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ONE-SHOT PUBLIC MEDIATED TALK

by

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and

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Abstract: We show that any correlation device with rational coefficients can be generated by a mechanism where each player sends a private message to a mediator who in turn makes a public deterministic announcement. Moreover, the mechanism suggested is immunized against individual deviations.

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1. Introduction

In a mediated talk (see Lehrer (1994)) players are allowed to communicate through a mediator. Each one of them transmits a private signal to the mediator. The latter, in turn, produces a public announcement which depends (deterministically) on the individual private signals. Then, each player may take an action relying on his private signal as well as on the public announcement. It is shown in Lehrer (1994) that if such a talk is not bounded in time then any correlated distribution (with rational coefficients) can be generated by a mediated public-deterministic talk. Here we improve the result and show that a one-shot communication suffices to generate any correlated distribution.

The main idea of the construction is to use different types of a jointly controlled lottery. All of them but one remain latent. The active device is selected by the realized set of the private inputs. Thus, the encoding made by the mechanism depends on the private inputs. Moreover, the decoding of the public announcement by each individual depends also on the private input.

2. Information Structure

Inspired by Aumann (1987), we adopt the following definition:

Definition 1: A (finite) information structure (or a correlation device) for n players is a list of n random variables X_i, $i = 1,...,n$, defined on the same probability space and ranged to finite action sets A_i, $i = 1,...,n$.

One may think of the probability space as the state space. The knowledge of each agent is modeled by a (finite) partition. A policy (or a strategy) of an agent is a function from states to action which is compatible with his knowledge.

In simple words, an information structure is a distribution Q over $A = \times_{i=1}^{n} A_i$.

An element $a \in A$ is chosen with probability $Q(a)$ and a_i is informed to i.

Definition 2: A public mediated talk is defined by (finite) private signal sets, S_i, $i = 1,\dots,n$, and an announcement map f from $S = \times S_i$ to some finite set X (the set of public announcements). Each player i chooses a private signal s_i to be sent to a mediator who makes the public announcement $x = f(s_1,\dots,s_n)$.

The goal of the paper is to describe a mechanism that mimics the information structure by independent random variables σ_i (call mixed messages) which take values in S_i and then by decoding maps θ_i from $S_i \times X$ to A_i. The map θ_i allows player i to interpret the public announcement x according to his private signal s_i. The maps θ_i are sometimes called strategies.

Let $\sigma = (\sigma_1,\dots,\sigma_n)$ and $\theta = (\theta_1,\dots,\theta_n)$ and denote by $P_{\sigma,\theta}$ the distribution induced on A by σ and θ. Now we are ready to state the main result of the paper.

Theorem: Assume Q is a distribution over A that assumes only rational values. Then there exist a public mediated talk, σ and θ, that satisfy the following.

(1) $$P_\sigma(a|s_i) = Q(a) \text{ for every } a \in A \text{ and } s_i \in S_i.$$

(2) $$P_\sigma(a_{-i}|s_i,x) = Q(a_{-i}|a_i),$$

where $a_i = \theta_i(s_i,x)$, for every a_{-i},s_i.

Remark 1: Notice that for any random variable τ_i ranged to S_i, and for every s_i,

$$P_{\tau_i,\sigma_{-i}}(\cdot|s_i) = P_\sigma(\cdot|s_i) \text{ and}$$

$$P_{\tau_i,\sigma_{-i}}(\cdot|s_i,x) = P_\sigma(\cdot|s_i,x).$$

Therefore, any unilateral deviation may not affect the distribution over A given s_i, and the distribution over A_{-i} given (s_i,x); both match the corresponding distributions defined by Q.

