Discussion Paper No. 1120

EXISTENCE AND UNIQUENESS OF EQUILIBRIUM IN FIRST PRICE AUCTION AND WAR OF ATTRITION WITH AFFILIATED VALUES

by

Alessandro Lizzeri Nicola Persico

Northwestern University

Existence and Uniqueness of Equilibrium in First Price Auction and War of Attrition with Affiliated Values

Alessandro Lizzeri [*] Nicola Persico [†]

First version, October 20 1994. This version, March 1, 1995. [‡]

Abstract

We prove existence and uniqueness of equilibrium in a 2-bidder asymmetric first price auction with affiliated values. The techniques used to prove uniqueness are different from the ones used in analyses of private values environments. Moreover the proof of existence is constructive. For comparison purposes we also consider the war of attrition and show that there is a continuum of equilibria in that game

1 Introduction

There are two strands in the theoretical literature on auctions. One deals with the question of optimality, the other with bidder behavior under specific auction rules. This paper is an attempt at a contribution to this second branch of the literature.

First price sealed bid auctions are an important class of auctions. The case of independent private values with symmetric bidders has been studied fairly extensively ([13], [7]). The asymmetric case has recently been studied as well ([9], [5], [6]). There are existence, uniqueness and characterization results.

However, the assumption of independent private values is very restrictive. Milgrom and Weber [11] have introduced the concept of affiliation: this allows a discussion of a more general class of environments, in particular, cases in which a bidder's valuation for an object may depend on other bidders information and in which some correlation

[*]KGSM, Northwestern University, Evanston, IL 60208 *e-mail* lizzeri@merle.acns.nwu.edu

[†]Department of Economics, Northwestern University, Evanston, IL 60208 *e-mail* npersico@casbah.acns.nwu.edu

[‡]We would like to thank Asher Wolinsky for useful comments. The second author also wishes to thank MURST and Bocconi University for financial support.

between the signals of different bidders is present. However their paper only discusses symmetric equilibria and does not attempt a characterization of the equilibrium set.

We prove that, in the case of two bidders, equilibrium exists and is unique. Existence has already been shown very explicitly by [1] for the symmetric case: they solve the differential equations that characterize bidders' behavior. It turns out to be impossible to do this for the asymmetric case (the same is true in private values) but we are still able to provide a constructive proof of existence.

With respect to uniqueness, the affiliated values environment introduces one new dimension relative to the case of private values. Bidder 1 cares about how aggressive bidder 2 is in his bidding for two reasons: (a) it affects the probability of winning with any particular bid (this effect is also present in private values), (b) it affects the value of the object conditional on winning with such a bid (this effect only exists if there is some common value element). In the war of attrition with private values, effect (a) is sufficient to introduce additional equilibria in which one bidder is more aggressive than the other ([12]). On the other hand there is a unique equilibrium in undominated strategies in the second price auction and in the first price auction ([8], [5]). However, [10], [1] show that the combination of effects (a) and (b) leads to the existence of a continuum of equilibria in second price auctions with common values. Our objective is to show that in first price auctions multiplicity is not a problem even in comon value environments.

So far we were able to give a complete proof of uniqueness only for the case of independent signals. In the case signals are correlated we prove uniqueness in the class of non-decreasing strategies equilibria.

There is one big technical difference between our work and the papers that consider the question of existence and uniqueness in independent private values environments ([5], [8]). For our case, as well as in the case of private values, equilibrium is described by a pair of differential equations characterizing optimality for inverse bidding functions and an initial condition that describes the set of types who bid more than a reserve price r. In the case of private values, this entails a failure of the Lipschitz condition, and this is the technical challenge. In our environment, if r is such that there is a set of types that prefers not to bid, we obtain Lipschitz continuity in a natural way that suggests that this should be a common feature of models with some common value elements.

The proof of existence of equilibrium is not straightforward because existence of a solution to the pair of differential equations is not sufficient to guarantee existence of equilibrium. This is because there is an additional requirement imposed by equilibrium: the highest bid ever played has to be the same for both bidders. This requirement makes it harder to prove existence, but helps with uniqueness.

Proving uniqueness involves some work because, in contrast to the case of private values, the initial conditions are not uniquely determined by the fact that a bidder who values the object at r bids r: the value of bidding r for player 1 depends on which types of bidder 2 bid less than r. However, the presence of the "final condition" imposing equality of the highest bids for both bidders, allows us to pick a unique initial condition.

In order to show the importance of this final condition, we discuss the war of attrition with affiliated values (its symmetric version is analyzed in [4]). In this game, the differential equations characterizing bidder behavior are Lipschitz continuous and display unbounded bidding. This means that there is no additional final condition imposed by equilibrium. We show that there is a continuum of equilibria in this game. This highlights the importance of the final condition as opposed to the failure of the Lipschitz condition in guaranteeing uniqueness: the first price auction with independent private values has a unique equilibrium despite the failure of the Lipschitz condition because of the final condition, the war of attrition with affiliated values has a continuum of equilibria despite Lipschitz continuity. A paper that makes a similar point is that of [2]: this adds a final condition to the war of attrition with independent private values and obtains a unique equilibrium.

In section 2.1 we introduce the model for the first price auction, some notation and definitions and state two theorems: 1) the uniqueness result in two parts, first uniqueness in the class of equilibria with non-decreasing strategies, next uniqueness overall in the case of independent signals; 2) the existence result.

The proofs proceed through a sequence of lemmas some of which are useful to prove both theorems. We start by restricting to equilibria in non-decreasing strategies and first prove a result (often called "no gaps" in the literature) We then show continuity of equilibrium strategies, then the fact that they are strictly increasing. We then describe the set of types that does not actively bid above the reserve price r: this is defined by an "entry" condition that requires that every type obtain non-negative payoff in the bidding. This allows identification of a pair of types θ_1^r, θ_2^r, one for each bidder, that will function as initial conditions for a pair of differential equations. The set of these types is characterized and is potentially very large. We then prove that inverse bidding functions are differentiable for bids above r and this allows us to characterize equilibrium inverse bidding strategies as solutions to a pair of differential equations and an initial condition. We then prove a Lipschitz condition for inverse bidding strategies which allows us to prove that there is a unique solution for every initial condition. This is not enough because, as mentioned before, the set of candidate initial conditions is very large. However, we prove that the trajectories describing different candidate equilibria cannot cross. The last step uses the fact that the initial conditions are ordered in such a way that, if there were several equilibria, the trajectories would have to cross. We conclude the subsection with a result that relaxes the restriction to the class of non decreasing strategy equilibria: we show that in the case of independent signals, equilibrium strategies must be non-decreasing.

Section 2.3 is devoted to the proof of existence of an equilibrium in nondecreasing pure strategies. Because of the Lipschitz condition, existence of a solution to the pair of differential equations is not a problem. However we need to prove that there are initial conditions such that the highest bid for both bidders is the same. We use the structure of the set of initial conditions to prove this.

Section 3 considers the war of attrition. Again, we characterize equilibrium inverse bidding strategies as solutions to a pair of differential equations and initial conditions: we describe the set of candidate initial conditions and then prove that equilibrium bidding strategies are unbounded. This implies that there is no final condition that pins down a unique pair of types as initial conditions. Instead all candidate initial conditions can be associated with an equilibrium. Thus the war of attrition has a continuum of equilibria in non-decreasing strategies.

Section 4 concludes.

2 First Price auction

2.1 The model and our results

In this section, we introduce the model, the notation and some definitions.

We denote $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$.

