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POPULATION PROCESSES: A MARKOV RENEWAL THEORETIC APPROACH

by

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1. INTRODUCTION

Consider a population of asexually reproducing organisms. When an organism dies, it produces offspring. The number of offspring produced and their life-lengths are determined by a probability law that depends only on the population size. The death and reproduction cycle is repeated as long as organisms are present. We will be interested in the number of organisms, and their ages, alive at any point in time. The process thus describing the evolution of the population through time is a generalization of an age-dependent branching process. It is a generalization because a branching process requires the production of offspring and their life-lengths to be independent of the size of the population; however, the growth of most biological populations is dependent on the population size, especially populations controlled by intraspecific competition. See Harris (1963) for a review of branching processes.

Kesten (1971) and (1972) has results on multitype population processes whose change from generation to generation is governed by a non-linear transformation. This transformation determines future composition (fraction of types) as a function only of current composition. In a process involving competition where both composition and density are needed to determine future composition, Kesten's transformation is not applicable. Labkovski (1972) also has some results of a generalized Galton-Watson process but he deals with a special limit theorem which is not of broad

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enough application for this paper.

Moyal (1962) formulated two equivalent probability measure spaces for a general stochastic population process: the exponential space and the counting measure space. A form of the latter will be used in this paper; however, a third way to picture the state space is given below because it is a more intuitive description.

Let $\mathbb{E} = \{(x,y): x \geq 0, y > 0\}$. For $A \subset \mathbb{E}$ let $|A|$ denote the cardinality of A. Let $S = \{A: A \subset \mathbb{E} \text{ and } |A| < \infty\}$. Let $X(t)$ be a stochastic process with state space S. If $X(t_0) = \{(x_1,y_1),(x_2,y_2)\}$, then we say that $X(t_0)$ is a population of two individuals whose ages are x_1 and x_2 and whose remaining life-lengths are y_1 and y_2. Note that life-length is assumed known immediately after birth; thus, for this process, the only important characteristics of an individual are its lifetime since birth and its lifetime remaining until death. To visualize the process, start with an initial set of points in $\mathbb{E}$. Translate the origin up and to the left along the line $y = -x$. Whenever the x-axis moves through a point of the population, that point produces its offspring along the translated y-axis. The configuration of the points on the y-axis is determined by a probability law that depends on the size of the population. (See Figure 1.)

In the remaining sections, the set of points representing a state of the process will be denoted by a purely atomic measure. In Section 2, the process is formalized as a semi-regenerative process and the extinction probabilities are shown to satisfy a system of Markov renewal equations; in Section 3, the semi-Markovian kernel of the process is derived; in Section 4, using certain assumptions, the existence of an invariant measure is proved; and in Section 5, some of the limiting properties are investigated. For a review of semi-regenerative processes see Çinlar (1969) and (1975).