Example 1: Consider the following 2 × 2 game:

	ℓ	r
t	7,7	3,8
b	8,3	0,0

where $A_1 = \{t,b\}$ and $A_2 = \{\ell,r\}$, and the following correlated equilibrium

Q =

	ℓ	r
t	1/2	1/4
b	1/4	0

The payoff associated with the correlated equilibrium cannot be sustained by any Nash equilibrium nor by any combination of Nash equilibria. Thus, the players might want to resort to some external mediating device that will generate the distribution Q. They can do it obeying the following procedure. Each player selects privately a number in {1,...,4} with probability 1/4 each and then transmits it to a deterministic machine which produces a public message according to the following matrix:

		1	2	3	4
	1	c	c	c	b
	2	c	c	b	c
Player I	3	c	b	b	b
	4	b	c	b	b

Figure 1: The Signaling Matrix

The machine publicly announces x if players I and II selections were i,j, respectively, and if the (i,j) cell of the matrix is x, x = c,b. In other words, S_i = {1,...,4} and σ_i assigns each symbol a probability of 1/4. After receiving the public announcement, the players play the following strategies.

The private selected signal was. . .	. . .and the public announcement is. . .	. . .then play
1	c	t
1	b	b
2	c	t
2	b	b
3	c	b
3	b	t
4	c	b
4	b	t

Figure 2: The Strategy of Player I (θ_1)

The private selected signal was. . .	. . .and the public announcement is. . .	. . .then play
1	c	ℓ
1	b	r
2	c	ℓ
2	b	r
3	c	r
3	b	ℓ
4	c	r
4	b	ℓ

Figure 3: The Strategy of Player II (θ_2)

One can check that if the players play the strategies just defined, then indeed Q is generated. Moreover, given these strategies, and the uniform selection of player II, all the rows of the signaling matrix (Figure 1) are equivalent in the sense that all induce the same distribution over joint actions. The same observation holds for player II. Therefore, no player has an incentive to deviate neither in the communication phase nor in the play phase.

To see that this example satisfies (1) and (2) of the theorem, notice that, given θ_1 and θ_2, the conditional distribution, given any s_i, induced by σ_1 and σ_2 over A is Q. Moreover, given s_i and x, the probability of any a_{-i} is exactly $Q(a_{-i}|a_i)$, where $a_i = \theta_i(s_i,x_i)$. For instance, suppose that $s_1 = 1$ and $x = c$. Here, $\theta_1(1,c) = t$, $Q(\ell|t) = 2/3$, and $Q(r|t) = 1/3$. And, indeed, given $s_1 = 1$ and $x = c$, the probability that player 2 will play ℓ is 2/3, while the probability of r being played is 1/3.

Example 2: In generating Q we used public mediation that used only two symbols, a and b. In order to generate the distribution $Q' = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$ over the set of joint actions we must use three symbols. One way to do it is to use the following signaling matrix:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} d & d & b \\ d & c & c \\ b & c & b \end{pmatrix} \end{array}$$

Figure 4

Here each player chooses one of the numbers 1, 2 and 3 with equal probability. The strategies that induce Q′ over the 2 × 2 game are easy to construct.

3. The Mediated Talk Extension of a Game

Let G be an n-player game. We will extend the game G to G* by adding a pre-play communication phase. In this phase player i selects (possibly randomly) a signal, s_i, from a finite set S_i. Then a deterministic mediator publicly announces $f(s_1,...,s_n)$. In the play phase each player chooses an action which may depend on the signal s_i and on $f(s_1,...,s_n)$. G* is called a mediated talk extension of G.

Obviously, any joint strategy in such extension induces a correlated joint strategy in G. We are concerned here with the inverse question--whether any correlated equilibrium of G can be generated as a Nash equilibrium of a mediated talk extension of G. We answer this question in the affirmative.

Corollary: Let C be a correlated equilibrium of G. Assume that all the probabilities of C are rational numbers. Then there exists a mediated talk extension of G having a Nash equilibrium that induces the distribution C over the set of joint actions of G.

Remark 2: The mechanism described here defines only Nash equilibrium of the extended game and not a strong equilibrium. Thus, it is immunized only against unilateral deviations.