Two players, called 1 and 2, are playing a sealed bid first price auction, with reserve price $r \in \mathbf{R}$. If a player bids less than r, he does not get the object, while the object is awarded whenever r is played. Ties are broken with a coin toss. The von Neumann-Morgenstern utility of the players is given by the value of the object minus the price paid. The value of the object to player i is $V^i(\theta_i, \theta_{-i})$, which is assumed to be strictly increasing in both arguments.

θ_i, θ_{-i} are real-valued signals, the first of which is known to player i, whereas the second is not. However, there is a joint probability distribution with c.d.f. F and density

$$f(\theta_1, \theta_2) \text{ on } [\underline{\theta_1}, \overline{\theta_1}] \times [\underline{\theta_2}, \overline{\theta_2}]$$

that is known to both players. We assume f to be "nice", in a sense to be clarified in the following. Moreover, we assume that (θ_1, θ_2) are affiliated, in the sense introduced by [11].

In our case, this means

$$\theta_1' \geq \theta_1 \text{ and } \theta_2' \geq \theta_2 \Rightarrow f(\theta_1', \theta_2')f(\theta_1, \theta_2) \geq f(\theta_1', \theta_2)f(\theta_1, \theta_2').$$

We shall work with V, but all our results can be taken to apply to a model in which $U^i(S, \theta_i, \theta_{-i})$ is the value of the object to player i. Here S would be an additional random variable, unobserved by either player, and θ_i, θ_j, S should be affiliated. We would then define $V(\theta_i, \theta_j) := E[V(S, \theta_i, \theta_j) \mid \theta_i, \theta_j]$.

Pure strategies are functions b_i,

$$b_i : [\underline{\theta}, \overline{\theta}] \to (-\infty, +\infty),$$

assumed to be measurable in the usual sense.

The profits to player θ_i of playing b when his opponent's strategy is $b_j(\cdot)$ will be

$$\pi(b, \theta_i, b_j) = \begin{cases} \int_{b_j(\theta_j) < b} [V(\theta_i, \theta_j) - b] f(\theta_j \mid \theta_i)\, d\theta_j + \\ + \frac{1}{2} \int_{b_j(\theta_j) = b} [V(\theta_i, \theta_j) - b] f(\theta_j \mid \theta_i)\, d\theta_j & \text{for } b \geq r \\ 0 & \text{for } b < r \end{cases}$$

It will sometimes be useful to rewrite this as

$$\begin{aligned} \pi(b, \theta_i, b_j) &= P\{b_j(\theta_j) < b \mid \theta_i\}\, E[V(\theta_i, \theta_j) - b \mid \theta_i, b_j(\theta_j) < b] + \\ &+ \frac{1}{2} P\{b_j(\theta_j) = b \mid \theta_i\}\, E[V(\theta_i, \theta_j) - b \mid \theta_i, b_j(\theta_j) = b] \end{aligned}$$

when $b \geq r$.

We now define mixed strategies, using the concept of behavioural strategy.

Let $\mathcal{B}$ be the class of Borel subsets of the real line, and let $A \in \mathcal{B}$. We will define the function $\eta_j(\cdot, \cdot)$ to be a behavioural strategy for player j if

$$\eta_j : \mathcal{B} \times [\underline{\theta}, \overline{\theta}] \to [0, 1]$$

is such that

1. $\eta_j(\cdot, \theta_j) : \mathcal{B} \to [0, 1]$ is a probability measure $\forall \theta_j \in [\underline{\theta}, \overline{\theta}]$.
2. $\eta_j(A, \cdot) : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ is measurable.

Definition 1 *A behavioural strategy $\eta_j(\cdot, \cdot)$ is* nondecreasing *if whenever $\theta_j' > \theta_j$, every element of the support of $\eta_j(\cdot, \theta_j')$ is greater or equal than every element of the support of $\eta_j(\cdot, \theta_j)$.*

Let $T \in \mathcal{B} \cap [\underline{\theta}, \overline{\theta}]$. Then we can define

$$\mu_j(A, T \mid \theta_i) := \int_T \eta_j(A, \theta_j) f(\theta_j \mid \theta_i)\, d\theta_j$$

to be the probability measure on the space of actions and types of player j induced by the behavioural strategy η_j, in θ_i's opinion.

Analogously, we define

$$\mu_i(A, \Gamma) := \int_\Gamma \eta_i(A, \theta_i) f(\theta_i)\, d\theta_i .$$

Now, we can also denote

$$\mu_j(A \mid \theta_i) := \mu_j(A, [\underline{\theta}, \overline{\theta}] \mid \theta_i) ,$$

the marginal distribution of bids induced by the strategy η_j in θ_i's opinion, and similarly

$$\mu_i(A) := \mu_i(A, [\underline{\theta}, \overline{\theta}]).$$

Denote $B_i = \sup\{b : b \in support\, \mu_i(\cdot)\}$, and let $B = B_1 \vee B_2$. Hence, B is the maximum bid that any of the players will ever play.

The payoff to type θ_i of bidding b when his opponent plays according to η_j is

$$\pi(b, \theta_i, \eta_j) = \int_{(-\infty, b) \times [\underline{\theta}, \overline{\theta}]} (V^i(\theta_i, \theta_j) - b)\mu_j(ds, d\theta_j \mid \theta_i) +$$

$$+ \frac{1}{2} \int_{\{b\} \times [\underline{\theta}, \overline{\theta}]} (V^i(\theta_i, \theta_j) - b)\mu_j(ds, d\theta_j \mid \theta_i).$$

It will be useful to rewrite this as

$$\pi(b, \theta_i, \eta_j) = \mu_j((-\infty, b) \mid \theta_i) E_{\theta_j}[V^i(\theta_i, \theta_j) - b \mid \theta_i, (-\infty, b) \times [\underline{\theta}, \overline{\theta}]] +$$

$$+ \frac{1}{2} \mu_j(\{b\} \mid \theta_i) E_{\theta_j}[V^i(\theta_i, \theta_j) - b \mid \theta_i, \{b\} \times [\underline{\theta}, \overline{\theta}]].$$

One last remark: in what follows, we use interchangeably the sentences "some event happens with positive probability" and "some event happens with positive probability in θ_i's opinion". The first is a statement about marginal probabilities, while the second is about conditional probabilities. We are allowed to do this for almost all θ_i's, because of assumption 1 below.

Let us state the assumptions we are going to make:

A 1 *$f(\theta_1, \theta_2) > 0$ in the interior of its domain.*

A 2 *For all i, $j \neq i$, $V^i(\theta_i, \theta_j)$ is strictly increasing in both arguments, and is continuous.*

A 3 *$E[V^i(\theta_i, \theta_j) \mid \theta_i = \underline{\theta}] < r$ for all i.*

A 4 *$f(\cdot, \cdot)$ is continuous.*

In the case of pure strategies, this definition reduces to $B_i = \sup_{\theta_i} b_i(\theta_i)$.

A 5 θ_1, θ_2 *are affiliated.*

A 6 $f(\theta_1, \theta_2) > 0$ *on* $[\underline{\theta}_1, \overline{\theta}_1] \times [\underline{\theta}_2, \overline{\theta}_2]$.

A 7 *For all* i, $f(\cdot,\cdot)$ *and* $V_i(\cdot,\cdot)$ *are of class* C^1.

Many of these assumptions are technical in nature. There is some repetition (e.g. A 6 is stronger than A 1) because the weaker version is enough to prove some of our results.

Assumption 5 was introduced in [11], it imposes a monotone structure on conditional expectations. Assumption 3 is important in the proof of Lemma 7. It ensures that there is a set of types of both players that does not bid actively. Any condition that guarantees this is sufficient for our results, and this assumption seems reasonable if the auctioneer sets a reserve price.

We now state the two main results of this section.

