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ON THE RELATIONSHIP BETWEEN RISK-DOMINANCE AND STOCHASTIC STABILITY*

by

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ABSTRACT. In a 2×2 symmetric game with two symmetric equilibria in pure strategies, one *risk-dominates* another if and only if the equilibrium strategy is a unique best response to any mixture that gives it at least a probability of one half. In an $n \times n$ symmetric game, we call a strategy *globally risk-dominant* if it is a unique best response to any mixture that gives it at least a probability of one half. We show that if a finite coordination game has a globally risk-dominant equilibrium then this is the one that is selected by the stochastic equilibrium selection processes proposed by Young and by Kandori, Mailath, and Rob.

KEYWORDS. Coordination games, equilibrium selection, risk-dominant equilibria, stochastic stability.

JEL Clasification Numbers: C72, C73.

1. Introduction

In order to provide justification for Nash equiliblium concept and equilibrium selection criteria simultaneously, recent studies have proposed several kinds of 'evolutionary' or 'stochastic' dynamics for finite strategic form games (e.g., Foster and Young (1990), Blume (1993), Kandori, Mailath, and Rob (1993), Kandori and Rob (1992, 1993), Young (1993), among others). A common result derived in the literature is that those dynamics typically select risk-dominant equilibria à la Harsanyi and Selten (1988) when they are applied to 2×2 games.[1]

It is known that this relationship between risk-dominant equilibria and 'evolutionarily selected' equilibria does not extend beyond the class of 2×2 games (e.g., Young (1993, p.73ff)). Consequently, we do not have much knowledge about what kind of equilibria will be selected by those processes in general.

In trying to shed some light on this topic, we propose a generalization of the notion of risk-dominance in 2×2 symmetric games and study its stochastic and/or evolutionary stability. In an $n \times n$ symmetric game with no asymmetric strict equilibria, we call a strategy *globally risk-dominant* if it is a unique best response to any mixture that gives it at least a probability of one half. This is indeed a generalization of the notion of risk-dominance in 2×2 symmetric games since a strategy in a 2×2 symmetric game constitutes a risk-dominant equilibrium if and only if it is a unique best response to any mixture that gives it at least a probability of one half. We show that if a finite coordination game has a globally risk-dominant equilibrium then this is the one that is selected by the stochastic equilibrium selection processes proposed by Young (1993), Kandori, Mailath, and Rob (1993), and Kandori and Rob (1992).[2] As an application, we will show an efficiency result of the following kind. We say that a strategy in a coordination game is of *pure coordination* if on all miscoordination profiles involving that strategy the players' payoffs remain constant.[3] It follows from the main result that if a pure coordination equilibrium in a coordination game is efficient then it is stochastically stable.

One drawback of global risk-dominant equilibria is that, due to possible cyclic risk-dominance relations, they may fail to exist even in the class of coordination games. Stochastically stable equilibria, in contrast, always exist for that class of games and they are generically unique. We shall argue, however, by means of an example that stochastically stable equilibria need not always seem to be plausible in games with no global risk-dominant equilibria due to cycles.

[1]There are some exceptions, however. The selection process of Binmore and Samuelson (1994) does not always select risk-dominant equilibria even for 2×2 games.

[2]Kandori and Rob (1992) is an $n \times n$ version of Kandori, Mailath, and Rob (1993). Since we will work in $n \times n$ setup, hereafter we refer only to Kandori and Rob (1992).

[3]Here the term 'pure coordination' predicates a *strategy*, not a game.

Kandori and Rob (1993) have offered a detailed analysis on the relationship between pairwise risk-dominance and stochastically stable equilibria[4] for coordination games with special properties. Our analysis extends that of Kandori and Rod (1993) to general coordination games. Also, it should be noted that Ellison (1995) has independently shown the main result of this paper.

The next section offers the formal analysis and the last section discusses the global risk-dominance concept and its relationship to other works in the literature.

2. Definitions and the Result

Consider a symmetric $n \times n$ ($n \geq 2$) game G with a strategy set $S = \{s_1, \dots, s_n\}$. Given $(s_i, s_j) \in S \times S$, where the first (second, respectively) entry is the strategy played by the row (column, respectively) player, the payoff for the row (column, respectively) is denoted by u_{ij} (u_{ji}, respectively). See Figure 1. Also, we write $N = \{1, \dots, n\}$ and $N_{-i} = N - \{i\}$ for $i \in N$.

	s_1	$\cdots$	s_i	$\cdots$	s_n
s_1	u_{11}, u_{11}	$\cdots$	u_{1i}, u_{i1}	$\cdots$	u_{1n}, u_{n1}
$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$
s_i	u_{i1}, u_{1i}	$\cdots$	u_{ii}, u_{ii}	$\cdots$	u_{in}, u_{ni}
$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
s_n	u_{n1}, u_{1n}	$\cdots$	u_{ni}, u_{in}	$\cdots$	u_{nn}, u_{nn}

Figure 1. A symmetric game G.

G is said to be a *coordination game* if (s_i, s_i) is a strict Nash equilibrium for every $i \in N$. Notice that if G is a coordination game then G has no asymmetric Nash equilibria in pure strategies. We assume throughout this section that G is a coordination game.

It is well known that if G is 2×2 game the notion of *risk-dominant relation* of Harsanyi and Selten (1988) can be characterized by the following inequality. An equilibrium (s_k, s_k) *risk-dominates* another equilibrium (s_i, s_i) if $u_{kk} - u_{ik} > u_{ii} - u_{ki}$. In this case, (s_k, s_k) is called a *risk-dominant equilibrium* in the 2×2 game. One possible generalization of this concept to $n \times n$ games is the following. Consider an $n \times n$ coordination game G and let

[4] Kandori and Rob (1993) call stochastically stable equilibria *long run* equilibria.

$i, k \in N$ with $i \neq k$. We call an equilibrium (s_k, s_k) of G *pairwise risk-dominant* if (s_k, s_k) risk-dominates (s_i, s_i) in the 2×2 game $\{s_i, s_k\} \times \{s_i, s_k\}$ for every $i \in N_{-k}$.

We now introduce the main concept of the paper, which is another generalization of the risk-dominance in 2×2 games.

Definition. $s_k \in S$ is *globally risk-dominant* if

$$u_{kk} - u_{ik} > u_{ij} - u_{kj}$$

holds for every $i, j \in N_{-k}$.

The definition has some immediate consequencs. In order to state these, we need to introduce a special notation, which will also be used in the sequel.