4. The Proof of the Theorem

We will show the proof in the two player case. This proof extends easily to the n player case. Suppose that the Q is a distribution over A. $Q = (c_{ij}/d)_{0\leq i\leq n-1, 0\leq j\leq m-1}$, where all c_{ij} and d are integers. The signaling matrix to be constructed is of the size dn

$\times$ dm. Actually, it will be described as a n $\times$ m matrix where each cell is a d $\times$ d matrix.

Let $a_1,\dots,a_y$ be a string of y symbols. The <u>latin square</u> corresponding to this string is the following matrix

$$\begin{pmatrix} a_1 & \cdots & a_y \\ a_2 & \cdots & a_y a_1 \\ a_3 & \cdots & a_1 a_2 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_y a_1 & \cdots & a_{y-1} \end{pmatrix}$$

For a vector $b_{0,0},b_{0,1},\dots,b_{n-1,m-1}$ of nm symbols we denote by $K(b_{0,0},b_{0,1},\dots,b_{n-1,m-1})$ the latin square corresponding to the string which consists of $c_{0,0}$ times $b_{0,0}$ and then $c_{0,1}$ times $b_{0,1}$, and so forth. Thus, $K(b_{0,0},\dots,b_{n-1,m-1})$ is a d $\times$ d matrix (because $\sum c_{ij} = d$).

<u>Example 3</u>: As in Example 1, let $Q = \begin{pmatrix} 2/4 & 1/4 \\ 1/4 & 0 \end{pmatrix}$. In this case

$$K(a,b,c,d) = \begin{pmatrix} a & a & b & c \\ a & b & c & a \\ b & c & a & a \\ c & a & a & b \end{pmatrix}.$$

This is so because $c_{0,0} = 2$, $c_{0,1} = c_{1,0} = 1$ and $c_{1,1} = 0$. Therefore, in every row and column, a appears twice, b and c appear once, and d does not appear at all. //

Now fix n $\times$ m different symbols $(b_{ij})_{0\le i\le n-1, 0\le j\le m-1}$. In what follows, for any integers x and y, x(n) and y(m) will stand for the numbers x modulo n and y modulo m, respectively. Before we get to the announcement map defined by the signaling matrix we need one more convention. When $(b_{ij})_{i,j}$ is referred to a string, rather than a matrix, the string is defined in a natural way: the first row first then the second row and so forth.

The signaling matrix consists of an $n \times m$ grand matrix where for every $0 \leq k \leq n-1$ and $0 \leq \ell \leq m-1$ in the (k,ℓ) cell of it stands the matrix

$$K(b_{i+k(n),j+\ell(m)})_{0\leq i\leq n-1, 0\leq j\leq m-1}.$$

Let now $S_1 = \{0,\ldots,n-1\} \times \{1,\ldots,d\}$ and $S_2 = \{0,\ldots,m-1\} \times \{1,\ldots,d\}$. For every $(k,d_1) \in S_1$ and $(\ell,d_2) \in S_2$ define $f((k,d_1),(\ell,d_2))$ to be the (d_1,d_2) entry of the matrix standing in the cell (k,ℓ) of the grand matrix.

Example 3 (Continued): With the distribution of Example 1, the grand matrix is 2×2 (the size of the original game) consisting of cells which are 4×4 matrices. Let $b_{0,0}, b_{0,1}, b_{1,0}, b_{1,1}$ be four different symbols. According to the construction in the (0,0) cell of the grand matrix stands the matrix

$$K(b_{ij}) = K(b_{00},b_{01},b_{10},b_{11}) = \begin{pmatrix} b_{00} & b_{00} & b_{01} & b_{10} \\ b_{00} & b_{01} & b_{10} & b_{00} \\ b_{01} & b_{10} & b_{00} & b_{00} \\ b_{10} & b_{00} & b_{00} & b_{01} \end{pmatrix}.$$

