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COLLUSION OVER THE BUSINESS CYCLE

by

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I. INTRODUCTION

Collusion is a balancing act. Each colluding firm balances the short-term temptation to cut its price against the expected long-term cost of the price war that such an act might instigate. When the level of demand grows and fluctuates through time, as along a business cycle, the relationship between the short-term temptation to cheat on a collusive agreement and the expected long-term cost from doing so need not be constant, and maintaining a balance between the two may require periodic adjustments in the collusive price. In this way, it is possible to forge a link between the state of the business cycle and the price level of colluding firms.

In a pioneering paper, Rotemberg and Saloner (1986) offer one such theory. Taking a simple but illustrative view of the business cycle, they assume that the level of market demand is determined in an iid fashion each period, so that the expected level of future demand - and thus the expected long-term cost from cheating - is independent of the current demand level. Today's demand level, however, does affect the short-term incentive to cheat, since a price cut is more attractive when many consumers are in the market. Associating a business-cycle boom (recession) with a period of high- (low-) demand realization, Rotemberg and Saloner then argue that collusion is most difficult in booms, when the incentive to cheat is greatest. Expanding on this insight, they conclude that for moderate values of the discount factor, collusive pricing is countercyclical, i.e., firms set a lower price in periods in which the level of demand is higher.[1]

Unfortunately, while analytically attractive, the iid assumption maintained by Rotemberg and Saloner rules out the possibility that a relatively high-demand realization today might signal an "upturn" in business conditions that leads firms to expect further growth in future demand. This lack of persistence, or indeed of any notion whatsoever of an "expansionary phase," compromises the interpretation of their paper as a theory of collusion over the business cycle.

Aware of this problem, Haltiwanger and Harrington (1991) and Kandori (1991) explore the robustness of Rotemberg and Saloner's conclusions to alternative assumptions about the

[1]Green and Porter (1983) consider an alternative theory of collusion, in which firms are unable to perfectly observe current demand conditions; in this case, low prices may be required during downturns. Staiger and Wolak (1992) reach an analogous conclusion by introducing capacity constraints into the Rotemberg-Saloner set-up.

pattern of demand fluctuations. Haltiwanger and Harrington relax the iid assumption by assuming instead that the level of demand follows a deterministic cycle.[2] Thus, in their framework, both the temporary incentive to cheat and the expected long-term cost of a price war change as the business cycle is traversed. Defining a boom (recession) as a sequence of periods over which demand is rising (falling), they show that collusion is now most difficult in recessions, since in such periods the cost of a future price war are smallest, and they demonstrate that collusive prices will thus be higher in booms, holding all else equal.

The general relationship between the collusive price and the level of current market demand is less clear, and Haltiwanger and Harrington are unable to provide complete analytic results on this matter. Simulations reveal that collusive prices are often procyclical; however, as the discount factor drops relative to the number of firms, collusive prices become increasingly countercyclical, much as Rotemberg and Saloner originally predicted. Further support for this conclusion is found in Kandori's work. He shows that the prediction of countercyclical pricing is robust to a class of correlated shocks to demand levels provided that the discount factor falls close to a specific value that is determined by the number of firms.

The Haltiwanger-Harrington and Kandori papers are provocative and instructive; however, neither paper offers a complete characterization of the collusive prices in terms of the stochastic parameters that are commonly understood to describe the business cycle.[3] This limitation might be ignored if the models' central predictions were nevertheless strongly confirmed by empirical studies, but in fact empirical evidence supporting both pro- and countercyclical pricing exists, and on the whole the evidence appears to be somewhat mixed and inconclusive.[4] This suggests that scope may still exist for sharpening the predictions of the

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[2]See also Montgomery (1988), who explores a similar model.
[3]See also Rotemberg and Woodford (1992), who provide further support for the prediction of countercyclical pricing. Working with a dynamic general-equilibrium model, their approach is to linearize around a steady state, and they are only able to establish results for small shocks.
[4]The empirical literature includes the cross-sectional work of Domowitz, Hubbard and Petersen (1986) and Rotemberg and Woodford (1992), as well as the industry studies of Borenstein and Shephard (1993), Chevalier and Scharfstein (1994), Ellison (1994), Porter (1983) and Rotemberg and Saloner (1986). See also Suslow (1988) for evidence that cartels are most likely to break down during recessions.

collusive theory and raises the possibility that the omitted stochastic features of the business cycle - the correlation of demand growth rates through time, the expected duration of booms and recessions, etc - may be important and not yet well-understood determinants of collusive prices.

Our goal in this paper is thus to offer a complete characterization of collusive pricing within an empirically attractive stochastic model of the business cycle. Our approach shares with Haltiwanger and Harrington an emphasis on persistent movements in demand and the critical role of cyclical turning points, but we depart from Haltiwanger and Harrington in assuming that turning points are unpredictable. Instead, we follow Hamilton (1989) in modeling the business cycle as the outcome of a Markov process that switches between two distinct states, one representing expansions and the other contractions. While the unpredictability of cyclical turning points enhances the empirical plausibility of the business cycle model,[5] it also greatly simplifies the analysis of the collusive prices, and we are able to completely characterize in a simple and intuitive way their cyclical properties.

In particular, we adopt a model of the business cycle in which the level of market demand alternates stochastically between slow- and fast-growth states, where the transition from one state to the other is determined by a Markov process. Thus, in contrast to Rotemberg and Saloner and Haltiwanger and Harrington, who define booms in terms of high and increasing levels of market demand, respectively, we refer to a boom phase as a sequence of periods of *fast growth* in the level of market demand. A recession phase then corresponds to periods of slower growth, and we allow - but do not insist - that recessions entail negative growth. Given this representation of the business cycle, a definition of cyclical pricing in terms of the level of market demand would be misguided, and we therefore instead say that collusive

[5]See, for example, Zarnowitz (1992) on the ample evidence the business-cycle turning points are difficult to predict. While Haltiwanger and Harrington's deterministic approach is thus not ideal for analyzing collusive pricing over the business cycle, it should be noted that the assumption of a deterministic cycle is much more appealing for collusive markets that are subjected to seasonal demand fluctuations. Borenstein and Shephard, for example, provide evidence in support of the Haltiwanger-Harrington model in the context of the retail gasoline market, where demand fluctuations are primarily seasonal and broadly predictable.

prices are procyclical (countercyclical) when they are higher in fast-growth (slow-growth) periods, i.e., in boom (recession) phases.

Within this context, we provide a complete characterization of the collusive prices as functions of a rich set of parameters, and establish a new role for the parameters that determine the extent of correlation in demand growth rates through time and the expected duration of boom and recession phases, respectively. Our main results are two: (1). collusive pricing may be procyclical (countercyclical) when market demand growth rates are positively (negatively) correlated through time, and (2). the amplitude of the collusive pricing cycle is larger when the expected duration of boom phases decreases and when the expected duration of recession phases increases. These predictions are in principle testable, and especially so given the methods that Hamilton and others have devised for estimating the underlying parameters.

With these basic results in place, we next consider more sophisticated business-cycle models; specifically, we maintain the assumption that the market demand level switches stochastically between fast- and slow-growth phases, but we expand the basic model to allow for random fluctuations in demand within given phases. By broadening our analysis in this way, we are able to assess the robustness of our main predictions regarding collusive pricing in boom and recession phases, identify new predictions, and integrate our research more closely with the previous "single-phase" models of collusion described above.

In our first formulation, within-phase shocks to the level of demand are fully embedded into the base from which demand growth occurs in subsequent periods, and so we refer to these shocks as permanent. When within-phase demand fluctuations are of this form, we show that collusive prices are completely unresponsive to within-phase demand shocks; in fact, collusive pricing is exactly the same as in the basic model in which within-phase fluctuations are absent.

A second formulation entails a combination of our modeling approach with that of Rotemberg and Saloner. In particular, we assume that an iid process generates shocks to the level of market demand in each period, and that such shocks occur outside of the Markov-growth process for demand, in that a shock to current-period demand has no affect on future-period demand levels. We thus refer to shocks of this form as being transitory. We then show that a higher transitory shock to demand results in a (weakly) lower collusive price, regardless of whether the market is in a boom or a recession phase. In this extended model, therefore, Rotemberg and Saloner's theory of collusive pricing can be interpreted in terms of the response of collusive prices to transitory demand shocks that occur within broader business cycle phases. We also demonstrate that our predicted association between procyclical (countercyclical) collusive prices and positively (negatively) correlated demand growth rates is robust to - and in fact strengthened by - the inclusion of transitory demand shocks.

A final remark concerns the significance of the relationship between the discount factor and the number of firms. This relationship emerged as a central determinant of the cyclical nature of collusive pricing in the literature reviewed above. In the present paper, however, demand follows a stochastic trend, sometimes growing fast and other times growing slow, and the relationship between the discount factor and the number of firms plays no role in determining whether collusive pricing will be pro- or countercyclical with respect to these distinct growth phases. Instead, the key determinants of the cyclical properties of collusive prices are the parameters defining the correlation of growth rates through time and the expected durations of boom and recession phases.

The paper is organized as follows. Section 2 lays out the basic assumptions of the oligopoly setting and considers the benchmark case of a market demand level that grows at a stationary rate. In Section 3, the basic growth model is developed and the incentive constraints for collusion are derived. The most-collusive prices are fully characterized in Section 4, while Section 5 extends our analysis of collusive pricing to business-cycle models that include within-phase shocks. Concluding remarks are offered in Section 6.

II. THE STATIONARY BENCHMARK

A. Basic Assumptions

We analyze a Bertrand-pricing supergame, in which a fixed set of $n \geq 2$ firms sells the same nondurable good in each period $t \in \{1,..., \infty\}$. The total mass or number of consumers in any period t is G_t, which is also called *the level of market demand* in period t. Within any period t, the firms select their respective prices simultaneously, and consumers observe these prices and divide up evenly over the lowest-priced firms. A firm earns zero profit for the period if it is not a lowest-priced firm, and it earns a profit of $\pi(P) = (P-c)D(P)$ per consumer when its price P is among the lowest, where $c \geq 0$ is a cost parameter and D(P) is the consumers' common demand function. We assume π has a unique maximizer, P_m, and that π is strictly increasing and differentiable over $P \in [0,P_m]$. Note that $\pi(P_m) > \pi(c) = 0$ and that the monopoly price P_m is acyclic, since its value is independent of the number of consumers to whom the firm sells. It is convenient to assume that firms select prices from the set $[0,P_m]$.

As is well known, in any Nash equilibrium of the Bertrand stage game all sales occur at the competitive price, $P = c$. Firms may be able to earn positive equilibrium profit in a dynamic model, however, as the short-term temptation to undercut rivals is then balanced against the long-term price war that such "cheating" might trigger. To model this idea, we assume that each firm observes all past prices, so that a firm's period-t price is a function of the prices charged by other firms in periods $\tau \in \{1,..., t-1\}$. We also introduce a common discount factor parameter, $\delta \in (0,1)$, as a measure of firms' patience with regard to future profit.

Finally, the tradeoff between short- and long-term profit is also affected by the current and the projected future levels of market demand. In other words, business cycle conditions can influence a firm's decision about whether or not to cheat on a collusive agreement. We do not develop an endogenous business cycle model here; rather, we assess the implications of exogenously-imposed business cycles for collusive pricing. In particular, we allow the state of the business cycle to determine the level of market demand, G_t. We assume further that each firm knows the current and all past levels of market demand when selecting current prices.

Formally, firm i's period-t strategy, σ_{it}, is a function that maps from the set of all possible past prices, all possible past market demand levels and the current period-t market demand level into the set of possible current prices, $[0,P_m]$. Firm i's objective is to select its strategy, $\sigma_i \equiv \{\sigma_{it}\}_{t=1,\ldots,\infty}$, to maximize its expected discounted profit, given the described consumer behavior, the process through which G_t evolves, and the strategies of opponent firms. Specifically, we require that firms select their strategies in a manner that generates a subgame perfect equilibrium.

We select among the set of such equilibria with two additional requirements. First, we assume that firms adopt symmetric strategies, so that $\sigma_i = \sigma_j$ for all $i \neq j$. Second, we characterize the most-collusive prices, which we define as the highest prices that can be supported in a symmetric subgame perfect equilibrium.[6] Following the arguments of Abreu (1986), we find such prices by supposing that a deviation induces a maximal punishment: if any firm i in any period t selects a deviant price $P \neq \sigma_{it}$, then in all future periods $\tau \geq t+1$ the firms' symmetric strategy specifies that pricing be competitive with $\sigma_{j\tau} = c$ for all $j = 1,\ldots,n$.

B. The Stationary-Growth Game

We begin with a very simple game, in which the level of market demand grows according to a stationary growth rate. Specifically, in the *stationary-growth game*, $G_0 > 0$ and $G_{t+1} = gG_t$, where $0 < \delta g < 1$. Thus, the level of market demand expands (contracts) over time if $g > 1$ ($g < 1$), and the stationary-growth game includes the familiar case of stationary demand as a special case when $g = 1$. The assumption that $\delta g < 1$ ensures that the discounted growth is finite.

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[6]Such prices always exist and are easy to characterize, because a higher equilibrium price in any one state raises the cost of a price war and thus makes possible higher equilibrium prices in other states as well.

A firm now faces a tradeoff between cheating today and sacrificing future profits in a market that grows at rate g. Since the growth rate is stationary, this tradeoff is the same at every date, and so a single most-collusive price, P^c, will be charged in all periods of a most-collusive equilibrium. To characterize this price, let

$$\Omega(P) = \pi(P) - \pi(P)/n = \pi(P)(n-1)/n$$

denote a firm's per-consumer incentive to cheat from a collusive agreement specifying that all firms select the price P. Note that a firm captures the entire market if it deviates and selects a price just below P. Next, let

$$\omega(P) = \pi(P)/n - \pi(c)/n = \pi(P)/n$$

give the per-period and per-consumer cost of a price war. When a firm defects, it sacrifices future profit as the subsequent price war forces the collusive price P to be abandoned and replaced with the competitive price c.

With these definitions in place, the central incentive constraint that each firm faces in any period t may be represented as:

$$G_t\Omega(P) \leq G_t \sum_{\tau=t+1}^{\infty}(\delta g)^{\tau-t}\omega(P),$$

which may be written in the simpler form, $\Omega(P) \leq [\delta g/(1-\delta g)]\omega(P)$. Since Ω and ω are both proportional to $\pi(P)$, it is easily verified that the incentive constraint holds for any $P > c$ if and only if $\delta g \geq (n-1)/n$. By contrast, when this inequality fails, the incentive constraint is violated for all $P > c$. We may summarize as follows: for the stationary-growth game, $P^c = P_m$ when $\delta g \geq (n-1)/n$ and $P^c = c$ otherwise.

The stationary-growth model is simple and instructive. It reveals that collusion is easier when firms are more patient (higher δ), the number of firms is smaller (lower n), and the growth rate of the market level of demand is higher (higher g); furthermore, the ability to collude is a discontinuous function of the model's parameters, with the key relationship being the sign of $\delta g - (n-1)/n$. With these points in mind, we turn next to a more complicated nonstationary model and explore the relationship between the most-collusive prices and the state of the business cycle.

III. THE MARKOV-GROWTH MODEL

A. Basic Assumptions

We now assume that the growth rate of the level of market demand is stochastic and determined by a Markov process. The purpose of this section is to formally define this Markov process and to derive and interpret the corresponding incentive constraints for collusion. A characterization of the most-collusive prices is deferred until the next section.