Theorem 1 *Assume A 1–A 7. If a nondecreasing strategies equilibrium for the first price auction exists, then*

1. *it is unique in the class of nondecreasing strategies equilibria, and it will be in pure strategies.*
2. *if in addition* θ_1, θ_2 *are independent, such an equilibrium is unique.*

Theorem 2 *Under A 1–A 7, an equilibrium in nondecreasing pure strategies exists for the first-price auction.*

2.2 Uniqueness

We prove Theorem 1 through a sequence of lemmas.

The first lemma is a result often called no gaps in equilibrium strategies.

Lemma 1 *Assume A 1–A 3. Consider any open interval* $(\alpha, \beta) \subseteq (r, B)$. *Then, in a nondecreasing strategies equilibrium,* $\mu_j((\alpha, \beta)) > 0 \quad \forall j$.

Proof: Suppose not. Then there is an open interval (α, β) such that either

I $\mu_i((\alpha, \beta)) > 0$ but $\mu_j((\alpha, \beta)) = 0$ for some $i \neq j$,

or

II $\mu_i((\alpha, \beta)) = 0$ for all i.

Case I : Let $b \in (\alpha, \beta)$ be a bid that is played by some θ_i; then player θ_i prefers $(b+\alpha)/2$ to b, since this does not lower his probability of winning, leaves unchanged the value when winning, and lowers his payment when winning. Since this can't happen in equilibrium, we obtain a contradiction.

Case II: Take the largest open interval containing (a, b), call it (γ, δ), such that $\mu(\gamma, \delta) = 0$. We must distinguish three subcases:

i) $\mu_i(\{\delta\}) > 0$ and $\mu_j(\{\delta\}) = 0$, i.e. there is an atom at δ in the distribution of player i's bids.

If this is the case, it must be true that there exists a positive measure of types of player i bidding δ with positive probability. But any such type will prefer to bid $(\delta + \gamma)/2$, since this does not lower his probability of winning, leaves unchanged the value when winning, and lowers his payment when winning. Contradiction.

ii) $\mu_1(\{\delta\}) \cdot \mu_2(\{\delta\}) > 0$, i.e. both players have an atom at δ.

Again, consider any type θ bidding δ with positive probability. For any such type, the expected value of winning the atom,

$$E_{\theta_j}[V_i(\theta_i, \theta_j) \mid \theta_j \in (\underline{\theta}_j, \overline{\theta}_j) \cap \{\delta\}],$$

must be strictly greater than δ (otherwise this type may deviate to $(\gamma + \delta)/2$, losing only the atom and saving money).

But, if this is the case, this same type would find it convenient to deviate to $\delta + \epsilon$, and gain the other 1/2 of the atom. This would also involve winning against the types of player j bidding in $(\delta, \delta + \epsilon)$ but, choosing ϵ small, this happens with negligible probability. The cost of such a deviation is negligible, but the gain is discrete. Contradiction.

iii) $\mu_1(\{\delta\}) \cdot \mu_2(\{\delta\}) = 0$.

By definition of δ, for all $\zeta > \delta$ there are an i and a positive mass of types $\tilde{\theta}$ such that $\eta_i((\delta, \zeta), \tilde{\theta}_i) > 0$. However, picking ζ arbitrarily close to δ implies that $\mu_j((\delta, \zeta), \tilde{\theta})$ is arbitrarily small. Hence, any such type $\tilde{\theta}_i$ is better off transferring the mass $\eta_i((\delta, \zeta), \tilde{\theta}_i)$ on $(\gamma + \delta)/2$, losing to a negligible mass of opponent's types and saving a nonnegligible cost. Contradiction.

We have exhausted all possible cases, and we have therefore proved the lemma.

Observe that in the proof we have made use of A.1, A.3 and the fact that strategies are nondecreasing, when we have implicitly assumed that "prefers" meant "strictly prefers" for the players. In fact, the three assumptions together guarantee that there is a mass of types of each player which bid below r. Thereby, the probability of winning when bidding at or above r is strictly positive. □

Remark: the above Lemma extends to all intervals with positive Lebesgue measure, observing that any such interval contains an open interval.

Remark: Lemma 1 implies, in particular, $B_1 = B_2$.

Remark: Lemma 1 implies that, in equilibrium, bidding starts at, or just above, r.

Corollary 1 *Assume A1–A3, and suppose we have an equilibrium in nondecreasing strategies. Then this equilibrium is in pure strategies.*

Proof: Straightforward, given Lemma 1 and the definition of nondecreasing strategies. □

We define $b_i^{-1}(\cdot)$ to be the (set-valued) **inverse correspondence** associated to $b_i(\cdot)$.

Corollary 2 *Assume A1–A3, and that equilibrium strategies are nondecreasing. Then equilibrium strategies $b_i(\cdot)$ are continuous on $b_i^{-1}((r, B])$.*

Proof: Observe that, by Corollary 1, we can restrict attention to pure strategies. Then the statement follows immediately from the assumption that strategies are nondecreasing, coupled with Lemma 1. □

Lemma 2 *Assume A1–A3, and that equilibrium strategies $b_i(\cdot)$ are nondecreasing. Then equilibrium strategies are strictly increasing on $b_i^{-1}((r, B])$.*

Proof: Suppose not. Then there is a $b \in (r, B]$ such that $P\{b_i(\theta_i) = b \mid \theta_j\} > 0$ for every θ_j.

If this is the case, we will show that no type of player i will want to bid in $(b - \epsilon, b]$ for some small ϵ, and this will contradict Lemma 1.

Considering the payoff of some θ_j who bids b,

$$
\begin{aligned}
\pi(b, \theta_j, b_i(\cdot)) = {} & P\{b_i(\theta_i) < b \mid \theta_j\} E(V(\theta_j, \theta_i) - b \mid \theta_j, b_i(\theta_i) < b) + \\
& + \frac{1}{2} P\{b_i(\theta_i) = b \mid \theta_j\} E(V(\theta_j, \theta_i) - b \mid \theta_j, b_i(\theta_i) = b),
\end{aligned}
$$

we can observe that the second expectation is (strictly) greater than the first, due to the monotonicity of $b_i(\cdot)$. Because θ_j must have nonnegative payoff in equilibrium, this implies that the second expectation, i.e. the profits of winning the atom, must be strictly positive. But, in this case, θ_j will prefer to raise his bid by a small amount, win the other 1/2 of the atom (plus something more with negligible probability) and pay negligibly more. The same reasoning applies, all the more, to any type θ_j bidding in $(b - \epsilon, b)$. □

Like Lemma 1, the previous result is typically proved in the independent private values case as well. Our proofs are somewhat more involved because the presence of the common value element means that a bidder has to condition on winning when determining the optimal bid.

Corollary 2 implies that every equilibrium inverse bidding correspondence will indeed be a function. This allows us to define the **inverse bidding functions** $\phi_i(\cdot)$, when $B > r$:

$$\phi_i : (r, B] \to [\underline{\theta}, \overline{\theta}], \qquad \phi_i(\cdot) = b_i^{-1}(\cdot).$$

We are aiming to a characterization of equilibrium as the solution to a pair of differential equations plus an initial condition. we now start to describe the set of potential initial conditions. Let θ_i^r be defined as

$$\theta_i^r = \lim_{b \downarrow r} \phi_i(b). \tag{1}$$

We can observe that, whenever strategies are nondecreasing, it will be true that $\theta_i^r = \sup\{\theta_i : b_i(\theta_i) \le r\}$.

Let us now characterize the pair (θ_1^r, θ_2^r) defined in (1).