Recall that $K(\bullet)$ is a latin square where b_{ij} is replicated c_{ij} times in each row and column. In the (0,1) cell of the grand matrix stands the matrix

$$K(b_{i,j+1(2)})_{i,j} = K(b_{01},b_{00},b_{11},b_{10}) = 10\begin{pmatrix} b_{01} & b_{01} & b_{00} & b_{11} \\ b_{01} & b_{00} & b_{11} & b_{01} \\ b_{00} & b_{11} & b_{01} & b_{01} \\ b_{11} & b_{01} & b_{01} & b_{00} \end{pmatrix}.$$

To facilitate the reading set $b_{00} = x$, $b_{01} = y$, $b_{10} = w$, $b_{11} = z$. The signaling matrix is therefore

$$\begin{pmatrix} x & x & y & w & y & y & x & z \\ x & y & w & x & y & x & z & y \\ y & w & x & x & x & z & y & y \\ w & x & x & y & z & y & y & x \\ \\ w & w & z & x & z & z & w & y \\ w & z & x & w & z & w & y & z \\ z & x & w & w & w & y & z & z \\ x & w & w & z & y & z & z & w \end{pmatrix}.$$

For instance, if $d_1 = 2$, $d_2 = 3$, $k = 1$ and $\ell = 0$, then f equals x. One can see that any symbol appears in any row and column only once or exactly three times. Moreover, if, for instance, x appears only once in a certain column, then it appears two more times in its row. Such an x, that appears once in its column will later be associated with the right action of player II. Since it appears only once player II knows what player I is going to do; player I will play top because according to the distribution Q the probability of top given right is 1. Furthermore, when player I is prescribed to play top he should assign probability 2/3 on left and 1/3 on right (these are the conditional probabilities), and therefore in the same row there appear two more x's which correspond to the left column. //

Now that S_1, S_2 and f are defined, in order to complete the description of the mediated talk it remains to define σ_i and θ_i, $i = 1,2$. σ_i is uniform over S_i and θ_i is defined as follows. If the public announcement is b_{ij} then θ_1 is $a_1 = (i - k)(n)$ and θ_2 is $a_2 = (j - \ell)(n)$, where k and ℓ are the respective signals sent. Notice that θ_1 does not depend on the second index of the public announcement and θ_2 does not depend on the first.

We first show that if both players follow the decoding maps, θ_i, just described then the distribution over A given any s_i is exactly Q. For any cell (k,ℓ) the corresponding matrix is K $(b_{i+k(n),j+\ell(m)})$, where in each row there are c_{ij} times the symbol $b_{i+k(n),j+\ell(m)}$. Moreover, each symbol in any row is assigned the same probability. In the case where $b_{i+k(n),j+\ell(m)}$ is the public announcement then by the above decoding map, player I's action is $a_1 = ((i + k)(n) - k)(n) = i$ and player II's action is $a_2 = ((j + \ell)(m) - \ell)(m) = j$. Therefore, the joint action (i,j) is prescribed c_{ij} times out of a total of d. In other words, the joint action (i,j) is prescribed with probability c_{ij}/d. Since this is true for any cell, (i,j) is assigned the probability c_{ij}/d for any row (namely, for any s_1). This shows (1).

Next we show (2). In other words, for every signal s_1 and public announcement x, we show that the probability of $a_{-1} \in A_{-1}$ is $Q(a_{-1}|a_1)$, where $a_1 = \theta_1(s_1,x)$. Suppose that indeed player II abides by σ_2. Thus the choice of players II is uniformly distributed over the columns of the signaling matrix. Fix an arbitrary (k,d_1). We now take a look at the d_1 row of the matrices that stand in the cells $(k,0),(k,1),\ldots,(k,m - 1)$ of the grand matrix. In the (k,0) matrix there are c_{ij} times $b_{i+k(n),j(m)}$; in the (k,1) matrix there are c_{ij} times $b_{i+k(n),j+1(m)}$, and so on. Thus, the symbol $b_{i+k(n),j}$ appears $\sum_\ell c_{i,j-\ell(m)}$ times, out of which (recall σ_2) $c_{i,j-\ell(m)}$ times are associated with the action of choosing $j - \ell(m)$ column.

