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A LINEAR PROGRAMMING FRAMEWORK FOR NETWORK GAMES

by

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and

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ABSTRACT

In this paper we present a linear programming game that is motivated by the assignment game of Shapley and Shubik. This new game is a very natural generalization of many of the network optimization games that have been well studied in the past. We first show that for this general class of games the core is nonempty. In fact any dual optimal solution of the underlying linear programming problem gives rise to a core allocation. We also show that for a particular subclass of games (which include the assignment, max flow and location games) the core exactly coincides with the set of optimal dual solutions. Additionally we study the relationship between this linear programming game and the production game of Owen.

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1. Introduction

Cooperative games that arise from various optimization problems have been extensively studied. Two very important works in this area are the papers by Shapley and Shubik [1972] and Owen [1975]. Shapley and Shubik consider an assignment game based on the matching problem in a bipartite graph. Each player controls a node of the graph and the value of a coalition is defined to be the maximum weight of a matching on the corresponding subset of the nodes. They show that the core is nonempty for this class of games and further that the set of core vectors coincides with the set of optimal dual solutions of the assignment problem. Owen introduced a production game based on a linear programming formulation. In this case each player i controls a vector b(i) and the value of a coalition is the value of the linear program where the right hand side is the sum of the appropriate b(i)'s. For Owen's game, any optimal solution of the dual LP will generate a core solution, though the two sets may not coincide. The assignment game is one of many special cases of this linear programming game. A great deal of additional work has been done on optimization games. Many of these are based on network optimization problems. See for example Granot [1986], Dubey and Shapley [1984], Kalai and Zemel [1982a], Kalai and Zemel [1982b], Granot and Hojati [1990], Deng and Papadimitiou [1994] and Tamir [1989],[1991] and [1992].

In this paper we consider a new linear programming game that is motivated by Shapley and Shubik's assignment game. This new game seems in some ways to be a more natural LP generalization of network based games than the Owen model. As is well known, the assignment problem can be modeled as a linear program where the constraints correspond to nodes and the variables correspond to edges. The assignment game has an interesting interpretation on this linear program with the players now corresponding to the constraints of the LP. In the assignment game the value of a coalition is obtained by eliminating nodes not in the coalition and solving the resulting assignment problem. In the LP this is equivalent to eliminating the constraints (nodes) not in the coalition as well as the variables (edges) incident with any such constraint. This leads us to a generalization of the Shapley and Shubik game that can be defined on any linear program.

In many linear programming models of network problems, the constraints correspond to nodes of the network, so this LP game provides a very natural generalization of many network games in which players control nodes of the network. Other examples of such network games can be found in Kalai and Zemel [1982a] and [1982b], Granot [1986] and Tamir [1991]. In fact the class of games that this model generalizes is much larger as will be shown later.

In the next section we will introduce the details of this LP model. In Section 3 we will present some results related to this model. In particular, we will examine the relationship between core vectors and optimal dual solutions. Section 4 is devoted to cases were the core and the set of dual optimal solutions coincide. Section 5 deals with games that arise from integer linear programs. In section 6 we look at the relationship between this model and Owen's model.

2. The Linear Programming Game

In this section we describe the Linear Programming Game (LPG) to be considered in this paper. We begin with some basic definitions and notation. A *game* (a cooperative game with transferable utility) is denoted by an ordered pair (N,v) where $N=\{1,..,n\}$ is a nonempty finite set of players and v , the characteristic function, is a real valued function defined on the set 2^N of subsets of players with $v(\emptyset)=0$. The value v(S) expresses the worth or profitability of the coalition $S\subseteq N$.

The game is called :

- *monotonic increasing (decreasing)* if $v(S) \geq (\leq)\ v(T)$ for every $T\subseteq S$.

- *super additive (sub additive)* if v(S) + v(T) ≤ (≥) v(S∪T) for every S,T such that S∩T=∅.
- *convex (concave)* if v(S) + v(T) ≤ (≥) v(S∪T) + v(S∩T) for every S,T ⊆N.

Given a game (N,v), a *feasible allocation,* $w \in \Re^N$, is a division of the worth of the grand coalition among its members, thus it must satisfy $\sum_{i \in N} w_i \leq v(N)$. An allocation is called *efficient* if it satisfies the above condition with equality.

The *Core* of a game (N,v) is the set of efficient allocations that can not be improved upon by any coalition i.e.

Core (N,v) = $\left\{ w \in \Re^N \mid \sum_{i \in N} w_i = v(N),\ \sum_{i \in S} w_i \geq v(S) \quad \forall S \subset N \right\}$.

Monotonically decreasing games are sometime referred to as *cost allocation* games. (One can view the characteristic function, denoted by c(S), as the cost associated with forming the coalition S). For those games we define the core as:

Core (N,c) = $\left\{ w \in \Re^N \mid \sum_{i \in N} w_i = c(N),\ \sum_{i \in S} w_i \leq c(S) \ \ \forall S \subset N \right\}$.

In games with nonempty cores the players have an incentive to cooperate and form the grand coalition N. By choosing a core allocation as a distribution of the worth v(N) (cost c(N)) we achieve a certain stability as the formation of smaller coalitions is discouraged.

Given a game (N,v), a collection **B** of coalitions S of N is said to be *balanced* if there exist positive real numbers δ_S, for S∈**B**, such that for each $i \in N$; $\sum_{S \in B,\, i \in S} \delta_S = 1$. The game (N,v) is *balanced* if for every balanced collection **B**, with balancing weights δ_S, $\sum_{S \in B} \delta_S \cdot v(S) \leq v(N)$. The game (N,v) is *totally balanced* if (S, v^S) is balanced for every S⊆ N, where v^S is the restriction of v to subsets of S. It was proved by Bondareva[1963] and Shapley[1971], that (N,v) is balanced iff it has a nonempty core.

The Model

Network optimization problems with independent decision makers can often be modeled as cooperative games with side payments. In these games players control (several) nodes and the arcs adjacent to them. The worth (cost) associated with every coalition is the optimal value of the optimization problem restricted to nodes and arcs that are controlled by the coalition. Some well known examples are the Shapley - Shubik assignment game, minimum cost spanning tree games, location games and max flow games.

In order to generalize these games, we consider the following linear optimization problem:

(P) Max $\{ cx \mid Ax \le b \, ; \, x \ge 0 \}$

Where $c \in R^p$, $b \in R^m_+$ and A is an $m \times p$ matrix whose elements are a_{ij}.

