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EXTENDED POISSON GAMES
AND THE CONDORCET JURY THEOREM

by

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Abstract. The Poisson model of games with population uncertainty is extended, by allowing that expected population sizes and players' utility functions may depend on an unknown state of the world. Such extended Poisson games are applied to prove a generalization of the Condorcet jury theorem.

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1. Introduction

Most applications of game theory begin the construction of a game model by specifying a given number of players. This assumption might seem to be innocuous, because any real game situation must involve some finite number of players. Standard game-theoretic analysis requires, however, that the parameters of our model should be common knowledge among the players. Thus, if we specify the number of players in our model, then we are implicitly assuming that it is common knowledge that there are exactly this many players in the game. In many large games, players may have substantial uncertainty about the number of other players in the game, so assuming that this number is common knowledge may be quite unrealistic. To avoid this assumption, we must consider game models with population uncertainty. That is, instead of specifying a given number of players, we should specify that the number of players is a random variable drawn from some given distribution.

In Harsanyi's (1967-8) model of Bayesian games, each player's private information is represented by a denoting a random variable for each player, which is called the player's type. In a game model with population uncertainty, we must instead describe, for each possible type, a random variable that represents the number of players of this type who are in the game.

In Harsanyi's Bayesian games, a strategy profile is a function that specifies, for each possible type of each player, a probability distribution over the set of possible actions. But a model of population uncertainty describes a world in which players' individual identities are not

globally recognized, and so we cannot formulate theories that make distinct predictions about different players of the same type. Thus, in a game model with population uncertainty, our solutions will instead be strategy functions that specify, for each possible type, a probability distribution over the set of possible actions.

In a previous paper (Myerson, 1994), I defined a Poisson game to be a game in which the number of players is a random variable that has a Poisson distribution with some mean n , and each player's type is then drawn independently from some fixed probability distribution, where we may let $r(t)$ denote the probability of type t . In such a Poisson game, the number of players of each type t is then an independent Poisson random variable with mean $nr(t)$. The reason for focusing on such Poisson games, among all games with population uncertainty, is because they have some very convenient technical properties, the most important of which is the independent-actions property.

We may say that a game with population uncertainty induces independent actions iff, for any strategy function, the numbers of players who choose the various actions will be independent random variables. To express this idea more formally, let C denote the set of actions among which each player must choose, and let the random variable $\tilde{X}(c)$ denote the number of players whose choice is c , for each action c in C . Then the game induces independent actions iff, for any strategy function, these random variables $(\tilde{X}(c))_{c \in C}$ will be independent of each other. Of course, $\sum_{c \in C} \tilde{X}(c)$ must equal the total number of players in the game, so these random variables cannot be independent if there is no population uncertainty. In fact, it can be shown (see Myerson, 1994) that Poisson games are the only games that have this independent-actions property. Furthermore, in a Poisson game, the distributions of these action counts $\tilde{X}(c)$ are themselves

Poisson, and so their probability distributions can be completely specified by their means (because the Poisson distributions are a one-parameter family).

This paper introduces a generalization of the Poisson game model, called extended Poisson games. In such extended Poisson games, we allow that there might be some dependence among the numbers of players who choose different actions, but we insist that this dependence must be completely explainable by some underlying state variable that can affect the numbers of players of each type in the game. That is, we assume that the basic structure of the game includes a random variable, which we call the state of the world, that has the following property: For any strategy function, the number of players choosing any one action would be conditionally independent of the numbers of players choosing the other possible actions if the state of the world were known. So in an extended Poisson game, learning the number of players who choose an action c can affect our beliefs about how many players are choosing other actions, but only to the extent that learning the number who choose action c provides some information about this unknown state which has influenced the overall distribution of players' types in the game.

The independent-actions theorem of Myerson (1994) directly implies that an extended Poisson game must become a Poisson game once the state of the world has been determined. Thus, extended Poisson games can be defined as games that have the following two-stage structure: First a random state variable is drawn from some given distribution, and then a Poisson game is played, where the parameters of the Poisson game are a function of the state. In this paper, we introduce the general analysis of such extended Poisson games, and we apply this model to prove a general formulation of the Condorcet jury theorem with strategic voting.

There is a vast literature on the Condorcet jury theorem, which asserts that majority

voting in large electorates should reach "correct" decisions with high probability, when the voters have the same fundamental preferences but have different information. (See for example McLean and Hewitt, 1994; Grofman, Owen and Feld, 1983; Nitzan and Paroush, 1985; Miller, 1986; Grofman and Feld, 1988; Young 1988; Ladha, 1992, 1993; and Berg, 1993.) Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1994,1996a,1996b) have recently showed, however, that the previous literature ignored the crucial question of what a rational voter should infer from the event that his (or her) vote could actually make a difference in the outcome of the election. As Austen-Smith and Banks and Feddersen and Pesendorfer have shown, this inference leads to rational behavior which is very different from the sincere behavior that was assumed in the previous literature, but they also showed that such rational behavior is compatible with a more general version of the Condorcet jury theorem. Proving such a general jury theorem with many types and arbitrary distributions seemed a daunting task, however, because of the complexity of these large games. We show in this paper that the simplifying structure of the extended Poisson model makes these large games tractable. That is, we prove here a general jury theorem with strategic voting by combining the ideas of Condorcet and Poisson.

2. Global uncertainty and extended Poisson games

We formally define an extended Poisson game to be any (Ω,q,T,n,r,C,U) that has the following interpretations and properties.