Rearranging the parameter we obtain that the symbol $b_{i+k(n),j+\ell(m)}$ appears $\sum_r c_{ir}$ times. So all the symbols with the first index $i + k(n)$ appear $\sum_r c_{ir}$ times.

Moreover, out of these $\sum_r c_{ir}$ appearances c_{ij} are associated with the j−th column. For each one of these symbols θ_1 obtains the value i (i.e., $\theta_1(s_1,x) = i$, where $s_1 = (k,d_1)$ and $x = b_{i+k(n),j+\ell(m)}$). Therefore, given (s_1,x), the probability of (i,j) being prescribed by (θ_1,θ_2) is $c_{ij}/\sum_r c_{ir}$, which is $Q(j|i)$. Since the same argument holds for player II, it proves (2). //

Proof of the Corollary: Let C be a correlated equilibrium in G. The theorem states that there exists a mediated talk which induces the distribution C over A (this is a consequence of (1)). In order to show that this mediated talk defines a Nash equilibrium of the extension we show that no player can gain by adopting another mix message $\bar{\sigma}_i$ or by adopting another decoding map $\bar{\theta}_i$. By (2), for every s_i and x which satisfy $\theta_i(s_i,x) = a_i$, $P(a_{-i}|s_i,x) = Q(a_{-i}|a_i)$ given σ_{-i} and θ. Since C is a correlated equilibrium, a_i is a best response against $Q(a_{-i}|a_i)$. Therefore, given σ_i, σ_{-i} and θ_{-i}, θ_i (which prescribes a_i) is a best response. By Remark 1, any alternative τ_i does not change properties (1) and (2) and therefore whatever the alternative:

1. the probability of playing a is the one assigned by C for any $a \in A$, and
2. whenever a_i is prescribed it is indeed an optimal response.

We have proved that (σ_i,θ_i) is a best response $(\sigma_{-i},\theta_{-i})$ and therefore it is a Nash equilibrium. //

Remark 3: In the construction of the signaling matrix we introduce the grand matrix which consists of the submatrices K(•). The first components of the private messages (sent by the players to the mediator) select the specific K(•) that becomes active. The second components (d_1 and d_2) determine the public announcement from the K(•) already chosen.

One may consider all the K's as jointly controlled devices. The active device is jointly chosen by the players. Then, the players jointly control the lottery by d_1 and d_2.

5. From Correlated to Communication Equilibrium

Forges (1986) introduced the concept of communication equilibrium. In a game where players have different types, the correlation applied may depend on the specific types of the players. Thus, before the mediator correlates between the players, he should receive some information from the players that reveals (or at least partially reveals) their types.

Example 4: Suppose that player I may be of two types: 1 and 2, which are equally likely. Player I knows his realized type while player II knows only the prior distribution over player I's type: (1/2,1/2). Let the payoffs be

		ℓ	r	Probability
type 1:	t	6,6	3,8	1/2
	b	8,3	0,0	

		ℓ	r	Probability
type 2:	t	0,0	8,3	1/2
	b	3,8	6,6	

Figure 5

Consider now the following mediation. If player I tells the mediator that he is of type 1, the latter chooses one of the joint actions (t,ℓ), (t,r), and (b,ℓ) with probability 1/3 each. However, if player I reports that he is of the second type, then the mediator chooses each of (r,b), (r,t) and (ℓ,b) with probability 1/3 each. Whatever the choice of the mediator, he informs player I of the row chosen and player II of the column chosen.

One can see that, given the mediator just described, player I has the incentive to reveal his true type. Notice that, once player II receives some information from the mediator, his prior over player I's type changes. For instance, if ℓ is sent, then the posterior ascribes type 1 the probability 2/3 while type 2 is ascribed only 1/3 (as opposed to the prior 1/2).