Using this problem one can construct a cooperative linear programming game (LPG) in the following way: Let $N=\{1,\dots,n\}$ be a set of players ($n \le m$). Each player controls a set of constraints (out of $Ax \le b$) such that every constraint is owned by one and only one player. The value v(S) of every coalition will be determined by solving (P) restricted to the set , m_S ,of constraints that are controlled by some member of S and using only variables which can not be found outside m_S , we denote this set of variables by I_S.

Formally, let A_j be the j^{th} row of A and:

$$(P_S) \qquad v(S) = \text{Max} \left\{ cx \middle| A_j \cdot x \le b_j \;\; \forall j \in m_s \, ; \, x_k = 0 \;\; \forall x_k \notin I_S \, ; \, x \ge 0 \right\}$$

Note that if A is the incidence matrix of a graph, controlling constraints is equivalent to controlling nodes and the restriction of using only arcs which are adjacent to members of S requires using variables which only appear in m_S . Throughout this paper we assume that (P) has an optimal solution. Note that if a coalition S controls no variables, then v(S)=0. We will also define v(S)=0 if (P_S) has no feasible solution.

In the case that A is non negative, v(S) will equal the value of the LP obtained by setting b_j to zero for every constraint j not controlled by S. Thus for these matrices our formulation coincides with the controlled programming problems of Dubey and Shapley [1984].

This framework is quite general as one can represent any totally balanced cooperative game using a maximization problem similar to (P). Given a game (N,v) let x_S be a variable associated with the coalition S and define the following linear program:

(G) $$\text{Max}\left\{\sum_S x_S V(S) \mid \sum_{S_i \in S} x_S = 1,\ x_S \geq 0\right\}$$

It is easy to see that (N,v) is totally balanced if and only if the game induced by (G) is identical to it.

The above LPG possesses the following properties:

- *monotonicity* - every feasible solution of (P_S) must be feasible in (P_T) for every $S \subseteq T$.
- *Super additivity* - Let S,T be disjoint ($S \cap T = \varnothing$) and let x^*_S , x^*_T be the optimal solutions to (P_S) and (P_T) respectively. From the definitions of v(S) and v(T) we know that $\left\{j \mid \left(x_S^*\right)_j > 0\right\} \cap \left\{j \mid \left(x_T^*\right)_j > 0\right\} = \varnothing$ hence $x_{S \cup T}^* \equiv x_S^* + x_T^*$ is feasible (but not necessarily optimal) in ($P_{S \cup T}$). In other words $v(S) + v(T) \leq v(S \cup T)$.

In the following sections we will focus on the core which is one of the most appealing and robust solution concepts used in cooperative cost allocation/surplus sharing models.

3. The Core of LP Games

In this section we will show that the core (LPG) is nonempty. In fact we show that any optimal dual solution to the underlying LP gives rise to an allocation in the core. Finally in some special cases we are able to characterize the entire core.

If we could show that (LPG) is convex we would know that it has a nonempty core (The Shapley value is a member of the core - Shapley [1971]). As the following example will illustrate the game might not be convex:

$$
\begin{aligned}
\text{Max}\quad & x_1+x_2+x_3 \\
\text{S.t.}\quad & x_1+x_2 \leq 2 \\
& x_1+x_2+x_3 \leq 3 \\
& x_2+x_3 \leq 2 \\
& x_1,\, x_2,\, x_3 \geq 0
\end{aligned}
$$

There are 3 players each controls exactly one constraint. Let S={1,2} , T={2,3} then S∪T=N and S∩T={2}. Clearly v(N)=3, v(S)=v(T)=2 and v({2})=0 so 4=v(S)+v(T) ≥ v(S∪T) + v(S∩T) =3 and the game is not convex.

Therefore we will use a different approach. Assume that there are n players and for now suppose that each controls a single constraint.

Theorem 1: (LPG) defined by (P) is balanced.

Note that every subgame of an (LPG) is also an (LPG), so Theorem 1 will imply that (LPG) is in fact totally balanced.

Proof: Let **B** be a balanced collection, with balancing weights δ_S and let x(S) be the optimal solution to (P_S), so $v(S) = \sum_{j=1}^{p} c_j \cdot x_j(S)$.

We have to show that $\sum_{S \in B} \delta_S \cdot v(S) \leq v(N)$.

$$\sum_{S\in B}\delta_S \cdot v(S) = \sum_{S\in B}\delta_S \sum_{j=1}^{p} c_j \cdot x_j(S) = \sum_{j=1}^{p} c_j \cdot \hat{x}_j \quad \text{where } \hat{x}_j \equiv \sum_{S\in B}\delta_S \cdot x_j(S).$$

It is enough to show that $\hat{x}$ is feasible (but not necessarily optimal) in (P), for then $v(N) \geq \sum_{j=1}^{p} c_j \cdot \hat{x}_j$.

To show that, notice that if A_k is the k[th] row of A, then clearly every coalition S that does not control the k[th] constraint can not use any variables which appear in it. In other words $A_k x = 0$ for every x feasible in (P_S), in particular $A_k \cdot x(S) = 0$.

Consider the following:

$$A_k \cdot \hat{x} = \sum_{j=1}^{p} a_{kj} \cdot \sum_{S\in B}\delta_S \cdot x_j(S) = \sum_{S\in B}\delta_S \cdot \sum_{j=1}^{p} a_{kj} \cdot x_j(S) = \sum_{S\in B}\delta_S \cdot A_k \cdot x(S) =$$

$$= \sum_{S\in B\, k\in S}\delta_S \cdot A_k \cdot x(S) \qquad (1)$$

$$\leq \sum_{S\in B\, k\in S}\delta_S \cdot b_k \qquad (2)$$

$$= b_k$$

Where (1) follows from the previous remark ($A_k \cdot x(S) = 0 \;\; \forall S | k \notin S$) and (2) is due to the feasibility of x(S) in (P_S) .

Hence $A \cdot \hat{x} \leq b$. $\hat{x}$ is just a positive linear combination of x(S)'s thus it is non negative and therefore $\hat{x}$ is feasible in (P). Q.E.D.

