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COPELAND METHOD II : MANIPULATION, MONOTONICITY AND PARADOXES

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ABSTRACT. An important issue for economics and the decision sciences is to understand why allocation and decision procedures are plagued by manipulative and paradoxical behavior once there are $n \geq 3$ alternatives. Valuable insight is obtained by exploiting the relative simplicity of the widely used Copeland method (CM). By use of a geometric approach, we characterize all CM manipulation, monotonicity, consistency, and involvement properties while identifying which profiles are susceptible to these difficulties. For instance, we show for $n = 3$ candidates that the CM reduces the negative aspects of the Gibbard-Satterthwaite theorem.

This paper continues our investigation into the properties and flaws of the widely used Copeland's method (CM). The purpose of this decision procedure is to extend the majority (or Condorcet [Cn]) rule where an alternative (a candidate, a sport's team, etc.) is selected should it be preferred by a majority of the voters whenever it is compared with any other alternative. When a majority winner does not exist, the CM [C] provides a natural choice that is compatible with Condorcet's approach. With the CM, the winning alternative from each pairwise contest receives one point, the losing candidate receives zero points, and, with a tie, each receives half a point. The sum of assigned points defines a candidate's CM score where the scores determine the CM ranking of the candidates. (To simplify proofs, we use the equivalent weights $(1, 0, -1)$.) In spite of its wide use (e.g., this procedure is commonly used to rank sports teams), surprisingly little is known about it.

Our first paper [SM], examining single profile (i.e., a listing of the voters' preferences) CM properties, contrasted the CM rankings with those of positional procedures, described how the CM rankings can vary as candidates enter or drop out of the election, and compared the "natural" and CM rankings associated with certain profiles. In this paper, we discuss those *multiple profile* issues that arise when CM outcomes for two or more profiles are compared. As examples, while the first profile $\mathbf{p}_1$ could represent the current, sincere preferences of the voters, the second profile $\mathbf{p}_2$ could model situations where some voters now decide to vote for the $\mathbf{p}_1$-CM winner, or they vote strategically to try to alter the outcome, or they forget to vote.

Key words and phrases. Copeland, monotonicity, voting, consistency, strategic voting, manipulation, voting paradoxes.

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The goal is to understand how the two outcomes are related. As such, multiple profile issues include not only the currently faddish topic of strategic voting, but also the traditional normative themes that question what should happen in a fixed population when the preferences change or new voters enter. The first normative concern leads to all of the monotonicity-manipulation issues; the second captures the consistency conditions.

These multiple profile topics plague all decision and economic procedures. Consequently, the relatively simple structure and wide use of CM makes it a valued system to be analyzed to understand, illustrate, identify, and explain the general source of these difficulties. Indeed, the simpler CM geometry even allows the profile space to be separated into regions where unexpected and strategic behavior can, or cannot, arise.

Typically, each multiple profile theme requires specialized methods of analysis, so questions about monotonicity, manipulation, effects of truncated ballots and so forth almost always are treated as separate issues. In this paper we demonstrate that this separated approach is conceptually misleading and unnecessary; instead, these topics can and should be treated as special cases of one analysis. To do so, we use a geometric approach ([S1, Chap. 4]) to unify these topics (and others that may arise) so that each issue can be addressed in the same manner. We demonstrate this approach by starting with three-candidate elections so that the geometric techniques can be illustrated with simple figures from a common three-dimensional cube. After the basic approach is established, it becomes easy to quickly describe the n-candidate situation in a unified manner.

Another important advantage of this geometric approach involves the difficulty of these issues that can be NP-hard [BO, BTT]. Reflecting this severe complexity is the fact that it is typical for conclusions to merely specify whether or not a procedure can suffer a particular (multiple profile) difficulty. While it is understandable why these conclusions are so severely limited, the assertions are unsatisfying if only because common sense and experience teaches us that these are not universal problems. For instance, surely by failing to vote, a voter will not always be rewarded with a better election outcome. Surely a particular voter cannot always successfully manipulate the election outcome.

Rather than being told that a procedure might suffer a problem, we really want to know when and where it can occur; i.e., we want to learn whether the difficulty is serious or just a minor annoyance. This requires characterizing which profiles can, or cannot, experience these difficulties. After all, only after we can specify *all* profiles susceptible to various problems can we hope to recognize which difficulties can be dismissed as mere anomalies, to identify which situations should cause worry, to understand why these problems occur, and to be able to transfer what we learn to other allocation or decision systems. Using the geometric approach, this identification problem is surprisingly easy for the CM, and, as we indicate, the basic ideas transfer to other systems.

Our approach (from [S1, Chap. 4]) is straight-forward. Once a particular two-profile issue is specified, we know the relationship between the profiles $\mathbf{p}_1$ and $\mathbf{p}_2$. To illustrate with manipulative behavior, if c_1 is the $\mathbf{p}_1$-CM winner, then all voters who have her top-ranked are content. Other voters, however, may vote strategically to try to elect a personally more preferred candidate. So, $\mathbf{p}_2$ is found by assuming that the strategic voters vote in an optimal manner. Thus, once we know $\mathbf{p}_1$, we also know who might vote strategically and how they should do it. Similarly, to determine what could happen when a voter forgets to vote, $\mathbf{p}_2$ is defined from $\mathbf{p}_1$ by omitting this voter. Indeed, for a multiple profile concern involving changes in an original profile $\mathbf{p}_1$; $\mathbf{p}_2$ is defined by the change.

After defining the profile change, we need to discover whether it affects the outcome. It does iff $\mathbf{p}_1$ and $\mathbf{p}_2$ belong to different profile sets where each set consists of those profiles supporting a particular conclusion. Consequently, different outcomes arise only if the profile change crosses the separating boundary of these sets. By finding all boundaries among the profile sets (which is surprisingly easy to do for most procedures), we can characterize which profile changes cross the boundary (i.e., which multiple profile concerns can occur) and identify where this happens (i.e., we can identify all choices for $\mathbf{p}_1$).

Intuitively, this approach is the same as shooting a water pistol. The nozzle corresponds to $\mathbf{p}_1$ while the weakly expelled water represents $\mathbf{p}_2$. Whether a target (i.e., the separating boundary between profile sets) is hit depends on the position of the pistol ($\mathbf{p}_1$) and how it is aimed ($\mathbf{p}_2 - \mathbf{p}_1$) relative to the target's orientation. The target's orientation (at each point) is determined by a vector perpendicular to the surface pointing in the desired direction of water flow. If the angle between the aimed pistol and the vector is less than 90^o, then the shot water does what we want. However, should this angle exceed 90^o, we are aiming in the wrong direction. More ambitiously, if the pistol is held in a fixed direction (so the direction $\mathbf{p}_2 - \mathbf{p}_1$ is fixed), by examining the normal vectors at each point, we can establish which portions of the surface (which profiles $\mathbf{p}_1$) satisfy the directional requirements. As this description is independent of dimension and choice of direction ($\mathbf{p}_2 - \mathbf{p}_1$), once we describe the geometric boundaries of the relevant sets in profile space, this "water pistol" approach characterizes how to analyze all multiple profile issues for any procedure.