The level of market demand is assumed to grow at one of two possible rates. We say that period t is a *boom period* if $G_t = bG_{t-1}$ and that period t is a *recession period* if $G_t = rG_{t-1}$, where $1 > \delta b > \delta r > 0$. In other words, if g_t denotes the period-t growth rate, then period t is a boom (recession) period if $g_t = b$ ($g_t = r$). The transition between boom and recession periods is assumed to be governed by a Markov process, in which:

$$\rho \equiv \text{Prob}(g_t = r \mid g_{t-1} = b) \in [0,1]$$

$$\lambda \equiv \text{Prob}(g_t = b \mid g_{t-1} = r) \in [0,1]$$

$$\mu \equiv \text{Prob}(g_1 = b) \in [0,1]$$

Thus, ρ is the transition probability associated with moving from a boom to a recession, while λ is the transition probability corresponding to moves from recessions to booms. The parameter μ describes how the system begins. Assume further that $G_0 > 0$.

The parameters ρ and λ also may be interpreted in terms of the expected duration of boom and recession phases, respectively. Suppose that $g_{t-1} = r$ and $g_t = b$, so that a switch to a boom period occurs at period t, and define $t^* \equiv \min\{\tau > t \mid g_\tau = r\}$. We then define a *boom phase* as a sequence of boom periods, $\{t, ..., t^*-1\}$, and the *expected duration of a boom phase* is given by

$$\sum_{z=1}^{\infty} z\rho(1-\rho)^{z-1} = 1/\rho$$

In the same manner, we may define a *recession phase* and derive that the *expected duration of a recession phase* is $1/\lambda$.

With the Markov-growth process now fully specified, we define the *Markov-growth game* as the Bertrand supergame for the case in which G_t evolves in the implied manner. Observe that the Markov-growth game includes the stationary-growth game as a special case (e.g., $g = b$, $\mu = 1$, $\rho = 0$).

B. The Incentive Constraints

The key task now is to find a tractable representation of the incentive constraints for collusion. The Markov structure is especially helpful here, since it implies that the incentives for collusion are the same in any boom period regardless of the specific date, and similarly for any recession period. The most-collusive prices thus now emerge as a pair, with P_b^c denoting the most-collusive price in boom periods and P_r^c representing the most-collusive price during recessions. An additional benefit of the Markov structure is that it admits a simple recursive structure, once the appropriate definitions are put forth. We now exploit these advantages and provide a simple representation for the incentive constraints associated with collusion in boom and recession periods, respectively.

Let P_b be the price that firms charge in boom periods and let P_r be the price that firms charge in recession periods. We are interested in whether this pair of prices can be supported as part of a collusive agreement. To this end, we define $\bar{\omega}_b(P_b,P_r)$ as the expected discounted profit per market consumer to a firm in period t+1 and thereafter, if period t+1 is a boom period and the prices P_b and P_r are charged in the future. Analogously, we may define $\bar{\omega}_r(P_b,P_r)$ when period t+1 is a recession period. Observe that $\bar{\omega}_b(P_b,P_r)$ and $\bar{\omega}_r(P_b,P_r)$ also provide a measure of the cost of a price war, since firms earn zero profit once such a war commences.

With these definitions in place, the incentive constraint for collusion when period t is a boom period appears as

$$G_t\Omega(P_b) \leq \delta\{\rho(rG_t)\bar{\omega}_r(P_b,P_r) + (1-\rho)(bG_t)\bar{\omega}_b(P_b,P_r)\},$$

since $G_{t+1} = rG_t$ with probability ρ and $G_{t+1} = bG_t$ with probability $1-\rho$, given that period t is a boom period. Observe now that the current-period level of market demand, G_t, cancels, enabling us to write the incentive constraint in the simpler form:

$$\Omega(P_b) \leq \delta\{\rho r\bar{\omega}_r(P_b,P_r) + (1-\rho)b\bar{\omega}_b(P_b,P_r)\}.$$

Intuitively, the future level of market demand is always proportional to the current level, and so the current demand level is simply a scaling factor that is irrelevant for the incentive to collude.

The incentive constraint given above is clearly incomplete, both because the counterpart incentive constraint for recession periods is not presented and because explicit representations for the terms $\bar{\omega}_b(P_b,P_r)$ and $\bar{\omega}_r(P_b,P_r)$ are not given. Suppressing notation slightly, a complete system of incentive constraints is given in the following four inequalities:

(1). $\Omega(P_b) \leq \delta\{\rho r\bar{\omega}_r + (1-\rho)b\bar{\omega}_b\}$

(2). $\Omega(P_r) \leq \delta\{\lambda b\bar{\omega}_b + (1-\lambda)r\bar{\omega}_r\}$, where

$$(3).\quad \bar{\omega}_b = \omega(P_b) + \delta\{\rho r\bar{\omega}_r + (1-\rho)b\bar{\omega}_b\}$$

$$(4).\quad \bar{\omega}_r = \omega(P_r) + \delta\{\lambda b\bar{\omega}_b + (1-\lambda)r\bar{\omega}_r\}.$$

Notice that (1) and (2) reflect the tension between the current incentive to cheat and the expected discounted future profit that cheating would sacrifice, while through (3) and (4) the recursive nature of the model may be exploited so as to explicitly calculate the cost of a price war.

Formally, solving (3) and (4) for $\bar{\omega}_b$ and $\bar{\omega}_r$, one obtains:

$$(5).\quad \bar{\omega}_b = \{\omega(P_b)[1-(1-\lambda)\delta r]/\delta + \omega(P_r)\rho r\}\Delta$$

$$(6).\quad \bar{\omega}_r = \{\omega(P_r)[1-(1-\rho)\delta b]/\delta + \omega(P_b)\lambda b\}\Delta$$

where

$$(7).\quad \Delta = \delta/\{[1-(1-\lambda)\delta r][1-(1-\rho)\delta b] - \delta^2\lambda b\rho r\}.$$

It is easy to show that $\Delta > 0$ and that Δ increases strictly in δ for $\delta \in (0,1/b)$.[7] Substituting (5), (6) and (7) back into (1) and (2), we are now able to write the two incentive constraints in terms of the known functions, Ω and ω:

$$(8).\quad \Omega(P_b) \le \{\omega(P_r)\rho r + \omega(P_b)b[1-\rho-\delta r(1-\lambda-\rho)]\}\Delta$$

$$(9).\quad \Omega(P_r) \le \{\omega(P_b)\lambda b + \omega(P_r)r[1-\lambda-\delta b(1-\lambda-\rho)]\}\Delta.$$

[7]Let $\Delta \equiv \delta/D(\delta)$, where D is the denominator of the expression in (7). Simple calculations reveal that $D(0) = 1$, $D'(0) \le 0$, $D(1/b) \ge 0$ and $\text{sign}\{D''(\delta)\} = \text{sign}\{1-\lambda-\rho\}$. Thus, if $1-\lambda-\rho \le 0$, then $D''(\delta) \le 0$ over $(0,1/b)$ and so $D(\delta) > 0$ follows necessarily. Consider next the case in which $1-\lambda-\rho > 0$, implying that $D''(\delta) > 0$. Observe that $D(1/r) \le 0$, where $1/r > 1/b$. Given the convexity of $D(\delta)$ and the fact that $D(1/b) \ge 0$, it follows that $D'(\delta) < 0$ for $\delta \in [0,1/b]$. This in turn implies that $D(\delta) > 0$ over $(0,1/b)$.

Intuitively, inequality (8) indicates that the incentive to cheat in a boom period must be no greater than the expected discounted loss in future profit that would occur once a price war commenced; moreover, this loss is a weighted average of the profit lost in future boom and recession periods, with the associated weights reflecting the expected duration in each of the respective types of periods. Inequality (9) may be interpreted similarly for recession periods.

As it will sometimes be more convenient to express these incentive constraints in terms of the underlying profit function, we note finally that (8) and (9) may be rewritten as:

(10). $\pi(P_b)B \leq \pi(P_r)\rho r\Delta$

(11). $\pi(P_b)\lambda b\Delta \geq \pi(P_r)R$

where Δ is defined in (7) and

(12). $B = n-1 - b\Delta[1-\rho-\delta r(1-\lambda-\rho)]$

(13). $R = n-1 - r\Delta[1-\lambda-\delta b(1-\lambda-\rho)]$.

C. Correlation

While we now have the incentive constraints represented in a manageable form, the model still embodies several parameters (n, δ, b, r, λ and ρ), and it is not clear how best to organize the space of parameters. Before proceeding to a characterization of the most-collusive prices, we therefore first offer and interpret a partial organizational scheme.

An important ingredient in the stationary-growth game is the growth rate, g, of the level of market demand. Reasoning by analogy for the Markov-growth game, we might expect that the ability to collude in a given period would be influenced by the expected growth in the level of market demand in the following period. Since this expectation may be in turn sensitive to whether the current period is a boom or a recession, we perform the following calculation:

$$E(G_{t+1} \mid g_t = b) - E(G_{t+1} \mid g_t = r)$$
$$= [E(g_{t+1} \mid g_t = b) - E(g_{t+1} \mid g_t = r)]G_t$$
$$= (1-\lambda-\rho)(b-r)G_t.$$

Thus, the expected rate of growth in period t+1, and hence the expected level of market demand in period t+1, is higher (lower) when period t is a boom as opposed to recession period if and only if $1-\lambda-\rho > 0$ $(1-\lambda-\rho < 0)$. Intuitively, when λ and ρ are small, the current rate of growth is likely to persist into the next period, and thus the expected rate of growth for the subsequent period is higher if current growth is at the boom rate.

These calculations are drawn from the comparison of the current market demand conditions with their expected values in the *next* period. For firms attempting to collude, a more fundamental consideration is the relationship between the current demand conditions and the expected discounted level of market demand in all future periods. To better understand this relationship, let us set $G_t = 1$ for simplicity and define $\bar{G}_b$ as the expected discounted level of market demand in period t and all subsequent periods when period t is a boom period. With $\bar{G}_r$ defined analogously for the situation in which period t is a recession period, we have that

$$\bar{G}_b = 1 + \delta[\rho r\bar{G}_r + (1-\rho)b\bar{G}_b]$$
$$\bar{G}_r = 1 + \delta[\lambda b\bar{G}_b + (1-\lambda)r\bar{G}_r].$$

Solving these equations, one finds that

(14). $\bar{G}_b - \bar{G}_r = \Delta(b-r)(1-\lambda-\rho),$

from which it follows that the expected discounted level of market demand in future periods is higher when the current period is a boom if and only if $1-\lambda-\rho > 0$.

The relation of the current period demand conditions to those expected in the next period, and to those expected in the discounted future, is thus entirely governed by the sign of $1-\lambda-\rho$. We therefore organize the sequel around the following three cases: $1-\lambda-\rho > 0$, $1-\lambda-\rho < 0$ and $1-\lambda-\rho = 0$. When $1-\lambda-\rho > 0$, a higher value for the growth rate at period t leads to a higher expected growth rate in period t+1, and so we say that growth rates exhibit *positive correlation*. Likewise, growth rates exhibit *negative correlation* when $1-\lambda-\rho < 0$, and finally there is *zero correlation* between the growth rates if $1-\lambda-\rho = 0$.

IV. THE MOST-COLLUSIVE PRICES

A. Extreme Cases

Before exploring the implications of correlation for collusion, we first examine a pair of extreme cases for which the form of most-collusive pricing is clear. The identification of these cases will in turn indicate the interesting range for the parameters n, δ, b and r.

In the Markov-growth game, the growth rate is sometimes at the boom rate b and other times at the recession rate r. As faster growth rates have been linked to better collusion, it might be expected that, if perfect collusion is possible even in a stationary-growth game with the slow growth rate r, then perfect collusion should also occur in the Markov-growth game. Likewise, if the most-collusive price is the competitive price in a stationary-growth game with the fast rate of growth b, then competitive pricing would also be expected in the Markov-growth game. This intuition is confirmed in the following theorem:

Theorem 1: In the Markov-growth game,

(i). If $\delta r \geq (n-1)/n$, then $P_b^c = P_r^c = P_m$.

(ii). If $\delta b < (n-1)/n$, then $P_b^c = P_r^c = c$.

A proof of this theorem is found in the Appendix.

In a similar way, it can be demonstrated that when $\delta b = (n-1)/n$, we have $P^c_b = P^c_r = c$ provided only that $\lambda < 1$ and $\rho > 0$; i.e., if competitive pricing is just averted in the stationary-growth game with growth rate b, and if booms are not certain to last indefinitely ($\rho > 0$) while recessions do not always return immediately to booms ($\lambda < 1$), then there is some chance that a recession will be experienced in the Markov-growth game, and so competitive pricing occurs in both boom and recession periods. It follows that the interesting case for the Markov-growth game is when

(15). $\delta b > (n-1)/n > \delta r$.

Henceforth, we therefore maintain the assumption that n, δ, b and r are such that (15) holds.[8]

With Theorem 1 in place, we now have a complete organizational scheme for our parameters: if n, δ, b and r fail (15) then the conclusions of Theorem 1 apply, while if as we assume below n, δ, b and r are such that (15) is satisfied, then the three correlation cases for λ and ρ will be considered.

B. Zero Correlation

Among the three kinds of correlation, the case of zero correlation ($1-\lambda-\rho = 0$) is the most simple with which to begin, since it can be understood as representing a special case of the already-examined stationary-growth game. The key idea here is that under zero correlation, the expected future growth rate is independent of whether the current period is a boom or a recession; as a consequence, it is possible to think of the zero-correlation case in terms of a stationary rate of growth, g, that satisfies

$$\rho r + (1-\rho)b = g = \lambda b + (1-\lambda)r.$$

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[8]See, however, footnote 12 for a relaxation of this constraint in an extended model.

This parallel with the stationary-growth game in turn indicates that the most-collusive prices are the same in booms and recessions ($P^c_b = P^c_r$) when there is zero correlation.

Drawing on the earlier analysis for the stationary-growth game, it follows that the growth rate g will support perfect collusion if and only if $\delta g \geq (n-1)/n$. Equivalently, $P^c_b = P^c_r = P_m$ when correlation is zero if and only if

(16). $\delta[\rho r + (1-\rho)b] \geq (n-1)/n.$

Letting ρ^* solve (16) with equality, and putting $\lambda^* \equiv 1 - \rho^*$, we find that

(17). $\rho^* = [\delta b - (n-1)/n]/[\delta(b-r)]$

(18). $\lambda^* = [(n-1)/n - \delta r]/[\delta(b-r)]$

where $\rho^* \in (0,1)$ and $\lambda^* \in (0,1)$ under the maintained assumption (15). Since the lefthand side of (16) is decreasing in ρ, we conclude that perfect collusion is possible when there is zero correlation if and only if $\rho \leq \rho^*$, while competitive pricing occurs otherwise.

Our results for the zero-correlation case now may be summarized as follows:

Theorem 2: In the Markov-growth game with zero correlation,

(i). If $\rho \leq \rho^*$, then $P^c_b = P^c_r = P_m$.

(ii). If $\rho > \rho^*$, then $P^c_b = P^c_r = c$.

Intuitively, if the expected duration of a boom phase is sufficiently long, then the associated zero-correlation growth rate is high enough to support perfect collusion.

C. Positive Correlation

The zero-correlation case serves to illustrate a relationship between the Markov- and stationary-growth games, but it does not deliver cyclical pricing. This possibility arises in the more interesting case of positive correlation ($1-\lambda-\rho > 0$), since then the expected discounted market demand level for the future is sensitive to the current state of the business cycle. In particular, when the evolution of market demand is characterized by positive correlation, it may be especially difficult to collude in recessions, as the expected discounted level of future market demand is then lower, implying that there is less to lose from a price war. This suggests that the most-collusive prices then might be procyclical (i.e., $P_b^c > P_r^c$), as firms reduce the collusive price in recessions so as to diminish the incentive to cheat and bring incentives back in line. We sketch here arguments supporting this conclusion, leaving a complete proof for the Appendix.