We can view (LPG) games as non cooperative games too. Each player charges a price to use their constraint. If all prices are compatible, i.e. they sum up to less than the worth of the grand coalition each player will get their price. If not no one gets anything. For this game every core allocation is a Nash equilibrium (even a strong one) and conversely every nontrivial strong Nash equilibrium must be in the core. Thus one can view the core as a set of stable prices in the non cooperative game. The non emptiness of the core is thus equivalent to the existence of strong equilibria.

We have proved that (LPG) is balanced and therefore has a nonempty core. It is, however, important to actually find allocations in the core.

To do this, we have to define the dual LP associated with (P):

(D) $\quad \text{Min}\ \{b^T y \mid A^T y \ge c\ ,\ y \ge 0\}$

We assumed the existence of an optimal solution for (P) therefore (D) must have an optimal solution call it y^*.

<u>*Theorem 2:*</u> $w \equiv (b_1 \cdot y_1^*, \cdots, b_n \cdot y_n^*)$ is in the core of (LPG) generated by (P).

<u>Proof:</u> The strong duality theorem of linear programming implies the efficiency of w. Since $\sum_{i=1}^{n} w_i = b' y^* = v(N)$. Let S be a coalition of size k<n. Without loss of generality, assume that S controls the first k constraints of A and that S can <u>not</u> use the first r $(x_1, x_2, \ldots, x_r)$. (Note that both these conditions can be satisfied by permuting the rows and columns of A.)

We have to show that $v(S) \le \sum_{i=1}^{k} b_i \cdot y_i^*$. Note that A must be of the form $A = \begin{pmatrix} F & G \\ H & 0 \end{pmatrix}$ where F and G are $k \times r$ and $k \times (p-r)$ matrices respectively, H is an $(n-k) \times r$ matrix with at least one non zero element in every column and 0 is an $(n-k) \times (p-r)$ matrix of zeros.

Define $\tilde{c} \equiv (c_{r+1}, \cdots, c_p)$, $\tilde{b} \equiv (b_1, \cdots, b_k)$ and $\tilde{x} \equiv (x_{r+1}, \cdots, x_p)$

to find v(S) we have to solve:

(P_S) $\quad Max\{\tilde{c} \cdot \tilde{x} \mid G \cdot \tilde{x} \le \tilde{b},\ \tilde{x} \ge 0\}$

or its dual

(D_S) $\quad Min\{\tilde{b}^T \cdot \tilde{y} \mid G^T \cdot \tilde{y} \ge \tilde{c},\ \tilde{y} \ge 0,\ \tilde{y} \in \Re^k\}$

Let $y^\# \equiv (y_1^*, \cdots, y_k^*)$, the feasibility of y^* in (D) implies

$$A^T \cdot y^* = \begin{pmatrix} F^T & H^T \\ G^T & 0 \end{pmatrix} \cdot y^* \ge c \quad or \quad G^T \cdot y^\# \ge \tilde{c}\ .$$

Hence $y^\#$ is feasible in (D_S), by applying weak duality we get :

$$v(S) \leq b' \cdot y^* = \sum_{i=1}^{k} b_i^T \cdot y_i^* = \sum_{i=1}^{k} w_i .$$ Q.E.D.

Some remarks:

1. In addition to showing that the core of (LPG) is not empty we are able to demonstrate an efficient way to find core allocations, i.e. every optimal solution of (D) induces a core allocation. In other words in order to find a core allocation one need not solve for the value of every coalition (Solving 2^N distinct LP problems) as is necessary in general for cooperative games, it suffices to solve just one.
2. The same cooperative game can be represented in many ways as an LP game. The following example will show that the core allocations induced by the dual solutions are a strict subset of the core and are very sensitive to the chosen LP.

   Consider a 2 player game, each controlling just one constraint:

$$\begin{aligned} \text{Max } & (1+\alpha)x_1+(2-\beta)x_2 + x_3 \\ \text{S.t.} \quad & 2x_1 + x_2 + x_3 \leq 1 \\ & x_1 + 2x_2 + x_3 \leq 1 \\ & x \geq 0 \end{aligned}$$

Where $0 \leq \alpha \leq \beta \leq 1$. Clearly $v(N)=1$, $v(\{1\})=v(\{2\})=0$ and $Core(N,v)=\{(w,1-w)|0 \leq w \leq 1\}$.

If we look at the Dual problem:

$$\begin{aligned} \text{Min } & y_1 + y_2 \\ \text{S.t. } & 2y_1 + y_2 \geq 1+\alpha \\ & y_1 + 2y_2 \geq 2 - \beta \\ & y_1 + y_2 \geq 1 \\ & y \geq 0 \end{aligned}$$

The dual optimal solutions constitutes the interval $[w,1-w]$ where $\alpha \leq w \leq \beta$.

By changing α and β we can get any sub interval of the core.

3. The proof of Theorem 2 can be modified to show that if each player, i, controls m_i constraints, then the allocation: $w_i = \sum_{j \in m_i} b_j \cdot y_j^*$ for i=1,...,n is in the core.

4. Again the same proof holds if we introduce equality constraints:

$$(P^=) \qquad \text{Max } \{ cx \mid A_1x \le b_1 \ ; \ A_2x = b_2; \ x \ge 0 \}$$

Where $c \in R^p$, $b_i \in R_+^{m_i}$ and A_i is an $m_i \times p$ matrix. And as in remark 3, players can control several constraints.

Many network problems have upper bounds on the capacity of the arcs, therefore we will introduce upper bounds to our, more general, framework. Consider the following underlying LP problem:

$$(P^u) \qquad \text{Max } \{ cx \mid Ax \le b \ ; \ x \le u; \ x \ge 0 \} \quad \text{where } u \in \Re_+^p.$$

Define a game in the same manner as before where each of the n players owns only constraints from $Ax \le b$. For now suppose that each player owns exactly one (m=n). Again we assume that (P^u) has an optimal solution. The value of a coalition will be determined as before. Let (D^u) be the corresponding dual problem:

$$(D^u) \qquad \text{Min } \{ b^Ty + u^Tz \mid A^Ty + Iz \ge c; \ y,z \ge 0 \}$$

and let (y^*, z^*) be an optimal dual solution.