2. Three-candidate CM results

To introduce the geometry of profiles and the basic approach, start with the $n = 3$ candidates $\{c_1, c_2, c_3\}$. The six ways these candidates can be linearly ranked (without indifference) define the six voter types

(2.1)

1	$c_1 \succ c_2 \succ c_3$	3	$c_3 \succ c_1 \succ c_2$	5	$c_2 \succ c_3 \succ c_1$
2	$c_1 \succ c_3 \succ c_2$	4	$c_3 \succ c_2 \succ c_1$	6	$c_2 \succ c_1 \succ c_3$

Instead of listing how many voters are of each type, it suffices to list the fraction of voters with a particular ranking. Thus, if p_j is the fraction of all voters with a type-j preference, $j = 1, \ldots, 6$, then, because the sum of the fractions equals unity, a profile can be identified with a point in the simplex

$$Si(6) = \{(x_1, \ldots, x_6) \mid \sum_{j=1}^{6} x_j = 1, x_j \geq 0\}. \tag{2.2}$$

Each rational point in $Si(6)$ defines several integer profiles because the total number of voters can be any common denominator for the six fractions.

It follows from Table 2.1 that the pairwise election between c_1 and c_2 is determined by the sign of

$$x_{1,2} = p_1 + p_2 + p_3 - p_4 - p_5 - p_6 \tag{2.3}$$

where a positive value means that over half of the voters prefer c_1, a negative value means that c_2 wins this pairwise competition, and a zero value corresponds to a tie. Thus, the hyperplane $H^3_{1,2}$ defined by $p_1 + p_2 + p_3 - p_4 - p_5 - p_6 = 0$ divides the profile space into three regions: $H^3_{1,2}$ is the profile set where the candidates are pairwise tied, and each side of $H^3_{1,2}$ defines the profile set supporting a different candidate. (The superscript "3" denotes the number of candidates.)

As a vector orthogonal to an hyperplane is determined by the coefficients of the defining linear equation, vector

$$\mathbf{N}^3_{1,2} = (1, 1, 1, -1, -1, -1) \tag{2.4}$$

is orthogonal to $H^3_{1,2}$ and points into the profile set supporting c_1. Thus, a profile change in this direction helps c_1 and hurts c_2. In a similar fashion, by using Table 2.1 to determine who would vote for whom, the equations

$$\begin{aligned} x_{2,3} &= p_1 - p_2 - p_3 - p_4 + p_5 + p_6 \\ x_{1,3} &= p_1 + p_2 - p_3 - p_4 - p_5 + p_6 \\ \mathbf{N}^3_{2,3} &= (1, -1, -1, -1, 1, 1) \\ \mathbf{N}^3_{1,3} &= (1, 1, -1, -1, -1, 1) \end{aligned} \tag{2.5}$$

represent the remaining pairwise elections where $x_{j,k} > 0$ means that c_j beats c_k, $x_{j,k} = 0$ defines the separating hyperplane $H^3_{j,k}$ that divides the profiles into the three profile sets for the different outcomes, and $\mathbf{N}^3_{j,k}$ is the normal vector for $H^3_{j,k}$ pointing into the profile set ensuring the c_j pairwise victory. Notice that $x_{j,k} = -x_{k,j}$ and that $\mathbf{N}^3_{j,k} = -\mathbf{N}^3_{k,j}$.

Three candidates define three pairs, so all pairwise outcomes can be depicted as points in a three-dimensional figure called the *representation cube* where the usual (x, y, z) coordinates are replaced with $(x_{1,2}, x_{2,3}, x_{3,1})$ values. As $-1 \leq x_{j,k} \leq 1$, start with the three-dimensional cube defined by these values. The actual pairwise election outcomes are the cube points further restricted by $-1 \leq x_{1,2} + x_{2,3} + x_{3,1} \leq 1$. These inequalities, which determine the slanted sides of the representation cube in Fig.1, exclude values that never could arise with transitive preferences. (For details, see ([S1, Chap. 2.5].)

With the neutrality of CM, we can assume without loss of any generality that c_1 is the CM winner. So, to interpret Fig. 1, observe that c_1 is the Condorcet winner in the region where $x_{1,2} > 0$, $x_{3,1} < 0$ (which, in standard coordinates is the $x > 0$, $z < 0$ region). Cyclic rankings are the positive and negative orthants where $\{x_{1,2} > 0, x_{2,3} > 0, x_{3,1} > 0\}$ and $\{x_{1,2} < 0, x_{2,3} < 0, x_{3,1} < 0\}$. A CM ranking $c_1 \succ c_2 \succ c_3$ is in the ranking region given by the orthant $x > 0$, $y > 0$, $z < 0$ (i.e., $x_{1,2}, x_{2,3} > 0, x_{1,3} < 0$). For our purposes, c_1 is a CM top-ranked candidate iff the outcome is in $\mathcal{D}^3(c_1)$ – the union of the two cyclic regions, the c_1-Condorcet region, specific portions of the two bounding planes for the c_1-Condorcet region $\{x_{1,2} = 0, x_{3,1} < 0\}$, $\{x_{1,2} > 0, x_{3,1} = 0\}$, and the origin $x_{1,2} = x_{2,3} = x_{3,1} = 0$. The $\mathcal{D}^3(c_j)$ sets supporting the selection of other alternatives are symmetric. Notice that outside of the c_1-Condorcet region, the $\mathcal{D}^3(c_1)$ outcomes require c_1 to share the honor of being CM top-ranked (e.g., in the two cyclic regions all candidates are CM winners).

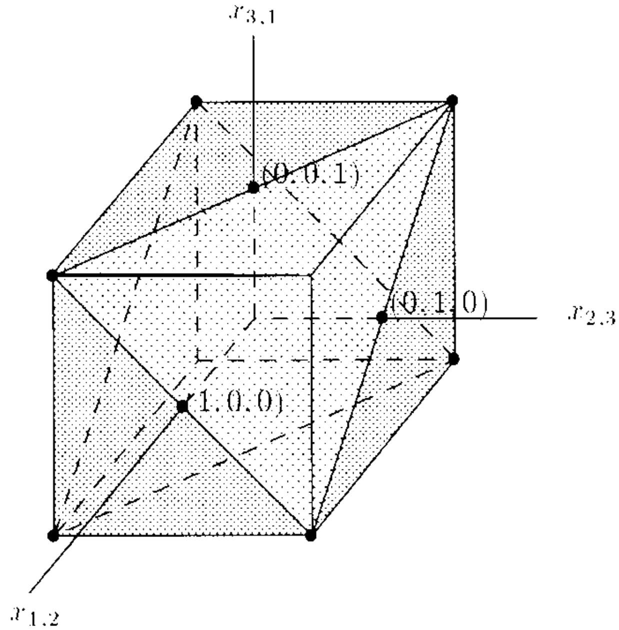

Fig. 1. The representation cube for the three pairs of candidates

Because the $\mathcal{D}^3(c_1)$ geometry is basic for what follows, it is displayed from a more convenient perspective in Fig. 2. This figure, which is a 90^o rotation of Fig. 1, shows the four relevant orthants of $\mathcal{D}^3(c_1)$ including the two cyclic regions. (To keep the figure simple, we ignored the slanted faces.)