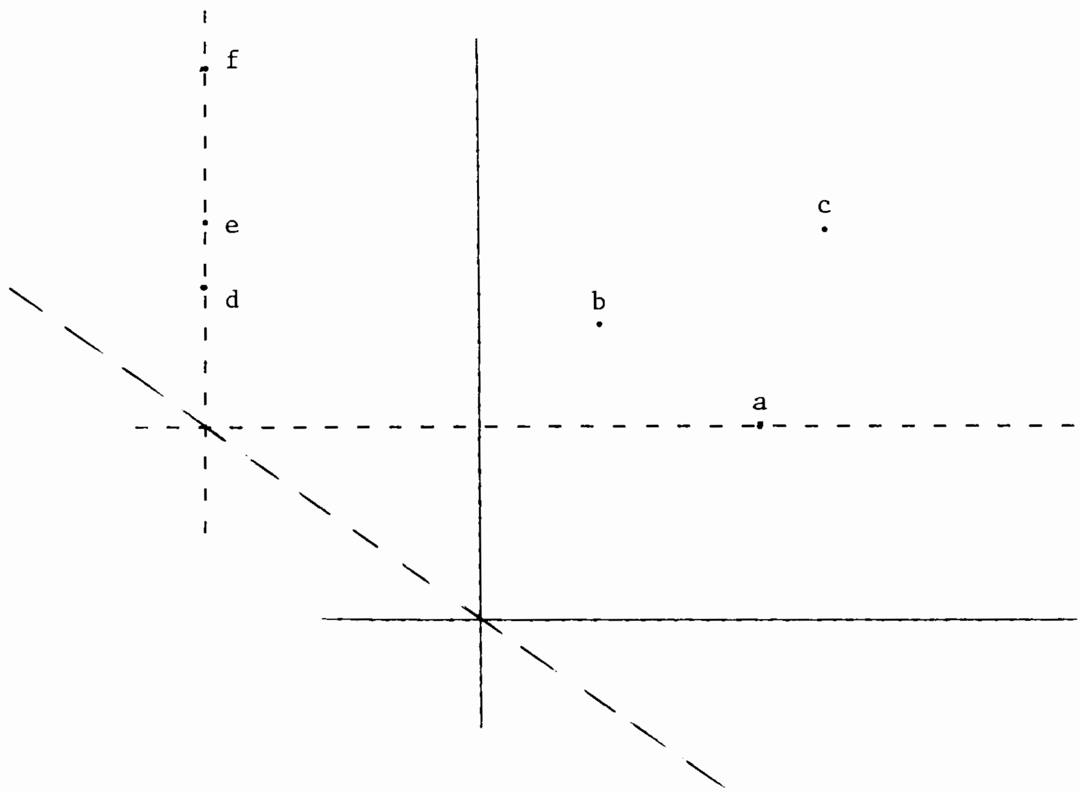

Figure 1: Points a, b, c are elements of the initial state.
Points d, e, f are offspring of a.

2. THE POPULATION PROCESS

Let $\mathbb{R}_+ = [0,\infty)$, $\mathbb{E} = [0,\infty)\times(0,\infty)$, $\mathbb{N} = \{0,1,2,\dots\}$, and $\mathbb{N}_+ = \{1,2,\dots\}$. Let $\underline{\underline{\mathbb{R}}}_+$ and $\underline{\underline{\mathbb{E}}}$ be the set of all borel subsets of $\mathbb{R}_+$ and $\mathbb{E}$ respectively. Let μ be a purely atomic finite measure on $\mathbb{E}$ such that $\mu(\{x\}) = 0$ or 1 for all $x \in \mathbb{E}$. Let $\mathbb{M}$ be the set of all such measures μ and let $\underline{\underline{\mathbb{M}}}$ be the σ-field generated by sets of the form $\{\mu \in \mathbb{M}:\mu(B_1) \geq 1, \dots, \mu(B_k) \geq 1, |\mu| = k\}$ as k varies over $\mathbb{N}$ and $B_1,\dots,B_k$ varies over $\underline{\underline{\mathbb{E}}}$.

Let $(\Omega,\mathcal{F},P)$ denote a probability space and let $Z = \{Z_t;\ t \geq 0\}$ be a stochastic process defined over $(\Omega,\mathcal{F},P)$ with state space $(\mathbb{M}, \underline{\underline{\mathbb{M}}})$. For $t \in \mathbb{R}_+$, $\omega \in \Omega$, and $\mu \in \mathbb{M}$, the equality $Z_t(\omega) = \mu$ indicates that μ is the state of the process at time t for realization ω. If $x \in \mathbb{E}$ where $x = (x_1, x_2)$ and if $\mu(\{x\}) = 1$, then there is an individual in the population at time t with age x_1 and remaining life-length x_2.

The process Z is governed by a deterministic rule as long as no deaths occur. To express this rule, two mappings are defined. The first mapping gives the time until the next death will occur for a given state μ. The second mapping is a translation producing the "aging" process (see Figure 2).