We begin by characterizing the parameter region for which perfect collusion ($P_b^c = P_r^c = P_m$) can be supported. To this end, let $\hat{\lambda}(\rho)$ satisfy $\lambda b\Delta = R$ when $1-\lambda-\rho \geq 0$, and observe from (11) that at $\lambda = \hat{\lambda}(\rho)$ perfect collusion is just consistent with the recession-period incentive constraint. Straightforward calculations then yield

$$(19). \quad \hat{\lambda}(\rho) = [1-(1-\rho)\delta b]/(1/\lambda^* - \delta b)$$

from which it is easily verified that $\hat{\lambda}$ is a linear function of ρ with $\hat{\lambda}(0) > 0$, $\hat{\lambda}'(\rho) > 0$ and $\hat{\lambda}(\rho^*) = \lambda^*$, as Figure 1 illustrates.

It is now easy to show that, in the case of positive correlation, $\lambda \geq \hat{\lambda}(\rho)$ implies $P_b^c = P_r^c = P_m$. This result may be understood as confirming the following intuitive argument. Under positive correlation, collusion is especially difficult to support in recession periods, and $\hat{\lambda}(\rho)$ is defined so that perfect collusion is just possible during such a period. Thus, if the expected duration of a recession phase is even lower (i.e., if $\lambda > \hat{\lambda}(\rho)$), then perfect collusion can be maintained in both boom and recession periods with slack.

Suppose next that $\lambda < \hat{\lambda}(\rho)$, in which case the expected duration of a recession phase is large enough to preclude perfect collusion in a recession period. In this case, firms nevertheless may be able to collude imperfectly. To understand, consider Figure 2, which depicts the boom- and recession-period incentive constraints for a situation in which perfect collusion is supportable only in boom periods. Incentive constraints are upward sloping in this figure, with a higher price in one state raising the cost of a price war and thus making possible a higher price in the other state as well, and all prices to the southeast of (10) (northwest of (11)) satisfy the boom- (recession-) period incentive constraint.[9] In the case depicted, along the recession-period incentive constraint (11), $P_r < P_m$ even for $P_b = P_m$, so perfect collusion cannot be sustained in a recession period; consequently, (11) lies left of the 45 degree line. Even so, a region of profitable prices exists at which both incentive constraints are satisfied, provided (as depicted) that the boom-period incentive constraint lies on or left of that of the recession-period. Moreover, the most-collusive prices are procyclical with $P_b^c = P_m > P_r^c$.

To characterize the boundary case where the incentive constraints lie atop of one another in Figure 2, and can therefore both just be satisfied with profitable prices, we observe from (10) and (11) that the incentive constraints are redundant when $RB = (\Delta\lambda b)(\Delta\rho r)$, and we thus define a function $\hat{\rho}(\lambda)$ as the ρ value for which this equation holds. Calculations reveal that

(20). $\hat{\rho}(\lambda) = [(b-r)/b]\rho^* + [\rho^* r/(b\lambda^*)]\lambda.$

Note that $\hat{\rho}$ is a linear function of λ with $\hat{\rho}(0) > 0$, $\hat{\rho}'(\lambda) > 0$ and $\hat{\rho}(\lambda^*) = \rho^*$. Further, and as Figure 1 illustrates, $\hat{\rho}(\hat{\lambda}(\rho)) > \rho$ for $\rho < \rho^*$ and so $\hat{\rho}(\lambda)$ lies below $\hat{\lambda}(\rho)$.

With the $\hat{\rho}$ now defined, it is possible to state a second result: under positive correlation, if $\lambda < \hat{\lambda}(\rho)$ and $\rho \leq \hat{\rho}(\lambda)$, then the most-collusive prices are procyclical with $P_b^c = P_m > P_r^c$. Intuitively, even if firms are unable to collude perfectly in recessions, they may be

[9]It is possible that $B \leq 0$, in which case the boom period incentive constraint (10) does not have positive slope: see the Appendix for this case.

able to collude imperfectly by reducing the price in recession periods to below-monopoly levels; in this way, they reduce the incentive to cheat in recession periods and thus make credible an imperfect collusive agreement.

The remaining possibility is that $\rho > \hat{\rho}(\lambda)$. In terms of Figure 2, this case corresponds to the situation in which the boom period incentive constraint (10) lies right of the recession period incentive constraint (11), indicating that both incentive constraints hold only when both prices are set at competitive levels. In fact, the following general result is easily confirmed: under positive correlation, if $\rho > \hat{\rho}(\lambda)$, then $P_b^c = P_r^c = c$. This result, too, rests on a simple intuition: if the expected duration of a boom phase is too short relative to the expected duration of a recession phase, then even imperfect collusion is impossible.

Our findings are summarized in Figure 1, which pinpoints the regions of monopoly, procyclical and competitive pricing for the positive-correlation case. A remaining point of interest concerns the behavior of the recession-period collusive price over the procyclical region. We find that, in the procyclical region, P_r^c satisfies the following properties:

(i). P_r^c is continuous, increasing in λ and decreasing in ρ.

(ii). $P_r^c \to P_m$ as $\lambda \to \hat{\lambda}(\rho)$ and $P_r^c \to c$ as $\lambda \to 0$.

(iii). P_r^c is increasing in λ along $\hat{\rho}(\lambda)$, with $P_r^c = c$ when $\lambda = 0$ and $P_r^c \to P_m$ as $\lambda \to \lambda^*$.

With $P_b^c = P_m$ holding throughout the procyclical region, it follows from (i) that the *amplitude* (i.e., $|P_b^c - P_r^c|$) of the price cycle increases as the expected duration of a recession (boom) phase lengthens (shortens). Such changes make collusion more difficult, and so a breakdown of the collusive agreement in recession periods can be averted only if the recession-period price is depressed further. Notice also from (ii) that P_r^c continuously climbs to the monopoly price as the monopoly region is approached. But, when the competitive region is approached, the most-collusive prices experience a discontinuity at the boundary. As (iii) reveals, P_r^c varies along this boundary, increasing from the competitive to the monopoly price as λ and ρ rise along $\hat{\rho}(\lambda)$.

The main conclusions of the positive-correlation case pertain to the predictions of procyclical ($P_b^c > P_r^c$) versus *countercyclical* ($P_b^c < P_r^c$) pricing and to the determinants of the amplitude of pricing cycles. We therefore summarize this case as follows:

Theorem 3: In the Markov-growth game with positive correlation,

(i). The most-collusive prices are sometimes procyclical but never countercyclical.

(ii). When the most-collusive prices are procyclical, the amplitude of the price cycle is increasing (decreasing) in the expected duration of a recession (boom) phase.

D. Negative Correlation

We turn last to the case of negative correlation ($1-\lambda-\rho < 0$). In this case, expected future market demand conditions are less favorable when the current period is a boom period, suggesting that collusion is now most difficult to maintain in *boom* periods. The implication is then that the most-collusive price must be depressed in boom periods, in order to reduce the incentive to defect, and so a prediction of countercyclical pricing is anticipated. The formal analysis of this case is sketched below, with further details again provided in the Appendix.

We begin by characterizing the parameter space over which perfect collusion is possible. To this end, we now analyze the boom-period incentive constraint (10), and define the function $\tilde{\rho}(\lambda)$ as the solution to $B = \Delta\rho r$ when $1-\lambda-\rho < 0$. Calculations reveal that

(21). $\tilde{\rho}(\lambda) = [1-(1-\lambda)\delta r]/[1/\rho^* - \delta r]$,

from which it can be verified that $\tilde{\rho}$ is a linear function of λ with $\tilde{\rho}'(\lambda) > 0$, $\tilde{\rho}(\lambda^*) = \rho^*$ and $\tilde{\rho}(\lambda) < \hat{\rho}(\lambda)$ over the negative-correlation range, as Figure 1 illustrates. The curve $\tilde{\rho}(\lambda)$ represents the combinations of parameters at which a firm is just indifferent between perfectly colluding and cheating when currently in a boom period. When the expected duration of a boom phase is greater, perfect collusion can be maintained over this phase with slack.

Consistent with the idea that boom periods are the most difficult time to collude in the negative-correlation case, it is in fact true for this case that $\rho \leq \tilde{\rho}(\lambda)$ implies $P_b^c = P_r^c = P_m$.

Consider next the region that lies between $\tilde{\rho}(\lambda)$ and $\hat{\rho}(\lambda)$. Here, the expected duration of a boom phase is sufficiently short that perfect collusion cannot be supported over this phase. But, as Figure 3 illustrates, the situation does not preclude imperfect collusion, if the boom-period incentive constraint lies on or above the recession-period incentive constraint. In fact, we establish the following: under negative correlation, if $\tilde{\rho}(\lambda) < \rho \leq \hat{\rho}(\lambda)$, then $P_r^c = P_m > P_b^c$. Thus, imperfect collusion now takes the form of countercyclical pricing, as the collusive price is depressed in booms in order to reduce the corresponding incentive to cheat. Finally, the last region has $\rho > \hat{\rho}(\lambda)$, and as before prices above cost can no longer be found that satisfy both incentive constraints; consequently, we have $P_b^c = P_r^c = c$ when $\rho > \hat{\rho}(\lambda)$.

Our remaining results for the negative-correlation case concern the behavior of the boom-period collusive price in the countercyclical region . We find that:

(i). P_b^c is continuous, increasing in λ and decreasing in ρ.

(ii). $P_b^c \to P_m$ as $\rho \to \tilde{\rho}(\lambda)$.

(iii). P_b^c is decreasing in λ along $\hat{\rho}(\lambda)$, with $P_b^c \to P_m$ as $\lambda \to \lambda^*$.

The amplitude of the collusive pricing cycle is again increased by a lengthening (shortening) in the expected duration of a recession (boom) phase, since the cost of a price war is then reduced, forcing a greater drop in the boom-period collusive price. Note also that collusive prices move continuously between the monopoly and countercyclical regions, and a discontinuity remains once the competitive region is encountered.

With these results in place, we summarize the main features of the negative-correlation case as follows:

Theorem 4: In the Markov-growth game with negative correlation,

(i). The most-collusive prices are sometimes countercyclical but never procyclical.

(ii). When the most-collusive prices are countercyclical, the amplitude of the price cycle is increasing (decreasing) in the expected duration of a recession (boom) phase.

The prediction of countercyclical pricing reverses that of the positive-correlation case.

Inspecting Figure 1, it is now clear that perfect collusion is sustainable if booms are sufficiently long and recessions are sufficiently short in expected duration, regardless of the precise nature of correlation. Similarly, whatever the level of correlation, competitive pricing is required whenever booms are sufficiently short and recessions are sufficiently long in expected duration. In the intermediate situation, however, monopoly pricing is sustainable in only one phase, and the the collusive price in the other phase must be reduced, in order to thwart the incentive to cheat. The designation of the "weak" state then depends upon the sign of correlation, with procyclical (countercyclical) collusive pricing occurring under positive (negative) correlation. More generally, predictions regarding the kind and amplitude of cyclical pricing in a collusive industry necessitate information about the expected durations of boom and recession phases in that industry.

A final remark concerns the manner in which our predictions vary with the parameters δ, b, r and n. Using (17) and (18), it is easily verified that $\lambda^* \to 0$ as $\delta r \to (n-1)/n$ and that $\rho^* \to 0$ as $\delta b \to (n-1)/n$. In words, the region of perfect collusion dominates as it becomes almost feasible to perfectly collude even in a stationary-growth game with the slow rate of growth r, and the region of competitive pricing similarly expands as it becomes almost infeasible to perfectly collude even in a stationary-growth game with the fast rate of growth b. The predicted most-collusive prices for the extreme parameter values described in Theorem 1 are thus approached continuously as parameters are varied over the region for which (15) holds.

V. COLLUSION AND WITHIN-PHASE DEMAND FLUCTUATIONS

A. Overview

In the business-cycle model adopted above, the demand growth rate is stochastic, and whether the market is in a fast- or slow- growth phase is determined by a Markov process. The model also specifies that, within any given phase, the level of market demand at any given period is a deterministic function of its value in the preceding period. The purpose of the present section is to analyze collusive pricing in more sophisticated models of the business cycle, where the level of market demand also fluctuates randomly within a given phase.

There are a number of ways in which such within-phase shocks might be modeled, and we will not examine all of the candidates here. Instead, we bracket the range of possibilities by considering two extreme cases. In our first formulation, within-phase shocks are permanent, in that they become fully embedded into the market demand level from which all future demand growth derives. Our second formulation highlights the opposite extreme: within phase shocks are transitory, as a shock in any period affects the level of market demand in that period only.[10] In either event, we continue to assume that firms observe all current-period demand conditions prior to selecting prices.

B. Permanent Shocks

Suppose now that the process through which the level of market demand evolves is described as $G_t = g_t G_{t-1} \varepsilon_t$, where $g_t = b$ ($g_t = r$) when period t is a boom (recession) period, with the transition between boom and recession periods following the Markovian structure presented in Section III, and where at every date t the expected value of ε_t is unity, $E\{\varepsilon_t\} = 1$.

This model of the business cycle is the same as the Markov-growth process defined above, except that we have now included the possibility of within-phase shocks through the

[10]The distinction can be made at a more formal level. In the first formulation, within any given phase, the log of the market demand level follows a random walk with drift, for which the autocorrelation function is unity for all orders. The second formulation addresses the opposite extreme, in which current-period shocks lack any correlation with future market demand levels. Specifically, within any given phase, the log of the market demand level is described by an ARMA (1,1) process with an autocorrelation function that is zero at all orders.

realization of the values for ε_t. Note moreover that these shocks are permanent, as they become embedded into the evolution of future demand levels and their influence on this evolution does not decay through time.[11] With the processes governing the evolution of the market demand level now fully specified, we define *Markov-growth game with permanent shocks* as the Bertrand supergame for the case in which G_t evolves in the described manner.

The next step is to derive the incentive constraints for this game. A key observation is that the most-collusive prices are a function of phases but not within-phase shocks; that is, the most-collusive price in any period t is either P_b (if period t is a boom period) or P_r (if period t is a recession period), regardless of the specific value that ε_t takes. Intuitively, for any period t and corresponding market demand level G_t, the levels of market demand expected in the future are proportional to Gt, and so as before the incentive to collude is independent of G_t. But since G_t embodies the within-phase shock ε_t, it follows immediately that the most-collusive prices are not influenced by within-phase shocks of this form.

To examine the incentive constraints more rigorously, we may again define $\bar{\omega}_b(P_b,P_r) \equiv \bar{\omega}_b$ as the expected discounted profit per market consumer to a firm in period t+1 and all future periods, if period t+1 is a boom period and P_b (P_r) is charged in all boom (recession) periods. With $\bar{\omega}_r$ defined analogously, the incentive constraints may be represented as:

(22). $G_t\Omega(P_b) \leq \delta E\{\rho(rG_t\varepsilon_{t+1})\bar{\omega}_r + (1-\rho)(bG_t\varepsilon_{t+1})\bar{\omega}_b\}$

(23). $G_t\Omega(P_r) \leq \delta E\{\lambda(bG_t\varepsilon_{t+1})\bar{\omega}_b + (1-\lambda)(rG_t\varepsilon_{t+1})\bar{\omega}_r\}$

where

(24). $\bar{\omega}_b = \omega(P_b) + \delta E\{\rho(r\varepsilon_{t+1})\bar{\omega}_r + (1-\rho)(b\varepsilon_{t+1})\bar{\omega}_b\}$

(25). $\bar{\omega}_r = \omega(P_r) + \delta E\{\lambda(b\varepsilon_{t+1})\bar{\omega}_b + (1-\lambda)(r\varepsilon_{t+1})\bar{\omega}_r\}$

[11] To see this more directly, observe that $G_t = G_0\Pi(g_\tau\varepsilon_\tau)$, where the product is taken from $\tau=1$ to $\tau=t$.

and all expectations are taken over ε_{t+1}.