The set of possible states of the world is denoted by Ω . Our prior beliefs about the state

of the world are denoted by $q = (q(\omega))_{\omega \in \Omega}$, which is a probability distribution on Ω . That is, $q(\omega)$ denotes the probability that ω is the true state of the world.

The expected number of players in each possible state is represented by the function $n: \Omega \rightarrow \mathbb{R}_+$. For each ω in Ω , if ω is the state of the world then the total number of players in the game will be a random variable drawn from a Poisson distribution with mean $n(\omega)$. The set T denotes the set of possible types. For each ω in Ω , if ω is the state of the world then each player has a type that is independently drawn from T according to the probability distribution $r(\cdot | \omega)$, which assigns a nonnegative probability $r(t | \omega)$ to every t in T . Thus, given any state ω , the random number of type- t players in the game has a conditional probability distribution that is Poisson with mean $n(\omega)r(t | \omega)$, and this random number is conditionally independent of the numbers of all other types of players.

The set of feasible actions for each player in the game is denoted by C . The utility payoff to each player can depend on the state of the world, the player's type, the player's action, and on the numbers of other players who choose each of the possible actions in C . For any set of players, a vector that lists how many of these players are choosing each action in C is called the action profile of these players. We let $Z(C)$ denote the set of all vectors $x = (x(c))_{c \in C}$ in \mathbb{R}^C such that each component $x(c)$ is a nonnegative integer. So $Z(C)$ is the set of all possible action profiles for the players in the Poisson game. Then the players' utility functions in the Poisson game are denoted by the function $U: Z(C) \times C \times T \times \Omega \rightarrow \mathbb{R}$. Here $U(x, b, t, \omega)$ denotes the utility payoff to a type- t player who chooses action b when ω is the true state of the world and when x is the action profile of the other players in the game (that is when, for each c in C , there are $x(c)$

other players who choose action c , not counting this player in the case of $c = b$).

Throughout this paper, we will assume that Ω and C are nonempty finite sets, T is a nonempty countable set, the function U is bounded, and $n(\omega) > 0$ and $q(\omega) > 0$ for every possible state ω in Ω . The distributions r and q must also satisfy $\sum_{\omega \in \Omega} q(\omega) = 1$, and

$$\sum_{t \in T} r(t|\omega) = 1, \quad \forall \omega \in \Omega.$$

A strategy function is any mapping σ that specifies a probability distribution over the set of actions C for each type in T , where $\sigma(c|t)$ is interpreted as the probability that a player will choose action c if his (or her) type is t . So we must have

$$\sum_{c \in C} \sigma(c|t) = 1, \quad \forall t \in T.$$

Our predictions about players' behavior must be written in terms of such a strategy function, because each player's predicted action can depend only on his (or her) own type.

If the players' behavior is characterized by the strategy function σ , then we let $\lambda(c|\omega)$ denote the conditionally expected number of players who will choose action c when the true state is ω ; that is,

$$(1) \quad \lambda(c|\omega) = \sum_{t \in T} n(\omega) r(t|\omega) \sigma(c|t), \quad \forall c \in C, \quad \forall \omega \in \Omega.$$

(Notice that λ is implicitly a function of σ here.) Given the state ω , the random number of players choosing action c is a Poisson random variable with mean $\lambda(c|\omega)$, and this random number is conditionally independent of the numbers of players choosing all other actions. The vector $\lambda = (\lambda(c|\omega))_{c \in C, \omega \in \Omega}$ in $\mathbb{R}^{C \times \Omega}$ satisfying equation (1) may be called the expected results vector corresponding to the strategy σ . For any possible state ω , we let $\lambda(\omega)$ denote the profile of expected results in state ω ; that is,

$$\lambda(\omega) = (\lambda(c|\omega))_{c \in C} \in Z(C).$$

So if ω is the state of the world then the conditional probability that some x in $Z(C)$ will be the players' action profile is

$$P(x | \lambda(\omega)) = \prod_{c \in C} \left(\frac{e^{-\lambda(c|\omega)} \lambda(c|\omega)^{x(c)}}{x(c)!} \right).$$

In most game models with finitely many players, we must carefully distinguish the following two numbers: (i) the probability that we should assign to the event that $x(c)$ players choose action c ; and (ii) the probability that a player in the game should assign to the event that $x(c)$ other players (not counting himself) choose action c . In a Poisson game, however, each player's probabilistic model of his environment is the same as our probabilistic model of the overall game. When a player in the game looks at his environment in the game, he sees one fewer person than we see when we look at the whole game from the outside. On the other hand, in a game with population uncertainty, a player's knowledge that he is in the game is itself a bit of private information that favors a larger population of players. In a Poisson game, these two considerations exactly cancel out. In fact, this environmental-equivalence property uniquely characterizes the Poisson games, as shown in Myerson (1994). Thus, in our extended Poisson game, if we would assign conditional probability $P(x | \lambda(\omega))$ to the event that x is the profile of all players' actions given that ω is the state, then any player should in the game should also assign the same conditional probability $P(x | \lambda(\omega))$ to the event that x is the profile of all other players' actions given that ω is the state, regardless of his own type and action.

So in state ω , the state-conditional expected utility of a type- t player who chooses action b is

$$U^*(\lambda, b, t, \omega) = \sum_{x \in Z(C)} P(x | \lambda(\omega)) U(x, b, t, \omega).$$

The probabilities $P(x_i | \lambda(\omega))$ depend continuously on the vector $\lambda(\omega)$, and the function $U^*(\lambda, b, t, \omega)$ is continuous in λ , because U is bounded.