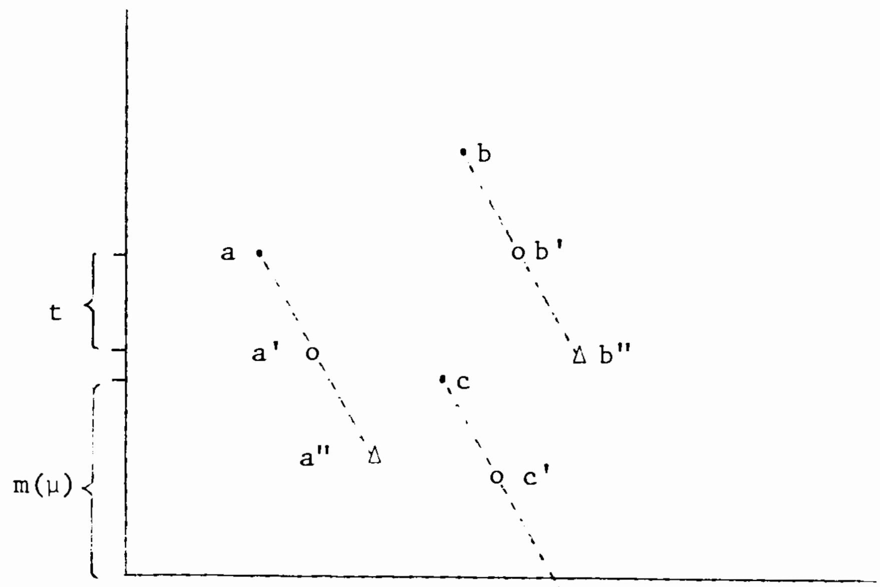

Figure 2: a, b, c are the atoms of μ.
a', b', c' are the atoms of $\tau_t(\mu)$.
a'', b'' are the atoms of $\theta(\mu)$.

The mapping $m: \mathbb{M} \to (0,\infty)$ is defined by

$$m(\mu) = \sup\{y : \mu(\mathbb{R}_+ \times (0,y)) = 0\}. \tag{2.1}$$

For each $t \in \mathbb{R}_+$, the mapping $\tau_t : \mathbb{M} \to \mathbb{M}$ is defined, for $x = (x_1, x_2)$, by

$$\tau_t\, \mu(x) = \begin{cases} 1 & \text{if } \mu\{(x_1 - t,\ x_2 + t)\} = 1 \text{ and } x_1 - t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

These mappings help define the process in the following manner: if

$Z_t(\omega) = \mu$ then for all $s < m(\mu)$ we have $Z_{t+s}(\omega) = \tau_s(\mu)$. As will be seen below, a common translation will be $\tau_t(\mu)$ for $t = m(\mu)$. For notational convenience define

$$\theta(\mu) = \tau_{m(\mu)}\,\mu. \tag{2.3}$$

Using the vague topology defined on $\mathbf{M}$, the function $t \to Z_t(\omega)$ is right continuous. (With the vague topology, $\mu_n \to \mu$ if $\int f\, d\mu_n \to \int f\, d\mu$ for all $f: \mathbf{E} \to \mathbb{R}_+$ bounded continuous.) The first, second,... points of discontinuities, denoted by $T_1(\omega), T_2(\omega), \ldots$, are the epochs at which deaths occur. Since the process is right continuous, $\{T_n;\ n \in \mathbb{N}_+\}$ is a sequence of stopping times. Define $T_0(\omega) = 0$ and $X_n(\omega) = Z_{T_n}(\omega)$ for all $n \in \mathbb{N}$. Thus, X_n denotes the population immediately after the n^{th} death. The assumption throughout this paper is that the probability law governing the number of offspring and their life-lengths depends only on the population size at the time of their birth.

For $\mu \in \mathbf{M}$, let $|\mu| = \mu(\mathbf{E})$. We can now write $P\{X_n \in A | Z_t,\ t < T_n\} = P\{X_n \in A \,|\, |Z_{T_n - 0}|\} = P\{X_n \in A \,|\, |Z_{T_{n-1}}|\} = P\{X_n \in A \,|\, |X_{n-1}|\}$ for $A \in \underline{\underline{M}}$. Thus $X = \{X_n;\ n \in \mathbb{N}\}$ is a Markov chain. We also have that given $X_n(\omega)$, the value of $T_{n+1}(\omega) - T_n(\omega)$ is fixed since $T_{n+1}(\omega) - T_n(\omega) = m(X_n(\omega))$.

(2.4) PROPOSITION. The stochastic process $(X,T) = \{X_n, T_n;\ n \in \mathbb{N}\}$ is a Markov renewal process with state space $(\mathbf{M}, \underline{\underline{M}})$.