Notation. For each $i, j \in N$ with $i \neq j$, *fix* a bijection $[ij] : N \longrightarrow N$ on N that satisfies

$$[ij](1) = i \quad \text{and} \quad [ij](2) = j.$$

Now let $x_h \in [0, 1]$ $(h \in N)$ with $\sum_h x_h = 1$. We denote by $(x_1, \ldots, x_n)_{[ij]}$ the mixed strategy profile in G that gives $s_{[ij](h)}$ the weight x_h. When we do not need to specify the second argument, we denote $(x_1, \ldots, x_n)_{[ij]}$ by $(x_1, \ldots, x_n)_{[i\cdot]}$, where j is some arbitralily fixed element in N_{-i}.

Observation.

(1) *$s_k \in S$ is globally risk-dominant if and only if s_k is a unique best response to the mixture $(1/2, 1/2, 0, \ldots, 0)_{[ki]}$ for every $i \in N_{-k}$.*

(2) *$s_k \in S$ is globally risk-dominant if and only if s_k is a unique best response to any mixture $(p_1, \ldots, p_n)_{[k\cdot]}$ such that $p_1 \geq 1/2$.*

(3) *If G has a globally risk-dominant strategy s_k then (s_k, s_k) is a pairwise risk-dominant equilibrium.*

It is clear that (1) follows from $u_{kk} + u_{kj} > u_{ik} + u_{ij}$ and that (1) implies (2). (3) follows from taking $i = j$. By (3), we can call (s_k, s_k) a *globally risk-dominant equilibrium* if s_k is globally risk-dominant.

In view of (3) above, any game that fails to have a pairwise risk-dominant equilibrium cannot have a globally risk-dominant equilibrium. We recall that nonexistence of a pairwise risk-dominant equilibrium can result not only from ties like $u_{kk} - u_{ik} = u_{ii} - u_{ki}$ but also from a cyclic risk-dominance relation.[5] There are coordination games that fail to have pairwise risk-dominant equilibria due to such cycles. The game in Figure 7 of Kandori and

[5]See Harsanyi and Selten (1988, p.217). Notice, however, that a mere existence of cycles does not necessarily preclude existence of a risk-dominant equilibrium.

Rob (1993, p.18) is such an example. Also, there are coordination games with pairwise risk-dominant equilibria that fail to be globally risk-dominant. The game in Example 3 of Young (1993, p.73) is such an example.

We shall show that if G has a globally risk-dominant strategy s_k then the stochastic equilibrium selection processes proposed by Young (1993) and Kandori and Rob (1992) select the globally risk-dominant equilibrium (s_k, s_k). Discussion of the global risk-dominance is deferred to the next section.

Because of the fact that both of the equilibrium selection processes of Young (1993) and of Kandori and Rob (1992) are built on the work of Freidlin and Wentzel (1984),[6] essentially the same argument holds for both of them. For definiteness, however, here we work in the setup of Young (1993). The case of Kandori and Rob (1992) is treated in Appendix 1. Thus we start with a brief review of the stochastic eqilibrium selection process of Young (1993). For details, the reader is referred to Young (1993).

Let m and κ be nonnegative integers with $\kappa < m$. Let H be the m-fold direct product of $S \times S$. Namely, H is the set of all sequences of strategy profiles with length m. H is the state space of Young's (1993) Markov chains.

Let $h = ((s_{i_1}, s_{j_1}), \ldots, (s_{i_n}, s_{j_n})) \in H$ and $h' = ((t_{i_1}, t_{j_1}), \ldots, (t_{i_n}, t_{j_n})) \in H$ be two states. A state h' is a *successor* of h if $(s_{i_k}, s_{j_k}) = (t_{i_{k-1}}, t_{j_{k-1}})$ for $k = 2, \ldots, n$. Let h' be a successor of h. t_{i_n} is a *mistake* in the transition from h to h' if t_{i_n} is not a best response to any sample of size κ from $(s_{j_1}, \ldots, s_{j_n})$. Define a mistake for t_{j_n} analogouly. For every $h, h' \in H$ define the *resistance* $r(h, h')$ *of the transition from* h *to* h' as follows.

$$r(h, h') = \begin{cases} \text{the total number of mistakes in the transition from } h \text{ to } h' & \text{if } h' \text{ is a successor of } h, \\ \infty & \text{otherwise.} \end{cases}$$

Since we are dealing with a 2-person game, the value of $r(h, h')$ is either 0, 1, 2, or ∞.

G is said to be *weakly acyclic* if from every strategy profile there exists a finite sequence of best responses by one player at a time that ends in a strict Nash equilibrium. Notice that if G is a coordination game, as assumed in this section, then G is weakly acyclic since there is a one-step path to a strict Nash equilibrium from every strategy profile (s_i, s_j). Now let us assume that m, the length of a history, and κ, the sample size, satisfy $\kappa \leq m/3$. Then by the Corollary to Theorem 2 of Young (1993), the selection process of Young (1993) will generically single out a unique strict Nash equilibrium of G. In addition, in this case the absorbing states of the adaptive play without mistakes are precisely those states of the form $h_i = ((s_i, s_i), \ldots, (s_i, s_i))$, where $i \in N$.

For $i, j \in N$ with $i \neq j$, consider an *ordered* tuple $\tau = \langle h^1, \ldots, h^v \rangle$ in H. We call such a tuple τ a *path from* i *to* j if $h^1 = h_i$ and $h^v = h_j$. Define the *resistance* $r(\tau)$ *of a*

[6] Strictly speaking, both of them are based on a discrete version of the development of Freidlin and Wentzel (1984) that is proved by Young (1993).

path $\tau = \langle h^1, \ldots, h^v \rangle$ *from* i *to* j by $r(\tau) = \sum_{u=1}^{v-1} r(h^u, h^{u+1})$. Finally define the *resistence from* i *to* j by

$$r_{ij} = \min \{ r(\tau) \mid \tau \text{ is a path from } i \text{ to } j \}.$$

We notice that a resistance r_{ij} is an integer. It will prove useful to define a 'continuous' version of it as follows. For $i, j \in N$ with $i \neq j$, define the *exit resistance* γ_{ij} *from* i *via* j by

$$\gamma_{ij} = \min \left\{ p \in [0, 1] \;\middle|\; \begin{array}{l} s_j \in BR(1-p,\ pq_1,\ \ldots,\ pq_{n-1})_{[ij]} \text{ for some} \\ q_h \in [0, 1] \text{ such that } \sum_{h=1}^{n-1} q_h = 1. \end{array} \right\},$$

where $BR(\cdot)$ is the best response correspondence of G. Further, define the *exit resistance* γ_i *from* i by

$$\gamma_i = \min \{ \gamma_{ij} \mid j \in N_{-i} \}.$$

The motivation for introducing γ_{ij} and γ_i is the following. Imagine that the current state is h_i and we want to know how much the resistance r_{ik} from i to another state k is. In order to move away from h_i it is necessary to have some mistakes by some player since (s_i, s_i) is a strict Nash equilibrium. Moreover, it is necessary to have enough mistakes so that some strategy different from s_i, but not necessarily s_k, can be played against the disturbed history. In other words, the process have to *exit* from the basin of attraction of h_i under the adaptive play without mistakes.[7] γ_{ij} corresponds to the minimum proportion of deviation from s_i in a history that is required to have a desired exit where the first strategy different from s_i that can be played during the exit is s_j.[8] Accordingly, γ_i corresponds to the minimum proportion of deviation to get an exit at all, regardless of the destination.