By Bayes's rule, when a player knows that his type is t , his conditional probability that the state of the world is ω is

$$q^*(\omega | t) = \frac{q(\omega) n(\omega) r(t | \omega)}{\sum_{y \in \Omega} q(y) n(y) r(t | y)}$$

Knowing his own type only, the expected utility of a type- t player who chooses action c is then

$$\sum_{\omega \in \Omega} q^*(\omega | t) U^*(\lambda, c, t, \omega).$$

So in equilibrium, σ should satisfy the following rationality condition:

$$(2) \quad \sum_{b \in C} \sigma(b | t) \sum_{\omega \in \Omega} q^*(\omega | t) U^*(\lambda, b, t, \omega) = \max_{c \in C} \sum_{\omega \in \Omega} q^*(\omega | t) U^*(\lambda, c, t, \omega), \quad \forall t \in T.$$

We say that a strategy function σ is an equilibrium iff the rationality condition (2) is satisfied with the expected results vector λ that corresponds to σ as specified by the mean equations (1). The general existence of equilibria for extended Poisson games is a straightforward application of well known fixed-point theorems.

Theorem 1. For any extended Poisson game $(\Omega, q, T, n, r, C, U)$ as above, the set of equilibria is nonempty.

Proof. Existence can be proven by a fixed-point argument on the set of vectors λ in $\mathbb{R}^{C \times \Omega}$ such that

$$\sum_{c \in C} \lambda(c, \omega) = n(\omega), \quad \forall \omega \in \Omega.$$

This set is a compact convex subset of a finite-dimensional vector space, because C and Ω are

assumed to be finite sets. The Kakutani fixed-point theorem is then applied to the correspondence $F(\bullet)$ such that $v \in F(\lambda)$ if and only if there exists some strategy function σ such that σ satisfies the rationality condition (2) for λ and

$$v(c|\omega) = \sum_{t \in T} n(\omega) r(t|\omega) \sigma(c|t), \quad \forall c \in C, \quad \forall \omega \in \Omega.$$

For any fixed point λ such that $\lambda \in F(\lambda)$, there must exist some equilibrium σ that satisfies conditions (1) and (2). Q.E.D.

To characterize limits of equilibria as $k \rightarrow \infty$, it is useful to renormalize the expected results vector by dividing the expected results in each state by the expected number of players in that state. That is, we may let $\tau = (\tau(c|\omega))_{c \in C, \omega \in \Omega}$ denote the vector in $\mathbb{R}^{C \times \Omega}$ such that

$$(3) \quad \tau(c|\omega) = \sum_{t \in T} r(t|\omega) \sigma(c|t), \quad \forall c \in C, \quad \forall \omega \in \Omega.$$

Then $\tau(c|\omega) = \lambda(c|\omega)/n(\omega)$, and $\tau(c|\omega)$ can be interpreted as the conditional probability that any randomly sampled voter will choose action c if the state of the world is ω . These conditional probabilities must satisfy

$$\sum_{c \in C} \tau(c|\omega) = 1, \quad \forall \omega \in \Omega.$$

The expected action profile $\lambda(\omega)$ in any possible state ω can then be rewritten in terms of τ as

$$\lambda(\omega) = n(\omega)\tau(\omega) = (n(\omega)\tau(c|\omega))_{c \in C}.$$

Similarly, the overall expected results vector λ may be rewritten in terms of τ as

$$\lambda = n\tau = (n(\omega)\tau(c|\omega))_{c \in C, \omega \in \Omega}.$$

3. Application: the Condorcet jury theorem

The Condorcet jury theorem asserts that majority voting in large electorates should reach "correct" decisions with high probability. Proofs of this theorem generally rely on an assumption that each voter has an independent probability greater than 1/2 of having information favorable to the correct decision. Feddersen and Pesendorfer (1994,1996a,1996b) and Austen-Smith and Banks (1996) have shown, however, that these arguments were limited by an assumption that voters would vote sincerely, which is not necessarily true in a rational equilibrium of a voting game. When this sincere-voting assumption is dropped, then a formulation of the Condorcet jury theorem can be proven even without the restrictive assumption about likelihoods being greater than 1/2. In this section, we show how extended Poisson games can be used to formulate and prove such a generalized version of the Condorcet jury theorem.

Consider an extended Poisson game in which there are two possible states of the world, $\Omega = \{1,2\}$. The players in the game are voters in a "jury" which must vote on the question: what is the true state of the world? Each voter can vote for state 1 or 2, so the set of actions is also $C = \{1,2\}$. There is no cost of voting, and we assume here that abstention is not allowed. (Feddersen and Pesendorfer, 1996a, show that if abstention is allowed then the strategic use of abstention can be remarkably important in such games. We rule out abstention here only for simplicity.) Each voter's utility payoff $U(x,c,t,\omega)$ is +1 when a majority votes for the true state of the world, -1 if a majority votes for the other possible state, and 0 if there is a tie vote. (We may suppose that a tie vote would leave the decision to a fair coin toss, which has a 50% chance of being correct.) The prior probabilities of each state ($q(1),q(2)$) are given positive numbers. We assume that the probability distribution $r(\bullet|\omega)$ out of which the players' types are drawn is not the

same in both states. That is, there exist some types t such that $r(t|1) \neq r(t|2)$.

The Condorcet jury theorem is about the accuracy of majority outcomes in large elections, so let us consider a sequence of such games in which (Ω, q, T, r, C, U) are fixed as above, and the expected numbers of voters in each state are parameters going to infinity in some fixed ratio. That is, there exists some fixed positive number θ such that $n(2) = \theta n(1)$ and $n(1) \rightarrow \infty$ in this sequence of games. We use the parameter $k = n(1)$ as the index for this sequence of jury voting games.