Cancelling Gt across the inequalities in (22) and (23) and imposing $E\{\varepsilon_{t+1}\} = 1$, it is apparent that the incentive constraint system given in (22) - (25) is in fact exactly that derived for the Markov-growth game and presented in (1) - (4). We therefore have that:

Theorem 5: In the Markov-growth game with permanent shocks, collusive pricing is exactly the same as in the Markov-growth game.

We conclude that our predictions are robust to the inclusion of within-phase permanent shocks.

C. Transitory Shocks

1. Basic Assumptions

We now investigate the opposite extreme, supposing that the process through which the market demand level evolves is described by $G_t = g_t(G_{t-1}/\varepsilon_{t-1})\varepsilon_t$, where $g_t = b$ ($g_t = r$) when period t is a boom (recession) period, with the Markovian transition probabilities as described in Section III, and where ε_t is iid through time with full support over $[\underline{\varepsilon},\bar{\varepsilon}]$ and $E\{\varepsilon_t\} = 1 \in (\underline{\varepsilon},\bar{\varepsilon})$.

Once again, we have the basic Markov-growth process, except that now shocks to the level of market demand occur in each period. Observe moreover that these shocks are transitory, since future levels of market demand are completely independent of ε_t.[12] We thus define the *Markov-growth game with transitory shocks* as the Bertrand supergame for the case in which G_t evolves in the described way.

As in the Rotemberg-Saloner (1986) model, such temporary shocks pose interesting problems for collusive agreements, since they affect the short-term incentive to cheat but not the

[12]To see this more directly, observe that $G_t = G_0\Pi(g_\tau)\varepsilon_t$, where the product is taken from $\tau=1$ to $\tau=t$.

long-term cost of a price war. We now combine methods presented above with those developed by Rotemberg and Saloner (1986) and characterize the most-collusive price at any period t as a function of the business-cycle phase (boom or recession) *and* the within-phase shock (ε_t) experienced in that period. In so doing, we generalize the analysis of Rotemberg and Saloner (1986) to a business-cycle specification that allows for multiple phases (i.e., a stochastic trend), and thereby offer a formal interpretation of their results in terms of the transitory shocks to demand that occur within broader business-cycle phases.

2. Solution Method

Our first step is to exploit the methods developed above in order to get the incentive constraints in a tractable form. Given that ε_t is iid, we now drop the time subscript and let $P_b(\varepsilon)$ and $P_r(\varepsilon)$ represent the prices charged in boom and recession periods, respectively, when the current period within-phase demand shock is given by ε. Let us next define $\bar{\omega}_b(P_b(\varepsilon),P_r(\varepsilon))$ as the expected discounted profit per market consumer to a firm in period t+1 and thereafter, if period t+1 is a boom period, $P_b(\varepsilon)$ and $P_r(\varepsilon)$ are the pricing functions, and the value for ε_{t+1} has not yet been determined. We define $\bar{\omega}_r(P_b(\varepsilon),P_r(\varepsilon))$ analogously when period t+1 is a recession period.

Consider next the incentive constraint facing a firm in period t, when period t is a boom period and the period-t within-phase shock is given by $\varepsilon_t = \varepsilon$. Simplifying notation slightly, we may represent this incentive constraint as

$$G_t\Omega(P_b(\varepsilon)) \leq \delta\{\rho(rG_t/\varepsilon)\bar{\omega}_r + (1-\rho)(bG_t/\varepsilon)\bar{\omega}_b\},$$

or more simply

$$\varepsilon\Omega(P_b(\varepsilon)) \leq \delta\{\rho r\bar{\omega}_r + (1-\rho)b\bar{\omega}_b\}.$$

Thus, the current-period "base" level of demand, G_t, again cancels, since all future demand growth will be proportional to this base, but the current-period within-phase shock, ε, is not represented in future demand growth, and its value remains in the collusive incentive constraint, with higher values for ε having the effect of raising the incentive to cheat.

Building on these insights, we now represent the complete incentive system as

(26). $\varepsilon\Omega(P_b(\varepsilon)) \leq \delta\{\rho r\bar{\omega}_r + (1-\rho)b\bar{\omega}_b\}$

(27). $\varepsilon\Omega(P_r(\varepsilon)) \leq \delta\{\lambda b\bar{\omega}_b + (1-\lambda)r\bar{\omega}_r\}$,

where

(28). $\bar{\omega}_b = E\{\omega(P_b(\varepsilon))\varepsilon\} + \delta\{\rho r\bar{\omega}_r + (1-\rho)b\bar{\omega}_b\}$

(29). $\bar{\omega}_r = E\{\omega(P_r(\varepsilon))\varepsilon\} + \delta\{\lambda b\bar{\omega}_b + (1-\lambda)r\bar{\omega}_r\}$.

But this is the same incentive system as represented in (1) - (4), except that $\varepsilon\Omega$ replaces Ω and $E\{\omega(\cdot)\varepsilon\}$ replaces $\omega(\cdot)$. Proceeding as before, we thus may rewrite the incentive constraints in the more useful form,

(30). $\varepsilon\Omega(P_b(\varepsilon)) \leq E\{\omega(P_r(\varepsilon))\varepsilon\}\rho r\Delta + E\{\omega(P_b(\varepsilon))\varepsilon\}[(n-1) - B]$

(31). $\varepsilon\Omega(P_r(\varepsilon)) \leq E\{\omega(P_b(\varepsilon))\varepsilon\}\lambda b\Delta + E\{\omega(P_r(\varepsilon))\varepsilon\}[(n-1) - R]$

.

Having derived the incentive constraints, we next follow Rotemberg and Saloner (1986) and exploit a two-step solution process for the most-collusive prices, $P_b^c(\varepsilon)$ and $P_r^c(\varepsilon)$. The initial step involves viewing the right-hand sides of (30) and (31) as fixed values, defined as

(32). $\tilde{\omega}_b \equiv E\{\omega(P_r(\varepsilon))\varepsilon\}\rho r\Delta + E\{\omega(P_b(\varepsilon))\varepsilon\}[(n-1) - B]$

(33). $\tilde{\omega}_r \equiv E\{\omega(P_b(\varepsilon))\varepsilon\}\lambda b\Delta + E\{\omega(P_r(\varepsilon))\varepsilon\}[(n-1) - R]$.

Using (30) - (33), the incentive constraints now appear as $\varepsilon\Omega(P_b(\varepsilon)) \leq \tilde{\omega}_b$ and $\varepsilon\Omega(P_r(\varepsilon)) \leq \tilde{\omega}_r$, and so, after substituting for Ω, the incentive constraints may be rewritten as

(34). $\pi(P_b(\varepsilon)) \leq [n/(n-1)]\tilde{\omega}_b/\varepsilon$

(35). $\pi(P_r(\varepsilon)) \leq [n/(n-1)]\tilde{\omega}_r/\varepsilon$.

We may now define $P_b(\tilde{\omega}_b, \varepsilon)$ and $P_r(\tilde{\omega}_r, \varepsilon)$ as the most-collusive prices when $\tilde{\omega}_b$ and $\tilde{\omega}_r$ are taken as fixed values; i.e, $P_b(\tilde{\omega}_b, \varepsilon)$ is the most-profitable price satisfying (34) and $P_r(\tilde{\omega}_b, \varepsilon)$ is defined analogously for (35). These prices can be represented as follows:

(36). $P_b(\tilde{\omega}_b, \varepsilon) = P^*(\tilde{\omega}_b/\varepsilon)$

(37). $P_r(\tilde{\omega}_r, \varepsilon) = P^*(\tilde{\omega}_r/\varepsilon)$,

where $P^*(\tilde{\omega}/\varepsilon)$ is defined by

(38). $P^*(\tilde{\omega}/\varepsilon) \equiv P_m$, if $\pi(P_m) \leq [n/(n-1)]\tilde{\omega}/\varepsilon$

(39). $P^*(\tilde{\omega}/\varepsilon) \equiv \min\{P \mid \pi(P) = [n/(n-1)]\tilde{\omega}/\varepsilon\}$, if $\pi(P_m) > [n/(n-1)]\tilde{\omega}/\varepsilon$.

In short, each price is set as close to the monopoly price as possible, while still being consistent with the corresponding incentive constraint.

We now proceed to the next step in this process, and present a fixed point technique through which the most-collusive values for $\tilde{\omega}_b$ and $\tilde{\omega}_r$ may be endogenously determined. Specifically, consistency requires that the most-collusive values for $\tilde{\omega}_b$ and $\tilde{\omega}_r$ lead through (38) and (39) to prices which in turn generate through (32) and (33) the originally specified values for $\tilde{\omega}_b$ and $\tilde{\omega}_r$. This requirement is captured by the following two fixed point equations:

(40). $\tilde{\omega}_b = E\{\omega(P^*(\tilde{\omega}_r/\varepsilon))\varepsilon\}\rho r\Delta + E\{\omega(P^*(\tilde{\omega}_b/\varepsilon))\varepsilon\}[(n-1) - B]$

(41). $\tilde{\omega}_r = E\{\omega(P^*(\tilde{\omega}_b/\varepsilon))\varepsilon\}\lambda b\Delta + E\{\omega(P^*(\tilde{\omega}_r/\varepsilon))\varepsilon\}[(n-1) - R]$.

It is straightforward to see that one consistent solution has $\tilde{\omega}_b = \tilde{\omega}_r = 0$, corresponding to competitive pricing in all states. We are interested instead in the *most-collusive fixed point solution*, $(\hat{\omega}_b, \hat{\omega}_r)$, which represents the largest values for $(\tilde{\omega}_b, \tilde{\omega}_r)$ that satisfy (40) and (41). Once these values are determined, the most-collusive prices are then defined by

(42). $P_b^c(\varepsilon) \equiv P^*(\hat{\omega}_b/\varepsilon)$

(43). $P_r^c(\varepsilon) \equiv P^*(\hat{\omega}_r/\varepsilon)$.

In this way, the problem of solving for the most-collusive price functions is reduced to the alternative task of solving for the most-collusive fixed point values.[13]

C. Results

The possibility of transitory within-phase demand shocks suggests that attempts to collude will be frustrated by high transitory shocks, as the incentive to cheat is then high as compared to the cost of a price war. It thus may be anticipated that, while the comparison of collusive prices across boom and recession phases will still hinge upon the correlation of growth rates through time, higher transitory demand shocks should require lower collusive prices within any given phase. We argue below that this is indeed the case and, further, that the disruptive effect of transitory demand shocks serves to expand the region over which cyclical pricing occurs, since perfect collusion (for both phases and all within-phase shocks) is possible over a reduced range when large transitory demand shocks are possible.

[13]The approach pursued here thus presumes that the fixed point solutions to (40) and (41) are pointwise ranked, so that a maximal solution can be unambiguously identified. The approach also presumes that the most-collusive prices are found by raising price as high as possible in each state, as is evident from (38) and (39) and (42) and (43). These presumptions are appropriate in the present model, because incentive constraints are complementary, with more collusion in any one state fostering greater collusion in the other as well.

Of course, perfect collusion is sure to fail if $\bar{\varepsilon}$ is sufficiently big, as the temptation to cheat is then irresistible when the within-phase shock is near its upper bound. To create at least the possibility of perfect collusion, we thus restrict the size of $\bar{\varepsilon}$ with the following assumption:

(44). $\delta b > \bar{\varepsilon}(n-1)/[1+\bar{\varepsilon}(n-1)]$

This assumption ensures that even a maximal transitory shock would not disrupt perfect collusion, if the growth rate were stationary at the boom rate .

A complete analysis of the model is found in the Appendix. We show there that, under the assumption given in (44), the parameter space can be carved into the nonempty regions depicted in Figure 4. We now illustrate some general themes and sketch the method of proof for the parameter region B in the case of positive correlation. Comparing Figures 1 and 4, we see that this region supports perfect collusion in the Markov-growth game, whereas we will argue that it gives procyclical collusive pricing in the Markov-growth game with transitory shocks. After making this argument, we discuss in a more informal way the nature of collusive pricing in other regions, before concluding the section with a final theorem.

It is useful to think of the fixed point equations (40) and (41) as defining two respective implicit functions, with $\tilde{\omega}_b$ being a function of $\tilde{\omega}_r$ in each case. These functions are naturally expected to have upward-sloping regions, whereby better collusion in one phase complements collusive efforts in the other phase as well. Figure 5 depicts the two fixed point equations for the parameter set corresponding to region B under positive correlation. The most-collusive fixed point solution, $(\hat{\omega}_b, \hat{\omega}_r)$, satisfies two interesting properties: $\hat{\omega}_b$ exceeds $\hat{\omega}_r$ under positive correlation, and both values fall between $\underline{\varepsilon}\pi(P_m)(n-1)/n$ and $\bar{\varepsilon}\pi(P_m)(n-1)/n$.

These properties may be interpreted in terms of the most-collusive prices as follows. Looking to (38)-(39) and (42)-(43), it is apparent that $P_b^c(\varepsilon)$ and $P_r^c(\varepsilon)$ are determined by the same monotonic pricing function, $P^*(\tilde{\omega}/\varepsilon)$, and so $\hat{\omega}_b > \hat{\omega}_r$ implies $P_b^c(\varepsilon) \geq P_r^c(\varepsilon)$. (Ties are possible since both prices may sometimes rest at the monopoly level.) Thus, our earlier prediction that positive correlation yields procyclical pricing is preserved. Observe also from these equations that the indicated range for $(\hat{\omega}_b, \hat{\omega}_r)$ implies that, in both phases, monopoly pricing is possible when $\varepsilon = \underline{\varepsilon}$ but below-monopoly prices must be charged when $\varepsilon = \bar{\varepsilon}$. The most-collusive pricing functions are illustrated in Figure 6a.

Continuing with the positive-correlation case, consider next regions A and C. In both of these regions, it continues to be true that under positive correlation, $P_b^c(\varepsilon) \geq P_r^c(\varepsilon)$ with the inequality being strict for large shocks. A novel feature of region A not shared by Region B is that the most-collusive fixed point solution may entail $\hat{\omega}_b \geq \bar{\varepsilon}\pi(P_m)(n-1)/n$, indicating that the monopoly price can always be supported in the boom phase. At the border between Region A and the region designated as the monopoly region, the monopoly price can just be supported in the recession phase when the transitory shock is at its most disruptive level, $\varepsilon = \bar{\varepsilon}$. Above this border, therefore, perfect collusion is sustainable. Region C describes a setting in which the prospects for collusion are less favorable, and it may be true here that $\hat{\omega}_r < \underline{\varepsilon}\pi(P_m)(n-1)/n$, meaning that the monopoly price cannot be sustained for any transitory shock in the recession phase. The lower boundary of region C is defined by the function, $\hat{\rho}(\lambda)$, and along this border monopoly pricing can be supported in the boom phase only under the smallest transitory demand shock, $\varepsilon = \underline{\varepsilon}$. Below this border, monopoly pricing is thus infeasible even in this best-case scenario, and the most-collusive prices switch discontinuously to the competitive solution, $P_b^c(\varepsilon) \equiv P_r^c(\varepsilon) \equiv c$.