By definition, γ_{ij} solves the minimization problem that defines γ_{ij}. In general, the $n-1$ probability vector $\langle q_1, \ldots, q_{n-1} \rangle$ associated with γ_{ij} may have a support of size two or more. In addition, its support need not include s_j. It will prove useful, however, to define resistances that ignore these two complications. For $i \in N$ and $j \in N_{-i}$, define

$$\beta_{ij} = \min \{ p \in [0, 1] \mid s_j \in BR(1-p,\ p,\ 0, \ldots, 0)_{[ij]} \}.$$

By definition, $\gamma_{ij} \leq \beta_{ij}$ for every $i, j \in N$ with $i \neq j$.

Given a real number x, denote by $\lceil x \rceil$ the minimum integer weakly exceeding x. We make the following assumptions on the length m of a history and the sample size κ. The first of these has been already stated above.

Assumption.

(1) $\kappa \leq m/3$.

[7] By the basin of attraction of the state h_i we mean the set of states that are absorbed into h_i with probability one under the adaptive play without mistakes.

[8] Thus we call γ_{ij} the exit resistance from i *via* j, not *to* j.

(2) κ is sufficiently large so that $x \geq y$ if and only if $\lceil x\kappa \rceil \geq \lceil y\kappa \rceil$ for every $x, y \in \{ \gamma_{ij}, \beta_{ij} \mid i, j \in N \text{ with } i \neq j \} \cup \{1/2\}$.

We are now ready to present our result.

Lemma 1. *For every $i \in N$ and $j \in N_{-i}$, $\lceil \gamma_i \kappa \rceil \leq r_{ij}$.*

Proof. First, it follows from the definition of γ_i that if $p < \gamma_i$ then

$$\{s_i\} = BR(1-p,\ pq_1,\ \ldots,\ pq_{n-1})_{[i\cdot]}$$

for every $q_h \in [0, 1]$ with $\sum_h q_h = 1$. Now by the definition of r_{ij} there is a path $\tau = \langle h_i, h^1, \ldots, h^v, h_j \rangle$ such that $r_{ij} = r(\tau)$. At state h_j, the best responses of both players to any sample from h_j is s_j, which is different from s_i. Thus, during the path τ, there is a *first* state h^* that includes a sample of size κ to which either row's or column's best response differs from s_i. Assume that at h^* the row can optimally play a strategy that is different from s_i. Then at h^* the number μ of periods in which the column has played strategies different from s_i must satisfy $\mu/\kappa \geq \gamma_i$, by the argument given in the beginning of the proof. On the other hand, by the choice of h^*, all non s_i strategies played by the column up to h^* have been mistakes. Thus, at h^*, the column has made mistakes at least μ times. Therefore $r_{ij} \geq \mu$. Consequently, $r_{ij} \geq \lceil \gamma_i \kappa \rceil$. □

Lemma 2. *Assume that G has a globally risk-dominant equilibrium (s_k, s_k). If (s_i, s_i) is a strict Nash equilibrium that is different from (s_k, s_k) then $r_{ik} < \left\lceil \frac{1}{2}\kappa \right\rceil < r_{ki}$.*

Proof. Pick any strict Nash equilibrium $(s_i, s_i) \neq (s_k, s_k)$. The proof takes two steps. First, we show that $1/2 < \gamma_{ki}$. Subsequently, we show that $r_{ik} < \lceil (1/2)\kappa \rceil < r_{ki}$.

Let us prove that $1/2 < \gamma_{ki}$. Pick $p \in [0, 1]$ and assume that $p \leq 1/2$. Then $1-p \geq 1/2$. Thus by the global risk-dominance of s_k,

$$\{s_k\} = BR(1-p,\ pq_1,\ \ldots,\ pq_{n-1})_{[ki]}.$$

Therefore

$$s_i \notin BR(1-p,\ pq_1,\ \ldots,\ pq_{n-1})_{[ki]}.$$

Thus we have shown that

$$\text{if } s_i \in BR(1-p,\ pq_1,\ \ldots,\ pq_{n-1})_{[ki]} \text{ then } 1/2 < p,$$

which implies that $1/2 < \gamma_{ki}$.[9]

[9] Since $BR(\cdot)$ is upper-hemicontinuous, the argument does show $1/2 < \gamma_{ki}$, rather than $1/2 \leq \gamma_{ki}$.

Since our proof of $1/2 < \gamma_{ki}$ does not depend on i, we have $1/2 < \gamma_k$ by the definition of γ_k. Thus by Lemma 1 and the Assumption above, we have $\lceil (1/2)\kappa \rceil < \lceil \gamma_k \kappa \rceil \leq r_{ki}$.