Notice that, if the players play the action sent by the mediator, then the process describes an equilibrium and no player has an incentive to deviate. This conclusion depends strongly on the specific posteriors. Therefore, any mechanism should always generate the same posteriors.

In order to generate this communication equilibrium by a mediated talk we adopt the matrix of Example 2 and define two signaling matrices, one for each type:

$$\begin{pmatrix} d & d & b \\ d & c & c \\ d & c & b \end{pmatrix},$$

as in Figure 4 for type 1, and

$$\begin{pmatrix} d & d & c \\ b & c & c \\ b & d & b \end{pmatrix},$$

for type 2. The private messages of player II are as in Example 2: {1,2,3}. However, player I has also to inform the mediator of his type. If the type is 1, then the active signaling matrix is the first. Otherwise, it is the second matrix. One can confirm that the posteriors of player II are either (2/3,1/3) or (1/3,2/3), as needed.

Notice that, if instead of $\begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix}$, the distribution on the second type's matrix would have been $\begin{pmatrix} 0 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$, for instance, then the matrix of Figure 4 could have not been used for the first type's matrix. //

Our construction enables one to design a mediated talk that applies to any communication equilibrium. The idea is to find a common denominator to all probabilities involved in all matrices and then to define for each correlated distribution (attached to each configuration of types) a mediated talk described in Section 4. Each player sends a private message (possibly a stochastic signal which depends on his type) and then the mechanism selects accordingly a mediated talk and chooses from it a public announcement.

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The goal of the paper is to describe a mechanism that mimics the information structure by independent random variables σ_i (call mixed messages) which take values in S_i and then by decoding maps θ_i from $S_i \times X$ to A_i. The map θ_i allows player i to interpret the public announcement x according to his private signal s_i. The maps θ_i are sometimes called strategies.

Let $\sigma = (\sigma_1,...,\sigma_n)$ and $\theta = (\theta_1,...,\theta_n)$ and denote by $P_{\sigma,\theta}$ the distribution induced on A by σ and θ. Now we are ready to state the main result of the paper.

Theorem: Assume Q is a distribution over A that assumes only rational values. Then there exist a public mediated talk, σ and θ, that satisfy the following.

(1) $$P_\sigma(a|s_i) = Q(a) \text{ for every } a \in A \text{ and } s_i \in S_i.$$

(2) $$P_\sigma(a_{-i}|s_i,x) = Q(a_{-i}|a_i),$$

where $a_i = \theta_i(s_i,x)$, for every a_{-i},s_i.

Remark 1: Notice that for any random variable τ_i ranged to S_i, and for every s_i,

$$P_{\tau_i,\sigma_{-i}}(\cdot|s_i) = P_\sigma(\cdot|s_i) \text{ and}$$

in {1,...,4} with probability 1/4 each and then transmits it to a deterministic machine which produces a public message according to the following matrix:

		1	2	3	4
	1	c	c	c	b
	2	c	c	b	c
Player I	3	c	b	b	b
	4	b	c	b	b

Figure 1: The Signaling Matrix

The machine publicly announces x if players I and II selections were i,j, respectively, and if the (i,j) cell of the matrix is x, x = c,b. In other words, S_i = {1,...,4} and σ_i assigns each symbol a probability of 1/4. After receiving the public announcement, the players play the following strategies.

One can check that if the players play the strategies just defined, then indeed Q is generated. Moreover, given these strategies, and the uniform selection of player II, all the rows of the signaling matrix (Figure 1) are equivalent in the sense that all induce the same distribution over joint actions. The same observation holds for player II. Therefore, no player has an incentive to deviate neither in the communication phase nor in the play phase.

To see that this example satisfies (1) and (2) of the theorem, notice that, given θ_1 and θ_2, the conditional distribution, given any s_i, induced by σ_1 and σ_2 over A is Q. Moreover, given s_i and x, the probability of any a_{-i} is exactly $Q(a_{-i}|a_i)$, where $a_i = \theta_i(s_i,x_i)$. For instance, suppose that $s_1 = 1$ and $x = c$. Here, $\theta_1(1,c) = t$, $Q(\ell|t) = 2/3$, and $Q(r|t) = 1/3$. And, indeed, given $s_1 = 1$ and $x = c$, the probability that player 2 will play ℓ is 2/3, while the probability of r being played is 1/3.