In this case we will arbitrarily distribute the value of an upper bound dual variable $\left(u_j \cdot z_j^*\right)$ among those players whose primal constraint uses the associated variable x_j. More formally let $Z_{n \times p}$ be an arbitrary real valued matrix satisfying: $Z_{ij} \ge 0$ for every i,j; $a_{ij}=0 \Rightarrow Z_{ij}=0$ and $\sum_{i=1}^{n} Z_{ij} = u_j \cdot z_j^* \quad \forall j = 1,..,p$.

Theorem 3: $w_i \equiv b_i \cdot y_i^* + \sum_{j=1}^{p} Z_{ij}$ for i=1,..n constitutes a core allocation of the LPG generated by (P^u).

Note that the analogous statement applied to Owen's production game would not be true since adding upper bounds to the production game can create a game with an empty core.

Proof: The allocation w is efficient, since:

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n}\left(b_i \cdot y_i^* + \sum_{j=1}^{p} Z_{i,j} \right) = b' y^* + \sum_{j=1}^{p} u_j z_j^* = b' y^* + u' z^* = v_u(N)$$

where the last equality is due to the strong duality theorem.

Again as before assume (without loss of generality) that a coalition $S \subset N$ of size k controls the first k constraints and does not use the first r variables. The matrix A must have the same block formation as previously shown $A = \begin{pmatrix} F & G \\ H & 0 \end{pmatrix}$.

Using the same arguments as before we can show that $(y^\#, z^\#) \equiv ((y_1^*, \cdots, y_k^*), (z_{r+1}^*, \cdots, z_p^*))$

is a feasible solution to (D^u_s), namely it satisfies: $G^T y^\# + z^\# \geq \tilde{c}$.

The dual objective value is:

$$\tilde{b}^T y^\# + \tilde{u}^T \cdot z^\# = \sum_{i=1}^{k} b_i \cdot y_i^* + \sum_{j=r+1}^{p} u_j \cdot z_j^* = \sum_{i=1}^{k} b_i \cdot y_i^* + \sum_{j=r+1}^{p} \sum_{i=1}^{n} Z_{i,j}$$

Since A^T has a lower right block of zeros we know that

$Z_{ij}=0$ for $i = k+1,\ldots,n$ and $j = r+1,\ldots,p$;

hence

$$\sum_{j=r+1}^{p} \sum_{i=1}^{n} Z_{ij} = \sum_{j=r+1}^{p} \sum_{i=1}^{k} Z_{ij} \leq \sum_{j=1}^{p} \sum_{i=1}^{k} Z_{ij}$$

where the inequality follows from the non negativity of Z_{ij}.

Combining the last equations we get:

$$v(S) \leq \tilde{b}' \cdot y^\# + \tilde{u}' \cdot z^\# \leq \sum_{i=1}^{k} b_i \cdot y_i^* + \sum_{i=1}^{k} \sum_{j=1}^{p} Z_{,j} = \sum_{i=1}^{k} w_i$$

where the first inequality is due to weak duality.
This conclu des the proof as $\sum_{i \in S} w_i \geq v(S) \quad \forall S \subset N$. Q.E.D

Remarks:

1. Every dual solution generates a *set* of core allocations as we can divide $u' \cdot z^*$ arbitrarily between the participants of each constraint.

2. As before, nothing changes if we allow players to control several constraints or add equality constraints.

Public constraints, are constraints that are not controlled by any player but must be satisfied by any coalition . If we add public constraints we obtain similar results. The underlying LP is:

(P^{pub}) Max { cx | $Ax \le b_1$; $Bx \le 0$; $x \ge 0$ }

Where A,b and c are as before and B is a $q \times p$ matrix. To explicitly determine the value of a coalition S one must solve ($P^{pub}{}_S$):

$$v(S)= \text{Max } \left\{ cx| A_j \cdot x \le b_j \ \ \forall j \in m_s ;\ B_j x \le 0 \ \forall j = 1..q,\ x_k = 0 \ \ \forall x_k \notin I_S ;\ x \ge 0 \right\}$$

Where A_j,B_j are the j^{th} rows of A and B and m_S and I_S are defined as before. Let (y^*,e^*) be an optimal solution to (D^{pub}).

<u>*Theorem 4:*</u> $w \equiv (b_1 \cdot y_1^*, \cdots, b_n \cdot y_n^*)$ is in the core of (N,V_{pub})

<u>proof:</u> Mutatis mutandis as Theorem 2.

The statement of Theorem 4 will no longer be true if we allow the public constraints to have a nonzero right hand side.

To illustrate this consider the following example:

$$\begin{aligned} \text{Max } & x_1 + x_2 + x_3 \\ \text{S.t. } & x_1 + x_2 \le 2 \\ & x_2 + x_3 \le 2 \\ & x_1 + x_2 + x_3 \le 3 \\ & x \ge 0 \end{aligned}$$

There are 2 players each controls one constraint and let the third constraint be public then, v({1})=v({2})=2 ; v({1,2})=3 and the game has an empty core.

Comment:

For games with both upper bounds and public constraints the previous results still hold providing that every variable which has an upper bound has a nonzero entry in the controlled constraint matrix (A).

A network game may also represent some minimization problem on graphs.

(P^{min}) Min $\{cx|\ Ax\leq b,\ x\geq 0\}$

Where $c \in R^p$, $b \in R^m_+$ and A is an $m \times p$ matrix.

Clearly this defines a cost allocation game and the previous results and proof methods still hold. The same is true for

$(P^{min}{}_u)$ Min $\{cx|\ Ax\leq b,\ u\geq x\geq 0)$.

The results of this section generalize several known results. For example, as mentioned in the introduction, the Shapley and Shubik assignment game is a natural special case of (LPG). Also, Kalai and Zemel [1982b] consider a game based on maximum flows in networks where players control arcs of the network. The value of a coalition is the value of the maximum weighted flow through the induced subgraph controlled by that coalition. Kalai and Zemel show that the core of their game is nonempty and that core allocations can be obtained from dual optimal solutions of the standard max flow linear programming formulation.

To see that these results are special cases of the previous theorems, consider the standard LP formulation of max flow.

Given a graph G=(N, {s,t}, E , u}, Let A be the node arc incidence matrix of G with the rows for s and t deleted.

The Max Flow problem can be defined as :

$$\text{(P)} \qquad Max\left\{\sum_{Head(e)=t} c_e \cdot x_e \mid Ax \leq 0;\ x \leq u; x \geq 0\right\}$$

Where u is a vector of upper bounds on arc flows.