It remains to prove that $r_{ik} < \lceil (1/2)\kappa \rceil$. Consider β_{ik}. Clearly, $\lceil \beta_{ik}\kappa \rceil / \kappa \geq \beta_{ik}$. Thus by the definition of β_{ik},

$$s_k \in BR\left(\frac{\kappa - \lceil \beta_{ik}\kappa \rceil}{\kappa}, \frac{\lceil \beta_{ik}\kappa \rceil}{\kappa}, 0, \ldots, 0\right)_{[ik]}.$$

Since s_k is globally risk-dominant, $\beta_{ik} < 1/2$. Thus by the Assumption on sample size, $\lceil \beta_{ik}\kappa \rceil < \lceil (1/2)\kappa \rceil$. This corresponds to the situation where, starting from the state $h_i = ((s_i, s_i), \ldots, (s_i, s_i))$, the column (say) player has continued to play s_k by mistake $\lceil \beta_{ik}\kappa \rceil$ times in a row and it is now possible for the row player to play s_k optimally. Let us call this state h^*. From the state h^*, there is a positive probability to move into the state $h_k = ((s_k, s_k), \ldots, (s_k, s_k))$ without further mistakes. This can be seen as follows. Suppose that the row will continuously be given as her sample the right most sequence of size κ in h^*. This event can happen with a positive probability at least next 2κ periods since $\kappa \leq m/3$ by assumption. During these periods, the row can optimally play s_k each time. Assume, on the other hand, the column observes the most recent sequence of size κ every period. Under this event, the proportion of s_k in the column's sample is strictly increasing. In particular, after $\kappa - \lceil \beta_{ik}\kappa \rceil$ periods from h^* at the latest, the column can also play s_k optimally. Continuing this fasion, after 2κ periods from h^* the process moves into a state h^{**} whose right most sequence of size $\kappa + \lceil \beta_{ik}\kappa \rceil$ consists of (s_k, s_k). From h^{**}, by taking most recent samples, the process eventually moves into h_k without any mistakes. Therefore $r_{ik} \leq \lceil \beta_{ik}\kappa \rceil$ by the definition of r_{ik}. Consequently, $r_{ik} < \lceil (1/2)\kappa \rceil$. □

A binary relation T on N is an *i-tree* ($i \in N$) if for every $j \in N_{-i}$,

(1) $(i, j) \notin T$.
(2) There is a unique $h \in N_{-j}$ such that $(j, h) \in T$.
(3) There are $j_1, \ldots, j_h \in N$ with $j_1 = j$ and $j_h = i$ such that $(j_l, j_{l+1}) \in T$ for every $l = 1, \ldots, h-1$.

Given an i-tree T, we define the *stochastic potential* $\rho(T)$ of T by

$$\rho(T) = \textstyle\sum_{(j,h)\in T} r_{jh}.$$

Following Young (1993), we call a Nash equilibrium (s_i, s_i) of a coordination game G *stochastically stable* if for every j-tree T_j ($j \neq i$) there is an i-tree T_i such that $\rho(T_i) < \rho(T_j)$.

Theorem. *If G has a globally risk-dominant equilibrium (s_k, s_k) then the stochastically stable equilibrium is (s_k, s_k).*

Proof. Fix any strict Nash equilibrium $(s_i, s_i) \neq (s_k, s_k)$ and an i-tree T_i. Consider the binary relation T_k on N defined by

$$T_k = (T_i - \{(k, j)\}) \cup \{(i, k)\}.$$

One can verify that T_k is a k-tree. Moreover it follows from the construction of T_k and Lemma 2 that

$$\rho(T_i) - \rho(T_k) = r_{kj} - r_{ik} > 0,$$

which shows that (s_k, s_k) is stochastically stable. □

Before closing this section, we state an immediate application of the Theorem. A strategy s_k in a coordination game G is a *pure coordination strategy* if there are numbers $\zeta, \eta \in \mathbb{R}$ with $\zeta < \eta$ such that for every $i \in N$,

$$u_{ik} = u_{ki} = \begin{cases} \eta & \text{if } i = k, \\ \zeta & \text{otherwise.} \end{cases}$$

This means that under all miscoordination situations involving the strategy s_k the players get the same payoff ζ. In other words, miscoordination payoffs do not depend on *how* palyers fail to coordinate. See Figure 2. We notice that if a coordination game G has a pure coordination equilibrium then its stochastic stability relates to its efficiency in the following way.

Corollary. *Assume that G has a pure coordination strategy s_k. Then (s_k, s_k) is globally risk-dominant if and only if $u_{kk} > u_{ij}$ for every $i, j \in N$ with $(i, j) \neq (k, k)$. In particular, if (s_k, s_k) is efficient then it is stochastically stable.*

Proof. If s_k is a pure coordination strategy then the defining inequality of global risk-dominance for s_k reduces to $u_{kk} > u_{ij}$. □

There are coordination games with pure coordination equilibria that are both stochastically stable and inefficient. Thus efficiency is not a necessary condition for pure coordination equilibria to be stochastically stable.

Since the seminal argument by Schelling (1960, p.291ff), it has been a controversial issue that whether efficiency of an equilibrium in a coordination game can make the equilibrium focal. For discussions of this issue in the context of 'non-evolutionary' equilibrium selection theories, see, for example, Harsanyi and Selten (1988, p.88ff, p.355ff), Aumann

	s_1	$\cdots$	s_i	$\cdots$	s_n	s_{n+1}
s_1	u_{11}, u_{11}	$\cdots$	u_{1i}, u_{i1}	$\cdots$	u_{1n}, u_{n1}	ζ, ζ
$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\vdots$
s_i	u_{i1}, u_{1i}	$\cdots$	u_{ii}, u_{ii}	$\cdots$	u_{in}, u_{ni}	ζ, ζ
$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$	$\vdots$
s_n	u_{n1}, u_{1n}	$\cdots$	u_{ni}, u_{in}	$\cdots$	u_{nn}, u_{nn}	ζ, ζ
s_{n+1}	ζ, ζ	$\cdots$	ζ, ζ	$\cdots$	ζ, ζ	η, η

Figure 2. A symmetric game. If $\zeta < \eta$ then s_{n+1} is a pure coordination strategy.

(1990), and Carlsson and van Damme (1993, p.1005ff). The Corollary suggests that, although mere efficiency is not enough to be focal in the sense of stochastic stability, an efficient equilibrium in pure coordination strategies is indeed focal. Constant payoffs in the corresponding row and column, together with efficiency, make the equilibrium conspicuous enough for the players so that they can successfully coordinate on it without worrying about any other sophisticated strategic consideration. In particular, an efficient pure coordination equilibrium can escape Aumann's (1990) argument, which casts doubts on the self-enforcing property of a certain kind of efficient equilibria. Because we are working in an evolutionary context where players are assumed to be myopic, the Corollary cannot be interpreted directly in the context of focal point arguments, where players are assumed to be fairly intelligent. Our point here is, therefore, that the above argument for efficient pure coordination equilibria in the context of focal points can be backed up by the evolutionary arguments of Young (1993) and of Kandori and Rob (1992).