Exactly analogous findings occur in the case of negative correlation. Consider for example region B under negative correlation. In terms of Figure 5, the most-collusive fixed point solution now occurs below the 45 degree line and satisfies $\bar{\varepsilon}\pi(P_m)(n-1)/n > \hat{\omega}_r > \hat{\omega}_b > \underline{\varepsilon}\pi(P_m)(n-1)/n$, so that the most-collusive prices appear as in Figure 6b. The prediction of countercyclical pricing is thus preserved, and higher transitory shocks require (weakly) lower most-collusive prices in both phases. Similarly, with zero correlation, the most-collusive prices are acyclic, as the most-collusive fixed point solution then rests on the 45 degree line, and higher transitory demand shocks necessitate lower most-collusive prices throughout region B.

Comparing Figures 1 and 4, we see that the monopoly region under Figure 4 is strictly smaller than that in Figure 1 while the regions with cyclical pricing are strictly larger. In particular, in the Markov-growth game, region C has cyclical pricing but regions A and B do not, whereas when transitory within-phase shocks are permitted, all three regions are characterized by cyclical pricing.[14]

Two main conclusions can be drawn from this analysis:

Theorem 6: In the Markov-growth game with transitory demand shocks,

(i). Under positive (negative) correlation, the most-collusive prices are sometimes procyclical (countercyclical) but never countercyclical (procyclical), and the range of cyclical pricing is larger than in the Markov-growth game.

(ii). Regardless of the nature of correlation, and in both boom and recession phases, a higher transitory demand shock results in a (weakly) lower most-collusive price.

We also note that point (ii) provides an interpretation of the Rotemberg-Saloner (1986) theory in terms of the transitory demand shocks that occur within broader business-cycle phases.

VI. CONCLUSION

We develop a simple theory of collusive pricing over the business cycle. The most-collusive prices are completely characterized as functions of business-cycle parameters, and a variety of specific predictions are offered. The most-collusive prices can be procyclical when growth rates are positivly correlated through time, the amplitude of the collusive pricing cycle is increased when recession phases are longer and boom phases are shorter in expected duration, and transitory demand shocks have the effect of lowering the most-collusive price regardless of whether the market is in a boom or a recession phase.

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[14]Moreover, if (44) is relaxed, the monopoly region may disappear altogether. Note also that (15) now may be relaxed as well: e.g., if $\delta r \geq (n-1)/n$, then cyclical pricing is impossible in the Markov-growth game, but it will be possible when transitory demand shocks are permitted, provided that sufficiently large shocks are allowed.

An interesting direction for future empirical research would be to examine the cyclical properties of collusive prices with industry-level data. Our theory characterizes the most-collusive prices in an industry as functions of the business-cycle parameters that describe the evolution of demand in that industry. Given that Hamilton (1989) has developed econometric techniques for estimating these same business-cycle parameters, our predictions thus may be especially well-suited for empirical analysis. Ideally, industry-specific price and sales data could be used to estimate for each industry the expected duration of boom and recession phases, so that the pricing predictions of our model could then be compared with industry pricing data.

As an empirical analyses of this kind for industry-level data has not yet been performed, it is at this point premature to predict whether collusive prices in specific industries will be pro- or countercyclical. Nevertheless, it is perhaps instructive to refer to Hamilton's (1989) description of the aggregate data. Working with U.S. real G.N.P. data, he finds that growth rates are positively correlated across quarters and that boom phases last longer in expectation than do recession phases. According to our theory, if these broad features of the U.S. business cycle are representative of the demand cycles experienced by a given industry, then the prediction of procyclical collusive pricing appears most salient.

Our model is constructed to isolate and characterize the consequences of stochastic demand growth cycles for collusive pricing. Interesting future work might further assess the robustness of our conclusions. We mention here three possibilities. First, we assume throughout that marginal costs are acyclic. While this assumption simplifies the analysis considerably, it may be interesting to explore an extended model that also accounts for cyclical movements in marginal costs. The related empirical literature seems somewhat inconclusive on this matter, as evidence exists that offers some support for acyclic (Miron and Zeldes, 1988) and also procyclical (Bils, 1987) marginal costs.[15] This suggests that an extended theory may be more useful in some industries than others.

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[15]Yet another possibility is that booms are caused by technology shocks, in which case marginal costs may be countercyclical.

Second, we have also maintained the assumption that firms can distinguish between transitory demand shocks and turning points in the business cycle, an assumption that underlies our sharp distinction between the response of collusive prices to these different movements. If firms cannot directly observe the state of the business cycle but instead only know the current demand realization and the underlying demand-generating process, then they must draw inferences about whether a given movement in current demand reflects a cyclical turning point or simply a transitory shock.[16] In this environment, the sharp distinction we have drawn between collusive price responses to transitory demand shocks versus cyclical turning points is likely to be softened, and opportunities for richer cyclical price movements may arise.

Finally, we adopt a partial-equilibrium perspective and take the business cycle as exogenous. This modeling approach enables us to highlight a number of intuitive effects in a simple framework, but the simplification is not without costs. Following the program set forth by Rotemberg and Woodford (1992), future research might embed the collusion model explored here in a dynamic general-equilibrium model, in order to explore more fully the macroeconomic implications of our work. The broad effects identified in this paper should arise also in a general-equilibrium framework, but new counter effects may appear there as well. For example, the distinction between boom and recession phases may be less significant for colluding firms, once entry and exit are allowed. Also, with endogenous interest rates, it could be that discounting is more severe in booms, and this could partially reverse the collusion-enhancing effect of persistent booms.

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[16]Hamilton (1989) develops techniques with which the econometrician can draw such inferences, and these techniques should be useful for modeling the inferences of colluding firms as well.

APPENDIX

Before proving our theorems, we first establish the following lemmas:

Lemma 1: In the Markov-growth game, if $\min\{\Delta\lambda b - R, \Delta\rho r - B\} \geq 0$, then $P_b^c = P_r^c = P_m$.

Proof: Observe first that

(A1). $[\Delta\rho r - B] - [\Delta\lambda b - R] = \Delta(b-r)(1 - \lambda - \rho)$

and so, e.g., the minimum value is $\Delta\lambda b - R$ under positive correlation. In any event, under the condition of the lemma, we have that $\Delta\lambda b - R \geq 0$ and $\Delta\rho r - B \geq 0$, and thus (10) and (11) are satisfied at $P_b^c = P_r^c = P_m$.

Lemma 2: In the Markov-growth game, if

(i). $\min\{\Delta\lambda b - R, \Delta\rho r - B\} < 0$, and

(ii). $RB > (\Delta\lambda b)(\Delta\rho r)$,

then $P_b^c = P_r^c = c$.

Proof: From (i) and (ii), it follows that $R > 0$ and $B > 0$. Suppose first that $\lambda = 0$ or $\rho = 0$. Then (10) and (11) require $P_b \leq c$ and $P_r \leq c$, and so $P_b^c = P_r^c = c$. If $\lambda > 0$ and $\rho > 0$, then under (ii) $P_b > c$ can satisfy (10) and (11) only if

$$\pi(P_b) \geq \pi(P_r)R/(\Delta\lambda b) \geq \pi(P_b)RB/[(\Delta\lambda b)(\Delta\rho r)] > \pi(P_b),$$

a contradiction. Thus, $P_b \leq c$ and so (11) implies $P_r \leq c$. Hence $P_b^c = P_r^c = c$.

Proof of Theorem 1: It is easy to confirm that

(A2). $\bar{G}_b \leq 1/(1 - \delta b)$

(A3). $\bar{G}_r \geq 1/(1 - \delta r)$,

with the former inequality strict if $\rho > 0$ and the latter strict if $\lambda > 0$. We also have that

(A4). $b\bar{G}_b - r\bar{G}_r = (b-r)\Delta/\delta > 0$

(A5). $\Delta\lambda b - R = \delta\{r(1-\lambda)\bar{G}_r + b\lambda\bar{G}_b\} - (n-1)$

(A6). $\Delta\rho r - B = \delta\{b(1-\rho)\bar{G}_b + r\rho\bar{G}_r\} - (n-1)$.

To prove part (i) of Theorem 1, assume $\delta r \geq (n-1)/n$ (i.e., $n-1 \leq \delta r/(1-\delta r)$). Using (A4), it is immediate that (A3), (A5) and (A6) then yield

$$\min\{\Delta\lambda b - R, \Delta\rho r - B\} \geq \delta r\bar{G}_r - (n-1) \geq \delta r/(1-\delta r) - (n-1) \geq 0$$

and so Lemma 1 implies $P^c_b = P^c_r = P_m$.

As for part (ii) of Theorem 1, if $\delta b < (n-1)/n$ (i.e., $n-1 > \delta b/(1-\delta b)$), then using (A4), it is direct from (A2), (A5) and (A6) that

$$\max\{\Delta\lambda b - R, \Delta\rho r - B\} \leq \delta b\bar{G}_b - (n-1) \leq \delta b/(1-\delta b) - (n-1) < 0$$

and so $\Delta\lambda b - R < 0$ and $\Delta\rho r - B < 0$. Lemma 2 thus applies, and hence $P^c_b = P^c_r = c$.

Lemma 3: In the Markov-growth game, if

(i). $1 - \lambda - \rho \geq 0$ and $\lambda \geq \hat{\lambda}(\rho)$ or

(ii). $1 - \lambda - \rho \leq 0$ and $\rho \leq \tilde{\rho}(\lambda)$,

then $P^c_b = P^c_r = P_m$.

Proof: Note that $\lambda \geq \hat{\lambda}(\rho)$ is equivalent to $\Delta\lambda b - R \geq 0$ and that $\rho \leq \tilde{\rho}(\lambda)$ is equivalent to $\Delta\rho r - B \geq 0$. Using (A1), Lemma 3 is now an immediate consequence of Lemma 1.

Lemma 4: In the Markov-growth game, if $\rho > \hat{\rho}(\lambda)$, then $P^c_b = P^c_r = c$.

Proof: Note that $\rho > \hat{\rho}(\lambda)$ is equivalent to $RB > (\Delta\rho r)(\Delta\lambda b)$. Further, under positive correlation, $\hat{\rho}(\hat{\lambda}(\rho)) > \rho$ implies $\lambda < \hat{\lambda}(\rho)$ and so $\Delta\lambda b - R < 0$. Similarly, under negative correlation, $\hat{\rho}(\lambda) > \tilde{\rho}(\lambda)$ implies $\Delta\rho r - B < 0$. The proof is thus direct from (A1) and Lemma 2.

Lemma 5: In the Markov-growth game with positive correlation, if $\lambda < \hat{\lambda}(\rho)$ and $\rho \leq \hat{\rho}(\lambda)$, then

(i). $P_b^c = P_m > P_r^c$.

(ii). P_r^c is continuous, increasing in λ and decreasing in ρ.

(iii). $P_r^c \to P_m$ as $\lambda \to \hat{\lambda}(\rho)$ and $P_r^c \to c$ as $\lambda \to 0$.

(iv). P_r^c is increasing in λ along $\hat{\rho}(\lambda)$, with $P_r^c = c$ when $\lambda = 0$ and $P_r^c \to P_m$ as $\lambda \to \lambda^*$.

Proof: We begin with (i). Under the conditions of the lemma, $\Delta\lambda b < R$ and $RB \leq (\Delta\lambda b)(\Delta\rho r)$; thus, $R > 0$ and $\Delta\rho r > B$. When (10) and (11) hold with equality, we have

$$\text{(A7).}\quad \frac{\partial P_b}{\partial P_r}\Big|_{(10)} = \quad [\pi'(P_r)/\pi'(P_b)][\Delta\rho r/B]$$

$$\text{(A8).}\quad \frac{\partial P_b}{\partial P_r}\Big|_{(11)} = \quad [\pi'(P_r)/\pi'(P_b)][R/\Delta\lambda b].$$

Since $\pi'(P) > 0$ over the range of possible prices and since $R > 0$ under the conditions of the lemma, the recession-period incentive constraint (11) binds at the price pair (c,c) and continues to bind along an upward sloping path of prices. (The curve is vertical if $\lambda = 0$.) Note also that under the conditions of the lemma (11) fails when $P_b = P_r$, except in the case of competitive pricing, and so (11) when binding crosses the 45 degree line only at (c,c). Finally, observe that (11) holds with slack for higher values of P_b or lower values of P_r. Let $\bar{P}_r \in [c, P_m)$ denote the value for P_r that solves (11) with equality when $P_b = P_m$. Figure 2 illustrates.

Consider now the boom-period incentive constraint (10). Clearly, if $B \leq 0$, then any $P_b \geq c$ and $P_r \geq c$ satisfy (10), and so $(P_b^c, P_r^c) = (P_m, \bar{P}_r)$ and (i) holds. If instead $B > 0$, then the incentive constraint (10) emanates from (c,c) when binding and slopes upward, and (10) holds with slack for higher values of P_r and lower values of P_b. Furthermore, we have from (A7) and (A8) that at any point of intersection, (A9). $\frac{\partial P_b}{\partial P_r}\Big|_{(10)} - \frac{\partial P_b}{\partial P_r}\Big|_{(11)} = [\pi'(P_r)/\pi'(P_b)][(\Delta\lambda b)(\Delta\rho r) - RB]/(\Delta\lambda bB)$,

and so under the conditions of the lemma, either $RB < (\Delta\lambda b)(\Delta\rho r)$ and the two incentive constraints when binding cross one time only at the competitive price with the binding boom constraint (10) otherwise lying left of the binding recession constraint (11), or $RB = (\Delta\lambda b)(\Delta\rho r)$ and the two binding incentive constraints overlap and traverse exactly the same prices. In either event, $(P_b^c, P_r^c) = (P_m, \bar{P}_r)$ and so (i) holds.

To prove (ii), let us assume $1-\lambda-\rho > 0$, $\lambda < \hat{\lambda}(\rho)$ and $\rho < \hat{\rho}(\lambda)$, and consider first small changes in ρ. Since (11) holds with equality over this region when $P_b = P_m$, we may implicitly differentiate and, using $R > 0$, get that

$$\text{sign}\{\frac{\partial P_r^c}{\partial \rho}\} = \text{sign}\{\pi(P_m)\lambda b\frac{\partial \Delta}{\partial \rho} - \pi(P_r^c)\frac{\partial R}{\partial \rho}\}.$$

Deriving that

(A10). $$\frac{\partial \Delta}{\partial \rho} = -\Delta^2 b[1-\delta r] < 0$$

and using $\pi(P_m) > \pi(P_r^c)$ we then find that

$$\pi(P_m)\lambda b\frac{\partial \Delta}{\partial \rho} - \pi(P_r^c)\frac{\partial R}{\partial \rho} < \pi(P_r^c)\Delta^2 b\lambda(r - b) < 0,$$

from which it follows that P_r^c decreases in ρ.

Arguing in a similar manner, we find that

$$\text{sign}\{\frac{\partial P_r^c}{\partial \lambda}\} = \text{sign}\{\pi(P_m)b[\Delta + \lambda\frac{\partial \Delta}{\partial \lambda}] - \pi(P_r^c)\frac{\partial R}{\partial \lambda}\}.$$

Deriving that

(A11). $$\frac{\partial \Delta}{\partial \lambda} = -\Delta^2 r[1-\delta b] < 0,$$

and simplifying, we discover that

$$\pi(P_m)b[\Delta + \lambda\frac{\partial \Delta}{\partial \lambda}] - \pi(P_r^c)\frac{\partial R}{\partial \lambda}$$

$$= [1 - (1 - \rho)\delta b]\Delta^2\{\pi(P_m)b(1 - \delta r) - \pi(P_r^c)r[1 - \delta b]\}/\delta > 0,$$

where the inequality follows from $b > r$ and $\pi(P_m) > \pi(P_r^c)$. Thus, P_r^c increases in λ.