Define

$$H_i(\phi_1, \phi_2) := E(V(\theta_1, \theta_2) \mid \theta_i = \phi_i, \theta_j \le \phi_j).$$

Lemma 3 *Assume A.1–A.3, and that equilibrium strategies are nondecreasing. Then an equilibrium pair (θ_1^r, θ_2^r) solves*

$$\begin{cases} H_i(\theta_1^r, \theta_2^r) = r \\ H_j(\theta_1^r, \theta_2^r) \ge r \end{cases} \quad \textit{for } i \ne j, \quad i, j = 1, 2. \tag{2}$$

Proof: First, observe that in equilibrium it must be true that

$$\begin{cases} E(V(\theta_1, \theta_2) \mid \theta_i = \theta_i^r, \theta_j \le \theta_j^r) \ge r \\ E(V(\theta_1, \theta_2) \mid \theta_j = \theta_j^r, \theta_i \le \theta_i^r) \ge r \end{cases} \quad \text{for } i \ne j, \quad i, j = 1, 2. \tag{3}$$

Indeed, suppose this was not the case for type θ_1^r, say, i. e. the first inequality in (3) does not hold: then there would be a θ_1' slightly greater than θ_1^r who, by definition of θ_1^r is bidding actively, and is getting an expected payoff arbitrarily close to

$$P\{\theta_2 \le \theta_2^r \mid \theta_1\} E(V(\theta_1, \theta_2) - r \mid \theta_1 = \theta_1^r, \theta_2 \le \theta_2^r),$$

which would be negative. But this cannot be, since then θ_1' would better off not bidding.

Let us now observe, to finish the proof, that in equilibrium it cannot be that both inequalities in (3) hold strictly. If this was the case, then there would exist types $\tilde{\theta}_1 < \theta_1^r$ and $\tilde{\theta}_2 < \theta_2^r$ who would have a positive payoff from bidding r, and hence would be doing so rather than bid below r, which is their only other possibility, given nondecreasing strategies. This implies that there would be an atom of player 1 types and an atom of player 2 types both bidding r: but this is impossible in a nondecreasing strategies equilibrium.

□

Remark: observe the role of A.3 in the above proof, in ensuring that θ^* satisfies $\theta_i^* > \underline{\theta}_i$ for all i.

We now state a result that will allow us to better describe the set of possible initial conditions.

Lemma 4 *Assume A.2–A.5. Then*

$$H_i(\phi_1,\phi_2) = E(V_i(\theta_1,\theta_2) \mid \theta_i = \phi_i, \theta_j \le \phi_j)$$

is increasing in (ϕ_1,ϕ_2).

Proof: The proof of this claim is an application of Theorem 5 in [11]. □

Lemma 3 simply says that the set of possible initial conditions are what we may call the **"North-East envelope"** of the level curves $H_1(\cdot,\cdot) = r$ and $H_2(\cdot,\cdot) = r$ in the $\phi_1 - \phi_2$ plane. More precisely, the set of possible initial conditions are the points where the level curves $H_1(\cdot,\cdot) = r$ and $H_2(\cdot,\cdot) = r$ cross, plus the points on the $H_1 = r$ curve but above the $H_2 = r$ curve, plus the points on the $H_2 = r$ curve but above the $H_1 = r$ curve. Fig. 1 gives a sketch of the North-East envelope. The horizontal distance between x and y represents the atom of player 1-types bidding r in an equilibrium with initial condition x.

Also, because of Lemma 4, the level curves $H_i(\cdot,\cdot)$ have negative slope. This means that their North-East envelope will have negative slope, too, so any two pairs of possible initial conditions, call them (θ_1^*,θ_2^*) and $(\hat\theta_1,\hat\theta_2)$ have the following property (see fig. 1):

$$\text{if } \theta_i^* > \hat\theta_i, \text{ then } j \neq i \text{ implies } \theta_j^* < \hat\theta_j. \qquad (4)$$

This observation will be useful in the proof of Lemma 8.

This concludes our description of the set of potential initial conditions.

Lemma 5 (Maskin and Riley '86). *Assume A.1–A.4, and that equilibrium strategies are nondecreasing. Then equilibrium inverse bidding functions are everywhere differentiable in the interior of their domain.*

Proof: Fix any $\theta_i \in (\underline{\theta}_i, \overline{\theta}_i)$, consider any increasing sequence $\{\theta_i^n\} \uparrow \theta_i$; let $b_i^n := b_i(\theta_i^n)$, $b_i := b_i(\theta_i)$. Continuity of $b_i(\cdot)$ (Corollary 2) implies $b_i^n \uparrow b_i$.

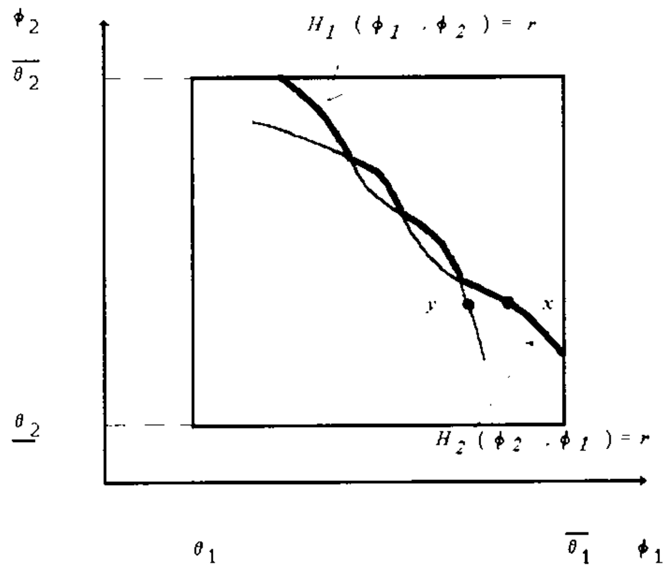

Figure 1: Level curves $H_i(\varphi_i, \varphi_j) = r$ and North-East envelope (thickened).

Because types θ^n prefer to bid b^n rather than b, we can write

$$\int^{\phi_j(b^n)} [V(\theta^n, \theta_j) - b^n] f(\theta_j | \theta^n)\, d\theta_j \geq \int^{\phi_j(b)} [V(\theta^n, \theta_j) - b] f(\theta_j | \theta^n)\, d\theta_j .$$

Subtracting $\int^{\phi_j(b^n)} [V(\theta^n, \theta_j) - b] f(\theta_j | \theta^n)\, d\theta_j$ from both sides, we obtain

$$\int^{\phi_j(b^n)} (b - b^n) f(\theta_j | \theta^n)\, d\theta_j \geq \int_{\phi_j(b^n)}^{\phi_j(b)} [V(\theta^n, \theta_j) - b] f(\theta_j | \theta^n)\, d\theta_j .$$

Dividing both sides by $b - b^n$, and taking limsup yelds

$$\int^{\phi_j(b)} f(\theta_j | \theta_i)\, d\theta_j \geq \limsup_{n \to \infty} \int_{\phi_j(b^n)}^{\phi_j(b)} [V(\theta^n, \theta_j) - b] f(\theta_j | \theta^n)\, d\theta_j \frac{1}{b - b^n} .$$

The RHS, by continuity of V and of f, can be replaced by

$$[V(\theta_i, \phi_j(b)) - b] f(\phi_j(b) | \theta_i) \limsup_{n \to \infty} \int_{\phi_j(b^n)}^{\phi_j(b)} d\theta_j \frac{1}{b - b^n} .$$

Therefore,

$$\limsup_{b^n \uparrow b} \frac{\phi_j(b) - \phi_j(b^n)}{b - b^n} \leq \frac{F(\phi_j(b) | \theta_i)}{[V(\theta_i, \phi_j(b)) - b] f(\phi_j(b) | \theta_i)} .$$

A symmetric reasoning leads to

$$\liminf_{b^n \uparrow b} \frac{\phi_j(b) - \phi_j(b^n)}{b - b^n} \geq \frac{F(\phi_j(b) | \theta_i)}{[V(\theta_i, \phi_j(b)) - b] f(\phi_j(b) | \theta_i)} .$$

Because the same exercise can be carried out choosing a decreasing sequence $\{\theta^n\}$ to θ, we conclude that $\phi_j(\cdot)$ is differentiable everywhere inside its domain. $\square$

We now characterize an equilibrium as a solution to a pair of differential equations and an initial condition.