Another implication of the Corollary is that for any coordination game G one can construct, by adding just one strategy to G, another game $\overline{G}$ that has a stochastically stable equilibrium with payoff exceeding the maximum payoff of G. This is clear from Figure 2. We should notice that, in Figure 2, any η of the form

$$\eta = \max\{\, u_{ij} \mid i, j \in \{1, \ldots, n\} \,\} + \varepsilon \quad \text{with} \quad \varepsilon > 0$$

can be used for this construction. On the other hand, in addition to $\zeta < \eta$, ζ needs to satisfy $\zeta < u_{ii}$ for every $i \in N$ to ensure $\overline{G}$ to be a coordination game.

3. Discussion

The analysis in the last section shows that the notion of global risk-dominance has some strong implications. Therefore it is important to know what classes of games have globally risk-dominant equilibria and what kinds of interpretations can be given to the global risk-dominance. Roughly, we claim that a game is more likely to have a globally risk-dominant equilibrium if the game desribes a coordinating situation of a certain kind. In addition, there are at least two interpretations, one is 'rationality' based and the other is 'evolutionary', that make sense of the global risk-dominance. We also argue by means of an example that, in games with no pairwise risk-dominant equilibria due to cycles, stochastically stable equilibria need not always seem to be plausible.

For a non 2×2 symmetric game G (not necessarily a coordination game), there are possible properties of G that G could not have possessed if it had been 2×2. One such property is that, even when a player is playing a mixture of particular two strategies, a third strategy might be a best response for the opponent to that mixture. Also, the pairwise risk-dominance relation of G might exhibit cycles, and these cycles in turn can result in nonexistence of a pairwise risk-dominant equilibrium. It turns out that if G has *neither* of these then it has a globally risk-dominant strategy s_k. Consider the condition

(CD) For every distinct $i, j, k \in N$ and $p \in [0, 1]$,

$$pu_{ik} + (1-p)u_{ij} < \max\{pu_{kk} + (1-p)u_{kj}, pu_{jk} + (1-p)u_{jj}\}.$$

This simply says that, as long as your opponent is playing a mixture of s_k and s_j, your best response to the mixture is also either s_k or s_j but never a third strategy, s_i. We regard **(CD)** as a formal description of a coordinating situation where pairwise mixtures are allowed. A coordinating problem arises when we are in a situation where we want to do a particular action if, and only if, others do the same action. In other words, in a coordinating situation, you want to do the *same* thing as others do. You want to drive left if, and only if, your fellow drivers drive left. It is this intuition that **(CD)** expresses for in any situation where **(CD)** is violated you have an incentive to act *differently* from what others do. In fact, one can show that

Claim.

(1) *If G satisfies* **(CD)** *then G is a coordination game.*
(2) *If G satisfies* **(CD)** *and has a pairwise risk-dominant equilibrium then the pairwise risk-dominant equilibrium is globally risk-dominant.*

Proof of the Claim is given in Appendix 2.[10] It follows from (2) that in the class of

[10] We do not need full strength of **(CD)** to get global risk-dominance. We need **(CD)** to hold just for every $i, j \in N_{-k}$ with $i \neq j$, where (s_k, s_k) is the risk-dominant equilibrium.

coordination games with (**CD**) a stochastically stable equilibrium is pairwise risk-dominant if and only if it is globally risk-dominant. As a consequence, our Theorem tells us that, in coordinating situations described by (**CD**), the notion of pairwise risk-dominance does incorporate some global properties of the game and can be used to determine relative stability between 'conventions'.

Kandori and Rob (1993) have formulated a condition called *total bandwagon property*. It dictates that, given a mixture of strategies in G, every pure best response to that mixture is in the carrier of that mixture. With an additional and independent condition, Kandori and Rob (1993) have shown that the 'long run' equilibrium coincides with the pairwise risk-dominant equilibrium, if the latter exists. Our result is stronger than theirs in two ways. First, (**CD**) is a restricted version of the total bandwagon property to mixtures between two pure strategies. Second, the additional condition is not needed.[11] Further, since the defining condition of global risk-dominance involves no probability calculation, it has a desirable property that it is easy to check if a given game has a globally risk-dominant equilibrium. The Corollary of the last section illustrates this point.

Working in a different setup and using a different stochastic selection technique from those of Young (1993) and of Kandori and Rob (1992), Blume (1993, p.413) has shown that if a pairwise risk-dominant equilibrium of a coordination game satisfies a certain condition then it is selected by the selection process of Blume (1993). Since Blume's (1993) condition implies global risk-dominance, the corresponding results in Kandori and Rob's (1992) and Young's (1993) settings follow from our result.[12]

We can give another interpretation to (**CD**), which in turn makes the global risk-dominance itself meaningful by the Claim. In their book, Harsanyi and Selten (1988) write

> Imagine a hypothetical situation where it is common knowledge that all players think that either U or V must be the solution without knowing which of both equilibrium points is the solution. Risk dominance tries to capture the idea that in this state of confusion the players enter a process of expectation formation that may lead to the conclusion that in some sense one of both equilibrium points is less risky than the other. (Harsanyi and Selten (1988, p.204))

As such, the notion of risk-dominance is indeed based on the idea of pairwise comparisons between equilibria. In general, however, it would be unreasonable to determine the risk-dominance relation between $U = (U_1, U_2)$ and $V = (V_1, V_2)$ by restricting attention to the corresponding 2×2 game $\{U_1, V_1\} \times \{U_2, V_2\}$. This can be seen as follows. In the process of expectation formation (i.e., the tracing procedure), it may happen that the players' beliefs fall in a region where (**CD**) is violated. In such a region, the supposed

[11] It should be noted, however, using the total bandwagon property and an additional condition, Kandori and Rob (1993) have succeeded to approximate the cost of transition, which corresponds to our resistance, for each ordered pair of equilibria.

[12] A global risk-dominant equilibrium need not satisfy Blume's (1993) condition.

common knowledge of the restriction of the solution to U or V conflicts with the best response principle. That is, the expectation that either U or V will be played, coupled with a rationality assumption, leads to the expectation that other strategies can be played. Consequently, it appears unreasonable to restrict attention to U or V in the first place.

In order to avoid this kind of inconsistency, Harsanyi and Selten (1988, p.198ff) introduce the notion of a *formation*. Roughly, the formation spanned by U and V is the minimal restricted game including U_i and V_i that is closed under best responses. Harsanyi and Selten (1988, p.208) then determine risk-dominance relation by applying the tracing procedure to the formation spanned by two equilibria. Now the significance of the condition (**CD**) in the selection theory of Harsanyi and Selten (1988) is clear. For each pair of equilibria in a coordination game, the corresponding 2×2 game is qualified as the formation if and only if (**CD**) is satisfied. Consequently, in the class of symmetric games with (**CD**), risk-dominant equilibria reduce to pairwise risk-dominant equilibria. Moreover, it follows from the main theorem and the claim above that if a symmetric game with (**CD**) has a risk-dominant equilibrium in the sense of Harsanyi and Selten (1988) then it is stochastically stable.