PROOF. The defining property for (X,T) to be a Markov renewal process with state space $(\mathbf{M}, \underline{\underline{M}})$ is that the equality $P\{X_{n+1} \in A,\ T_{n+1} - T_n \leq t | X_0, \ldots, X_n, T_0, \ldots, T_n\} = P\{X_{n+1} \in A,\ T_{n+1} - T_n \leq t | X_n\}$ is true for all $A \in \underline{\underline{M}}$, $t \in \mathbb{R}_+$, and $n \in \mathbb{N}$. From the discussion in the preceding paragraph, this property obviously holds. □

(2.5) PROPOSITION. The population process $Z = \{Z_t;\ t \in \mathbb{R}_+\}$ is a semi-regenerative process with the imbedded process (X,T).

PROOF. The proof follows immediately from the assumption that the future is conditionally independent of the past given the population size at the time of the last death. See Çinlar (1975) for the definition of semi-regenerative processes. □

Let Q and R be the semi-Markovian kernel and the Markov renewal function respectively corresponding to (X,T). That is,

(2.6) $$Q(\mu,A,t) = P\{X_1 \in A,\ T_1 \leq t \mid X_0 = \mu\} = P_\mu\{X_1 \in A,\ T_1 \leq t\},$$

and

(2.7) $$R(\mu,A,t) = \sum_{n=0}^{\infty} Q^n(\mu,A,t)$$

for all $t \in \mathbb{R}_+$, $\mu \in \mathbb{M}$, and $A \in \underline{\underline{\mathbb{M}}}$. The term $Q^n(\mu,A,t)$ is the n-fold convolution of Q for a fixed μ evaluated at A and t. That is, $Q^0(\mu,A,t) = 1_A(\mu)\cdot 1_{[0,\infty)}(t)$ and for $n \in \mathbb{N}$

$$Q^{n+1}(\mu,A,t) = \int_{\nu\in\mathbb{M}} \int_{s\in[0,t]} Q(\mu,d\nu,ds)Q^n(\nu,A,t-s).$$

In order to obtain probabilities associated with Z, a "reverse" mapping corresponding to τ_t must be defined. For each $t \in \mathbb{R}_+$, the mapping $\hat{\tau}_t\colon \underline{\underline{\mathbb{M}}} \to \underline{\underline{\mathbb{M}}}$ is defined by

(2.8) $$\hat{\tau}_t(A) = \{\mu : \tau_t(\mu) \in A \text{ and } t < m(\mu)\}.$$

A similar "reverse" mapping $\hat{\tau}\colon \underline{\underline{\mathbb{M}}} \to \underline{\underline{\mathbb{M}}}$ is defined by

(2.9) $$\hat{\tau}(A) = \{\mu : \tau_t(\mu) \in A \text{ for some } t < m(\mu)\}.$$

An argument similar to that used in the proof of Lemma (3.5) can be used to show that, for $A \in \underline{\underline{\mathbb{M}}}$, $\hat{\tau}_t(A)$ and $\hat{\tau}(A)$ belong to $\underline{\underline{\mathbb{M}}}$.

(2.10) THEOREM. For $\mu \in \mathbb{M}$, $t \in \mathbb{R}_+$, and $A \in \underline{\underline{\mathbb{M}}}$,

$$P_\mu\{Z_t \in A\} = \int_{s\in[0,t]} \int_{\nu\in\hat{\tau}_{t-s}(A)} R(\mu,d\nu,ds).$$

PROOF. The standard Markov renewal argument is used.

$$P_\mu\{Z_t \in A\} = P_\mu\{Z_t \in A,\ T_1 > t\} + P_\mu\{Z_t \in A,\ T_1 \leq t\}$$

$$= 1_A(\tau_t(\mu))[1 - Q(\mu,\mathbb{M},t)] + \int_{\nu\in\mathbb{M}} \int_{s\in[0,t]} Q(\mu,d\nu,ds)P_\nu\{Z_{t-s} \in A\}$$

$$= \int_{\nu\in\mathbb{M}} \int_{s\in[0,t]} R(\mu,d\nu,ds)1_A(\tau_{t-s}(\nu))[1 - Q(\nu,\mathbb{M},t-s)]$$

$$= \int_{\nu\in\mathbb{M}} \int_{s\in(t-m(\nu),t]} R(\mu,d\nu,ds)1_A(\tau_{t-s}(\nu))$$

$$= \int_{\nu\in\hat{\tau}(A)} \int_{s\in(t-m(\nu),t]} R(\mu,d\nu,ds)1_A(\tau_{t-s}(\nu))$$

$$= \int_{s\in[0,t]} \int_{\nu\in\hat{\tau}_{t-s}(A)} R(\mu,d\nu,ds).$$

The third equality comes from Markov renewal equation theory; the fourth equality is true since $Q(\nu,\mathbb{M},t) = 0$ if $t < m(\nu)$ and 1 if $t \geq m(\nu)$, and the last two equalities come from Definitions (2.8) and (2.9). □