Lemma 6 *Assume A.1–A.5, and that equilibrium strategies are nondecreasing. Then every equilibrium inverse-function pair $(\phi_1(\cdot), \phi_2(\cdot))$ solves*

$$\phi_i(r) = \theta_i^* \tag{5}$$

and

$$\phi_i'(b) = \frac{F(\phi_j(b) | \phi_i(b))}{f(\phi_j(b) | \phi_i(b))} \frac{1}{V(\phi_i(b), \phi_j(b)) - b} \qquad for\ b \in (r, B) \tag{6}$$

for $i, j = 1, 2$ and $i \neq j$.

Proof: Equation (5) is just a restatement of the definition of ϕ, equation (1). Equation (6) is derived from first-order conditions. Let us show equation (6).

Recall that the objective function of type θ when he plays b is

$$\int_{\underline{\theta}}^{\phi_j(b)} [V(\theta, \theta_j) - b] f(\theta_j|\theta)\, d\theta_j.$$

Because $\phi_j(\cdot)$ is differentiable (Lemma 5), we may take the derivative with respect to b, and equate it to 0.

$$\phi_j'(b)[V(\theta, \phi_j(b)) - b] f(\phi_j(b)|\theta) - F(\phi_j(b)|\theta) = 0, \tag{7}$$

which gives (6).

In order to make sure that (6) indeed picks out a maximum, let us consider the second order conditions; substitute (6) into (7), obtaining

$$\frac{F(\phi_j(b)|\phi_i(b))}{f(\phi_j(b)|\phi_i(b))} \frac{[V(\theta, \phi_j(b)) - b]}{V(\phi_i(b), \phi_j(b)) - b} f(\phi_j(b)|\theta) - F(\phi_j(b)|\theta).$$

This has the same sign of

$$\frac{F(\phi_j(b)|\phi_i(b))}{f(\phi_j(b)|\phi_i(b))} \frac{[V(\theta, \phi_j(b)) - b]}{V(\phi_i(b), \phi_j(b)) - b} - \frac{F(\phi_j(b)|\theta)}{f(\phi_j(b)|\theta)}. \tag{8}$$

Suppose now that type θ bids more than his equilibrium bid, i.e. $b > b_i(\theta)$. This is the same as saying $\phi_i(b) > \theta$. Recalling that V_i is strictly increasing in its first argument, and that $F(\cdot|\theta)/f(\cdot|\theta)$ is nonincreasing in θ by affiliation (see [11], Lemma 1), we observe that (8) is strictly less than 0. This means that it is convenient for θ to reduce his bid.

Reasoning similarly for the case when $b < b_i(\theta)$, we conclude that (6) indeed picks out a maximum. $\square$

We have then shown that all nondecreasing equilibria of this auction are characterized by the same differential equation, with possibly different initial conditions.

In what follows we shall sometimes write $\phi_i(\theta_1, \theta_2, b)$ to denote the trajectory solving (5) and (6).

Define

$$\overline{b}(\theta_1, \theta_2) := min\{b : \phi_1(\theta_1, \theta_2, b) = \overline{\theta_1} \text{ or } \phi_2(\theta_1, \theta_2, b) = \overline{\theta_2}\}.$$

The function $\overline{b}(\theta_1, \theta_2)$ may be seen as the "hitting time" of the first, among the two trajectories starting from (θ_1, θ_2), to reach the upper bound of its range.

We now come to the result that shows that trajectories have bounded derivative.

Lemma 7 *Assume A.1–A.6. Then any pair of trajectories described by (5) and (6), where the pair $(\underline{\theta}_1, \underline{\theta}_2)$ solves (2), have bounded derivative on $[r, \bar{b}(\underline{\theta}_1, \underline{\theta}_2)]$.*

Proof: Let $b \in [r, \bar{b}(\underline{\theta}_1, \underline{\theta}_2)]$: we shall prove that $\phi_i(\underline{\theta}_1, \underline{\theta}_2, b)$ has bounded derivative. This involves shoing that the denominator in the RHS of (6) is bounded away from 0. Because f is bounded away from 0 (A.6), it is enough to prove that $V_i'(\phi_i(b), \phi_j(b)) - b$ is, too. Let us do this.

By A.6 and A.3 there exists an $\epsilon > 0$ independent of b, such that

$$P\{\theta_j \le \phi_j(b) \mid \theta_i = \phi_i(b)\}[V_i'(\phi_i(b), \phi_j(b)) - b + \epsilon] >$$

$$P\{\theta_j \le \phi_j(b) \mid \theta_i = \phi_i(b)\}[E(V_i'(\theta_i, \theta_j) - b \mid \theta_i = \phi_i(b), \theta_j \le \phi_j(b))] \ge$$

$$\ge \lim_{b \to r} P\{\theta_j \le \phi_j(b) \mid \theta_i = \phi_i(b)\}[E(V_i'(\theta_i, \theta_j) - b \mid \theta_i = \phi_i(b), \theta_j \le \phi_j(b))] =$$

$$= E(V_i'(\theta_i, \theta_j) - r \mid \theta_i = \underline{\theta}_i, \theta_j \le \underline{\theta}_j) \lim_{b \to r} P\{\theta_j \le \phi_j(b) \mid \theta_i \le \phi_i(b)\} \ge 0$$

where the second inequality follows from the second order conditions in Lemma 6, the third from continuity of the expected values and (5). The last inequality from the fact that the pair $(\underline{\theta}_1, \underline{\theta}_2)$ solves (2). Because the "probability term" is strictly positive (recall the Remark after Lemma 3), the chain of inequalities proves the claim. □

Remark: In connection with Theorem 2, it is important to notice that Lemma 7 does not make use of equilibrium assumptions: we are going to show that only one of the trajectories is an equilibrium, the previous result says that all trajectories starting from one of the potential initial conditions has bounded derivatives.

Corollary 3 *Assume A.1–A.7. Then for every pair $(\underline{\theta}_1, \underline{\theta}_2)$ there is a unique pair $\phi_1(\cdot), \phi_2(\cdot)$ solving (5) and (6).*

Proof: This is a consequence of standard results about uniqueness of solutions to differential equations, of Lemma 7 and A.7 (see [3], pp. 162 ff.). □

An important implication of (4) is the following result.

Lemma 8 *Assume A.1–A.7, and suppose equilibrium strategies are nondecreasing. Let $(\phi_1(\cdot), \phi_2(\cdot))$ and $(\hat{\phi}_1(\cdot), \hat{\phi}_2(\cdot))$ be two distinct equilibria. Then $\phi_i(\cdot)$ never crosses $\hat{\phi}_i(\cdot)$.*

Proof: First, observe that there cannot be a b such that for all i $\phi_i(b) = \hat{\phi}_i(b)$, because of uniqueness for a given initial condition (Lemma 7 and Corollary 3).

Define

$$\hat{b} := \min_i\{b \mid \phi_i(b) = \hat{\phi}_i(b)\}.$$

Thus, $\hat{b}$ is the first b at which the two trajectories meet.