Theorem 2. Given a sequence of jury voting games as above, there exists a sequence $\{\bar{\sigma}_k, \bar{\lambda}_k\}_{k=1}^{\infty}$ such that each $\bar{\sigma}_k$ is an equilibrium for the game when $n(1) = k$ and $n(2) = \theta k$, each $\bar{\lambda}_k$ is the expected results vector corresponding to $\bar{\sigma}_k$ in this game (satisfying equation (1)), and

$$\lim_{k \rightarrow \infty} \bar{\lambda}_k(1|1)/\bar{\lambda}_k(2|1) > 1 \text{ and } \lim_{k \rightarrow \infty} \bar{\lambda}_k(2|2)/\bar{\lambda}_k(1|2) > 1.$$

Thus, the probability of a correct majority decision in these equilibria goes to one in each state as the expected numbers of voters go to infinity. These equilibria also have the likelihood-ratio property that, for each k , there exists some number ρ_k such that, for each type t ,

$$\text{if } \rho_k r(t|1) > r(t|2) \text{ then } \bar{\sigma}_k(1|t) = 1; \text{ but if } \rho_k r(t|1) < r(t|2) \text{ then } \bar{\sigma}_k(2|t) = 1.$$

Furthermore, in the case where every type t has a strictly positive probability $r(t|\omega)$ in each state ω , these equilibria $\bar{\sigma}_k$ converge as $k \rightarrow \infty$ to a strategy function σ such that

$$(4) \quad 2\sqrt{\tau(1|1)\tau(2|1)} - \tau(1|1) - \tau(2|1) = \theta(2\sqrt{\tau(1|2)\tau(2|2)} - \tau(1|2) - \tau(2|2)),$$

where $\tau(c|\omega) = \sum_{t \in T} r(t|\omega)\sigma(c|t) = \lim_{k \rightarrow \infty} \bar{\lambda}_k(c|\omega)/n(\omega)$ for each c and ω .

Before proving this general theorem, let us consider a specific example. Suppose that

$\theta = 1$, so $n(1) = n(2) = k$, and there are two types, $T = \{1,2\}$. Let

$$q(1) = q(2) = 0.5, \quad r(1|1) = 0.9, \quad r(2|1) = 0.1, \quad r(1|2) = 0.8, \quad r(2|2) = 0.2.$$

So in each state, a player is more likely to be type 1 than type 2, but the chances of being type 2 are higher in state 2 than in state 1. By Bayes's rule, a type-1 player's assessment of the state probabilities should be

$$q^*(1|1) = .5 \times 100 \times .9 / (.5 \times 100 \times .9 + .5 \times 100 \times .8) = .529,$$

$$q^*(2|1) = 1 - q^*(1|1) = .471.$$

Similarly, a type-2 player's assessment of the state probabilities should be

$$q^*(1|2) = .5 \times 100 \times .1 / (.5 \times 100 \times .1 + .5 \times 100 \times .2) = .333, \quad q^*(2|2) = .667.$$

Let us now look for an equilibrium of the game where the expected number of players is $k = 100$. Every player wants to maximize the probability that the majority vote is for the true state. So it might seem natural to guess that each type of voter would vote for the state that he (or she) thinks is more likely. Such a strategy is called the sincere strategy, and it would be optimal if only one person were voting. Here, the sincere strategy is $\sigma(1|1) = 1$ and $\sigma(2|2) = 1$; that is, type-1 voters vote for 1, type-2 voters vote for 2. With sincere voting and $k = 100$, the expected vote totals in state 1 are

$$\lambda(1|1) = 90, \quad \lambda(2|1) = 10;$$

but in state 2 the expected vote totals are

$$\lambda(1|2) = 80, \quad \lambda(2|2) = 20.$$

Thus, if everyone voted sincerely then large majorities for 1 should be expected both in state 1 and in state 2.

But each voter knows that his vote only matters if it changes the majority outcome of the

election. We say that a vote would be pivotal iff adding this vote to the total votes of all others would change the outcome selected by the majority. Let $v(c|\lambda(\omega))$ denote the probability that a vote for c would be pivotal when ω is the true state. Myerson (1997) showed that, when $\lambda(1|\omega)$ and $\lambda(2|\omega)$ are large, the pivot probability $v(c|\lambda(\omega))$ can be approximated by the formula

$$(5) \quad \frac{e^{2\sqrt{\lambda(1|\omega)\lambda(2|\omega)} - \lambda(1|\omega) - \lambda(2|\omega)}}{4\sqrt{\pi\sqrt{\lambda(1|\omega)\lambda(2|\omega)}}} \left(\frac{\sqrt{\lambda(1|\omega)} + \sqrt{\lambda(2|\omega)}}{\sqrt{\lambda(1|\omega)}} \right)$$

Using this formula (5), we find that the probabilities that a vote for 1 would be pivotal depends on the true state as follows, when everyone votes sincerely:

$$v(1|\lambda(1)) = 1.46 \times 10^{-19}, \quad v(1|\lambda(2)) = 6.89 \times 10^{-11}.$$

So the probability of one vote making a difference is very small in either state, but it is larger in state 2 than in state 1 by a ratio of more than 10^8 . The reason for this difference is that the expected difference in vote totals is significantly closer in state 2 (80 versus 20) than in state 1 (90 versus 10), and so ties are much more likely in state 2. So in the event that a single vote can make a difference in the majority outcome, the conditional probability of state 1 (as assessed by either type of voter) would be less than 10^{-8} . If everyone else is expected to vote sincerely, then either type of voter should prefer to vote for 2, because if his vote makes a difference then the conditional probability of state 2 is more than $1 - 10^{-8} = .99999999$.