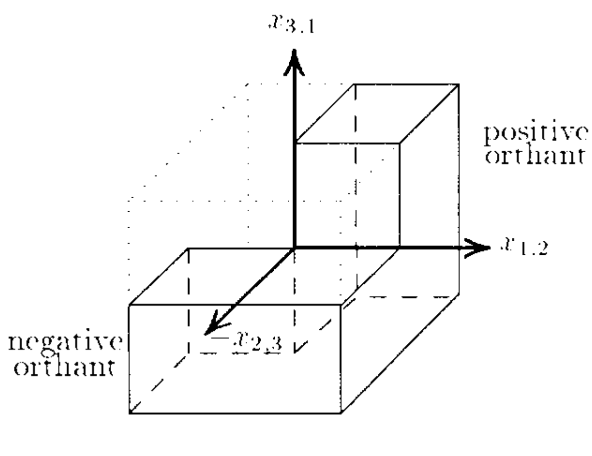

Figure 2. Rotated version of $\mathcal{D}(c_1)$: the c_1-CM region.

2.1. Illustrating examples. We use three-candidate examples to introduce the basic technique and to show how by exploiting the geometry of $\mathcal{D}^3(c_1)$, seemingly different concerns can be resolved with the same argument. While for simplicity and to avoid unlikely cases we assume no pairwise election ends in a tie, our conclusions easily extend to these tie-vote settings.

Monotonicity. A procedure is *monotonic* should a candidate still be elected after she improves her ranking in the individual preferences, while each voter's relative ranking of the other candidates remains unchanged. For example, suppose after a TV political debate, Mr. Smith decides to rank Ms. Young second instead of last where he keeps fixed his relative ranking of the remaining two candidates. We would not expect this preference change to hurt Ms. Young's chances. Indeed, procedures failing to be monotonic (such as, say, runoff elections [Sm, S1]) admit the perversity where we can beat a candidate by increasing her support!

Suppose c_1, Ms. Young, is a CM winner. Before the debate, Mr. Smith had c_1 bottom ranked, so in the original profile $\mathbf{p}_1$ his type was either four or five; assume he was type-four with the $c_3 \succ c_2 \succ c_1$ ranking. By changing his preferences as described, he changes from type-four to -three, so after the debate there is one less type-four voter and one more type-three out of the M voters. Letting $\mathbf{E}_j$ denote a profile of type-j, the profile change is

$$(2.6) \qquad \frac{1}{M}\mathbf{d}_{4-3} = \mathbf{p}_2 - \mathbf{p}_1 = \frac{1}{M}(0,0,1,-1,0,0) = \frac{1}{M}(\mathbf{E}_3 - \mathbf{E}_4).$$

To verify monotonicity, we need to discover whether *any* $\mathbf{p}_1$ supporting a $\mathcal{D}^3(c_1)$ outcome allows the $\mathbf{d}_{4-3}$ profile change (Smith's change of preferences) to dislodge c_1 (Young) from victory. Using the geometric water pistol approach, the equivalent question is to determine whether there is a normal vector $\mathbf{N}_{i,j}$, pointing in the direction of helping c_1 (i.e., toward the interior of $\mathcal{D}^3(c_1)$), where its angle with $\mathbf{d}_{4-3}$ exceeds 90^o. This angular information is determined by the sign of the scalar product of two non-zero vectors

$$(2.7) \qquad (\mathbf{a}, \mathbf{b}) = \sum_{j=1}^{6} a_j b_j$$

where positive, negative, and zero values require, respectively, the angle to be less than, greater than, and equal to 90^o. (As the boundaries are hyperplanes, a zero value requires the profile change to be parallel to the boundary, so it does not cross.) A computation shows that $(\mathbf{N}_{1,2}, \mathbf{d}_{4-3}) = 2$, $(\mathbf{N}_{1,3}, \mathbf{d}_{4-3}) = (\mathbf{N}_{2,3}, \mathbf{d}_{4-3}) = 0$. Thus, the $\mathbf{d}_{4-3}$ profile change fails to influence the $\{c_1, c_3\}$ and the $\{c_2, c_3\}$ pairwise outcomes, but it has a positive impact (for c_1) on the $\{c_1, c_2\}$ election. The same elementary computations prove this is true for any such profile change. Consequently, we recover the established (e.g., see [N]) fact.

Theorem 1. *For $n = 3$ candidates, CM is monotonic.*

Abstention. For a second scenario, suppose Mr. Smith not only missed the TV debate (so Ms. Young, c_1, remains his bottom-ranked candidate), but he forgot to vote. Surely, this helps her — or does it? As a type-four voter, Smith's actions change the original M-voter profile $\mathbf{p}_1$ by dropping one type-four voter to define the profile change $\mathbf{p}_2 - \mathbf{p}_1 = \frac{1}{M-1}\mathbf{p}_1 - \frac{1}{M-1}\mathbf{E}_4$. The $(\mathbf{N}^3_{i,j}, \mathbf{p}_1)$ value returns the $\{c_i, c_j\}$ pairwise election outcome (which is near zero whenever this pairwise election is close to a tie vote). Therefore, the relevant portion of this profile change — the part that indicates whether a different outcome will occur – is the $\mathbf{E}_4 = (0, 0, 0, 1, 0, 0)$ term. Computations prove that $(\mathbf{N}^3_{1,2}, -\mathbf{E}_4) = (\mathbf{N}^3_{1,3}, -\mathbf{E}_4) = (\mathbf{N}^3_{2,3}, -\mathbf{E}_4) = 1$.

As the positive values of the first two terms require Smith's forgetful action to help c_1, if the original profile $\mathbf{p}_1$ is far from the boundaries (so the profile change cannot reach a boundary) or near either hyperplanes $H^3_{1,j}$, $j = 2, 3$, then, as we would expect, Mr. Smith's action can assist, but not hurt, c_1. The impact of the $(\mathbf{N}^3_{2,3}, -\mathbf{E}_4) = 1$ computation is decided by whether there are $\mathbf{p}_1$ locations where a profile change in the $\mathbf{N}^3_{2,3}$ direction hurts c_1. This is equivalent to discovering whether $\mathcal{D}^3(c_1)$ has a $x_{2,3} = 0$ boundary where an c_2 improvement over c_3 moves the outcome outside of $\mathcal{D}^3(c_1)$: namely, can a change in the $x_{2,3}$ direction hurt c_1? (The $\frac{1}{M-1}$ scalar term requires such a profile to be close – a voter away – to the boundary.) It follows immediately from Fig. 2 that this happens only when the $\mathbf{p}_1$ outcome is in the negative orthant (a cyclic region) near the $x_{2,3} = 0$, $x_{1,2} < 0$, $x_{3,1} < 0$ boundary surface. Thus, only for profiles $\mathbf{p}_1$ supporting an outcome where the CM ranking is $c_1 \sim c_2 \sim c_3$ (so all three candidates are selected) and the $c_3 \succ c_2$ outcome is almost tied can Mr. Smith's forgetful actions help him. By not voting, the c_2, c_3 outcome is either $c_2 \sim c_3$ or $c_2 \succ c_3$, so the CM winner is Smith's second-ranked c_2 rather than Ms. Young, his bottom-ranked candidate. Other choices of initial preferences for Smith lead to the same conclusion.