Alternatively, it is also possible to give the global risk-dominance an 'evolutionary' interpretation. The global risk-dominance represents a kind of evolutionary stability against possibly *large* fraction of mutants. Imagine a population where everyone plays s_k. All of sudden, a considerable fraction of the population mutates into playing another strategy s_j. In reacting to this mutation, the natural selection process presumably begins to work and reselects a (possiblly third) strategy s_i that does best against the disturbed strategy frequency. Let us now suppose that the mutants are fixed. That is, the natural selection process does not operate on the mutated fraction but only on the nondeviating fraction. In this event, the global risk-dominance guarantees that the natural selection process reselects s_k again if the fraction of mutants is at most one half. The story can be exemplified in matrix notation as follows.

$$\begin{pmatrix} 1-\varepsilon & 0 & \varepsilon \end{pmatrix} \begin{pmatrix} u_{kk} & u_{ki} & u_{kj} \\ u_{ik} & u_{ii} & u_{ij} \\ u_{jk} & u_{ji} & u_{jj} \end{pmatrix} \begin{pmatrix} 1-\varepsilon \\ 0 \\ \varepsilon \end{pmatrix} - \begin{pmatrix} 0 & 1-\varepsilon & \varepsilon \end{pmatrix} \begin{pmatrix} u_{kk} & u_{ki} & u_{kj} \\ u_{ik} & u_{ii} & u_{ij} \\ u_{jk} & u_{ji} & u_{jj} \end{pmatrix} \begin{pmatrix} 1-\varepsilon \\ 0 \\ \varepsilon \end{pmatrix}$$

$$= (1-\varepsilon)^2(u_{kk} - u_{ik}) - (1-\varepsilon)\varepsilon(u_{ij} - u_{kj}).$$

The right hand side is positive if s_k is globally risk-dominant and $\varepsilon \leq 1/2$. Stability against large deviations seems to be a relevant idea in environments like those of Young (1993), and Kandori and Rob (1992),[13] where *any* amount of mutation (i.e., deviation) is possible all the time. The analysis in the last section verifies this intuition.

[13] That is, models built on the work of Freidlin and Wentzell (1984).

	s_1	s_2	s_3	s_4
s_1	10, 10	11, 7	3, 8	0, 7
s_2	7, 11	18, 18	2, 2	−9, 16
s_3	8, 3	2, 2	6, 6	0, 0
s_4	7, 0	16, −9	0, 0	1, 1

Figure 3. A coordination game.

	s_1	s_2
s_1	10, 10	0, 0
s_2	0, 0	10, 10

	s_1	s_2
s_1	6, 4	0, 0
s_2	0, 0	4, 6

Figure 4. 2×2 games from Schelling (1960).[14]

One drawback of global or pairwise risk-dominant equilibria is that, due to possible cyclic pairwise risk-dominance relations, they may fail to exist even in the class of coordination games. Stochastically stable equilibria, in contrast, always exist for that class of games and they are generically unique. There is no doubt that the general existence is a desirable property in itself. The existence result, however, need not imply that the selection process always singles out an equilibrium that is consistent with our intuitive notions of stability. Let us consider the game in Figure 3. The game in Figure 3 is a coordination game with (**CD**). There are two cycles in the pairwise risk-dominance relation. One consists of (s_1, s_1), (s_2, s_2), and (s_4, s_4) and the other involves (s_2, s_2), (s_3, s_3), and (s_4, s_4). Thus there is no pairwise (hence, global) risk-dominant equilibrium. One can verify, on the other hand, that the stochastically stable equilibrium is (s_3, s_3).[15]

In this example, the efficient equilibrium (s_2, s_2) might look attractive at first glance, but it is rather 'risky'. In this respect, s_3 does look 'safer' than s_2. The difficulty in this example is that s_1 and s_4 are also safer than s_2 and there seems to be no clue that would distinguish s_3 favorably from s_1 and s_4. On the face of it, one can argue that it is simply 'difficult' to achieve a successful coordination in the coordination problem described by the game. In this regard, we recall that Schelling (1960, p.291ff) has classifed games like those in Figure 4 as 'insoluble' on the ground that there is no clue in the payoff structure of the game that would distinguish an equilibrium from others. It is interesting to note that

[14] According to our definition in Section 2, the game on the right is not a coordination game.

[15] For details, see Appendix 3.

all games that Schelling (1960, p.291ff) has classified as insoluble are those that have no risk-dominant equilibria due to *ties*. Given the example above, our point here is that the existence of *cycles* in the pairwise risk-dominance relation might correspond to another kind of difficulty in a coordinating situation, which can arise only if there are more than two strategies. If this is so, the example suggests that this kind of difficulty need not always be overcome by the selection processes based on stochastic stability.

Alternatively, we can examine the game in Figure 3 from a formal point of view. Recall that we defined in Section 2 the exit resistance γ_i by

$$\gamma_i = \min\{\, \gamma_{ij} \mid j \in N_{-i} \,\}.$$

Roughly, γ_i measures the difficulty for the process to get out from the basin of attraction[16] of the equilibrium (s_i, s_i). The higher γ_i, more mistakes are needed to get out from (s_i, s_i). Given this intuition, we call an equilibrium (s_k, s_k) *γ-dominant* if

$$\gamma = \gamma_k = \max\{\, \gamma_i \mid i \in N \,\}.$$

That is, a γ-dominant equilibrium is one that is most difficult for the process to get out from. It is immediate that a globally risk-dominant equilibrium is a γ-dominant equilibrium with $\gamma > 1/2$ and that a γ-dominant equilibrium always exists and is generically unique. As mentioned earlier, in the class of coordination games with (**CD**) and with pairwise risk-dominant equilibria, globally risk-dominant equilibria and stochastically stable equilibria coincide. In particular, within this class, stochastically stable equilibria are always γ-dominant equilibria. Thus it is interesting to know if stochastically stable equilibria are γ-dominant equilibria in general. It turns out that these notions do not coincide in general as the unique γ-dominant equilibrium in the game in Figure 3 is (s_1, s_1) with $\gamma = 3/10$.[17] In summary, in the class of coordination game with (**CD**), although a stochastically stable equilibrium in a game with a risk-dominant equilibrium can be described as the one that is most difficult for the process to get out from, it may lose this intuitive characterization when the game has a cyclic risk-dominance relation that results in nonexistence of risk-dominant equilibria.