By associating with each constraint of $Ax \leq 0$ (i.e. each node of G) a player, we obtain the node version of the max flow game and the core solutions follow from Theorem 3.

Similarly, if we associate each constraint of $x \leq u$ (i.e. each arc of G) with a player and treat the $Ax \leq 0$ as public, we obtain the arc version of the game and the core solutions of Kalai and Zemel follow from Theorem 4.

Even more generally, we could consider a combination of the two max flow games in which players may own subsets of the arcs ($x \leq u$ constraints) **and** the nodes ($Ax \leq 0$ constraints) of G. The value of a coalition is the maximum flow using only arcs and nodes owned by the coalition. Theorem 3 gives us that dual LP solutions provide core allocations for this more general game. Furthermore, we can even allow a subset of the nodes and arcs to be publicly owned. The comment after Theorem 4 implies that so long as there is no public arc that joins two public nodes (s and t are considered public for these purposes) then core allocations can be obtained from optimal dual solutions.

In a similar manner, using standard LP formulations, Theorems 3 and 4 can be used to derive and generalize known results on other games. For example consider Minimum Cost Spanning Tree games. Those can be formulated in the following way:

(P^{MCST})

$$Min \sum_{(i,j)\in E} c_{ij} \cdot x_{ij}$$

$$S.t. \quad \sum_{\{j:(i,j)\in E\}} f_{ij}^{k0} - \sum_{\{j:(j,i)\in E\}} f_{ji}^{k0} = \begin{cases} 1 & if\ i = k \\ 0 & if\ i \neq k, 0, \end{cases} \qquad \forall k \neq 0, k \in N$$

$$0 \leq f_{ij}^{k1} \leq y_{ij}$$

$$y_{ij} \in \{0,1\}$$

where 0 is the root of the tree.

For each coalition $S \subseteq N$, let $c^*(S)$ be the optimal value of the LP relaxation of (P_S^{MCST}). Clearly $c^*(S) \leq c(S)$ for every $S \subseteq N$ and $c^*(N)=c(N)$ (Proven in Wong [1984]) thus we have $Core(N,c^*) \subseteq Core(N,c)$.

Using Theorem 4 we can find a core allocation For the MCST game by looking at the optimal solutions to the dual of the LP relaxation of (P^{MCST}). One can see that one of the dual optimal solutions corresponds to allocating to every player the cost of the unique arc, on the optimal tree, leaving his node. This allocation was the one given by Granot and Huberman [1981] in their original proof of the non emptiness of the core.

Other examples of results that can be obtained via these theorems are Tamir [1991] network synthesis games and Tamir [1989] Traveling salesman games. Using standard LP formulations the nonemptiness of the core of Steiner tree games and Shortest path games can also be proved.

4. Equivalence of the Core and Optimal Dual Solutions

In many of the games we have considered it is not difficult to show that the set of allocations produced by optimal dual solutions is in fact a strict subset of the core. In this section we will try to find sufficient conditions for the equivalence of the core and the set of allocations induced by dual optimal solutions. We first show that if A is a 0-1 matrix and the right hand side of the constraints is a vector of ones, then the core coincides with the set of optimal dual solutions.

Theorem 5: If A is a 0-1 matrix then every core allocation of the game induced by

$$(P^{01}) \qquad \text{Max } \{ cx \mid Ax \leq 1 \ ; x \geq 0 \},$$

is also a dual optimal solution.

Proof: Let $S^k \equiv \{i \mid a_{ik}=1\}$ where a_{ik} is the $(i,k)^{th}$ element of A. Then S^k is the minimal coalition (with respect to inclusion) that can use the variable x_k. Define $x_k^{\#}=1$; $x_j^{\#}=0$ for $j \neq k$. The vector $x^{\#}$ is feasible in the problem associated with S^k $\left(P^{01}_{S^k}\right)$. Hence $v(S^k) \geq c_k$. Now let w be a core allocation then by definition $\sum_{i \in S} w_i \geq v(S^k) \geq c_k$ in other words w satisfies the k^{th} dual constraint, by repeating the same argument for every k=1,...,p we show that w is dual feasible. Efficiency and strong duality imply that w is dual optimal.

Q.E.D.

Shapley and Shubik [1972] showed that for their assignment game, the core coincides with the optimal duals. Analogous theorems have been shown for location games (Tamir [1992]) and the path formulation of maximum flow games (Kalai and Zemel [1984]). These games are each defined by 0-1 LP's with a right hand side vector of ones, so the results follow directly from Theorem 5.

Recall that every totally balanced game can be represented in our framework by using a 0-1 matrix and a vector of ones for b:

$$\text{(G)} \qquad \text{Max}\left\{\sum_S x_S V(S) \mid \sum_{S i \in S} x_S = 1,\ x_S \geq 0\right\}$$

According to the last theorem, the set of dual optimal solutions and the core must coincide. The dual to this problem is :

$$\text{(D}^G\text{)} \qquad \text{Min}\left\{\sum_{i=1}^{n} y_i \mid \sum_{j \in S} y_j \geq v(S)\ \forall S \subseteq N\right\}$$

Which is just the familiar problem used to define the core of a balanced game.

Thus we can characterize the core of <u>every</u> totally balanced game as the set of dual optimal solutions to some LP, though with 2^N variables.

Define $$exc(S,x) \equiv v(S) - \sum_{i \in S} x_i$$

which is just the excess of S under the allocation x, and

$$\max exc(x) \equiv Max\{exc(S,x) | 0 \subset S \subset N\}$$

The Least Core (LC) of the game is the set of efficient allocations which have the lowest maximum excess, i.e.:

$$LC(N,v) \equiv \left\{ w \in \Re^n | \sum_{i\in N} w_i = v(n) \; ; \; \max exc(w) \le \max exc(w') \; \forall w' \; s.t. \sum_{i\in N} w'_i = v(n) \right\}$$

Note that the maximum excess of every core allocation is non positive, while the maximum excess of every efficient allocation outside the core is positive, hence $LC(N,v) \subseteq Core(N,v)$.

Theorem 6: If A is a 0-1 matrix then for the game induced by

(P^{01}) Max { cx | Ax≤1 ; x≥0 },

$LC(N,v_{01}) = Core(N,v_{01})$.