Thus, sincere voting is not an equilibrium. Voters need to evaluate the expected value of a vote given the information that could be inferred about the state if this vote actually made a difference in the majority outcome.

Of course, everyone voting for 2 ($\sigma(2|1) = 1 = \sigma(2|2)$) is not an equilibrium either. In that scenario, a vote could only make a difference if the total number of other voters is one or

zero, which occurs with the same probability in both states; and so sincere voting would then be optimal, because that the event of being pivotal would convey no information.

An equilibrium exists between these two scenarios, at

$$\sigma(1|1) = .594, \sigma(2|1) = .406, \sigma(1|2) = 0, \sigma(2|2) = 1.$$

Then the expected vote totals in state 1 are

$$\lambda(1|1) = 53.47, \lambda(2|1) = 46.53,$$

whereas the expected vote totals in state 2 are

$$\lambda(1|2) = 47.53, \lambda(2|2) = 52.47.$$

Notice that $\lambda(1|1) > \lambda(2|1)$ and $\lambda(2|2) > \lambda(1|2)$. So in each state, the majority is expected to be on the correct side. The probability that the majority is correct is about .75 in state 1 and is about .59 in state 2 (as can be computed using a Normal approximation to the Poisson distribution).

By formula (5), the probabilities of being pivotal in state 1 are

$$v(1|\lambda(1)) = .0303, v(2|\lambda(1)) = .0325,$$

in this equilibrium, and the probabilities of being pivotal in state 2 are

$$v(1|\lambda(2)) = .0362, v(2|\lambda(2)) = .0345$$

(Recall that $\lambda(\omega) = (\lambda(c|\omega))_{c \in C}$ here.) Notice that, in each state, the pivot probability is higher for the side that is more likely to lose.

A type- t voter should believe that his expected gain from voting for 1 is

$$G(1|t) = q^*(1|t) v(1|\lambda(1)) - q^*(2|t) v(1|\lambda(2)).$$

That is, a voter's expected payoff gain from contributing a vote for 1 is the probability that his vote is pivotal and the state is 1 (in which case his vote is changing the jury's outcome from wrong to right) minus the probability that his vote is pivotal and the state is 2 (in which case his

vote is changing the jury's outcome from right to wrong), using the state probabilities $q^*(\omega|t)$ that the voter assesses given the information that his type is t . Similarly, a type- t voter should believe that the expected gain from voting for 2 is

$$G(2|t) = q^*(2|t)v(2|\lambda(2)) - q^*(1|t)v(2|\lambda(1)).$$

In the equilibrium scenario, when a type-1 voter votes for 1 then his expected net gain is

$$G(1|1) = .529 \times .0303 - .471 \times .0362 = -.001$$

Similarly, when a type-1 voter votes for 2 in equilibrium then his expected net gain is

$$G(2|1) = .471 \times .0345 - .529 \times .0325 = -.001$$

The equality of these expected net gains implies that the type-1 voters are willing to randomize between voting for 1 and voting for 2, as the equilibrium scenario requires.

Notice that, in this equilibrium, $G(1|1)$ and $G(2|1)$ are equal but are negative. Thus, although the type-1 voters are indifferent between voting for 1 or 2 in this equilibrium, these type-1 would strictly prefer to abstain, rather than vote for either side. The result is a manifestation of the swing voter's curse, found by Feddersen and Pesendorfer (1996a). (If we allowed abstention, we would get a different equilibrium in which the type-1 voters randomize between voting for 1 and abstaining, but they do not vote for 2.)

In this equilibrium, a type-2 voter's expected net gains from voting for 1 would be

$$G(1|2) = .333 \times .0303 - .667 \times .0362 = -.014,$$

while a type-2 voter's expected net gains from voting for 2 are

$$G(2|2) = .667 \times .0345 - .333 \times .0325 = .012$$

Thus, the type-2 voters all strictly prefer to vote for 2, as the equilibrium scenario specifies.

Now let us consider what happens to the equilibrium strategy in this example when we

increase the expected number of voters $n(1) = n(2) = k$, taking the limit as $k \rightarrow \infty$. When σ denotes the limit of the equilibrium strategy as $k \rightarrow \infty$, Theorem 2 tells that us to consider the vector $\tau = (\tau(c|\omega))_{c \in C, \omega \in \Omega}$ that is derived from σ as in equation (3); that is,

$$\tau(c|\omega) = \sum_{t \in T} r(t|\omega) \sigma(c|t), \quad \forall c \in C, \quad \forall \omega \in \Omega.$$

In terms of this vector τ , the pivot-probability formula (5) can be rewritten:

$$v(c|n(\omega)\tau(\omega)) \approx \frac{e^{n(\omega)(2\sqrt{\tau(1|\omega)\tau(2|\omega)} - \tau(1|\omega) - \tau(2|\omega))}}{4\sqrt{\pi n(\omega)\sqrt{\tau(1|\omega)\tau(2|\omega)}}} \left(\frac{\sqrt{\tau(1|\omega)} + \sqrt{\tau(2|\omega)}}{\sqrt{\tau(1|\omega)}} \right)$$

Theorem 2 tells us that, at the limit of equilibria σ , this vector τ must satisfy equation (4); that is,

$$2\sqrt{\tau(1|1)\tau(2|1)} - \tau(1|1) - \tau(2|1) = \theta(2\sqrt{\tau(1|2)\tau(2|2)} - \tau(1|2) - \tau(2|2)).$$

Multiplying through by $n(1)$, it can be seen that this equation (4) is equivalent to requiring that the exponent of e in the pivot-probability formula should be the same in both states ω . If equation (4) were violated in the limit, then in large games the event of a vote being pivotal would be overwhelming evidence against one of the two states, which would make all voters want to vote for the other state.