Without loss of generality, set $\phi_1(\hat{b}) = \hat{\phi}_1(\hat{b})$ and $\phi_1(b) < \hat{\phi}_1(b)$ for $b < \hat{b}$.

This means that ϕ_1 crosses $\hat{\phi}_1$ at $\hat{b}$ from below. In order for this to be possible, it must be that $\phi_1'(\hat{b}) \geq \hat{\phi}_1'(\hat{b})$. But this is not possible in equilibrium. In fact, observe that (1), together with the definition of $\hat{b}$, yield that $\phi_2(b) > \hat{\phi}_2(b)$ for $b < \hat{b}$ (the first inequality is strict by our initial observation).

But then we can write the following string of inequalities

$$\begin{aligned}
\hat{\phi}_1'(\hat{b}) &= \frac{F(\hat{\phi}_1(\hat{b})|\hat{\phi}_2(\hat{b}))}{f(\hat{\phi}_1(\hat{b})|\hat{\phi}_2(\hat{b}))} \frac{1}{V_2(\hat{\phi}_2(\hat{b}), \hat{\phi}_1(\hat{b})) - \hat{b}} = \\
&= \frac{F(\phi_1(\hat{b})|\hat{\phi}_2(\hat{b}))}{f(\phi_1(\hat{b})|\hat{\phi}_2(\hat{b}))} \frac{1}{V_2(\hat{\phi}_2(\hat{b}), \phi_1(\hat{b})) - \hat{b}} > \\
&> \frac{F(\phi_1(\hat{b})|\phi_2(\hat{b}))}{f(\phi_1(\hat{b})|\phi_2(\hat{b}))} \frac{1}{V_2(\phi_2(\hat{b}), \phi_1(\hat{b})) - \hat{b}} = \\
&= \phi_1'(\hat{b}),
\end{aligned}$$

where to get the inequality we have used $F(\cdot|x)/f(\cdot|x)$ decreasing in x (a consequence of affiliation, see [11]), and strict monotonicity of V_2. Contradiction.

Hence, there is no $\hat{b}$, and no i such that $\phi_i(\hat{b}) = \hat{\phi}_i(\hat{b})$. $\square$

We finally come to the proof of Theorem 1.

Proof of Theorem 1, part 1.

Proof: We shall proceed by contradiction. Suppose there are two equilibria, let us call them $\phi(\cdot)$ and $\hat{\phi}(\cdot)$.

Because of (1), their initial conditions are related in a specific way. In particular, assume without loss of generality that $\phi_2(r) \geq \hat{\phi}_2(r)$ and $\phi_2(r) > \hat{\phi}_2(r)$. Then (1) implies that $\phi_1(r) < \hat{\phi}_1(r)$.

To each equilibrium is associated a "final condition", meaning a highest bid, B (recall that the Remark following Lemma 1 points out that this B is common to both bidders). Because equilibrium strategies are increasing, it has to be the case that $\phi_i(\overline{a}_i) = B$ for all i. Let B be associated to $\phi(\cdot)$, and $\hat{B}$ to $\hat{\phi}(\cdot)$.

There are several cases to examine, according to the relative position of the quantities $\overline{a}_1, \overline{a}_2, B$ and $\hat{B}$.

1. $B = \hat{B}$

   This case can be ruled out since the differential equation system (6) is C^1 at B, by Lemma 7. Hence, using A.7, there is a unique trajectory departing (backward) from B and satisfying (6).

2. $\overline{a}_1 \geq \overline{a}_2$ and $\hat{B} < B$

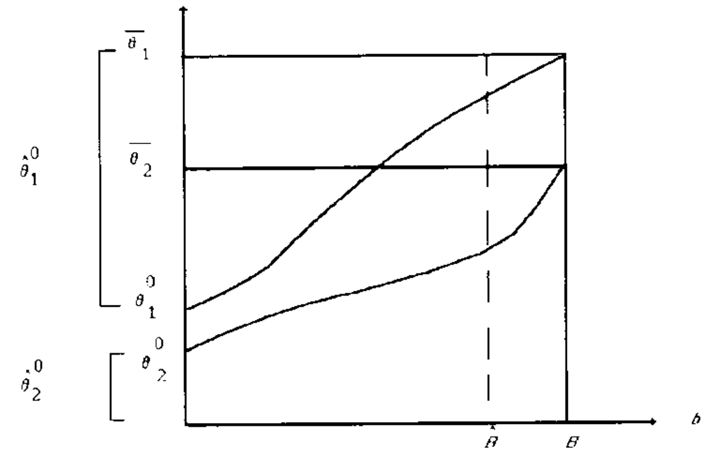

Figure 2: Inverse bidding functions

In order for this to be an equilibrium, $\hat\phi_2(\cdot)$ must cross ϕ_2 before $\hat B$, in order to reach $\overline{\theta_2}$ at $\hat B$. But this contradicts Lemma 8. See fig. 2

3. $\overline{\theta_1} \geq \overline{\theta_2}$ and $\hat B > B$

   In order for this to be an equilibrium, $\hat\phi_1(\cdot)$ must cross ϕ_1 before B, in order to reach $\overline{\theta_1}$ at $\hat B$. But this contradicts Lemma 8. See fig. 2

The cases in which $\overline{\theta_1} \leq \overline{\theta_2}$ are treated similarly.

Hence, it is not possible to have multiple equilibria.

$\square$

Remark: Observe that the proof relies heavily on the equality of the upper bounds of equilibrium bidding functions, B_1, B_2. This is a property of the equilibrium, not of the differential equations.

Next we prove a result about the case of independent signals. This will allow a full characterization of the equilibrium set for this particular case.

Lemma 9 *Assume A 1–A 3. Suppose θ_1, θ_2 are independent. Then equilibrium behavioural strategies are nondecreasing.*

Proof: We will prove that, if some type θ_i wins the object with some nonzero probability, then all types greater than θ_i will bid at least as much.

By contradiction, suppose $b \geq r$, and

$$\theta' > \theta \text{ but } support(\eta_i(\cdot,\theta')) \ni b' < b \in support(\eta_i(\cdot,\theta)).$$

Optimality of the bidding function $\eta_i(\cdot)$ implies

$$\pi(b,\theta,\eta) \geq \pi(b',\theta,\eta)$$

and

$$\pi(b',\theta',\eta) \geq \pi(b,\theta',\eta)$$

or, using independence,

$$\begin{aligned}
&\mu_j((-\infty,b))\,E_{\theta_j}(V(\theta_i,\theta_j) - b \mid (-\infty,b) \times [\underline{\theta}_j,\overline{\theta}_j]) + \\
&\quad \frac{1}{2}\,\mu_j(\{b\})\,E_{\theta_j}(V(\theta_i,\theta_j) - b \mid \{b\} \times [\underline{\theta}_j,\overline{\theta}_j]) \geq \\
&\geq \mu_j((-\infty,b'))\,E_{\theta_j}(V(\theta_i,\theta_j) - b' \mid (-\infty,b') \times [\underline{\theta}_j,\overline{\theta}_j]) + \\
&\quad \frac{1}{2}\,\mu_j(\{b'\})\,E_{\theta_j}(V(\theta_i,\theta_j) - b' \mid \{b'\} \times [\underline{\theta}_j,\overline{\theta}_j])
\end{aligned}$$

and

$$\begin{aligned}
&\mu_j((-\infty,b'))\,E_{\theta_j}(V(\theta',\theta_j) - b' \mid (-\infty,b') \times [\underline{\theta}_j,\overline{\theta}_j]) + \\
&\quad \frac{1}{2}\,\mu_j(\{b'\})\,E_{\theta_j}(V(\theta',\theta_j) - b' \mid \{b'\} \times [\underline{\theta}_j,\overline{\theta}_j]) \geq \\
&\geq \mu_j((-\infty,b))\,E_{\theta_j}(V(\theta',\theta_j) - b \mid (-\infty,b) \times [\underline{\theta}_j,\overline{\theta}_j]) + \\
&\quad \frac{1}{2}\,\mu_j(\{b\})\,E_{\theta_j}(V(\theta',\theta_j) - b \mid \{b\} \times [\underline{\theta}_j,\overline{\theta}_j]).
\end{aligned}$$