Proof: We will actually prove that the maximum excess of every core allocation is zero and therefore the core and the least core coincide. Assume that there exists a core allocation w, such that maxexc(w)<0 . Let $S^k \equiv \{i | a_{ik}=1\}$ where a_{ik} is the $(i,k)^{th}$ element of A. It was proved in Theorem 5 that $v(S^k) \ge c_k$ thus $\sum_{i\in S^k} w_i > c_k$. By repeating the same argument for every k=1,..,p we show that w obeys all the constraints in (D^{01}) with strict inequalities. This is a contradiction to the efficiency of w. Thus maxexc(w)=0 for every core allocation w.

For this game $LC(N,v_{01}) = Core(N,v_{01})$ = set of optimal solutions to (D^{01}) . Q.E.D.

The next theorem is similar to Theorem 5 but in the situation when public constraints are allowed. This theorem generalizes an equivalence result of Kalai and Zemel [1982b] and the proof method is similar to Samet and Zemel [1984]

Theorem 7: Let w be a core allocation of the game induced by

(P) Max { cx | Ax≤1 ;Bx≤0 ; x≥0 },

where A is an n×n positive matrix and B is a matrix of <u>public</u> constraints such that for every objective vector c, there exist an optimal solution x^* such that Ax^* is integer valued. Then there exists z^* s.t (w,z^*) is optimal in the dual LP,

(D) $\quad \text{Min} \{ 1^T y \mid A^T y + B^T z. \geq c \,;\, y,z \geq 0 \}$.

<u>Proof:</u> By making the substitution y=v + w in (D) we obtain the following equivalent LP:

(D^w) $\quad \text{Min} \{ 1^T v \mid A^T v + B^T z. \geq c - A^T w \,;\, v \geq -w;\, z \geq 0 \}$.

Let (y^*,z^*) be an optimal solution to (D), then $(v=y^*-w,z^*)$ is optimal for (D^w). Note also that since w is a core allocation, $1^T w$ is equal to $1^T y^*$ and thus the optimal objective value of (D^w) is zero. To prove the theorem we will show that there exists z^* such that $(v=0, z^*)$ is feasible for (D^w) which implies in turn that $(y=w,z^*)$ is optimal for (D) completing the proof of the theorem

In order to show the existence of a vector $(v=0,z^*)$ feasible for (D^w) we will show that the following LP,

$(P^{w'})$ $\quad \text{Max} \{ (c-wA)x \mid Ax \leq 1 \,; Bx \leq 0 \,;\, x \geq 0 \}$

has an optimal objective value of zero.

This will be sufficient since it implies that the dual problem,

$(D^{w'})$ $\quad \text{Min} \{ 1^T v \mid A^T v + B^T z. \geq c - A^T w \,;\, v \geq 0;\, z \geq 0 \}$.

Also has zero as an optimal objective value. Since $v \geq 0$ we conclude that the optimal solution of $(D^{w'})$ is $(0,z^*)$ for some z^* and thus $(0,z^*)$ is feasible for (D^w).

It remains only to show that $(D^{w'})$ has an optimal objective value of zero.

Assume to the contrary that the optimal solution is strictly positive. Let x^* be an optimal solution such that Ax^* is a 0-1 vector. Define $S \equiv \{ i \mid (Ax^*)_i = 1 \}$ then x^* is feasible in (P_s) (Note that A is positive) and $v(S) \geq cx^*$. Thus using the assumption we get:

$$0 < (c-wA)x^* = cx^* - wAx^* \leq v(S) - \sum_{i \in S} w_i$$

which contradicts the fact that w is in the core of v. Q.E.D.

Kalai and Zemel [1982b] showed that for the arc version of max flow games in which all arc bounds were set to one, every core allocation arose from some optimal dual solution. By setting A to be the identity matrix and B to be the incidence matrix, we see that this result is a special case of Theorem 7.

5. Integer Programming Games

To this point we have focused on games derived from linear programs. We note here, however, that cooperative games can be similarly defined when the base optimization problem is formulated as an integer program:

(IP) Max { cx | Ax≤b ; x≥0 ; x integer}

Unlike LP games, integer programming games do not necessarily have nonempty cores. Note, however, that if (P) is the LP relaxation of (IP), then $v_P(S) \geq v_{IP}(S)$ for every $S \subseteq N$. Thus in the case where $v_P(N) \geq v_{IP}(N)$, Theorem 2 implies that the optimal solutions to the dual of (P) are core allocations for the integer programming game. Results for IP games can also be obtained using the concept of a k-core (Kuipers [1994]).

The k-core of a game (N,v) is defined as:

$$\text{k-Core(N,v)} \equiv \left\{ w \in \Re^n \mid \sum_{i \in N} w_i = v(N) \ \text{and} \ k \cdot \sum_{i \in S} w_i \geq v(S) \ \forall S \subset N \right\}$$

Given a game (N,v), let k_{min}(N,v) be the minimum k for which k-Core(N,v) is nonempty. Let $k^* \equiv$ VLP/VIP where VIP and VLP are the values of the grand coalition in the integer program and its LP relaxation respectively.

Proposition 8: For every (N,v_{IP}) Game, $k_{min} \leq k^*$

Proof: Let y^* be an optimal solution to the dual of the LP relaxation, (D). Then by Theorem 2, $w \equiv (b_1 \cdot y_1^*, \cdots, b_n \cdot y_n^*)$ is in the core of (N,v_P). We claim that $\frac{1}{k^*}(w)$ is in the k^*-Core(N,v_{IP}). To see this, note that $\sum_{i \in N} \frac{1}{k^*} \cdot w_i = \frac{1}{k^*} v_P(N) = v_{IP}(N)$ and

$$k^* \cdot \sum_{i \in S} \tfrac{1}{k^*} \cdot w_i = \sum_{i \in S} w_i \geq v_P(S) \geq v_{IP}(S) \quad \forall S \subset N.$$

Hence the k^*-Core(N, v_{IP}) is not empty and $k_{min} \leq k^*$.

Theorem 9: If A is a 0-1 matrix, b is a vector of ones and in the game induced by

$$(IP^{01}) \qquad \text{Max } \{ cx \mid Ax \leq 1 \text{ ; } x \geq 0 \text{ ; } x \text{ integer} \}$$

at least one coalition (other then the grand one) has a non trivial value, then $k_{min} = k^*$.