Theorem 2. *For $n = 3$ candidates with c_1 as a CM top-ranked candidate, a CM abstention paradox can occur iff the CM outcome is a complete tie due to a cyclic outcome and the pairwise vote for (c_2, c_3) is nearly tied. The paradox is caused because, by a voter not voting, the ranking of this pair either is reversed, or ends in a tie. For the CM outcome to be personally better, such a voter must have c_1 bottom-ranked.*

Strategic voting. As a third scenario, suppose after Mr. Smith missed the TV debate, he learned that Ms. Young is projected to be a CM winner. Adopting an "anyone but Young (c_1)" attitude, Smith wishes to vote strategically to defeat her. His available options are $\frac{1}{M}\mathbf{d}_{4-j} = \frac{1}{M}(\mathbf{E}_j - \mathbf{E}_4)$. As $(\mathbf{N}^3_{1,k}, \mathbf{d}_{4-j})$ is either positive or zero for $k = 2, 3$ and all j, if the original profile is either more than a person away from the boundary of the profile set defining $\mathcal{D}^3(c_1)$, or near $H^3_{1,k}$, Smith must accept Young's CM victory. But, $(\mathbf{N}^3_{2,3}, \mathbf{d}_{4-j}) > 0$ for $j = 1, 5, 6$ and zero otherwise, so Smith can successfully frustrate Ms. Young's victory – only when crossing an $x_{2,3} = 0$ boundary in favor of c_2 leaves the $\mathcal{D}^3(c_1)$ set. From Fig.2, this requires the $\mathbf{p}_1$ outcome to be in the negative orthant where the CM defines a three-way tie caused by the cyclic $c_2 \succ c_1, c_3 \succ c_2, c_1 \succ c_3$ outcomes. Also, this outcome must be so close to a boundary that Smith's changed vote causes a $c_3 \sim c_2$ tie or $c_2 \succ c_3$ victory (and the CM ranking $c_2 \succ c_1 \succ c_3$).

These possibilities offer Smith several options; he even could pretend to be a type-one voter – he could *slyly manipulate* ([S1]) the outcome – by publicly announcing and voting for Ms. Young (using the ranking $c_1 \succ c_2 \succ c_3$) to defeat her! (This example demonstrates why in the definition of "monotonicity" the voters are required to improve the status of a particular candidate while keeping the relative rankings of all others fixed. Although improving c_1's status, $\mathbf{d}_{4-1}$ also changes the relative rankings of other candidates, so c_1 is hurt by the new profile that should enhance her standing.)

Notice the heavy price Smith pays for his obsession to defeat Young: his strategic vote causes his top-ranked candidate c_3 to lose her status as a CM winner. The new issue, then, is to determine whether other voters could enjoy more strategic success, or whether this example identifies a CM trait that significantly mitigates the effects of strategic voting. To address this question where c_1 is a CM winner, the potential strategic voters (identified by subscript k) and their strategic actions are given by $\mathbf{d}_{k-j} = \frac{1}{M}(\mathbf{E}_j - \mathbf{E}_k)$, $k = 3, 4, 5, 6$, $j = 1, \dots, 6$. We already have examined the effects of crossing a $x_{2,3}$ boundary, so it remains to investigate if a $x_{1,j}$ boundary can be crossed. But the only way $(\mathbf{N}^3_{1,2}, \mathbf{d}_{k-j}) < 0$ is if k and j identify, respectively, a positive and negative component of $\mathbf{N}^3_{1,2}$. Thus $k = 3$ (i.e., the ranking $c_3 \succ c_1 \succ c_2$) and $j = 4, 5, 6$. To see whether this type-three voter can enjoy more success than Mr. Smith by voting strategically, examine Fig. 2 to determine where crossing the $x_{1,2} = 0$ coordinate plane in the c_2 direction helps this voter. It surely hurts should the new profile end up in the $x_{1,2} < 0, x_{2,3} > 0$ region, as this change would elect c_2, the strategic voter's bottom-ranked candidate. The remaining possibility (where $\mathbf{p}_1$ has c_1 as the sole CM winner) is the separating boundary between where c_1 is the Condorcet winner and the negative orthant. Here, this voter's strategic action allows his top-ranked candidate c_3 to be included, but at the expense of also including his bottom ranked c_2. Accepting this cost is the only way a type-three voter can strategically influence the outcome.

Carrying this analysis a step further, recognize that $\mathbf{p}_1$ must be near the cyclic region for this type-three voter to have success. If this voter makes an ever so slight miscalculation because, in fact, $\mathbf{p}_1$ already is in a cyclic region near either a $c_1 \succ c_3$ boundary (of the negative orthant because $x_{3,1} < 0$) or a $c_2 \succ c_3$ boundary (of the positive orthant because $x_{2,3} > 0$), then the previously clever $\mathbf{d}_{3-6}$ strategy, which affects the outcome for all three pairs, backfires as either c_1 or c_2 becomes the sole CM winner — neither of these candidates is top-ranked for our strategic voter and his top-ranked candidate is dropped. In fact, it follows immediately from Fig. 2 that for a particular strategy, there are more opportunities (i.e., more choices of $\mathbf{p}_1$) where the strategy is counter-productive than strategically helpful.

These simple illustrations provide new insight into CM strategic behavior: insight that removes much of the sting of the Gibbard [G]- Satterthwaite [St] Theorem. They underscore, for instance, the critical fact that to avoid being counter-productive a strategic action must be accompanied by surprisingly precise information about the other voters' preferences and intended action. Namely, without knowing the precise $\mathbf{p}_1$ value, a strategic action can hurt, rather than assist, a voter's interests. But, with the exception of Congress and other settings where votes are announced in advance, it is not clear whether such precise information is commonly available. Consequently, it is not clear whether CM strategic action is, in general, wise. Furthermore, the examples prove that for a strategic action to be successful, the original outcome must satisfy exacting and highly unlikely conditions; so, except in contrived examples of the types used to illustrate academic articles, one must wonder whether strategic voting is a practical CM concern. Finally, at least for $n = 3$ candidates, just by carrying out a couple more simple computations, we learn that a penalty usually accompanies CM strategic attempts.

Theorem 3. *For $n = 3$ candidates, suppose c_1 is a CM top-ranked candidate with*

$\mathbf{p}_1$ where $\mathbf{p}_1$ does not admit any pairwise tied votes. For a voter without c_1 top-ranked to successfully manipulate the outcome, $\mathbf{p}_1$ must either define a cycle with near tie votes for the (c_2, c_3) pair, or c_1 as the sole winner with one pairwise election nearly tied. In the first case, a voter with c_1 bottom-ranked can strategically ensure the victory of his second-ranked candidate at the expense of depriving his top-ranked candidate the $\mathbf{p}_1$ status of a CM winner. In the second case, a voter with c_1 middle-ranked can vote strategically to include his top-ranked candidate as a CM winner, but this action ensures his bottom-ranked candidate also becomes a CM winner.

The remaining strategic situations are the highly unlikely settings with two or more pairwise ties. With two ties, certain voters can include their second-ranked candidate with the CM sincere winner c_1, and with all pairwise votes ending in ties, a voter can get his second-ranked candidate selected. As these regions correspond to edges of $\mathcal{D}^3(c_1)$ in Fig.2, we leave the simple analysis to the reader.