Consider next (iii). As $\lambda \to \hat{\lambda}(\rho)$, we have that $\lambda\Delta b \to R$. The binding constraint (11) then approaches the 45 degree line, and so $P_r^c = \bar{P}_r \to P_m$. Next, as $\lambda \to 0$, $P_r^c = \bar{P}_r \to c$, since as (A8) indicates the binding incentive constraint (11) then becomes vertical at $P_r^c = c$.

Finally, we consider (iv). Again, (11) must bind, and so we may differentiate (11) when $P_b = P_m$ and $\rho = \hat{\rho}(\lambda)$ to get that

$$\text{sign}\{\frac{\partial P_r^c}{\partial \lambda}\}\Big|_{\rho=\hat{\rho}(\lambda)} = \text{sign}\{\pi(P_m)b[\Delta + \lambda\frac{d\Delta}{d\lambda}] - \pi(P_r^c)\frac{dR}{d\lambda}\}.$$

Notice that we use total derivatives here, since $\hat{\rho}$ depends upon λ as well. After further calculations, we find that

$$\pi(P_m)b[\Delta + \lambda\frac{d\Delta}{d\lambda}] - \pi(P_r^c)\frac{dR}{d\lambda}$$

$$= [1 - \delta b + \delta(b - r)\rho^*]\Delta^2\{\pi(P_m)b(1 - \delta r) - \pi(P_r^c)r[1 - \delta b]\}/\delta > 0,$$

where the inequality uses $b > r$ and $\pi(P_m) > \pi(P_r^c)$. Thus, P_r^c increases in λ along $\rho = \hat{\rho}(\lambda)$.

Proof of Theorem 3: Follows immediately from Lemmas 3-5.

Lemma 6: In the Markov-growth game with negative correlation, if $\tilde{\rho}(\lambda) < \rho \le \hat{\rho}(\lambda)$, then

(i). $P_r^c = P_m > P_b^c$.

(ii). P_b^c is continuous, increasing in λ and decreasing in ρ.

(iii). $P_b^c \to P_m$ as $\rho \to \tilde{\rho}(\lambda)$.

(iv). P_b^c is decreasing in λ along $\hat{\rho}(\lambda)$, with $P_b^c \to P_m$ as $\lambda \to \lambda^*$.

Proof: Given the conditions of the lemma, we have that $B > \Delta\rho r$ and $RB \le (\Delta\lambda b)(\Delta\rho r)$, and so $B > 0$ and $R < \Delta\lambda b$. Looking at (A7), it follows that the boom incentive constraint (10) when binding is upward sloping. This constraint emanates from the point (c,c) and does not bind at any other point along the 45 degree line; furthermore, beyond the price pairs at which the constraint binds, higher values for P_r and lower values for P_b continue to satisfy (10). Let $\bar{P}_b \in (c,P_m)$ denote the value for P_b that solves (10) with equality when $P_r = P_m$. (Since $\rho > 0$ over the given range, $\bar{P}_b > c$.) Figure 3 illustrates.

Consider next the recession incentive constraint (11). If $R \leq 0$, then any $P_b \geq c$ and $P_r \geq c$ satisfy (11), and so $(P_b^c, P_r^c) = (\bar{P}_b, P_m)$, in which case (i) holds. If instead $R > 0$, then by (A8) we have that (11) when binding emanates from (c,c) and slopes upward, and (11) holds with slack for higher values of P_b and lower values of P_r. Furthermore, using (A9) and under the conditions of the lemma, the incentive constraint curve that is associated with (10) binding lies above (when $RB < (\Delta\lambda b)(\Delta\rho r)$) or on top of (when $RB = (\Delta\lambda b)(\Delta\rho r)$) the incentive constraint curve associated with (11). In either case, we have $(P_b^c, P_r^c) = (\bar{P}_b, P_m)$. This proves (i).

As for (ii), let us assume that $1-\lambda-\rho < 0$ and $\tilde{\rho}(\lambda) < \rho \leq \hat{\rho}(\lambda)$. To determine the effect of an increase in ρ on P_b^c, recall that (10) binds at the most-collusive price pair; using $B > 0$ and implicitly differentiating this constraint reveals that

$$\text{sign}\{\frac{\partial P_b^c}{\partial \rho}\} = \text{sign}\{\pi(P_m)r[\Delta + \rho\frac{\partial \Delta}{\partial \rho}] - \pi(P_b^c)\frac{\partial B}{\partial \rho}\}.$$

Using (A10) and simplifying, we find that

$$\text{sign}\{\frac{\partial P_b^c}{\partial \rho}\} = \text{sign}\{\pi(P_m)r(1 - \delta b) - \pi(P_b^c)b(1 - \delta r)\}.$$

But using (11) when $P_r^c = P_m$, we have that

$$\text{(A12)} \quad \pi(P_m)r(1 - \delta b) - \pi(P_b^c)b(1 - \delta r)$$
$$\leq \pi(P_m)\{r(1 - \delta b)\Delta\lambda - R(1 - \delta r)\}/(\Delta\lambda)$$
$$= \pi(P_m)\{n\delta r - (n - 1)\}/(\Delta\lambda)$$
$$< 0,$$

where the final inequality derives from (15). Thus, P_b^c decreases in ρ.

Next, suppose that λ increases. Working again with (10), we find that

$$\text{sign}\{\frac{\partial P_b^c}{\partial \lambda}\} = \text{sign}\{\pi(P_m)\rho r\frac{\partial \Delta}{\partial \lambda} - \pi(P_b^c)\frac{\partial B}{\partial \lambda}\}.$$

Using (A11) and simplifying,

$$\text{sign}\{\frac{\partial P_b^c}{\partial \lambda}\} = -\text{sign}\{\pi(P_m)r(1 - \delta b) - \pi(P_b^c)b(1 - \delta r)\},$$

and so (A12) implies that P_b^c increases with λ, completing the proof of (ii).

Next, for (iii), observe that as $\rho \to \tilde{\rho}(\lambda)$, $\Delta\rho r \to B$, and so the binding incentive constraint (10) then approaches the 45 degree line. It follows that $P_b^c = \bar{P}_b \to P_m$.

Finally, we establish (iv). If λ increases and $\rho = \hat{\rho}(\lambda)$, then both incentive constraints bind throughout, and so the effect on P_b^c can be determined by totally differentiating the recession incentive constraint (11) for the case in which $\rho = \hat{\rho}(\lambda)$ and $P_r^c = P_m$. We find that

$$\text{sign}\{\frac{\partial P_b^c}{\partial \lambda}\} = \text{sign}\{\pi(P_m)\frac{dR}{d\lambda} - \pi(P_b^c)b[\Delta + \lambda\frac{d\Delta}{d\lambda}]\}.$$

Simplifying then reveals that

$$\text{sign}\{\pi(P_m)\frac{dR}{d\lambda} - \pi(P_b^c)b[\Delta + \lambda\frac{d\Delta}{d\lambda}]\} = \text{sign}\{\pi(P_m)r(1 - \delta b) - \pi(P_b^c)b(1 - \delta r)\},$$

and thus (A12) implies that P_b^c decreases with λ along $\rho = \hat{\rho}(\lambda)$, completing the proof of (iv).

Proof of Theorem 4: Follows immediately from Lemmas 3, 4 and 6.

Section V Definitions and Facts:

Let us define $E(\tilde{\omega}) \equiv E\{\omega(P^*(\tilde{\omega}/\varepsilon)\varepsilon)\}$. Calculations and integration by parts reveals

(A13) $E(\tilde{\omega}) = \tilde{\omega}/(n-1)$, if $\tilde{\omega} \leq \underline{\varepsilon}\pi(P_m)(n-1)/n$

(A14) $E(\tilde{\omega}) = \tilde{\omega}/(n-1) - \pi(P_m)/n \displaystyle\int_{\underline{\varepsilon}}^{[n/(n-1)]\tilde{\omega}/\pi(P_m)} F(\varepsilon)d\varepsilon$, if $\underline{\varepsilon}\pi(P_m)(n-1)/n < \tilde{\omega} < \bar{\varepsilon}\pi(P_m)(n-1)/n$

(A15) $E(\tilde{\omega}) = \pi(P_m)/n$, if $\tilde{\omega} \geq \bar{\varepsilon}\pi(P_m)(n-1)/n$,

where $F(\varepsilon)$ is the distribution function for ε. It follows that $E(\tilde{\omega})$ is initially linear and increasing at the rate $1/(n-1)$, then increasing at a lower rate and (strictly) concave, and finally constant at the value $\pi(P_m)/n$. The fixed point equations (40) and (41) now may be rewritten as

(A16) $0 = E(\tilde{\omega}_r)\Delta\rho r + E(\tilde{\omega}_b)[(n-1) - B] - \tilde{\omega}_b$

(A17) $0 = E(\tilde{\omega}_b)\Delta\lambda b + E(\tilde{\omega}_r)[(n-1) - R] - \tilde{\omega}_r$.

Observe that $\tilde{\omega}_b = \tilde{\omega}_r = 0$ satisfies (A16) and (A17).

In correspondence with (A16) and (A17) when $\tilde{\omega}_b = \tilde{\omega}_r$, we may define

(A18) $f_b(\tilde{\omega}) \equiv E(\tilde{\omega})[\Delta\rho r + (n-1) - B] - \tilde{\omega}$

(A19) $f_r(\tilde{\omega}) \equiv E(\tilde{\omega})[\Delta\lambda b + (n-1) - R] - \tilde{\omega}$.

Thus, e.g., when $f_b(\tilde{\omega}) = 0$, it follows that the boom-period incentive constraint (A16) is satisfied on the 45 degree line at $\tilde{\omega}_b = \tilde{\omega}_r = \tilde{\omega}$. Note $f_b(\tilde{\omega})$ and $f_r(\tilde{\omega})$ are (weakly) concave.

We now differentiate the boom-period fixed point equation (A16) to get

(A20) $\frac{\partial\tilde{\omega}_b}{\partial\tilde{\omega}_r}\Big|_b = E'(\tilde{\omega}_r)\Delta\rho r \,/\, \{1 - E'(\tilde{\omega}_b)[(n-1) - B]\} = E'(\tilde{\omega}_r)\Delta\rho r \,/\, [E'(\tilde{\omega}_b)\Delta\rho r - f_b'(\tilde{\omega}_b)]$.

Similarly, the recession-period fixed point equation (A17) satisfies

(A21) $\frac{\partial\tilde{\omega}_b}{\partial\tilde{\omega}_r}\Big|_r = \{1 - E'(\tilde{\omega}_r)[(n-1) - R]\} \,/\, [E'(\tilde{\omega}_b)\Delta\lambda b] = [E'(\tilde{\omega}_r)\Delta\lambda b - f_r'(\tilde{\omega}_r)] \,/\, [E'(\tilde{\omega}_b)\Delta\lambda b]$.

Lemma 7: In the Markov-growth game with transitory shocks, if

(i). $\min\{\Delta\lambda b - R, \Delta\rho r - B\} > 0$

(ii) $\max\{\Delta\lambda b - R, \Delta\rho r - B\} < (\bar{\varepsilon}-1)(n-1)$,

then $\underline{\varepsilon}\pi(P_m)(n-1)/n < \hat{\omega}_b, \hat{\omega}_r < \bar{\varepsilon}\pi(P_m)(n-1)/n$ and $\text{sign}\{\hat{\omega}_b - \hat{\omega}_r\} = \text{sign}\{1-\lambda-\rho\}$.

Proof: Using (A13), (A18) and (A19), we have that condition (i) implies that $f_b'(\tilde{\omega}) > 0$ and $f_r'(\tilde{\omega}) > 0$ for all $\tilde{\omega} \in [0, \underline{\varepsilon}\pi(P_m)(n-1)/n]$; furthermore, it is direct from (A15), (A18) and (A19) that condition (ii) implies that $f_b(\tilde{\omega}) < 0$ and $f_r(\tilde{\omega}) < 0$ for all $\tilde{\omega} \geq \bar{\varepsilon}\pi(P_m)(n-1)/n$. It now follows easily that unique positive roots, $\hat{\omega}_b$ and $\hat{\omega}_r$, exist at which $f_b(\hat{\omega}_b) = f_r(\hat{\omega}_r) = 0$, and further that $\underline{\varepsilon}\pi(P_m)(n-1)/n < \hat{\omega}_b, \hat{\omega}_r < \bar{\varepsilon}\pi(P_m)(n-1)/n$. Given the concavity of f_b and f_r, we have also that $f_b'(\hat{\omega}_b) < 0$ and $f_r'(\hat{\omega}_r) < 0$. Finally, we have from (A1), (A18) and (A19) that

(A22) $f_b(\hat{\omega}_r) = f_b(\hat{\omega}_r) - f_r(\hat{\omega}_r) = E(\hat{\omega}_r)\Delta(b-r)(1-\lambda-\rho)$,

so that $E(\hat{\omega}_r) > 0$ implies

(A23) $\text{sign}\{\hat{\omega}_b - \hat{\omega}_r\} = \text{sign}\{1-\lambda-\rho\}$.

Thus, under conditions (i) and (ii), both fixed point equations cross the 45 degree line, and the boom-period fixed point equation crosses this line at a higher value under positive correlation.

Using (A20) and (A21), it is now a simple matter to see that

(A24) $$\frac{\partial \tilde{\omega}_b}{\partial \tilde{\omega}_r}\Big|_b \in [0,1) \text{ at } \tilde{\omega}_b = \tilde{\omega}_r = \bar{\omega}_b$$

(A25) $$\frac{\partial \tilde{\omega}_b}{\partial \tilde{\omega}_r}\Big|_r > 1 \text{ at } \tilde{\omega}_b = \tilde{\omega}_r = \bar{\omega}_r.$$

Both fixed point equations pass through the origin, and the signs of (A20) and (A21) at the origin depend on the respective signs of B and R, but eventually the respective curves slope upward through their respective 45 degree line crossings. In particular, there must exist values $\underline{\omega}_b$ and $\underline{\omega}_r$ with $0 \leq \underline{\omega}_b \leq \bar{\omega}_b$ and $0 \leq \underline{\omega}_r \leq \bar{\omega}_r$ such that (A16) is satisfied at $(\tilde{\omega}_b,\tilde{\omega}_r) = (\underline{\omega}_b,0)$ and slopes upward from $(\underline{\omega}_b,0)$ through $(\bar{\omega}_b,\bar{\omega}_b)$ and on, while (A17) holds at $(\tilde{\omega}_b,\tilde{\omega}_r) = (0,\underline{\omega}_r)$ and slopes upward from $(0,\underline{\omega}_r)$ through $(\bar{\omega}_r,\bar{\omega}_r)$ and on. The boom-period fixed point equation thus crosses the 45 degree line at $(\bar{\omega}_b,\bar{\omega}_b)$ from above, while the recession-period fixed point equation crosses at $(\bar{\omega}_r,\bar{\omega}_r)$ from below. Neither equation can cross the 45 degree at any other point, except the origin.

It may now be verified that $\underline{\varepsilon}\pi(P_m)(n-1)/n < \min\{\bar{\omega}_b,\bar{\omega}_r\} \leq \hat{\omega}_b, \hat{\omega}_r \leq \max\{\bar{\omega}_b,\bar{\omega}_r\} < \bar{\varepsilon}\pi(P_m)(n-1)/n$, where $\text{sign}\{\hat{\omega}_b - \hat{\omega}_r\} = \text{sign}\{\bar{\omega}_b - \bar{\omega}_r\}$. Combining this with (A23) then proves the lemma. Figure 5 illustrates the case where $1-\lambda-\rho > 0$.