For our example, the limit of equilibrium strategies σ and the corresponding vector τ are

$$\sigma(1|1) = .5882, \quad \sigma(2|1) = .4118, \quad \sigma(1|2) = 0, \quad \sigma(2|2) = 1,$$

$$\tau(1|1) = .5294 = \tau(2|2), \quad \tau(2|1) = .4706 = \tau(1|2).$$

These vectors σ and τ can be directly computed from equations (3) and (4) together with the ρ_k -condition in Theorem 2, which implies for our example that either $\sigma(2|1)$ or $\sigma(1|2)$ must equal 0 (because an equilibrium cannot simultaneously have type-1 voters voting for 2 and type-2 voters voting for 1).

In any case where $\theta = 1$, as in this example, equation (4) in Theorem 2 can be further

simplified to

$$\tau(1|1) - \tau(2|1) = \tau(2|2) - \tau(1|2).$$

That is, if the expected population size is the same in both states, and every type has positive likelihood in both states, then we can find a sequence of equilibria of the Condorcet jury game such that the expected margin (as a percentage of the overall voting population) in favor of the true state converges to the same positive limit in both states. This simplified formula does not extend to the games studied by Feddersen and Pesendorfer where abstention is allowed, because abstention can make expected vote totals different in the two states even when $n(1) = n(2)$.

Theorem 2 claims that majorities are asymptotically almost surely correct in some equilibria of the jury voting game, but not necessarily in all equilibria. If we altered the above example to have prior probabilities $q(1) = .7$ and $q(2) = .3$, then there would always exist an equilibrium in which both types plan to vote for 1, no matter how large the size parameter k gets. However, this altered example also has a sequence of equilibria which satisfy the Condorcet jury theorem, and which converge to the same limit σ described above.

Proof of Theorem 2. For any strategy function σ ; given any possible state ω , we let $\tau(\omega) = (\tau(c|\omega))_{c \in C}$ denote the conditional probability distribution on C that is derived from σ as in equation (3), using the notation that was introduced at the end of Section 2..

Let n_k denote the vector that lists the expected population sizes in each state as they depend on the parameter k ; that is,

$$n_k = (n_k(\omega))_{\omega \in \Omega}, \text{ where } n_k(1) = k, n_k(2) = \theta k.$$

Thus, $n_k(\omega)\tau(\omega) = (n_k(\omega)\tau(c|\omega))_{c \in C}$ denotes the expected action profile in state ω , with the size

parameter k and the strategy function σ . In this environment, a type- t player should believe that voting for 1 is optimal if and only if

$$\begin{aligned} & q^*(1|t) v(1 | n_k(1)\tau(1)) - q^*(2|t) v(1 | n_k(2)\tau(2)) \\ & \geq q^*(2|t) v(2 | n_k(2)\tau(2)) - q^*(1|t) v(2 | n_k(1)\tau(1)) \end{aligned}$$

Now let

$$V(n_k(\omega)\tau(\omega)) = v(1 | n_k(\omega)\tau(\omega)) + v(2 | n_k(\omega)\tau(\omega)).$$

That is, $V(n_k(\omega)\tau(\omega))$ is the probability that some single vote (for 1 or 2) could change the outcome if the state is ω , when the players use strategy function σ in the game of size k . Then the above inequality can be rewritten

$$q^*(1|t) V(n_k(1)\tau(1)) \geq q^*(2|t) V(n_k(2)\tau(2)).$$

But Bayes's rule implies

$$q^*(2|t)/q^*(1|t) = \theta q(2) r(t|2)/(q(1) r(t|1)).$$

Thus, a type- t voter should believe that voting for 1 is optimal if and only if

$$(6) \quad \frac{r(t|2)}{r(t|1)} \leq \frac{V(n_k(1)\tau(1)) q(1)}{V(n_k(2)\tau(2)) \theta q(2)}.$$

Let $\rho(n_k\tau)$ denote the critical ratio on the right-hand side of this inequality

$$\rho(n_k\tau) = \frac{V(n_k(1)\tau(1)) q(1)}{V(n_k(2)\tau(2)) \theta q(2)}.$$

So a type- t voter is willing to vote for 1 iff $r(t|2)/r(t|1) \leq \rho(n_k\tau)$. Similarly, a type- t voter is willing to vote for 2 iff $r(t|2)/r(t|1) \geq \rho(n_k\tau)$. A type- t voter is willing to randomize his vote iff these two inequalities are both satisfied by equality.