<u>Example 2</u>: In generating Q we used public mediation that used only two symbols, a and b. In order to generate the distribution $Q' = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$ over the set of joint actions we must use three symbols. One way to do it is to use the following signaling matrix:

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} d & d & b \\ d & c & c \\ b & c & b \end{pmatrix} \end{array}$$

Figure 4

$\times$ dm. Actually, it will be described as a n $\times$ m matrix where each cell is a d $\times$ d matrix.

Let $a_1,\ldots,a_y$ be a string of y symbols. The <u>latin square</u> corresponding to this string is the following matrix

$$\begin{pmatrix} a_1 & \cdots & a_y \\ a_2 & \cdots & a_y a_1 \\ a_3 & \cdots & a_1 a_2 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_y a_1 & \cdots & a_{y-1} \end{pmatrix}$$

For a vector $b_{0,0},b_{0,1},\ldots,b_{n-1,m-1}$ of nm symbols we denote by $K(b_{0,0},b_{0,1},\ldots,b_{n-1,m-1})$ the latin square corresponding to the string which consists of $c_{0,0}$ times $b_{0,0}$ and then $c_{0,1}$ times $b_{0,1}$, and so forth. Thus, $K(b_{0,0},\ldots,b_{n-1,m-1})$ is a d $\times$ d matrix (because $\sum c_{ij} = d$).

<u>Example 3</u>: As in Example 1, let $Q = \begin{pmatrix} 2/4 & 1/4 \\ 1/4 & 0 \end{pmatrix}$. In this case

$$K(a,b,c,d) = \begin{pmatrix} a & a & b & c \\ a & b & c & a \\ b & c & a & a \\ c & a & a & b \end{pmatrix}.$$

This is so because $c_{0,0} = 2$, $c_{0,1} = c_{1,0} = 1$ and $c_{1,1} = 0$. Therefore, in every row and column, a appears twice, b and c appear once, and d does not appear at all. //

Now fix n $\times$ m different symbols $(b_{ij})_{0\le i\le n-1, 0\le j\le m-1}$. In what follows, for any integers x and y, x(n) and y(m) will stand for the numbers x modulo n and y modulo m.

$$K(b_{i,j+1(2)})_{i,j} = K(b_{01},b_{00},b_{11},b_{10}) = \begin{matrix} 10 \end{matrix}\begin{pmatrix} b_{01} & b_{01} & b_{00} & b_{11} \\ b_{01} & b_{00} & b_{11} & b_{01} \\ b_{00} & b_{11} & b_{01} & b_{01} \\ b_{11} & b_{01} & b_{01} & b_{00} \end{pmatrix}.$$

To facilitate the reading set $b_{00} = x$, $b_{01} = y$, $b_{10} = w$, $b_{11} = z$. The signaling matrix is therefore

$$\begin{pmatrix} x & x & y & w & y & y & x & z \\ x & y & w & x & y & x & z & y \\ y & w & x & x & x & z & y & y \\ w & x & x & y & z & y & y & x \\ \\ w & w & z & x & z & z & w & y \\ w & z & x & w & z & w & y & z \\ z & x & w & w & w & y & z & z \\ x & w & w & z & y & z & z & w \end{pmatrix}.$$

For instance, if $d_1 = 2$, $d_2 = 3$, $k = 1$ and $\ell = 0$, then f equals x. One can see that any symbol appears in any row and column only once or exactly three times. Moreover, if, for instance, x appears only once in a certain column, then it appears two more times in its row. Such an x, that appears once in its column will later be associated with the right action of player II. Since it appears only once player II knows what player I is going to do; player I will play top because according to the distribution Q the probability of top given right is 1. Furthermore, when player I is prescribed to play top he should assign probability 2/3 on left and 1/3 on right (these are the conditional probabilities), and therefore in the same row there appear two more x's which correspond to the left column. //