Proof: Let y be a vector such that $\frac{1}{k_{min}}(y)$ is in the k_{min}-Core(N, v_{IP}^{01}). Note that the LP relaxation of this game is just (P^{01}) and define S^k as before. Using the notation of Theorem 5, $x^{\#}$ is clearly feasible in the integer problem associated with S^k. Thus $v_{IP}(S^k) \geq c_k$ and $k_{min} \cdot \sum_{i \in S} \frac{1}{k_{min}} \cdot y_i = \sum_{i \in S} y_i \geq v(S^k) \geq c_k$ where the first inequality is due to the definition of the k-Core. So y is a feasible solution for (D^{01}) and by weak duality $\sum_{i \in N} y_i \geq v_P(N)$. Hence $v_{IP}(N) = \frac{1}{k_{min}} \sum_{i \in N} y_i \geq \frac{1}{k_{min}} v_P(N) = \frac{k^*}{k_{min}} \cdot v_{IP}(N)$ or $k_{min} \geq k^*$. Proposition 8 completes the proof Q.E.D.

The above results generalize several known results on matching and assignment games. In a multi dimensional assignment problem we are given r sets $\{A_k\}_{k=1}^r$ and we need to use an element from each and every set in order to produce a certain finished product. The product's value depends on the specific parts used. The objective is to maximize the value of all finished goods produced. This can be formulated as an integer program. We can define a cooperative game in the natural way. Let $N = \bigcup_{k=1}^{r} A_k$ be the set of players and let the game be defined on the following LP,

$$(P^{GA}) \qquad \begin{aligned} &Max \quad \sum c_{a_1,\ldots,a_r} \cdot x_{a_1,\ldots,a_r} \\ &S.t \quad \sum_{a_{-j} \in N \setminus A_j} x_{a_{-j},a_j} \leq 1 \quad \forall a_j \in A_j \text{ ; } j = 1,\ldots,r \\ &\qquad x \in \{0,1\} \quad \forall a_1,\ldots a_r \in N; \end{aligned}$$

where each player controls one constraint.

If we relax the integer requirements in (P^{GA}) we know, from Theorem 5, that the core and the set of optimal solutions to (D^{GA}) coincide. Unfortunately with the binary constraints the core might be empty (See Kuipers [1994]). We can apply the k-core concept to this problem and using Theorem 9 we conclude that k_{min} , the minimal k for which the k-core is nonempty, must equal the ratio between the optimal values of the relaxed (P^{GA}) and the original one.

A different generalized assignment game is due to Curiel and Tijs [1986]. Here we have a group, B, of buyers and a group, S, of sellers. Each seller has r different objects to sell and each buyer needs all the objects but can decide form which buyer to purchase which object. Let $c_{i;k_1,\dots,k_r}$ be the total surplus from buyer i getting the first object from seller k_1 the second from k_2 etc. Define the following cooperative game:

$$(P^{r\text{-}GA}) \qquad \begin{aligned} &Max && \sum c_{i,k_1,\dots,k_r} \cdot x_{i,k_1,\dots,k_r} \\ &S.t && \sum_{k_1,\dots,k_r} x_{i,k_1,\dots,k_r} \le 1 \quad \forall i \in B \\ & && \sum_{i,k_{-j}} x_{i,k_1,\dots,k_r} \le 1 \quad \forall k_j \\ & && x \in \{0,1\} \quad \forall i \in B \;\; k_1,\dots k_r \in S^r ; \end{aligned}$$

As before, if we could relax the integer constraints we would conclude from Theorem 5 that the core and the set of optimal dual solutions coincide. However even if we keep the binary constraints, we can still conclude from theorem 9 that the minimal k such that the k-core is not empty is precisely the ratio between the optimal values of the relaxed problem to ($P^{r\text{-}GA}$).

The following example demonstrates that the core of the game might be empty:

Assume 2 buyers $\{b_1,b_2\}$, 2 sellers $\{s_1,s_2\}$ and 2 objects
let $c_{b_1;s_1,s_1} = c_{b_1;s_2,s_2} = c_{b_2;s_1,s_2} = c_{b_2;s_2,s_1} = 1$ while all the other transaction possibilities have zero surplus. Then clearly v(N)=1, $v(\{b_1,s_1\})= v(\{b_1,s_2\})= v(\{b_2,s_1,s_2\})= 1$.

The last value implies that b_1 must get 0 in every core allocation while the first two values imply that s_1 and s_2 must get at least 1 each and we reach a contradiction. In the LP relaxation the point (0.5,0.5,0.5,0.5) is in the core and is also an optimal dual solution.

Note that if we set r=2 in the two generalized games we get the original Shapley Subik [1972] assignment game. By using Theorem 4, we get the equality of the core and the set of dual optimal solutions for this simple game. Thus increasing the dimensionality of the problem, either by increasing the number of goods or the number of sets to be matched we can produce games possessing an empty core.

Another generalization of the assignment game is the Tamir [1994] b-Matching game. Given a graph G=(N,E) with weights c_e for every edge $e \in E$ and a number b_v for every node $v \in N$. Let players control nodes and define:

$$(P^{match}) \quad \begin{cases} Max \sum_{p \in P} c_e \cdot x_e \\ S.t. \quad \sum_{e:e \in \delta(v)} x_e \le b_v \quad \forall v \in N \\ x_e \ge 0 \ \ and\ integer. \end{cases}$$

Where $\delta(v)$ is the set of edges which meet v.

Note that if b_v=k for every $v \in N$ then Theorem 5 guarantees that every core allocation is k times some optimal solution of:

$$(D^{match}) \qquad Min\left\{ \sum_{v \in N} y_v \mid \sum_{v \mid e \in \delta(v)} y_v \ge c_e \quad \forall e \in E \ ; \ y \ge 0 \right\}$$

If b_v=1 for every $v \in N$ then the game reduces to the Shapley Shubik assignment game.

We now restrict our attention to b-matchings with a specific objective function. Let T=(N,E') be an undirected tree with positive edge lengths. For every pair of nodes $u,v \in N$ let d(u,v) be the length of the path on T connecting u and v. Let G=(N,E) be a complete graph and let c_e=d(u,v) for e=(u,v) and define a relaxed (P^{match}) (i.e. eliminate the integrality constraints). Tamir [1994] proved that if b_v is a positive integer for every $v \in N$. Then the relaxed (P^{match}) has an integer optimal solution. Using this fact and Theorem 1 we know that the core of the restricted b-matching game is not empty and if y^* is an optimal solution to the dual of the relaxed (P^{match}) then $w_v = b_v y_v^*$ is a core allocation. This is exactly Theorem 4.1 in Tamir [1994].