Switching the LHS with the RHS in the first inequality, and adding the second one gives

$$\begin{aligned}
&\mu_j((-\infty,b'))\,E_{\theta_j}(V(\theta',\theta_j) - V(\theta_i,\theta_j) \mid (-\infty,b') \times [\underline{\theta}_j,\overline{\theta}_j]) + \\
&\quad \frac{1}{2}\,\mu_j(\{b'\})\,E_{\theta_j}(V(\theta',\theta_j) - V(\theta_i,\theta_j) \mid \{b'\} \times [\underline{\theta}_j,\overline{\theta}_j]) \geq \\
&\geq \mu_j((-\infty,b))\,E_{\theta_j}(V(\theta',\theta_j) - V(\theta_i,\theta_j) \mid (-\infty,b) \times [\underline{\theta}_j,\overline{\theta}_j]) + \\
&\quad \frac{1}{2}\,\mu_j(\{b\})\,E_{\theta_j}(V(\theta',\theta_j) - V(\theta_i,\theta_j) \mid \{b\} \times [\underline{\theta}_j,\overline{\theta}_j]).
\end{aligned}$$

This inequality may be rewritten as

$$\int_{[\text{illegible}]} [V(\theta', \theta) - V(\theta, \theta)]\mu(ds, d\theta) +$$
$$\frac{1}{2} \int_{[\text{illegible}]} [V(\theta', \theta) - V(\theta, \theta)]\mu(ds, d\theta) \geq$$
$$\geq \int_{[\text{illegible}]} [V(\theta', \theta) - V(\theta, \theta)]\mu(ds, d\theta) +$$
$$\frac{1}{2} \int_{[\text{illegible}]} [V(\theta', \theta) - V(\theta, \theta)]\mu(ds, d\theta).$$

Recalling now that $b' < b$, the above is equivalent to

$$\begin{aligned} 0 \geq \quad & \int_{[\text{illegible}]} [V(\theta', \theta) - V(\theta, \theta)]\mu(ds, d\theta) + \\ & \tfrac{1}{2} \int_{[\text{illegible}]} [V(\theta', \theta) - V(\theta, \theta)]\mu(ds, d\theta) + \\ & \tfrac{1}{2} \int_{[\text{illegible}]} [V(\theta', \theta) - V(\theta, \theta)]\mu(ds, d\theta). \end{aligned}$$

But, strict monotonicity of V guarantees that the arguments of the integrals are strictly positive. To finish the proof, by reaching a contradiction, it is enough to prove that at least one the supports of integration on the RHS has nonzero μ-measure. However, this follows from the fact that, were it not so, type θ would strictly prefer to lower his bid to b', winning with the same probability and lowering his payment. Thus the RHS is strictly positive. Contradiction. □

Remark: the above argument also takes care of the mixed strategy issue, by showing that the minimal element in the support of a mixed strategy of a high type must be higher than the maximal element in the support of a mixed strategy of a low type. But then Lemma 1 implies that in equilibrium all strategies are pure.

The proof of Lemma 9 is similar to the corresponding one in the private-values case, but is slightly more involved in that we have to proceed by contradiction.

Proof of Theorem 1, part 2

Proof: Straightforward, given Lemma 9. □

2.3 Existence

In this section we show that an equilibrium exists, thereby proving Theorem 2.

We state the result we want to prove.

Under A 1–A 7, an equilibrium in pure strategies exists, and is characterized by (2), (5) and (6).

Notice that, in order to prove existence, it is not enough to show that a solution to the differential equations exists. Indeed, equilibrium strategies must also "arrive together". In other words, an equilibrium, besides solving (5) and (6) for a suitable set of initial conditions, must have the property that $\phi_1(b) = \overline{\theta_1}$ and $\phi_2(b) = \overline{\theta_2}$ for the same b.

We will call **candidate initial conditions** for some trajectory, a pair (θ_1^*, θ_2^*) such that local optimality is satisfied for all types below them, meaning that all such types are happy not to bid slightly above r, whenever the bidding above r is described by those trajectories.

Lemma 10 *Assume A 1–A 7. Then all interior points of the North-East envelope of the level curves $H_1 = r, H_2 = r$ are candidate initial conditions for a pair of trajectories solving (6).*

Proof: For a given pair (θ_1^*, θ_2^*), choose any $b \in (r, \overline{b}(\theta_1^*, \theta_2^*))$. Let $\theta_i < \theta_i' < \theta_i^*$, where θ_i' is chosen so that $b_i(\theta_i') < b$.

Affiliation and monotonicity of V_i give the following inequality:

$$\phi_j'(b)[V_i(\theta_i, \phi_j(b)) - b] - \frac{F(\phi_j(b)|\theta_i)}{f(\phi_j(b)|\theta_i)} <$$

$$\phi_j'(b)[V_i(\theta_i', \phi_j(b)) - b] - \frac{F(\phi_j(b)|\theta_i')}{f(\phi_j(b)|\theta_i')}.$$

Rewriting,

$$\frac{f(\phi_j(b)|\theta_i')}{f(\phi_j(b)|\theta_i)}\left\{\phi_j'(b)[V_i(\theta_i, \phi_j(b)) - b]f(\phi_j(b)|\theta_i) - F(\phi_j(b)|\theta_i)\right\} <$$

$$\phi_j'(b)[V_i(\theta_i', \phi_j(b)) - b]f(\phi_j(b)|\theta_i') - F(\phi_j(b)|\theta_i').$$

But the RHS of this inequality is negative, by choice of θ_i' (it is the derivative of his payoff at b). Therefore θ_i does not want to bid b, since the derivative of his payoff at b is negative too. $\square$

Proof of Theorem 2

Proof: First, it is clear that the function $\overline{b}(\theta_1^*, \theta_2^*)$ is continuous, since the functions $\phi_i(\theta_1^*, \theta_2^*, b)$ are continuous in their first two arguments (see [3]).

Our problem, in view of Lemma 10, reduces to finding a pair of initial conditions (θ_1^*, θ_2^*) on the North East envelope such that

$$\phi_1(\theta_1^*, \theta_2^*, \overline{b}(\theta_1^*, \theta_2^*)) = \overline{\theta_1} \text{ and } \phi_2(\theta_1^*, \theta_2^*, \overline{b}(\theta_1^*, \theta_2^*)) = \overline{\theta_2}.$$

So, consider the function

$$F(\theta_1, \theta_2) := [\overline{\theta_1} - \phi_1(\theta_1, \theta_2, \bar{b}(\theta_1, \theta_2))] - [\phi_2(\theta_1, \theta_2, \bar{b}(\theta_1, \theta_2)) - \overline{\theta_2}].$$

We are looking for zeros of this function lying on the North East envelope.

Observe that, for points on the North East envelope with θ_1 close enough to $\overline{\theta_1}$, $F(\theta_1, \theta_2)$ is negative, since $\phi_2' < \infty$ and $\phi_1' > 0$. For points on the North East envelope with θ_2 close enough to $\overline{\theta_2}$, $F(\theta_1, \theta_2)$ is positive, reasoning symmetrically.

Also, it is clear that $F(\cdot, \cdot)$ is a continuous function, because of continuity of trajectories with respect to initial conditions (see [3]).

