) = \int_0^\infty u_{t+1}(y_{t+1} | \omega_t) \cdot P_t(dx_T | \omega_t, \omega_t, \omega_t) \cdot \ldots$$

$$= \int_0^\infty \int_0^\infty u_{t+1}(y_{t+1} | \omega_t) \cdot P_t(dx_T | \omega_t, \omega_t, \omega_t) \cdot \ldots$$

Denote $C_{t+1}(x_T) = C_{t+1}$ for short. Then, using (11), the right side of the above equality is identified to be equal to $u_{t+1}(t^{(t+1)}(c_{t+1}))$. Hence,

$$u_{t+1}(c_{t+1}) = u_{t+1}(t^{(t+1)}(c_{t+1}))$$

and $t^{(t+1)}(c_{t+1}) = t^{(t+1)}(c_{t+1}) = t^{(t+1)} = t^{(t)} = y_t$ by definition. Recalling

that $t^{(t)} = \lambda_{t+1}^t$, we may rearrange the terms to yield the result

$C_{t+1}(x_T | \omega_t) = \lambda_{t+1}^t \cdot y_t$. The second part of (12) now follows by

repeating the above argument k times. $\Box$

**Corollary (Samuelson's Theorem 2, 1965):** For $t < \tau$, the sequence

$y_t, y_{t+1}, \ldots, y_k$ defined by (2) has the property

$$E_t(y_{t+k} | \omega_t) = \lambda_{t+k} \cdot y_t.$$

$$E_t(y_{t+k} | \omega_t) = \lambda_{t+1} \cdot \lambda_{t+2} \cdot \ldots \cdot \lambda_{t+k} \cdot y_t.$$  (13)