Lemma 8: In the Markov-growth game with transitory shocks, if

(i). $\min\{\Delta\lambda b - R, \Delta\rho r - B\} \in (0, (\bar{\varepsilon}-1)(n-1))$

(ii) $\max\{\Delta\lambda b - R, \Delta\rho r - B\} \geq (\bar{\varepsilon}-1)(n-1)$,

then $\underline{\varepsilon}\pi(P_m)(n-1)/n < \min\{\hat{\omega}_b, \hat{\omega}_r\} < \bar{\varepsilon}\pi(P_m)(n-1)/n$ and $\text{sign}\{\hat{\omega}_b - \hat{\omega}_r\} = \text{sign}\{1-\lambda-\rho\}$.

Proof: Suppose first that $1-\lambda-\rho > 0$. Then (A1) and conditions (i) and (ii) imply

$$\Delta\rho r - B \geq (\bar{\varepsilon}-1)(n-1) > \Delta\lambda b - R > 0$$

and so (A13) - (A15) and (A18) and (A19) yield that $f_b'(\tilde{\omega}) > 0$ and $f_r'(\tilde{\omega}) > 0$ for all $\tilde{\omega} \in [0, \underline{\varepsilon}\pi(P_m)(n-1)/n]$ and $f_b(\bar{\varepsilon}\pi(P_m)(n-1)/n) \geq 0 > f_r(\bar{\varepsilon}\pi(P_m)(n-1)/n)$. It then follows that $\bar{\omega}_r$ exists at which $f_r(\bar{\omega}_r) = 0 > f_r'(\bar{\omega}_r)$, and further that $\underline{\varepsilon}\pi(P_m)(n-1)/n < \bar{\omega}_r < \bar{\varepsilon}\pi(P_m)(n-1)/n$. There also exists a positive root at which $f_b(\bar{\omega}_b) = 0$, and this root satisfies $\bar{\omega}_b \geq \bar{\varepsilon}\pi(P_m)(n-1)/n$ and $f_b'(\bar{\omega}_b) < 0$. Clearly, $\bar{\omega}_b > \bar{\omega}_r$.

The recession-period fixed point equation then takes the same form as described in the proof of Lemma 7 (for the positive correlation case) and as presented in Figure 5. The boom-period fixed point equation is also described as before, except that now it is flat when it crosses the 45 degree line (since $\bar{\omega}_b \geq \bar{\varepsilon}\pi(P_m)(n-1)/n$ and $E'(\bar{\omega}_b) = 0$ by (A15)). It follows that $\hat{\omega}_b > \hat{\omega}_r > \bar{\omega}_r > \underline{\varepsilon}\pi(P_m)(n-1)/n$ and $\bar{\varepsilon}\pi(P_m)(n-1)/n > \hat{\omega}_r$. Related arguments apply when $1-\lambda-\rho \leq 0$.

Lemma 9: In the Markov-growth game with transitory shocks, if

$$\min\{\Delta\lambda b - R, \Delta\rho r - B\} \geq (\bar{\varepsilon}-1)(n-1)$$

then $(\hat{\omega}_b, \hat{\omega}_r) \geq (\bar{\varepsilon}\pi(P_m)(n-1)/n, \bar{\varepsilon}\pi(P_m)(n-1)/n)$.

Proof: In this case, we have from (A13) - (A15) and (A18) - (A19) that $f_b(\bar{\varepsilon}\pi(P_m)(n-1)/n) \geq 0$ and $f_r(\bar{\varepsilon}\pi(P_m)(n-1)/n) \geq 0$. Since f_b and f_r always have a positive root, it follows that these roots satisfy $\bar{\omega}_b \geq \bar{\varepsilon}\pi(P_m)(n-1)/n > 0 = f_b(\bar{\omega}_b)$ and $\bar{\omega}_r \geq \bar{\varepsilon}\pi(P_m)(n-1)/n > 0 = f_r(\bar{\omega}_r)$, respectively. Now use (A15) to see that $(\hat{\omega}_b, \hat{\omega}_r) = (\bar{\omega}_b, \bar{\omega}_r)$ satisfies (A16) and (A17).

Lemma 10: In the Markov-growth game with transitory shocks, if

(i). $\min\{\Delta\lambda b - R, \Delta\rho r - B\} \leq 0$

(ii) $\max\{\Delta\lambda b - R, \Delta\rho r - B\} \leq (\bar{\varepsilon}-1)(n-1)$

(iii). $\rho \leq \hat{\rho}(\lambda)$,

then $\underline{\varepsilon}\pi(P_m)(n-1)/n \leq \max\{\hat{\omega}_b, \hat{\omega}_r\} \leq \bar{\varepsilon}\pi(P_m)(n-1)/n$ and $\text{sign}\{\hat{\omega}_b - \hat{\omega}_r\} = \text{sign}\{1-\lambda-\rho\}$.

Proof: Suppose that $1-\lambda-\rho > 0$. Then (A1) and conditions (i) and (ii) imply

(A26) $\Delta\lambda b - R \leq 0$

(A27) $\Delta\rho r - B \leq (\bar{\varepsilon}-1)(n-1)$,

and so $f_r'(\tilde{\omega}) \leq 0$ for all $\tilde{\omega} \in [0, \underline{\varepsilon}\pi(P_m)(n-1)/n]$ and $f_b(\bar{\varepsilon}\pi(P_m)(n-1)/n) \leq 0$. Next, condition (iii) and $1-\lambda-\rho > 0$ imply that

(A28) $\Delta\rho r - B > 0$.

Otherwise, we have $\Delta\rho r - B \leq 0$, or equivalently $\rho \geq \tilde{\rho}(\lambda) > 0$, where the definition of $\tilde{\rho}(\lambda)$ is extended into the positive correlation range. Since $\rho > 0$, it follows that $B > 0$. Using (A1), we also have that $\lambda b\Delta - R < 0$. It then would follow that $RB > (\lambda b\Delta)(\rho r\Delta)$, or equivalently $\rho > \hat{\rho}(\lambda)$, contradicting condition (iii); thus, (A28) must hold. This implies that $f_b'(\tilde{\omega}) > 0$ for all $\tilde{\omega} \in [0, \underline{\varepsilon}\pi(P_m)(n-1)/n]$. It now follows that $\bar{\omega}_b$ exists at which $f_b(\bar{\omega}_b) = 0 > f_b'(\bar{\omega}_b)$, and further that $\underline{\varepsilon}\pi(P_m)(n-1)/n < \bar{\omega}_b \leq \bar{\varepsilon}\pi(P_m)(n-1)/n$. There are two cases for f_r. If (A26) holds strictly, then $f_r(\tilde{\omega}) < 0$ and $f_r'(\tilde{\omega}) < 0$ for all $\tilde{\omega} > 0$; alternatively, if (A26) holds with equality, then $f_r(\tilde{\omega}) = 0$ for all $\tilde{\omega} \in [0, \underline{\varepsilon}\pi(P_m)(n-1)/n]$, with $f_r(\tilde{\omega}) < 0$ and $f_r'(\tilde{\omega}) < 0$ for all $\tilde{\omega} > \underline{\varepsilon}\pi(P_m)(n-1)/n$.

Referring to (A20) and (A21), we thus have that the boom-period fixed point equation eventually slopes upward, crossing the 45 degree line at $\bar{\omega}_b$, as in Figure 5, while the recession-period incentive constraint slopes upward and is everywhere above the 45 degree line (if (A26) holds strictly), or runs along the 45 degree line up until $\tilde{\omega}_r = \underline{\varepsilon}\pi(P_m)(n-1)/n$ and then slopes upward and remains above the 45 degree line (if (A26) holds with equality). Moreover, for $\tilde{\omega} \in [0, \underline{\varepsilon}\pi(P_m)(n-1)/n]$, the former takes slope $\Delta\rho r/B$ while the latter takes slope $R/(\Delta\lambda b) \geq 1$. The boom-period fixed point equation thus departs the origin northwest of that of the recession period if $B \leq 0$, or if $B > 0$ and condition (iii) holds strictly. An intersection of the two curves is then necessary, and it must be that $\bar{\varepsilon}\pi(P_m)(n-1)/n \geq \bar{\omega}_b > \hat{\omega}_b > \hat{\omega}_r$; in addition, since the two curves are intially linear out of the origin, we have $\hat{\omega}_b \geq \underline{\varepsilon}\pi(P_m)(n-1)/n$, with equality when $\rho = \hat{\rho}(\lambda)$ and the two curves are collinear initially.

Finally, the case of negative correlation may be solved analogously. When there is zero correlation, the conditions of the lemma require $\rho = \hat{\rho}(\lambda)$ and so $\hat{\omega}_b = \hat{\omega}_r = \underline{\varepsilon}\pi(P_m)(n-1)/n$.

Lemma 11: In the Markov-growth game with transitory shocks, if

(i). $\min\{\Delta\lambda b - R, \Delta\rho r - B\} \leq 0$

(ii) $\max\{\Delta\lambda b - R, \Delta\rho r - B\} > (\bar{\varepsilon}-1)(n-1)$

(iii). $\rho \leq \hat{\rho}(\lambda)$,

then $\underline{\varepsilon}\pi(P_m)(n-1)/n \leq \max\{\hat{\omega}_b, \hat{\omega}_r\}$ and $\text{sign}\{\hat{\omega}_b - \hat{\omega}_r\} = \text{sign}\{1-\lambda-\rho\}$.

Proof: Suppose $1-\lambda-\rho > 0$. Then (A1) and conditions (i)-(ii) imply (A26) holds while (A27) now fails; hence, (A17) is as described in the proof of Lemma 10, but $f_b(\bar{\varepsilon}\pi(P_m)(n-1)/n) > 0$ is now true, whence $f_b'(\tilde{\omega}_b) > 0$ for all $\tilde{\omega}_b \in [0, \underline{\varepsilon}\pi(P_m)(n-1)/n]$. Thus, (A16) crosses the 45 degree line with zero slope at $\bar{\omega}_b > \bar{\varepsilon}\pi(P_m)(n-1)/n$. As in the proof of Lemma 10, it then follows that $\hat{\omega}_b > \hat{\omega}_r$ and $\hat{\omega}_b \geq \underline{\varepsilon}\pi(P_m)(n-1)/n$; note, though, that $\hat{\omega}_b > \bar{\varepsilon}\pi(P_m)(n-1)/n$ is now possible. Similar arguments apply when $1-\lambda-\rho < 0$, and conditions (i)-(ii) rule out $1-\lambda-\rho = 0$.

Lemma 12: In the Markov-growth game with transitory shocks, if $\rho > \hat{\rho}(\lambda)$, then $\hat{\omega}_b = \hat{\omega}_r = 0$

Proof: As in the proof of Lemma 4, $\rho > \hat{\rho}(\lambda)$ is equivalent to $RB > (\Delta\rho r)(\Delta\lambda b)$, and implies that $\min\{\Delta\lambda b - R, \Delta\rho r - B\} < 0$; thus, it must be that R and B are each positive, $\min\{R,B\} > 0$. Consider now the range $\tilde{\omega}_r \in [0, \underline{\varepsilon}\pi(P_m)(n-1)/n]$, over which (A13), (A20) and (A21) yield

$$\frac{\partial\tilde{\omega}_b}{\partial\tilde{\omega}_r}\bigg|_r = R/(\Delta\lambda b) > (\Delta\rho r)/B = \frac{\partial\tilde{\omega}_b}{\partial\tilde{\omega}_r}\bigg|_b$$

and so (A17) initially lies above (A16), with an intersection possible for the given range only at the origin. A subsequent interesection can occur only if (A16) crosses (A17) from below; this is impossible when $\tilde{\omega}_b \geq \bar{\varepsilon}\pi(P_m)(n-1)/n$, since then (A17) takes infinite slope, as is evident from (A21) and (A15). Further, we have that

$$\frac{\partial^2\tilde{\omega}_b}{\partial\tilde{\omega}_r^2}\bigg|_b = \frac{E''(\tilde{\omega}_r)\Delta\rho r + [\frac{\partial\tilde{\omega}_b}{\partial\tilde{\omega}_r}|_b]^2[(n-1) - B]E''(\tilde{\omega}_b)}{1 - E'(\tilde{\omega}_b)[(n-1) - B]}$$

$$\frac{\partial^2\tilde{\omega}_b}{\partial\tilde{\omega}_r^2}\Big|_r = \frac{E''(\tilde{\omega}_r)[(n-1) - R] + [\frac{\partial\tilde{\omega}_b}{\partial\tilde{\omega}_r}|_r]^2 E''(\tilde{\omega}_b)}{-E'(\tilde{\omega}_b)\Delta\lambda b}.$$

But since $(n-1) - B \geq 0$, $(n-1) - R \geq 0$, $E''(\tilde{\omega}) \leq 0$ and $E'(\tilde{\omega}) \leq 1/(n-1)$ are true generally, and noting $E'(\tilde{\omega}_b) > 0$ for $\tilde{\omega}_b < \bar{\varepsilon}\pi(P_m)(n-1)/n$, we have that (A16) defines a concave function while (A17) gives a convex function, and so (A16) can never cross (A17) from below. Hence, the unique intersection occurs at the origin, where $\hat{\omega}_b = \hat{\omega}_r = 0$.

Proof of Theorem 6: The lemmas give an exhaustive characterization of the most-collusive fixed point solutions. We now define parameter relationships that delineate the regions in Figure 4 for which the various lemmas apply. Let $\hat{\lambda}(\rho,\bar{\varepsilon})$ satisfy $\Delta\lambda b - R = (\bar{\varepsilon}-1)(n-1)$ and let $\tilde{\rho}(\lambda,\bar{\varepsilon})$ satisfy $\Delta\rho r - B = (\bar{\varepsilon}-1)(n-1)$. Calculations reveal that

$$\hat{\lambda}(\rho,\bar{\varepsilon}) = [1-(1-\rho)\delta b]/(1/\lambda^*(\bar{\varepsilon}) - \delta b) \text{ and } \tilde{\rho}(\lambda,\bar{\varepsilon}) = [1-(1-\lambda)\delta r]/[1/\rho^*(\bar{\varepsilon}) - \delta r],$$

where $\lambda^*(\bar{\varepsilon}) = 1 - \rho^*(\bar{\varepsilon})$ and

$$\lambda^*(\bar{\varepsilon}) = \{\bar{\varepsilon}(n-1)/[1+\bar{\varepsilon}(n-1)] - \delta r\}/[\delta(b-r)] \text{ and } \rho^*(\bar{\varepsilon}) = \{\delta b - \bar{\varepsilon}(n-1)/[1+\bar{\varepsilon}(n-1)]\}/[\delta(b-r)].$$

Under assumption (44), we find that $\lambda^*(\bar{\varepsilon}) \in (\lambda^*,1)$ and $\rho^*(\bar{\varepsilon}) \in (0,\rho^*)$, implying that $\hat{\lambda}(\rho,\bar{\varepsilon}) > \hat{\lambda}(\rho)$ and $\tilde{\rho}(\lambda,\bar{\varepsilon}) < \tilde{\rho}(\lambda)$, which corresponds to Figure 4. We note that $\tilde{\rho}(0,\bar{\varepsilon}) > 0$ is true, and so $\tilde{\rho}(\lambda,\bar{\varepsilon})$ intersects $\hat{\lambda}(\rho)$; however, it may or may not occur that $\hat{\lambda}(\rho,\bar{\varepsilon})$ intersects $\tilde{\rho}(\lambda)$.