But then, there is a point $(\tilde{\theta}_1, \tilde{\theta}_2)$ on the North East envelope such that

$$F(\tilde{\theta}_1, \tilde{\theta}_2) = 0.$$

Then $(\tilde{\theta}_1, \tilde{\theta}_2)$, coupled with the system (5) and (6), is an equilibrium for the auction.

□

3 War of Attrition

In this section we want to compare the results obtained for the First Price auction with those for the War of Attrition, which we are going to sketch below.

Because our results are negative, we are satisfied with spelling out the case of "independent common values". In the following we shall describe the model, and outline the proof of a theorem implying a continuum of equilibria for the War of Attrition.

Two players, called 1 and 2, are playing a War of Attrition with reserve price r: this means that each player bids a real number and, when his opponent's bid is lower than his, is awarded the object at the price of his opponent's bid (or r, whichever is higher). If the opponent's bid is higher than his, then the player does not get the object *and pays his own bid* if the latter is higher than r. If a player bids less than r, he does not get the object and pays nothing. The object is awarded whenever r is played, and ties are broken with a coin toss. The value that each player attaches to the object being auctioned is $V(\theta_1, \theta_2)$, and players are risk-neutral.

The assumptions on $V(\theta_1, \theta_2)$, θ_1, θ_2 are exactly the same as A 1 – A 7, except that we will restrict to the case of θ_1, θ_2 independent.

Pure strategies are functions b_i,

$$b_i : [\underline{\theta}_i, \overline{\theta}_i] \to (-\infty, +\infty),$$

assumed to be measurable in the usual sense.

The profits to player θ_i of playing b when his opponent's strategy is $b_j(\cdot)$ will be

$$\pi_i(b,\theta_i,b_j) = \begin{cases} \int_{b_j(\theta_j)<b} [V(\theta_i,\theta_j) - (b_j(\theta_j) \vee r)] f(\theta_j)\, d\theta_j + \\ + \frac{1}{2} \int_{b_j(\theta_j)=b} [V(\theta_i,\theta_j) - b] f(\theta_j)\, d\theta_j \\ -bP\{b_j(\theta_j) > b\} & \text{for } b \geq r \\ 0 & \text{for } b < r \end{cases}$$

Assume that there exists an equilibrium in strategies that are differentiable and strictly increasing over types who bid above r (we will presently construct one), and call $\phi(\cdot)$ its inverse bidding functions. In this case, the objective function becomes

$$\pi_i(b,\theta_i,b_j) = \int_{\underline{\theta}}^{\phi_j(b)} [V(\theta_i,\theta_j) - (b_j(\theta_j) \vee r)] f(\theta_j)\, d\theta_j - b[1 - F(\phi_j(b))].$$

Taking first-order condition for $b > r$ and rearranging yelds

$$\phi_j'(b) = \frac{1 - F(\phi_j(b))}{f(\phi_j(b))} \frac{1}{V(\phi_i(b),\phi_j(b))}. \tag{9}$$

Second-order conditions are readily verified to hold. Thus, this differential equation, if coupled with a suitable set of initial conditions, would describe an equilibrium in pure strategies.

In the war-of-attrition, too, it is possible to construct a set of candidate initial conditions on the North-East envelope, in the exact same way as done in Lemmas 3 and 10, which hold verbatim, if $H_i(\cdot,\cdot)$ is defined by

$$H_i(\phi_i,\phi_j) := P\{\theta_j \leq \phi_j\} E(V(\theta_i,\theta_j) \mid \theta_i = \phi_i, \theta_j \leq \phi_j)$$

and (9) replaces (6) in the statement of Lemma 10. Hence, all trajectories starting on the North-East envelope and described by (9) are candidate equilibria. In the first-price auction we disposed of this multiplicity making use of a final condition: no such procedure is allowed here. This is the import of the following

Lemma 11 *Any trajectory pair described by equation (9) displays unbounded bidding.*

Proof:

Let us rewrite (9) as follows

$$\frac{1}{V(\phi_i(b),\phi_j(b))} = \phi_j'(b) \frac{f(\phi_j(b))}{1 - F(\phi_j(b))} = \frac{\partial}{\partial b}(-\log(1 - F(\phi_j(b)))).$$

Integrating with respect to b up to c we obtain

$$\int_r^c \frac{1}{V(\phi_i(s),\phi_j(s))}\, ds = -\log(1 - F(\phi_j(c))) + \text{constant}.$$

Suppose now, by contradiction, that $b_j(\cdot)$ were bounded, that is $b_j(\theta) < K < \infty$. Then necessarily $F(\sigma_j(K)) = 1$ whence, evaluating the last expression at $c = K$, the RHS is infinite while the LHS is not. Contradiction. $\square$

So, in the absence of a final condition, all trajectories starting on the North-East envelope and described by (9) are equilibria of the war-of-attrition, and we have shown

Theorem 3 *The War of Attrition has a continuum of equilibria in nondecreasing pure strategies.*

This result may seem to contrast starkly with that obtained for a similar model in [4]. However, the uniqueness result they obtain is restricted to uniqueness among *strictly increasing* strategies equilibria.

4 Conclusions

We have proved existence and uniqueness of equilibrium in a 2-player asymmetric first-price auction with affiliated values. In doing so, we have generalized existing results in the context of private values. Our proof of existence is constructive. We have compared these results with those for the war of attrition, in which we find a continuum of equilibria, ranked according to the aggressiveness of the players: This latter feature seems to be a common thread across different models in this literature.

We have argued that the difference between the first-price auction and the war of attrition is due to the fact that the latter lacks a final condition. We believe this to be an insight that should prove valuable in the study of other auction mechanisms.

Lastly, we have provided a natural setting that may, we hope, accommodate further analyses on other mechanisms: one of its prominent features being that equilibrium strategies are Lipschitz-continuous, as a result of the common-value component.

References

[1] S. Bickchandani and J. G. Riley, Equilibrium in Open Common Value Auctions, *J. of Economic Theory*, **53** (1991), 101–130.

[2] D. Fudemberg and J. Tirole, A Theory of Exit in Duopoly, *Econometrica*, **54** (1986), pp. 943–960.

[3] M. Hirsch and S. Smale, "Differential Equations, Dynamical Systems and Linear Algebra" Academic Press, 1974.

[4] V. Krishna and J. Morgan, "An Analysis of the War of Attrition and the All-Pay Auction", MIMEO, September 16 1994.

[5] B. Lebrun. "First Price Auction: Properties of the Equilibria in the Asymmetric N Bidders Case", MIMEO, March 29 1994.

[6] R. Marshall and M. Meurer and J. F. Richard, and W. Stromquist, "Numerical Analysis of Asymmetric First-Price Auctions" MIMEO 1992.

[7] E. Maskin and J. G. Riley. "Existence and Uniqueness in High Sealed Bid Auctions", MIMEO 1986.

[8] E. Maskin and J. G. Riley. "Equilibrium in High Price Sealed Bid Auctions", MIMEO (1992).

[9] E. Maskin and J. G. Riley. " Asymmetric Auctions", MIMEO (1993).

[10] P. Milgrom, Rational Expectations, Information Acquisition and Competitive Bidding, *Econometrica* **49** (1981), 921–943.

[11] P. Milgrom and R. Weber, A Theory of Auctions and Competitive Bidding, *Econometrica* **50** (1982), 1089–1122.

[12] B. Nalebuff and J. G. Riley, Asymmetric Equilibria in the War of Attrition, *J. of Theoretical Biology*, **113**, 517–527.

[13] J. G. Riley and W. Samuelson, Optimal Auctions, *American Economic Review* **71** (1981), 381–392.