Weak consistency. Suppose there are only two election districts where, on election night, Ms. Young's pollwatcher from each district informs her that she is a CM winner in that district. Should her victory party start? Both profiles, $\mathbf{p}_1$, $\mathbf{p}_2$, define results in $\mathcal{D}^3(c_1)$ and the full profile is $\mathbf{p}_3 = \lambda\mathbf{p}_1 + (1-\lambda)\mathbf{p}_2$ where λ is the fraction of all voters that reside in the first district. The $\mathbf{p}_3$ CM outcome, then, is on the line connecting the CM outcomes for $\mathbf{p}_1$ and $\mathbf{p}_2$; the group outcome is determined by the λ value. (It is λ of the distance from the first outcome to the second.) To determine what can happen by combining the votes, it suffices to examine how lines can be drawn in Fig. 2 so that both endpoints are in $\mathcal{D}^3(c_1)$. If both endpoints are in a c_1-Condorcet winner region, then the convexity of this region requires the full line to remain in this region (so Young remains a CM winner). However, with endpoints in different cyclic regions (the positive and negative orthants), it is easy to position the line so that portions are in any desired region. Consequently, the combined outcome can be whatever is desired: Young could be the sole CM winner, she could share this status with either one or both other candidates, or she could be CM jth ranked candidate for $j = 2$ or 3. With more districts, the outcome is $\mathbf{p} = \sum_{j=1}^{k} \lambda_j \mathbf{p}_j$, where $\sum_{j=1}^{k} \lambda_j = 1$, $\lambda_j \geq 0$, so the same conclusion holds: e.g., Ms. Young could be a CM winner in all regions, but the overall CM loser. We leave it to the reader to experiment with Fig. 2 to show that even if $\mathbf{p}_1$ is in the c_1-Condorcet region, c_1 can lose should $\mathbf{p}_2$ be in a cyclic region (because the union of these regions loses convexity).

To summarize, a choice procedure is *weakly consistent* [S1, 3] if when $\mathbf{p}_1$ and $\mathbf{p}_2$ have the same outcome, then this common outcome holds for the profile defined by combining $\mathbf{p}_1$ and $\mathbf{p}_2$. (This definition extends Young's [Y] "strong consistency" which specifies what should occur should both profiles have outcomes that need not agree, but with some candidates in common.)

Theorem 4. *For $n = 3$ candidates, the CM is not weakly consistent. All illustrating examples must have at least one of the profiles in a cyclic region.*

2.2. Unified approach. Although these scenarios sample significantly different themes, conceptually and technically the analysis is the same. What emerges is the lesson (holding for all decision and allocation procedures) that whether a profile change can cause expected, or unexpected and even paradoxical outcomes, depends upon the location of the original profile and the geometry of the boundary set. What makes the CM so important to analyze is that its simpler geometry divides these effects into geometrically separate regions.

More specifically, the Condorcet region (the $\{x_{1,2} > 0, x_{3,1} \leq 0\} \cup \{x_{1,2} \geq 0, x_{3,1} < 0\}$ portion) of $\mathcal{D}^3(c_1)$ is where the consequences of profile changes are of the expected type. This benign behavior is preserved by the normal vectors $\mathbf{N}^3_{1,j}$ which only recognizes changes in c_1's status. The more unusual, strategic, or paradoxical outcomes, then, critically depend upon the geometric positioning of the cyclic regions (with the CM ranking $c_1 \sim c_2 \sim c_3$). For the most part, these conclusions occur because this portion of $\mathcal{D}^3(c_1)$ admits normal directions $\mathbf{N}^3_{j,k}$, $j, k \neq 1$ where changes in the status of *other* candidates can adversely affect c_1's CM status.

A similar explanation holds for other allocation and decision procedures ([S1, Chap. 4]). To illustrate, we show how this approach simplifies the analysis of positional voting defined by $(w_1 = 1, w_2, w_3 = 0)$, $0 \leq w_2 \leq 1$, where w_j points are assigned to a voter's jth ranked candidate. The election tallies for c_1, c_2, c_3 are, respectively, $p_1+p_2+w_2p_3+w_2p_6$, $w_2p_1+w_2p_4+p_5+p_6$, $w_2p_2+p_3+p_4+w_2p_5$. Thus, the equation representing a $c_1 \sim c_2$ vote is where the first summation minus the second is zero. From the coefficients of this expression, the normal vector favoring c_1 over c_2 is $\mathbf{N} = (1 - w_2, 1, w_2, -w_2, -1, w_2 - 1)$. So, for instance, the effect of Smith not voting, $\mathbf{d} = -\mathbf{E}_4$, helps c_1 as $(\mathbf{N}, -\mathbf{E}_4) = w_2 \geq 0$, and all of Smith's strategic choices to elect c_2 over c_1 are $\mathbf{d}_{4-j}$, where $j = 5, 6$ for $w_2 < \frac{1}{2}$ and $j = 5$ for $\frac{1}{2} \leq w_2 < 1$ (as $(\mathbf{N}, \mathbf{d}_{4-5}) = w_2 - 1$, $(\mathbf{N}, \mathbf{d}_{4-6}) = 2w_2 - 1$.) In other words, when other procedures are analyzed, the only difference in the approach is that different boundary normals are used.

In this same geometric manner all other multiple profile CM issues – whether they involve groups or individuals – can be examined. Specifying a topic determines the profile change vectors, $\mathbf{d}$, while the signs of scalar products (using Eq. 2.7 and the $\mathbf{N}^3_{j,k}$ normal vectors) determine what can occur and where. These signs indicate whether or not the profile change moves in the direction of $\mathbf{N}^3_{j,k}$; the choice of $\mathbf{N}^3_{j,k}$ at various profiles determines whether the change helps or hurts a particular candidate. Then, those profiles that admit such behavior can be completely classified. In this manner, for instance, it becomes easy to analyze the strategic action of a group of voters of different types; the resulting conclusion identifies the highly exacting and carefully coordinated conditions that must exist.

3. More candidates; more possibilities

These $n = 3$ conclusions extend in the same way to any number of candidates. However, as geometric diagrams are impossible, analytic descriptions of the geometry – of the $\mathcal{D}^n(c_1)$ boundaries and the associated directions of the normal vectors are required. The important fact to remember for $n > 3$ candidates is that $\mathcal{D}^n(c_1)$, the set of pairwise outcomes where c_1 is CM top-ranked, is the union of the region of pairwise votes where c_1 is a Condorcet winner, of portions of the bounding hyperplanes (plus portions of certain lower dimensional parts of the boundary) of this Condorcet region, along with many regions with cyclic rankings.

One fact is clear: by being the union of many regions (where several represent cycles), $\mathcal{D}^n(c_1)$ admits additional boundaries with different orientations. Each new boundary orientation provides added opportunities for profile changes to leave (or enter) $\mathcal{D}^n(c_1)$. Consequently, we must expect more kinds of multiple profile concerns. As these new orientations are introduced by the cyclic portions of $\mathcal{D}^n(c_1)$, unexpected or strategic outcomes tend to be caused by regions admitting cyclic rankings; e.g., when the voters have mixed views about the selection of the available substitutes (as manifested by the cycle), a ranking change of some seemingly unrelated pair can affect c_1's fate. In other words, precisely where we need the CM because a majority winner does not exist, troubles arise. Nevertheless, the CM geometry conveniently separates the effects of those outcomes that are to be expected from paradoxical or manipulative conclusions.

3.1. Technical description. The $n \geq 3$ candidates define $n!$ voter types; list them in some order. If p_j represents the fraction of voters with the jth ranking, then a profile can be represented by a point in the simplex

$$Si(n!) = \{\mathbf{p} \in R^{n!} \mid \sum_{j=1}^{n!} p_j = 1, p_j \geq 0 \quad \forall j\}.$$

The profile set supporting a pairwise tie vote between c_i and c_j defines the hyperplane $H^n_{i,j}$; with $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs there are $\binom{n}{2}$ such hyperplanes. Each side of $H^n_{i,j}$ defines the profile set supporting the pairwise victory of one of the two candidates. The normal vector $\mathbf{N}^n_{i,j}$ that points into the side of $H^n_{i,j}$ where c_i beats c_j has entries equally divided between 1 and -1; the positive values correspond to that half of the coordinates (voter types) representing the relative ranking $c_i \succ c_j$. As the CM depends upon the number of pairwise victories, ties, and defeats of each candidate, the boundaries of the profile sets supporting each CM outcome consist of portions of these $H^n_{i,j}$ hyperplanes. Therefore, the potential CM effects of a given profile change $\mathbf{d}$ is determined by the sign of the scalar product $(\mathbf{N}^n_{i,j}, \mathbf{d})$.