Using (A1), and looking at Figure 4, it may now be confirmed that Lemma 7 describes Region B, Lemma 8 corresponds to region A, Lemma 9 applies for the monopoly region, Lemma 10 represents that part of region C for which $\rho \geq \tilde{\rho}(\lambda,\bar{\varepsilon})$, Lemma 11 covers the remainder of region C (including the origin), and Lemma 12 describes the competitive region. The lemmas confirm the cyclical features of the most-collusive prices stated in the text, once the prices are recovered from the most-collusive fixed point solutions via (36)-(39) and (42)-(43).

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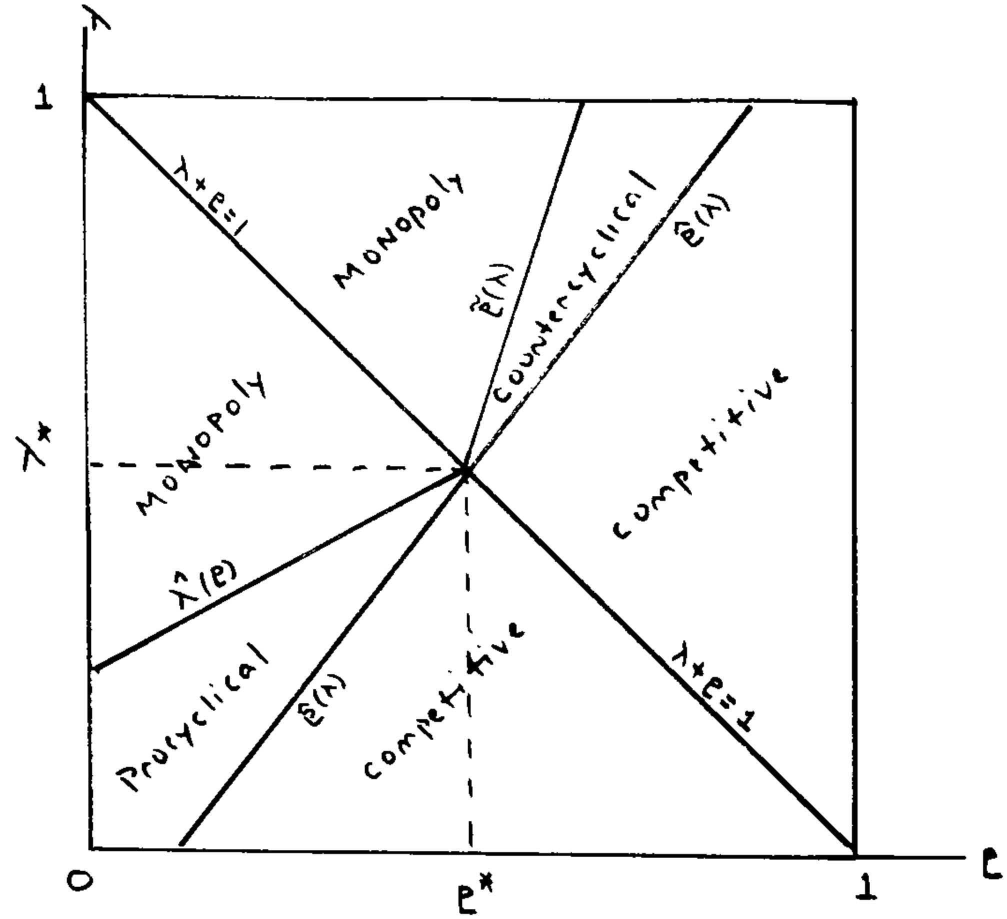


Figure 1

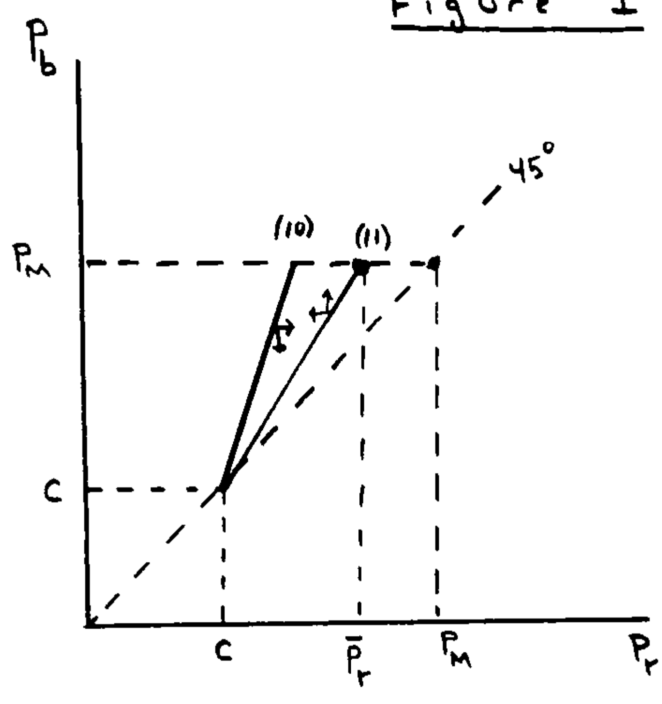


Figure 2

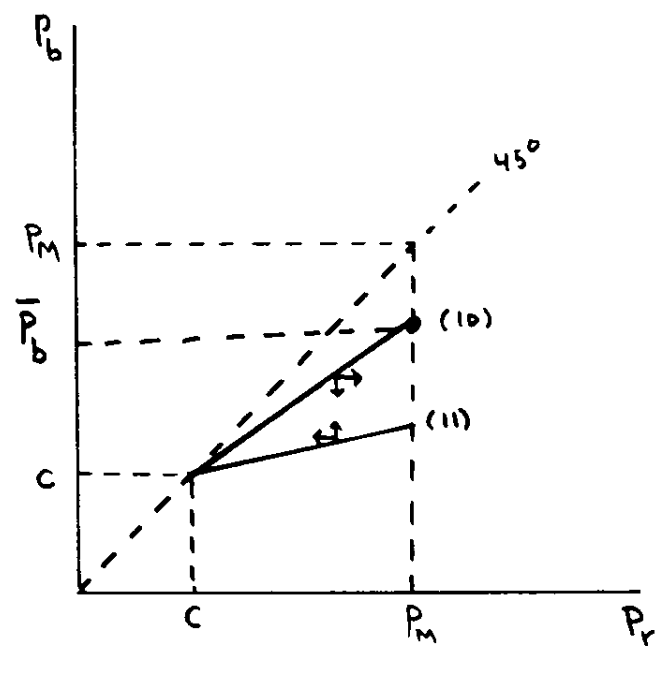
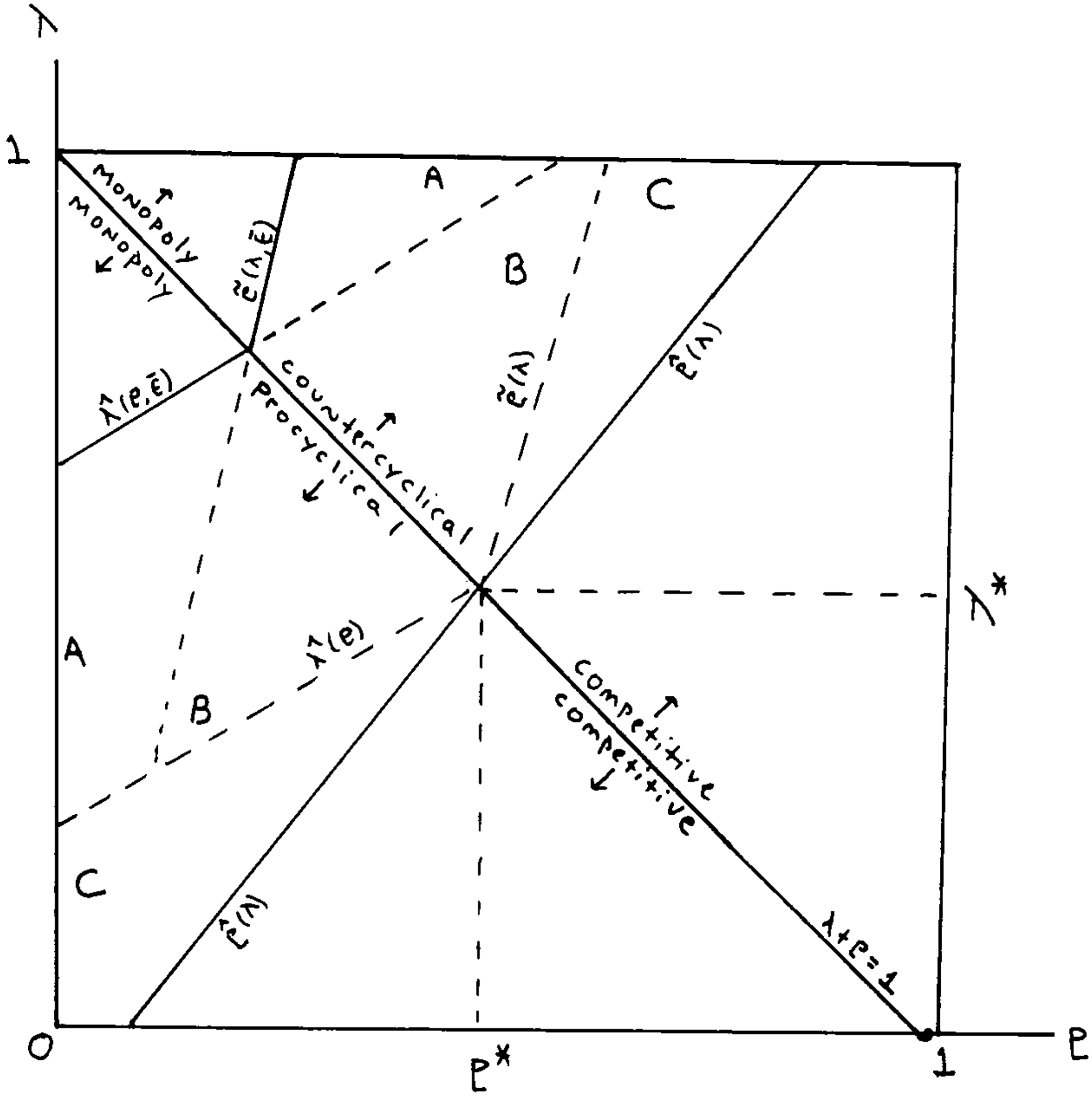


Figure 3


λ
1
A
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monopoly
monopoly
$\tilde{e}(\lambda,\bar{\varepsilon})$
$\hat{\lambda}(e,\bar{\varepsilon})$
countercyclical
procyclical
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$\hat{e}(\lambda)$
λ^*
A
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O
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e

Figure 4

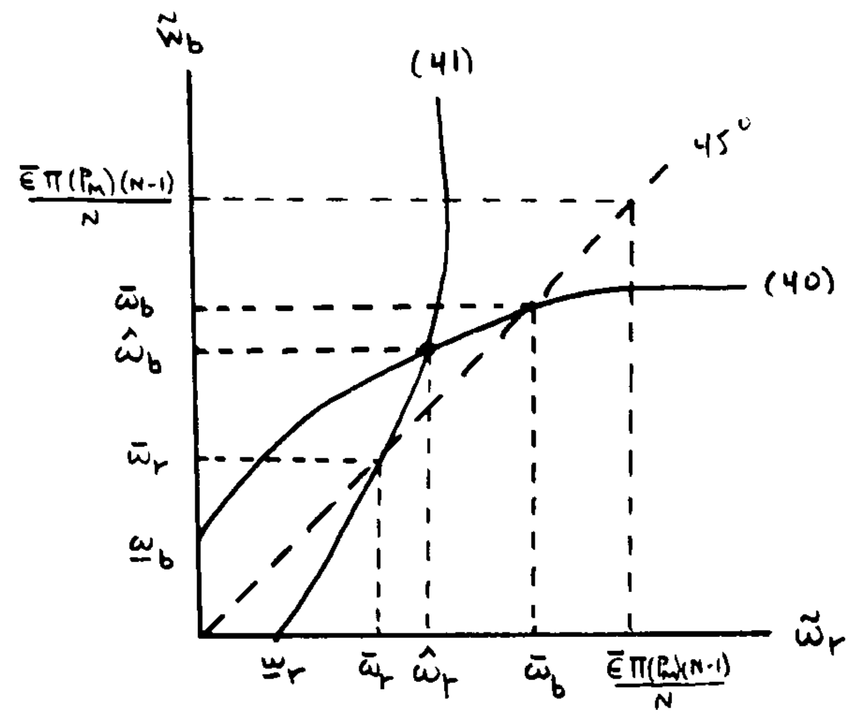

Figure 5

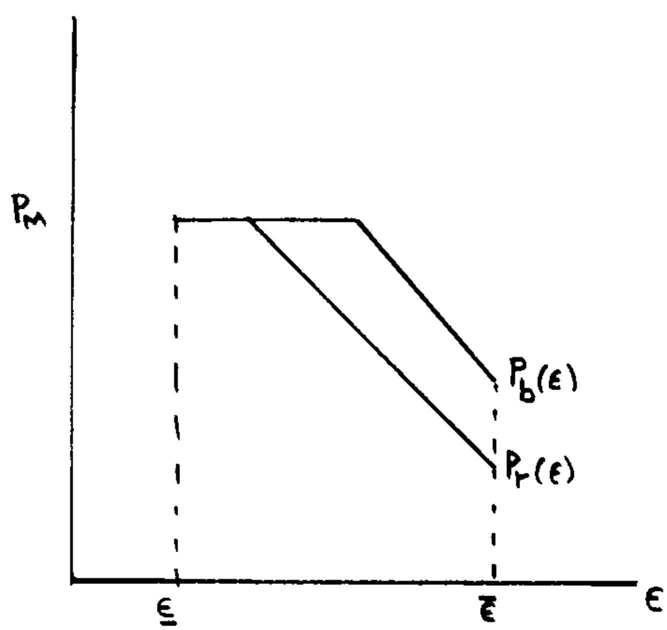

Figure 6a

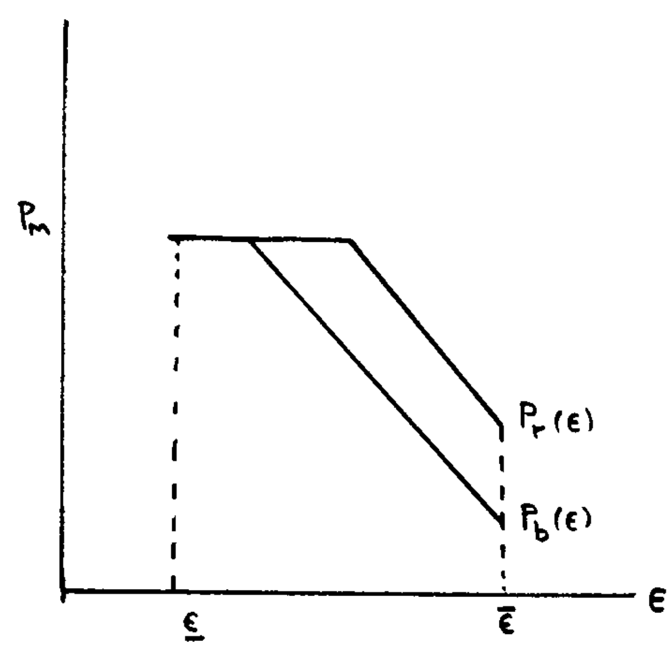

Figure 6b