Relabelling the types if necessary, we can assume without loss of generality that the types

are numbered $T = \{1, 2, \dots\}$ such that, for any two types s and t ,

$$\text{if } s \geq t \text{ then } r(s|2)/r(s|1) \geq r(t|2)/r(t|1).$$

That is, we can order the types so that the monotone likelihood ratio property is satisfied, and higher types are stronger evidence in favor of state 2. With this monotonicity structure, higher types always are never less inclined to vote for 2 than lower types. So we can restrict our attention to step strategies that are such that, for some type s (the step type),

$$\sigma(1|t) = 1, \quad \forall t < s, \text{ and}$$

$$\sigma(1|t) = 0, \quad \forall t > s.$$

(Of course, $\sigma(2|t) = 1 - \sigma(1|t)$ for all t .) We can now parameterize these step strategies by a real number h , ranging between 1 and $\#T + 1$. The h -step strategy, denoted by σ_h , is defined such that, when s_h denotes the type satisfying

$$s_h \leq h < s_h + 1$$

then, for each type t ,

$$\sigma_h(1|t) = 1 \text{ if } t < s_h,$$

$$\sigma_h(1|s_h) = h - s_h, \text{ and}$$

$$\sigma_h(1|t) = 0 \text{ if } t > s_h.$$

Notice that this parameterization makes σ_h continuous in the parameter h over the interval from 1 to $\#T + 1$. (We can allow the case where T is countably infinite, in which case $\#T + 1 = +\infty$.) At the extremes, σ_1 here denotes the strategy of always voting for 2, and $\sigma_{\#T+1}$ denotes always voting for 1.

Let $\tau_h(c|\omega)$ denote the expected fraction of voters who vote c in state ω , when σ_h is used;

that is,

$$\tau_h(c|\omega) = \sum_{t \in T} \sigma_h(c, t) r(t|\omega).$$

For each state ω , we let $\tau_h(\omega) = (\tau_h(c|\omega))_{c \in C}$.

Formula (5) (or a direct application of Theorem 1 of Myerson, 1997, as shown in Section 4 of that paper) implies that, if the voters use the step strategy σ_h then, in each state ω , then the conditional probability of a single vote being pivotal $V(n_k(\omega)|\tau_h(\omega))$ satisfies

$$\lim_{n_k(\omega) \rightarrow \infty} \log(V(n_k(\omega)|\tau_h(\omega)))/n_k(\omega) = 2\sqrt{\tau_h(1|\omega)\tau_h(2|\omega)} - \tau_h(1|\omega) - \tau_h(2|\omega).$$

Thus, with $n_k(1) = k$ and $n_k(2) = \theta k$,

$$\begin{aligned} (7) \quad & \lim_{k \rightarrow \infty} \frac{\log(V(n_k(1)|\tau_h(1))/V(n_k(2)|\tau_h(2)))}{k} \\ &= \lim_{k \rightarrow \infty} \left(\log(V(n_k(1)|\tau_h(1)))/n_k(1) - \theta \log(V(n_k(2)|\tau_h(2)))/n_k(2) \right) \\ &= \left(2\sqrt{\tau_h(1|1)\tau_h(2|1)} - \tau_h(1|1) - \tau_h(2|1) \right) - \theta \left(2\sqrt{\tau_h(1|2)\tau_h(2|2)} - \tau_h(1|2) - \tau_h(2|2) \right). \end{aligned}$$

Now let $f(h)$ be defined by the last formula in (7) above.

$$\begin{aligned} f(h) &= \left(2\sqrt{\tau_h(1|1)\tau_h(2|1)} - \tau_h(1|1) - \tau_h(2|1) \right) - \theta \left(2\sqrt{\tau_h(1|2)\tau_h(2|2)} - \tau_h(1|2) - \tau_h(2|2) \right) \\ &= \theta \left(\sqrt{\tau_h(1|2)} - \sqrt{\tau_h(2|2)} \right)^2 - \left(\sqrt{\tau_h(1|1)} - \sqrt{\tau_h(2|1)} \right)^2. \end{aligned}$$

Substituting this definition of $f(h)$ back into the definition of the critical ratio ρ that was defined after equation (6) above, we get

$$(8) \quad \lim_{k \rightarrow \infty} \log(\rho(n_k|\tau_h))/k = f(h),$$

because $q(1)$, $q(2)$, and θ are held fixed as $k \rightarrow \infty$.

We can now find numbers $I(1)$ and $I(2)$ such that

$$1 < I(1) < I(2) < \#T + 1,$$

$$\tau_{I(1)}(1|1) = \tau_{I(1)}(2|1), \text{ and } \tau_{I(2)}(1|2) = \tau_{I(2)}(2|2).$$

That is, the expected vote totals for 1 and 2 are equal in state 1 under strategy function $\sigma_{I(1)}$, and are equal in state 2 under $\sigma_{I(2)}$. To prove that such numbers exist, notice that $\tau_h(1|1) - \tau_h(2|1)$ increases continuously from -1 to 1 as h goes from 1 to $\#T + 1$, and so there exists a number $I(1)$ such that $\tau_{I(1)}(1|1) - \tau_{I(1)}(2|1) = 0$. Similarly, $\tau_h(1|2) - \tau_h(2|2)$ increases continuously from -1 to 1 , and so there exists a number $I(2)$ such that $\tau_{I(2)}(1|2) - \tau_{I(2)}(2|2) = 0$. We must have $I(1) < I(2)$ because, by the monotone likelihood ratio property and the assumption that the distributions $r(\bullet|1)$ and $r(\bullet|2)$ are different, $\tau_h(1|1) > \tau_h(1|2)$ and $\tau_h(2|1) < \tau_h(2|2)$ for all h between 1 and $\#T+1$. Furthermore, for any number h ,

$$\text{if } I(1) < h < I(2) \text{ then } \tau_h(1|1) > \tau_h(2|1) \text{ and } \tau_h(1|2) < \tau_h(2|2).$$

That is, all step strategies with steps between $I(1)$ and $I(2)$ generate expected majorities for the correct state in both states.