In the context of equilibrium selection based on 'global payoff uncertainty' à la Carlsson and van Damme (1993), a recent paper by Morris, Rob, and Shin (1995) have proposed the notion of *p-dominant* equilibria, which is similar to our γ-dominant equilibria just defined above. An action profile (s_1, s_2) in a two person finite strategic form game associated with a state ω in an information system is called p-dominant at ω if s_i is a unique best response for the player i as long as she believes at that state that player j will play s_j with

[16] See footnote 6.

[17] See Appendix 3.

probability at least p.[18] Morris. Rob. and Shin (1995) have shown that if an information system has an action pair (s_1, s_2) that is p-dominantat at every state with $p \leq 1/2$ and some additional conditions are met then the only rationalizable play in the incomplete information game is playing (s_1, s_2) at every state. Our main result in the last section and the example in Figure 3 show that the number 1/2 is a threshold for the stochastic equilibrium selection processes of Young (1993) and Kandori and Rob (1992). Similarly, the result of Morris, Rob. and Shin (1995) shows that the number 1/2 also palys a role of a threshold for the equilibrium selection based on the global payoff uncertainty of Carlsson and van Damme (1993).

Finally, we mention a work by Ellison (1995), where our main result has been derived independently. Ellison (1995) includes, among other things, applications of global risk-dominance concept to the local interaction model developed by Ellison (1993).[19]

[18] Thus, given a strategy s_i, Morris. Rob, and Shin's (1995) 'p' refers to the probability attached to s_i, whereas our 'γ' refers to the sum of probabilities attached to strategies different from s_i.

[19] Ellison (1995) calls a global risk-dominant equilibrium a *1/2-dominant equilibrium.*

APPENDIX 1

Here we verify that the analysis in Section 2 also works for the process of Kandori, Mailath, and Rob (1993) and Kandori and Rob (1992). Since the process of Kandori, Mailath, and Rob (1993) is a special case of Kandori and Rob (1992), we henceforth refer only to the latter. The reader is referred to Kandori and Rob (1992) for details.

Let G be a symmetric $n \times n$ coordination game with the strategy set $\{s_1, \ldots, s_n\}$. Since their model is one of random matching, we first have a population $\{1, \ldots, M\}$ of players. The state space is defined to be

$$Z = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{R}^n_+ \mid z_i \in \mathbb{N} \text{ for every } i \in N \text{ and } \textstyle\sum_{i=1}^n z_i = M \right\}.$$

Denote by e_i the state $(0, \ldots, 0, M, 0, \ldots, 0) \in Z$, where M lies on the i-th coordinate. Since we are considering a coordination game G and their process is also a best response dynamic in the case of without mistake, the recurrent communication classes (i.e., *limit sets*, as Kandori and Rob (1992) call them) of the case without mistake are just $\{e_1\}, \ldots, \{e_n\}$.

As in Section 2, we define for every $z, z' \in Z$ the *cost of transition* $c(z, z')$ *from* z *to* z' to be the minimum number of mistakes in the transition from z to z' (see Kandori and Rob (1992)). Again similarly to Section 2, for every $i, j \in N$ with $i \neq j$ we call an ordered tuple $\tau = \langle z^1, \ldots, z^v \rangle$ with $z^1 = e_i$ and $z^v = e_j$ a *path from an equilibrium* i *to another equilibrium* j. Given a path $\tau = \langle z^1, \ldots, z^v \rangle$, define $c(\tau) = \sum_{u=1}^{v-1} c(z^u, z^{u+1})$. Finally define the *cost of transition from* i *to* j by

$$c_{ij} = \min\{ c(\tau) \mid \tau \text{ is a path from } i \text{ to } j \}.$$

Assuming that G has a globally risk-dominant equilibrium (s_k, s_k), we try to evaluate c_{ij} in terms of γ_i and β_{ik}, which are defined in Section 2. Here we also make use of a 'large parameter' assumption. Recall that in the proof of the Lemma 2 in Section 2, we have shown that $\beta_{ik} < 1/2 < \gamma_k$ for every $i \in N_{-k}$.

Assumption. The population size M is sufficiently large so that for every $i \in N_{-k}$

$$\frac{M-1}{2} \geq \lceil \beta_{ik} M \rceil.$$

For example, if $\max\{ \beta_{ik} \mid i \in N_{-k} \} = 0.4999$ then $M = 10000$ will do. Given this, what we are going to show are the following.

(1) $c_{ij} \geq \gamma_i(M-1)$ for every $i \in N$ and $j \in N_{-i}$.

(2) $\lceil \beta_{ik} M \rceil \geq c_{ik}$ for every $i \in N$ with $i \in N_{-k}$.

It is indeed enough to show these since it follows from (1), (2), and the Assumption, together with $1/2 < \gamma_k$, that $c_{ik} < 1/2(M-1) < c_{ki}$.

(1) can be shown as follows. Let a path $\langle e_i, z^2, z^3, \ldots, e_j \rangle$ be a minimizer of the definition of c_{ij}. By the 'triangular inequality' shown by Kandori and Rob (1992, p.17), we can assume that there is $h \in N$ with $z_h^2 \neq 0$ such that s_i is not a best response to the adjusted mixture that s_h playing players are facing. In this event it must be the case that either

$$\sum_{l \neq i} \frac{z_l^2}{M-1} \geq \gamma_i \qquad \text{or} \qquad \frac{z_h^2 - 1}{M-1} + \sum_{\substack{l \neq i \\ l \neq h}} \frac{z_l^2}{M-1} \geq \gamma_i$$

by the definition of γ_i. Therefore $\sum_{l \neq i} \frac{z_l^2}{M-1} \geq \gamma_i$. Thus by the definition of c_{ij}, $c_{ij} \geq \gamma_i(M-1)$.

Now let us turn to (2). It suffices to show that starting from the state e_i, where everybody plays s_i, the process can move into e_k by just $\lceil \beta_{ik} M \rceil$ mistakes. Consider the state

$$z = (M - \lceil \beta_{ik} M \rceil, \lceil \beta_{ik} M \rceil, 0, \cdots, 0),$$

where the first (the second, respectively) entry denotes the number of players who play s_i (s_k, respectively). By the definition of β_{ik}, players who play s_i at z can optimally play s_k next period. Consequently, we can take as the next state a state z' where the number of players playing s_k is at least $M - \lceil \beta_{ik} M \rceil$. Now we notice that it follows from the large population assumption that

$$\frac{M - \lceil \beta_{ik} M \rceil - 1}{M-1} \geq \frac{1}{2},$$

which means that at z' every player is facing a mixture at least a one half of the population (excluding herself) playing s_k. Then it follows from the definition of global risk-dominance that everybody can optimally play s_k next period, thereby moving into e_k.