As true in Sect. 2, to avoid the higher dimensional geometry of $R^{n!}$ we emphasize the geometry of the space of pairwise votes. Following [S1], the election tallies for the $\binom{n}{2}$ pairs defines a point in $R^{\binom{n}{2}}$ where $x_{i,j} = (\mathbf{N}^n_{i,j}, \mathbf{p})$ specifies the (c_i, c_j) election outcome. (This $x_{i,j} = (\mathbf{N}^n_{i,j}, \mathbf{p})$ computation is equivalent to summing those p_k's corresponding to a voter type with $c_i \succ c_j$ and then subtracting the sum of the remaining voter types.) If C^n is the cube in $R^{\binom{n}{2}}$ where each $x_{i,j}$ can vary over the values $[-1, 1]$, a point representing election outcomes must be in C^n. As each unanimity profile defines a particular vertex of this cube, they determine $n!$ of the $2^{\binom{n}{2}}$ vertices. The *representation cube* (the space of pairwise election outcomes resulting from voters with transitive preferences) is the convex hull of these vertices. Any (rational) point in the representation cube is supported by a profile.

Let $\mathcal{D}^n(c_1)$ be the subset of the representation cube where c_1 is CM top-ranked. Geometrically, $\mathcal{D}^n(c_1)$ is the union of the c_1-Condorcet region (i.e., where $x_{1,j} > 0$ for all j), portions of certain boundaries of this region (as some $x_{1,j}$ values are zero, the admissible boundaries are where the number of positive coordinates $x_{1,i}$ is at least as large as the difference between positive and negative values of coordinates $x_{k,m}$ for any k), plus regions with cyclic rankings (where the pairwise victories from the cycle cancel one another to contribute CM scores of zero) where c_1's CM score is the largest.

To illustrate with $n = 4$, the scores from the cycle where $x_{1,2}, x_{2,3}, x_{3,4}, x_{4,1}$ all have positive values contributes a CM tally of zero for each candidate. Thus, the signs of the remaining two pairwise outcomes, $x_{1,3}, x_{2,4}$, determine the CM tally. If, for instance, both values are positive, the CM ranking is $c_1 \sim c_2 \succ c_3 \sim c_4$ supported by the tally $(1, 1, -1, -1)$. Thus, this region forms a portion of $\mathcal{D}^4(c_1)$. A related $\mathcal{D}^4(c_1)$ region is defined by keeping the $x_{1,3}$ and $x_{2,4}$ values, but reversing all other values. (The only difference is that the cycle circles in the other direction.) Similarly, $\mathcal{D}^4(c_1)$ also includes the orthant where $x_{1,2}, x_{2,3}, x_{3,1}$ are positive (so, the scores from this cycle contribute a CM score of zero to these three candidates), and where each of these candidates wins the remaining pairwise election to have the the CM ranking $c_1 \sim c_2 \sim c_3 \succ c_4$ and CM scores $(1, 1, 1, -3)$.

It is important to note that all (rational) points in $\mathcal{D}^n(c_1)$ are supported by some profile. (This is a trivial extension of the argument in [S1].) Therefore, we can concentrate on the geometry of this set rather than profile examples. (Examples, however, are easy to construct.) Also, the argument varies between $\mathcal{D}^n(c_1)$ and the supporting set of profiles; to avoid introducing new notation, both sets are denoted by $\mathcal{D}^n(c_1)$. All $H^n_{1,j}$ planes form portions of the boundary for $\mathcal{D}^n(c_1)$, so the more interesting issue is to determine the remaining boundaries. Again, to simplify the analysis, we consider only settings without pairwise ties; the reader should have no trouble analyzing this excluded case.

Proposition 1. *For $n \geq 3$ and for every $i, j \neq 1$, a portion of $H^n_{i,j}$ is a boundary for $\mathcal{D}^n(c_1)$. On different portions of this boundary both $\pm\mathbf{N}^n_{i,j}$ points to the interior of $\mathcal{D}^n(c_1)$. For $n \geq 4$ and without pairwise ties, this assertion can occur on $\mathcal{D}^n(c_1)$ regions where there are k CM top-ranked candidates (which include c_1). The restrictions on k are that $2 \leq k \leq n$ and $k \neq n-1$ for odd values of n, and $2 \leq k \leq n-1$ for even values of n. Similarly, for $n \geq 5$, when $\mathcal{D}^n(c_1)$ is divided into regions based on which candidates share the CM top-rank, it can be that the CM top-ranked candidates win with only one more pairwise victory than other candidates and that the same boundary assertion holds between these regions. (This applies even to regions where c_1 is the sole CM top-ranked candidate.)*

Proof. For $n = 3$, this assertion follows from the two cyclic regions of Fig. 2. For $n = 4$, modify the above example involving a three-candidate cycle by replacing c_2, c_3 with c_i, c_j, and then create a second example (a second region of $D^4(c_1)$) by reversing each ranking in the original cycle. A similar modification of the first $n = 4$ example shows how two candidates can be CM top-ranked. Although the argument for $n \geq 5$ is similar, we defer its proof to the Appendix. □

The importance of this proposition is that it tells us the kinds of profile changes that can affect c_1's CM top-ranked status.

Proposition 2. *Let $\mathbf{d}$ be a direction of profile change and let $i, j \neq 1$. If $(\mathbf{d}, \mathbf{N}^n_{i,j}) \neq 0$, then there are choices of $\mathbf{p}_1$ so that this profile change drops c_1 from being a CM top-ranked candidate, and there are other choices of $\mathbf{p}_1$ so that either c_i or c_j joins c_1 as being CM top-ranked.*

The first case arises when c_1 and other candidates are CM top-ranked; the second is when c_1 wins one more pairwise victory than some other candidate. The proof of Proposition 2 follows from Proposition 1. In fact, by using Proposition 1 and considering profiles near edges (representing where several pairwise elections are nearly tied), the reader should have no trouble constructing examples where several candidates either join or drop out of the CM top-ranks thanks to the actions of a single voter.

It remains to compute $(\mathbf{N}^n_{i,j}, \mathbf{d})$. Because components of $\mathbf{d}$ indicate how many voters change preferences, this analysis is simple. Namely, if $\mathbf{d}$ has more voters reversing the preference $c_j \succ c_i$ than voters reversing $c_i \succ c_j$, then $(\mathbf{N}^n_{i,j}, \mathbf{d})$ is positive; in the contrary case, $(\mathbf{N}^n_{i,j}, \mathbf{d})$ is negative. The remaining setting, with a balance between these numbers, has $(\mathbf{N}^n_{i,j}, \mathbf{d}) = 0$. Notice that because most choice issues are defined in terms of voters changing preferences in the same manner, this computation is simple.