6. Production Games

In this section we consider the relationship between (LPG) and the production games of Owen and Granot. Consider the following model in which production of p different commodities is possible with the use of m available resources. Let a_{ij} be the amount of resource i needed to produce one unit of commodity j. For a given resource constraint vector $b \in \Re^m_+$ the *production possibility set,* $Y_b \equiv \{x \in \Re^p_+ | Ax \leq b\}$. If a price vector $c \in \Re^p$ is given, the maximal profit that can be made is given by the following LP:

Max $\{cx \mid x \in Y_b\}$. One can define an n-person cooperative game by letting every coalition $S \subseteq N$ control a resource vector b^S ($b^\phi = 0$) thus having a value of:

$$v(S) = \text{Max } \{cx | \ Ax \leq b^S;\ x \geq 0\}.$$

Owen [1975] defined an additive resource game where $b^S = \sum_{j \in S} b^{\{j\}} \quad \emptyset \neq S \subseteq N$

and proved that this game is totally balanced.

Granot [1986], generalized Owen's results to games with any balanced collection of resource vectors.

Notice that those instances of (LPG) for which the entries of A are all non negative, can easily be shown to be special cases of Owen's production game. To see this, for each player j, set $b_j^{\{j\}} = b_j$ *and* $b_k^{\{j\}} = 0 \ \forall k \neq j$. This transformation does not work however if A contains negative entries as is the case in many network models. On the other hand we will next show that the production game results of Owen and Granot can be derived from the results of Section 3.

Given a production game with a collection of resource vectors,(N,v_p) ,let y^N be an optimal solution to the dual problem of the grand coalition:

(D) $\quad \text{Min } \{b^{N}{}'y \mid A'y \geq c \text{ , } y \geq 0\})$.

Define the following game $w(S) \equiv b^S y^N$ for every $S \subseteq N$.

Clearly $w(N)=v_p(N)$ and $w(S) \geq v_p(S)$ for every $S \subseteq N$ (y^N is feasible in the dual LP for finding the value of the coalition S, but it is not necessarily optimal). Thus if we can show that (N,w) has a nonempty core so will (N,v_p), in fact $\text{Core}(N,w) \subseteq \text{Core}(N,v_p)$.

Two games (N,v) and (N,w) are said to be *equivalent* if there exist

$\alpha \in \Re_+$ *and* $\{\beta_j\}_{j=1}^{n}$ *such that* $v(S) = \alpha \cdot w(S) + \sum_{j \in S} \beta_j \quad \forall S \subseteq N$.

a well known result is that if $x \in \text{Core}(N,w)$ then $(\alpha x+\beta) \in \text{Core}(N,v)$.

Now define the underlying LP:

(P) $\quad \text{Max } \{cx_1+ \ \dots \ +cx_n \mid Ax_1 \leq b^{\{1\}} ,\dots, Ax_n \leq b^{\{n\}} \text{ ; } x_1,\dots,x_n \geq 0\}$

where A,b,c are as defined in the production game and $x_1,\cdots,x_n \in \Re_+^p$. Define a game (N,v') using our formulation where the j^{th} player controls the constraints associated with $b^{\{j\}}$. Let $y^s \in R^m$ be the optimal solution of $\text{Min}\{b^{s}{}'y \mid A'y \geq c \text{ , } y \geq 0\})$.

And we have $v'(S) = \sum_{j \in S} b^{\{j\}}{}' \cdot y^{\{j\}}$.

For the Owen game let $\alpha=1$ and $\beta_j=(y^N - y^{\{j\}}) \cdot b^{\{j\}}$ then

$$\alpha \cdot v'(S) + \sum_{j \in S} \beta_j = \sum_{j \in S} b^{\{j\}} \cdot y^{\{j\}} + \sum_{j \in S} b^{\{j\}} \cdot \left(y^N - y^{\{j\}}\right) = \sum_{j \in S} b^{\{j\}} \cdot y^N = b^S \cdot y^N = w(S)$$

hence v' and w are equivalent.

Theorem 2 asserts that $u \equiv (b^{\{1\}}{}'y^{\{1\}},\dots,b^{\{n\}}{}'y^{\{n\}})$ is in the core of (N,v') which implies that $(\alpha u+\beta) \in \text{Core }(N,w) \subseteq \text{Core}(N,v_p)$. But $(\alpha u+\beta)= (b^{\{1\}}{}'y^N,\dots,b^{\{n\}}{}'y^N)$ which is exactly the allocation given by Owen [1975].

For the Granot generalized production game with balanced resource vectors, let $t(i) \equiv (t_1(i),\cdots t_n(i)) \in Core(N,b_i)$ *for* $i = 1,\ldots,m$; and let y^N be as before optimal dual for the grand coalitions problem (P). Granot [1986] showed that the allocation $x_j = \sum_{i=1}^{m} t_j(i) \cdot y_i^N$ j=1,...,n is in the core.

Let α=1 and $\beta_j = \sum_{i=1}^{m}\left(t_j(i) \cdot y_i^N\right) - b^{\{j\}} \cdot y^{\{j\}}$; define w'=$\alpha$v'+$\beta$. Then we have

$$w'(S) = v'(S) + \sum_{j \in S} \beta_j = \sum_{j \in S} b^{\{j\}} \cdot y^{\{j\}} + \sum_{j \in S}\left(\sum_{i=1}^{m} t_j(i) \cdot y_i^N + b^{\{j\}} \cdot y^{\{j\}} \right) = \sum_{i=1}^{m} y_i^N \cdot \sum_{j \in S} t_j(i) .$$

Since t(i) is a core allocation, it follows that $\sum_{j \in S} t_j(i) \geq b^S$. Also w'(S)$\geq$w(S) for every S$\subset$N and w'(N)=w(N). In other words Core(N,w')$\subset$Core(N,w). Using Theorem 2 we know that (αu+β)$\in$Core(N,w')$\subseteq$Core(N,w)$\subseteq$Core(N,v_p).

But since $(\alpha \cdot u + \beta)_j = \sum_{i=1}^{m} t_j(i) \cdot y_i^N$, Granot's result follows.

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