We can also find a number J such that

$$I(1) < J < I(2) \text{ and } f(J) = 0.$$

To verify the existence of this number J , notice that $f(I(1)) > 0$, $f(I(2)) < 0$, and the function f is continuously decreasing over the interval from $I(1)$ to $I(2)$. More generally, over the interval from $I(1)$ to $I(2)$, the function f satisfies:

$$\text{if } I(1) \leq h < J \text{ then } f(h) > 0,$$

$$\text{if } J < h \leq I(2) \text{ then } f(h) < 0.$$

Let us now write more carefully the conditions for a step strategy σ_h to be an equilibrium

in the voting game where the expected numbers of voters are $n_k(1) = k$ and $n_k(2) = \theta k$ in states 1 and 2 respectively. If h is not an integer then σ_h is an equilibrium in this m -game iff

$$\rho(n_k \tau_h) = r(2|s_h)/r(1|s_h),$$

where s_h denotes the greatest integer less than h , because voters of type s_h are randomizing their votes under the strategy function σ_h . If h is an integer then σ_h is an equilibrium iff

$$r(2|h)/r(1|h) \leq \rho(n_k \tau_h) \leq r(2|h+1)/r(1|h+1),$$

because σ_h stipulates that all voters with type h or below are supposed to vote for 1 while all voters with type $h+1$ and above are supposed to vote for 2. So let us define the point-to-set correspondence $\eta(\bullet)$ such that, if h is an integer then $\eta(h)$ is the closed interval from $r(2|h)/r(1|h)$ to $r(2|h+1)/r(1|h+1)$, and if h is not an integer then $\eta(h)$ contains only the value $r(2|s_h)/r(1|s_h)$. Then, in this notation, σ_h is an equilibrium of the game with expected population sizes $n_k(1) = k$ and $n_k(2) = \theta k$ iff

$$\rho(n_k \tau_h) \in \eta(h).$$

The $\eta(\bullet)$ correspondence is upper-hemicontinuous, convex valued, and increasing.

Notice also that $\eta(h)$ does not depend on the size parameter k . Furthermore, $\eta(I(2))$ must have a strictly positive lower bound, because a likelihood ratio of zero for the highest type to vote for 1 at $\sigma_{I(2)}$ would imply that the types that are supposed to vote for 1 have zero expected turnout in state 2, but the expected turnouts for the two sides are equal under $\sigma_{I(2)}$ in state 2. A similar argument shows that $\eta(I(1))$ must have a finite upper bound. But the limit equation (8) implies that, for any number h between $I(1)$ and $I(2)$:

$$\text{if } I(1) \leq h < J \text{ then } \lim_{k \rightarrow \infty} \rho(n_k \tau_h) = +\infty,$$

$$\text{if } J < h \leq I(2) \text{ then } \lim_{k \rightarrow \infty} \rho(n_k \tau_h) = 0.$$

So for all sufficiently large k , we have $\rho(n_k \tau_{I(2)}) < \min(\eta(I(2)))$ and $\rho(n_k \tau_{I(1)}) > \max(\eta(I(1)))$.

Furthermore, for any fixed k , the function $\rho(n_k \tau_h)$ is a continuous function of h . Thus, for all sufficiently large k , there must exist some $H(k)$ such that

$$I(1) < H(k) < I(2) \text{ and } \rho(n_k \tau_{H(k)}) \in \eta(H(k)).$$

and so $\sigma_{H(k)}$ is an equilibrium of the game with $n(1) = k$ and $n(2) = \theta k$. In the notation of the theorem, we can let $\bar{\sigma}_k = \sigma_{H(k)}$ and $\bar{\lambda}_k = n_k \tau_{H(k)}$.

Now let L be a limit point of these bounded numbers $H(k)$ as $k \rightarrow \infty$. Having L strictly greater than J would imply $0 = \lim_{k \rightarrow \infty} \rho(n_k \tau_{H(k)}) \in \eta(L)$ (using the upper-hemicontinuity of η);

and so L cannot equal $I(2)$. Similarly, having L strictly less than J would imply

$+\infty = \lim_{k \rightarrow \infty} \rho(n_k \tau_{H(k)}) \in \eta(L)$; and so L cannot equal $I(1)$. So $I(1) < L < I(2)$, and so

$$\tau_L(1|1) > \tau_L(2|1) \text{ and } \tau_L(1|2) < \tau_L(2|2).$$

Thus we get

$$\lim_{k \rightarrow \infty} \bar{\lambda}_k(1|1)/\bar{\lambda}_k(2|1) = \tau_L(1|1)/\tau_L(2|1) > 1,$$

$$\lim_{k \rightarrow \infty} \bar{\lambda}_k(1|2)/\bar{\lambda}_k(2|2) = \tau_L(2|2)/\tau_L(1|2) > 1.$$

So in each state ω , the difference between the number of votes for the correct state and the number of votes for the incorrect state has an expected value that increases linearly as a positive fraction of the expected total number of voters $n(\omega)$. But the standard deviation of this difference is $\sqrt{n(\omega)}$, because the vote totals for each side are independent Poisson random variables with variances equal to their expected values. So the expected value of this difference divided by its standard deviation goes to infinity as $k \rightarrow \infty$. Thus, the probability of a majority error goes to zero in each state as the expected number of voters goes to infinity.

Now suppose that $r(t|\omega) > 0$ for all t in T and all ω in Ω . In this case, $r(t|2)/r(t|1)$ must

be a finite and strictly positive number for each type t . Then the argument in the preceding paragraph can be sharpened to prove that $L = J$, because $\eta(L)$ cannot contain 0 or $+\infty$. Thus the limiting equilibrium strategy $\sigma = \sigma_L$ will satisfy $f(L) = 0$ and equation (4) in the theorem. Q.E.D.

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