3.2. Changes that make a difference. Propositions 1 and 2 allow us to analyze and quickly resolve multiprofile issues just by defining the profile change associated with a specified issue. This is illustrated with the following where we start with the particularly strong restriction on profile changes imposed by monotonicity. More precisely, monotonicity requires that if c_1 is selected with the original profile, then a voter changing his ranking of c_1 must rank her higher than previously while keeping the relative ranking of the remaining $n-1$ candidates unchanged. Thus, all admissible changes must be in a $\mathbf{N}_{1,j}$ direction. Although some pairwise methods, such as runoff elections and agendas, fail to be monotonic (so the boundary regions of their $\mathcal{D}(c_1)$ region admit interior vectors of the $-\mathbf{N}^n_{1,j}$ type), the following known statement (e.g., see [N]) indicates that the CM geometry is better behaved.

Theorem 5. *For $n \geq 3$ candidates, the CM is monotonic.*

Proof. All admissible changes are in the direction of $\mathbf{N}^n_{1,j}$ for some j. The conclusion follows. □

One way to appreciate the monotonicity conditions is to relax them until conditions emerge where c_1 can be hurt by receiving increased support. One change is to permit voters to improve c_1's ranking even if she is not top-ranked. Although this new definition requires sets other than $\mathcal{D}^n(c_1)$ to be analyzed, the boundaries of the profile sets still include portions of the hyperplanes $H^n_{j,k}$. As admissible changes remain in the direction $\mathbf{N}^n_{1,j}$, this increased support can only help, but never hurt, c_1's CM standing.

A second way to relax monotonicity is to admit changes in the directions other than $\mathbf{N}^n_{1,j}$. For instance, after hearing a debate between c_1 and c_2, a voter may rerank the candidates to c_1's advantage just by interchanging c_1 and c_2. If c_2 was ranked more than one above c_1 where, say, c_3 separated them, then this c_1-improvement creates a profile change in the $\mathbf{N}^n_{3,2}$ direction; a change where, according to Prop. 2, initial profiles can be found where the new outcome drops c_1 out of CM top-place or forces c_1 to share this status with another candidate.

Theorem 6. *For $n \geq 3$ candidates, the CM fails any definition of "monotonicity" with respect to c_1 where voters' relative rankings of the other $n-1$ candidates need not be fixed when improving c_1's ranking.*

The same analysis applies to all CM properties. With the propositions, just compute the profile changes defined by an issue. According to Prop. 2, any change not of the monotonicity type can either add or subtract candidates from the CM top-ranked class. Consequently there exist situations where any $\{c_i, c_j\}$ change can help or hurt c_1's status. This is true whether the profile changes occur when:

1. The number of voters remains the same. This includes issues such as monotonicity, manipulation, and Pareto conditions.
2. The number of voters change. This includes the effects of abstention, consistency, etc.
3. Preferences are not strict. This addresses issues such as truncated ballots where a voter refuses to vote for certain candidates because he knows nothing about them.

In each setting, with Prop. 2 we can identify both the profile sets allowing a specified multiple profile change and the kinds of results which follow.

3.3. Same number of voters. We already have discussed monotonicity, so the remaining issues include game theoretic changes and strategic voting. We start with Pareto efficiency. (Recall, $\mathbf{x}$ is a Pareto point if, for any proposal y, some voter prefers $\mathbf{x}$ to $\mathbf{y}$.)

Theorem 7. *All sincere CM top-ranked outcomes are Pareto points.*

This known conclusion ([N, p. 84]) does not hold for all voting procedures even those defined by pairwise outcomes. To see this, consider the agenda that matches the (c_3, c_2) majority winner against c_1, and that majority winner against c_4. Although c_4 wins with the three voter profile $c_1 \succ c_2 \succ c_3 \succ c_4$, $c_2 \succ c_3 \succ c_4 \succ c_1$, $c_3 \succ c_4 \succ c_1 \succ c_2$, this is not a Pareto outcome because all voters strictly prefer c_3 to c_4.

Proof. The assertion follows if no candidate is universally preferred to c_1, a CM top-ranked candidate. If this is false because all voters prefer $c_2 \succ c_1$, then c_1 loses the $\{c_1, c_2\}$ pairwise contest and, for every $\{c_1, c_j\}$ election that c_1 wins, c_2 wins the $\{c_2, c_j\}$ election. Thus, c_2 is CM ranked higher than c_1. □

For manipulative outcomes, it is to the advantage of a voter of type k to vote as type j if $\mathbf{d}_{k \to j}$ provides a personal improvement of the election outcome: that is, if it elects a candidate that this voter prefers to c_1. As it follows from Prop. 2 that this always is possible, it remains to characterize the choices of $\mathbf{p}_1$ allowing such a manipulation and to determine whether once $n \geq 4$ it is possible to manipulate a CM election without incurring the penalties described in Thm. 3.

Theorem 8. *There are situation where a voter with relative rankings $c_2 \succ c_1$ can strategically vote to eliminate c_1 as a CM top ranked candidate and ensure that c_2 is the sole CM top-ranked candidate. For this to occur, both c_1 and c_2 are CM top-ranked with $\mathbf{p}_1$ and either there is a candidate c_j that beats c_2 by one vote or c_1 beats c_j by one vote. In the first case, this strategic voter has the sincere relative ranking $c_j \succ c_2 \succ c_1$; in the second, this voter has the sincere ranking $c_1 \succ c_j$. Such conditions are possible only for $n \geq 4$.*

The strategic vote for the first case, of course, is to vote for c_2 rather than c_j; in the second it is to vote for c_j over c_1. According to the propositions, there always exist such situations where c_1 and c_2 are the only two-top-ranked candidates. Without ties, this is possible only with $n \geq 4$. The theorem now follows. □

This theorem proves that although the CM resists the consequences of the Gibbard-Satterthwaite Theorem with three candidates, the sting of this assertion returns once $n \geq 4$. Moreover, with Prop. 2, it now is easy to concoct all sorts of other manipulative settings. We leave the exploration of these options to the reader.

3.4. Changing numbers of voters. Consistency first was explored by Smith [Sm] and Young [Y]. Later, Saari [S1, 3] relaxed the conditions so that they would apply to a wider variety of procedures. The relaxed condition of "weak consistency" is to determine if the CM top-ranked candidates for $\mathbf{p}_1$ and $\mathbf{p}_2$ agree, must this be so when the profiles are combined? Again, the answer is no.

Theorem 9. *For $n \geq 4$, the CM is not weakly consistent. Examples must involve profiles not in the c_1-Condorcet region.*

The convexity of the c_1 Condorcet region means that if $\mathbf{p}_1$, $\mathbf{p}_2$ are in this region then so is $\lambda\mathbf{p}_1 + (1-\lambda)\mathbf{p}_2$. Thus, to find counter examples for weak consistency, we need regions with cyclic rankings. For instance, place $\mathbf{p}_1$ in a region with one cycle and $\mathbf{p}_2$ in the region defined by the reversed cycle but where all other pairwise rankings are the same for both regions. While both profiles define the same CM scores for all candidates, by choosing them appropriately, the points on the line connecting the two profiles can replace the cyclic rankings in any desired manner. The conclusion follows. □

We now consider what might happen should a voter vote, or not vote. Such an analysis includes the abstention paradox that Fishburn and Gerhlein [FG], Moulin [M], and Saari [S1,2] have analyzed for other procedures. Using the terminology of [S1], we consider the following natural requirements: *Positive involvement* – If c_1 is CM top-ranked, adding a voter who has her top-ranked should not weaken her CM status. *Negative involvement* – The status of c_1 should not improve when a new voter is added who has her bottom-ranked.