Appendix 2

Here we prove the Claim in Section 3. Let G be a symmetric $n \times n$ game. It suffices to show the following two lemmas.

A.1 Lemma. *Assume that G satisfies* (**CD**). *Then the set of pure strategy Nash equilibria of G is $\{(s_1, s_1), \dots, (s_n, s_n)\}$. Moreover, all of them are strict.*

Proof. Fix $k \in N$. It suffices to show that s_k is the unique best response to s_k. In other words, it suffices to show that $u_{ik} < u_{kk}$ and $u_{jk} < u_{kk}$ for every $i, j \in N_{-k}$ with $i \neq j$.

By (**CD**) with $p = 1$, we have

$$u_{ik} < \max\{u_{kk}, u_{jk}\} \leq \max\{u_{kk}, u_{ik}, u_{jk}\}. \tag{1}$$

Flipping i and j and then applying (**CD**) with $p = 1$,

$$u_{jk} < \max\{u_{kk}, u_{ik}\} \leq \max\{u_{kk}, u_{ik}, u_{jk}\}. \tag{2}$$

Now (1) and (2) together imply that $\arg\max\{u_{kk}, u_{ik}, u_{jk}\} = \{u_{kk}\}$. □

A.2 Lemma. *Assume that G satisfies* (**CD**) *and has a risk-dominant equilibrium (s_k, s_k). Then for every $i, j \in N_{-k}$, $u_{kk} - u_{ik} > u_{ij} - u_{kj}$.*

Proof. If $i = j$ then the inequality follows from the assumption that (s_k, s_k) is risk-dominant. Thus assume that $i \neq j$. Consider the 'subgames' $\{s_k, s_j, s_i\} \times \{s_k, s_j, s_i\}$ and $\{s_k, s_j\} \times \{s_k, s_j\}$. Applying (**CD**) to the completely mixed equilibrium of $\{s_k, s_j\} \times \{s_k, s_j\}$, which exists since both (s_k, s_k) and (s_j, s_j) are strict, we have

$$\frac{u_{ik}(u_{jj} - u_{kj})}{u_{kk} + u_{jj} - u_{kj} - u_{jk}} + \frac{u_{ij}(u_{kk} - u_{jk})}{u_{kk} + u_{jj} - u_{kj} - u_{jk}} < \frac{u_{kk}u_{jj} - u_{jk}u_{kj}}{u_{kk} + u_{jj} - u_{kj} - u_{jk}}.$$

First cancelling denominators and then rearranging terms,

$$u_{ij} - u_{kj} < \frac{u_{jj} - u_{kj}}{u_{kk} - u_{jk}}(u_{kk} - u_{ik}).$$

Since (s_k, s_k) risk-dominates (s_j, s_j), $\dfrac{u_{jj} - u_{kj}}{u_{kk} - u_{jk}} < 1$. Therefore $u_{ij} - u_{kj} < u_{kk} - u_{ik}$. □

Here we verify that the stochastically stable equilibrium of the game in Figure 3, which is reproduced below, is (s_3, s_3).

	s_1	s_2	s_3	s_4
s_1	10, 10	11, 7	3, 8	0, 7
s_2	7, 11	18, 18	2, 2	−9, 16
s_3	8, 3	2, 2	6, 6	0, 0
s_4	7, 0	16, −9	0, 0	1, 1

Figure 5. The game in Firure 3.

First, we compute γ_{ij} for each ordered pair $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Notice that since the defining minimization problem for γ_{ij} is a linear program, we can use, for example, the simplex algorithm to compute γ_{ij}. We have following values.

$$\gamma_{12} = 3/10 = .3000 \qquad \gamma_{13} = 2/5 = .4000 \qquad \gamma_{14} = 9/22 \approxeq .4091$$
$$\gamma_{21} = 29/44 \approxeq .6591 \qquad \gamma_{23} = 31/40 = .7750 \qquad \gamma_{24} = 1/6 \approxeq .1667$$
$$\gamma_{31} = 13/40 = .3250 \qquad \gamma_{32} = 1/5 = .2000 \qquad \gamma_{34} = 11/24 \approxeq .4583$$
$$\gamma_{41} = 1/4 = .2500 \qquad \gamma_{42} = 5/6 \approxeq .8333 \qquad \gamma_{43} = 1/7 \approxeq .1429$$

Now assume that the sample size κ is so large that applications of addition (+) and integer operator ($[\,\cdot\,]$) (in any order) to γ_{ij}s preserve order between γ_{ij}s. Then we have following resistances r_{ij}.

$$r_{12} = [(3/10)\kappa] = [.3000\kappa] \qquad r_{13} = [(2/5)\kappa] = [.4000\kappa] \qquad r_{14} = [(9/22)\kappa] \approxeq [.4091\kappa]$$
$$r_{21} = [(5/12)\kappa] \approxeq [.4167\kappa] \qquad r_{23} = [(13/42)\kappa] = [.3095\kappa] \qquad r_{24} = [(1/6)\kappa] \approxeq [.1667\kappa]$$
$$r_{31} = [(13/40)\kappa] = [.3250\kappa] \qquad r_{32} = [(1/5)\kappa] = [.2000\kappa] \qquad r_{34} = [(11/30)\kappa] \approxeq [.3667\kappa]$$
$$r_{41} = [(1/4)\kappa] = [.2500\kappa] \qquad r_{42} = [(12/35)\kappa] \approxeq [.3429\kappa] \qquad r_{43} = [(1/7)\kappa] \approxeq [.1429\kappa]$$

In increasing order, we have

$$r_{43} < r_{24} < r_{32} < r_{41} < r_{12} < r_{23} < r_{31} < r_{42} < r_{34} < r_{13} < r_{14} < r_{21}$$

or

$$[.1429\kappa] < [.1667\kappa] < [.2000\kappa] < [.2500\kappa] < [.3000\kappa] < [.3095\kappa] < [.3250\kappa] < [.3429\kappa]$$
$$< [.3667\kappa] < [.4000\kappa] < [.4091\kappa] < [.4167\kappa].$$

Now it is easy to see that the unique minimum tree is

$$1 \to 2 \to 4 \to 3.$$

Therefore (s_3, s_3) is the stochastically stable equilibrium.

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