Now that S_1, S_2 and f are defined, in order to complete the description of the

Moreover, out of these $\sum_r c_{ir}$ appearances c_{ij} are associated with the j−th column. For each one of these symbols θ_1 obtains the value i (i.e., $\theta_1(s_1,x) = i$, where $s_1 = (k,d_1)$ and $x = b_{i+k(n),j+\ell(m)}$). Therefore, given (s_1,x), the probability of (i,j) being prescribed by (θ_1,θ_2) is $c_{ij}/\sum_r c_{ir}$, which is $Q(j|i)$. Since the same argument holds for player II, it proves (2). //

Proof of the Corollary: Let C be a correlated equilibrium in G. The theorem states that there exists a mediated talk which induces the distribution C over A (this is a consequence of (1)). In order to show that this mediated talk defines a Nash equilibrium of the extension we show that no player can gain by adopting another mix message $\bar{\sigma}_i$ or by adopting another decoding map $\bar{\theta}_i$. By (2), for every s_i and x which satisfy $\theta_i(s_i,x) = a_i$, $P(a_{-i}|s_i,x) = Q(a_{-i}|a_i)$ given σ_{-i} and θ. Since C is a correlated equilibrium, a_i is a best response against $Q(a_{-i}|a_i)$. Therefore, given σ_i, σ_{-i} and θ_{-i}, θ_i (which prescribes a_i) is a best response. By Remark 1, any alternative τ_i does not change properties (1) and (2) and therefore whatever the alternative:

1. the probability of playing a is the one assigned by C for any $a \in A$, and
2. whenever a_i is prescribed it is indeed an optimal response.

We have proved that (σ_i,θ_i) is a best response $(\sigma_{-i},\theta_{-i})$ and therefore it is a Nash equilibrium. //

Remark 3: In the construction of the signaling matrix we introduce the grand matrix which consists of the submatrices K(•). The first components of the private messages (sent by the players to the mediator) select the specific K(•) that becomes active. The second components (d_1 and d_2) determine the public announcement from the K(•) already chosen.

One may consider all the K's as jointly controlled devices. The active device is chooses each of (r,b), (r,t) and (ℓ,b) with probability 1/3 each. Whatever the choice of the mediator, he informs player I of the row chosen and player II of the column chosen.

One can see that, given the mediator just described, player I has the incentive to reveal his true type. Notice that, once player II receives some information from the mediator, his prior over player I's type changes. For instance, if ℓ is sent, then the posterior ascribes type 1 the probability 2/3 while type 2 is ascribed only 1/3 (as opposed to the prior 1/2).

Notice that, if the players play the action sent by the mediator, then the process describes an equilibrium and no player has an incentive to deviate. This conclusion depends strongly on the specific posteriors. Therefore, any mechanism should always generate the same posteriors.

In order to generate this communication equilibrium by a mediated talk we adopt the matrix of Example 2 and define two signaling matrices, one for each type:

$$\begin{pmatrix} d & d & b \\ d & c & c \\ d & c & b \end{pmatrix},$$

as in Figure 4 for type 1, and

$$\begin{pmatrix} d & d & c \\ b & c & c \\ b & d & b \end{pmatrix},$$

for type 2. The private messages of player II are as in Example 2: {1,2,3}. However, player I has also to inform the mediator of his type. If the type is 1, then the active signaling matrix is the first. Otherwise, it is the second matrix. One can confirm that the posteriors of player II are either (2/3,1/3) or (1/3,2/3), as needed.

Notice that, if instead of $\begin{pmatrix} 0 & 1/3 \\ 1/3 & 1/3 \end{pmatrix}$, the distribution on the second type's matrix would have been $\begin{pmatrix} 0 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$, for instance, then the matrix of Figure 4 could have not

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