Theorem 10. *For $n \geq 3$, the CM satisfies neither positive nor negative involvement.*

This theorem shows that Moulin's conclusion [M] requiring all Condorcet methods to suffer the abstention paradox holds for the CM. Again, thanks to the propositions, the proof is trivial. To verify that positive involvement can be violated, consider where c_1 and c_2 are CM top-ranked and $\mathbf{N}^n_{3,2}$ is an interior normal. So, with $\mathbf{p}_1$ near this boundary, adding a voter with c_1 top-ranked who also has the relative ranking of $c_2 \succ c_3$ will cause c_1 to lose. For negative involvement, consider a setting where c_1 and c_2 are CM top-ranked, where $\mathbf{N}^n_{2,3}$ is the interior normal. A voter with $c_3 \succ c_2$ can drop c_2 from CM top-rank independent of how he ranks c_1. (Notice that this proof extends to many other procedures.) □

3.5. Loss of strict rankings. What we have in mind here are those situations where voters don't have a transitive ranking of the candidates, or don't rank all of them. If the group's ranking of pairs involves a cycle, the CM scoring approach cancels the cycle and ranks the candidates based on the remaining rankings of pairs.

This is not the situation with an individual ranking that includes a cycle: instead, it could be that a vote from the cycle could alter the ranking of a pair, and from this, the identity of the CM top-ranked candidates. A similar situation applies to where a voter has a transitive ranking for only a part of the candidates. Some of the notions here find their origins in the work of Fishburn and Brams [FB1, FB2] where they considered the consequences of voters casting truncated ballots with the Hare method and runoff elections. The reasons for casting such a ballot range from their setting where voters try to manipulate the outcome to more innocent explanations where voters refuse to vote when they don't know the candidates. In either case, a truncated ballot (where a voter does not vote in all pairwise elections) could help or hurt his cause.

Theorem 11. *There exist situation where, if a voter casts a truncated ballot, the CM outcome is personally more favorable than if the voter voted sincerely. There are other situations where the outcome is personally worse.*

In summary, once $n \geq 4$, the geometry of the $\mathcal{D}^n(c_1)$ region is such that the CM admits a host of problems and paradoxes. The important fact is that all can be easily analyzed in the same way. Moreover, this approach extends to other decision and voting procedures.

Appendix

Proof of Proposition 1. The basis of this proof is that all possible pairwise rankings are admissible. (This is suggested by the description of the representation cube. For an analytic proof, see the references for the construction of "dictionaries" in [S4].) We first show it is impossible (without pairwise ties) for all candidates to be CM top-ranked when n is even. The conclusion follows because the CM scores range from $n-1, n-3, \dots, 1, -1, \dots, -(n-1)$ so zero is not an admissible value. But as the sum of the CM scores equal zero, not all candidates can have the same CM score.

We need to show that, without ties, there can be precisely k CM top-ranked candidates. Start with a matrix where the i,j entry describes c_i's outcome from the $\{c_i, c_j\}$ election; thus, this is a skew-symmetric matrix where the i,j entry is the negative of the j,i entry and the diagonal elements are zeros. For an example where $n-1$ candidates are CM top-ranked when n is even, let all candidates beat c_n (so all non-zero entries in the last column are $+1$). For each row, have the first half of the remaining $n-2$ entries $+1$ and the last half -1 where entries wrap around the row to start in the c_1 column. Thus, up to the $\frac{n}{2}$th row, all entries to the left of the diagonal are negative; at this row, all entries to the right of the diagonal are positive. For rows beyond the $\frac{n}{2}$th, there are not enough columns to fit in all positive entries, so they are placed to the extreme right of the row. Notice that this construction leads to a consistent signs placement for the entries. The resulting skew-symmetric (actually, permutation) matrix is of the form

$$\begin{matrix} c_1 \\ c_2 \\ \dots \\ c_{n-1} \\ c_n \end{matrix} \begin{pmatrix} 0 & 1 & 1 & \dots & -1 & -1 & \dots & 1 \\ -1 & 0 & 1 & \dots & 1 & -1 & \dots & 1 \\ \dots & & & & & & & \\ 1 & 1 & \dots & 1 & \dots & -1 & 0 & 1 \\ -1 & -1 & \dots & -1 & -1 & \dots & -1 & 0 \end{pmatrix}$$

As the jth candidate's CM score is the sum of terms in the jth row, the conclusion holds.

For n even, a m CM top-ranked candidate example, $\frac{n}{2}+1 \leq m \leq n-1$, is constructed by reversing the values for the last m nonzero entries in the last column. (This also reverses the last m non-zero entries of the bottom row.) As these are the only candidates who lose another pairwise election (to c_n) and as c_n still has more losses than victories, the conclusion follows. For $1 \leq m \leq \frac{n}{2}-1$ CM top-ranked candidates, the analysis is slightly modified. For c_1 to be the only top-ranked candidate, change the first -1 in the first row to $+1$ (which means that the first $+1$ in the first column now becomes a -1 and that c_1 now beats $c_{\frac{n}{2}+1}$). Notice that c_1 has one more victory than $c_2, \ldots, c_{\frac{n}{2}}$. To have m top-ranked candidates, do the same for the first m rows (with corresponding changes for the first m columns). This argument has to be modified for $m = \frac{1}{2}$. Here, for the first m rows, let the first m entries after the diagonal be $+1$ and the rest -1. This gives each of these candidates a CM score of 1, where, with this assignment, c_{n-j} has $\frac{n}{2}-j-1$ victories. To ensure that all of these candidates end up with a negative score, fill in the rest of the rows to th right of the diagonal with -1.

For odd values of n, all candidates can be tied for top-rank. To prove this, let each row have $+1$ for the first $\frac{n-1}{2}$ entries after the diagonal and -1 for the rest. But, without ties, it is impossible to have $n-1$ CM top-ranked candidates when n is odd. This is because the smallest non-negative CM scores are 2 and 0. As the sum of CM scores equals zero, one candidate must have a negative score so all others must have at least 2 points. As this means there are at least $2(n-1)$ positive points and at most $-(n-1)$ for negative scores, this violates the CM summation constraint.

To show that any other k value is admissible, start where all entries in the last column are $+1$ (so all candidates beat c_n). Because an even number of candidates remain, the above analysis can be applied to this submatrix.

It remains to show that $H^n_{i,j}$, $i, j \neq 1$, is a boundary for one of these regions when $k \geq 2$. In the construction, both c_1 and c_2 are top-ranked, and there are candidates other than c_1 who beat c_2. Suppose this is c_s. So, if the $\{c_2, c_s\}$ election would be reversed, c_2 would have one more victory than c_1 and this would drop c_1 as a top-ranked candidate. Thus, portions of $H^n_{2,s}$ serve as part of the boundary of $\mathcal{D}^n(c_1)$ with interior normal $\mathbf{N}^n_{2,s}$. For i, j where we want $\mathbf{N}^n_{i,j}$ to be an interior normal, interchange the second and ith rows and columns, and the sth and jth rows and columns.

The same construction shows that $\mathcal{D}^n(c_1)$ can be divided into sections indicating the number of candidates that are CM top-ranked. Similarly, it shows that each $H^n_{i,j}$ is part of the separating boundary between these regions and that, in different sections, both $\pm\mathbf{N}^n_{i,j}$ are interior normal vectors. □

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