It is sometimes of interest to know the size of the population. For this purpose, let

(2.11) $$M^k = \{\mu \in \mathbb{M}: |\mu| = k\}.$$

(2.12) COROLLARY. For $\mu \in \mathbb{M}$, $k \in \mathbb{N}$, and $t \in \mathbb{R}_+$,

$$P_\mu\{Z_t \in M^k\} = \int_{\nu\in M^k} \int_{s\in(t-m(\nu),t]} R(\mu,d\nu,ds).$$

PROOF. The proof is immediate from the fifth equality in the proof of (2.10) using the fact that $M^k = \hat{\tau}(M^k)$. □

Using (2.12) the extinction probabilities are obtained. Let the measure in $\mathbb{M}$ with no atoms be denoted by 0.

(2.13) COROLLARY. For $\mu \in \mathbb{M}$ and $t \in \mathbb{R}_+$,

$$P_\mu\{Z_t = 0\} = R(\mu,0,t).$$

3. THE SEMI-MARKOVIAN KERNEL

At the time of death, new individuals are born. They are represented by a measure in $\mathbb{M}$ concentrated on $\{0\}\times(0,\infty)$. Any state μ can be decomposed into atoms representing new births and atoms representing older individuals still alive. Thus we define

$$(3.1) \qquad M_0 = \{\mu \in \mathbb{M}: \mu((0,\infty)\times(0,\infty)) = 0\}$$

and

$$(3.2) \qquad M_1 = \{\mu \in \mathbb{M}: \mu(\{0\}\times(0,\infty)) = 0\}.$$

Note that $M_0 \cap M_1 = \{0\}$. We now can uniquely represent any element of $\mathbb{M}$ as $\mu = \mu_0 + \mu_1$ where $\mu_0 \in M_0$ and $\mu_1 \in M_1$. By an abuse of notation we shall write $\mu_0(B)$ instead of $\mu_0(\{0\}\times B)$ for $B \in \underline{\underline{\mathbb{R}}}_+$ and $\mu_0 \in M_0$ when there can be no confusion.

Consider $Z_0(\omega) = \mu$ and let $|\mu| = n$. For $t < m(\mu)$, $Z_t(\omega) = \tau_t(\mu)$. For $t = m(\mu)$, $Z_t(\omega) = X_1(\omega) = \theta(\mu) + \mu_0$ for some $\mu_0 \in M_0$. (The functions m and θ are as defined by (2.1) and (2.3).) It is assumed that life-lengths are determined by a continuous distribution function so at an epoch of death only one individual dies (almost surely). The individual that died at $t = m(\mu)$ gives birth to k new individuals for some $k \in \mathbb{N}$. Thus at the time of their birth, we have k numbers $x_1, x_2, \ldots, x_k$ which represent the newly born individuals' lifetimes. The points $(0,x_1),\ldots,(0,x_k)$ are the atoms of μ_0.

Let $K(n,\cdot)$ be the given probability measure associated with a population of size n. That is, $K(n,A_0)$ is the probability that upon the death of an individual from a population of size n, the measure μ_0 representing the new offspring is in the set $A_0 \subset M_0$ for $A_0 \in \underline{\underline{\mathbb{M}}}$.

For $Q(\mu,A,t)$ defined by (2.6) let

$$(3.3) \qquad P(\mu,A) = \lim_{t\to\infty} Q(\mu,A,t).$$

(3.4) PROPOSITION. The transition probability function P of the Markov chain X is given by

$$P(\mu,A) = K(|\mu|, A - \theta(\mu)),\ \mu \in M,\ A \in \underline{\underline{M}},$$

where $A - \nu = \{\mu_0 \in M_0 : \mu_0 + \nu \in A\}$ for any given $\nu \in M_1$.

For the proof we will need the following

(3.5) LEMMA. The function $\theta: \mathbb{M} \to \mathbb{M}$ defined by (2.3) is measurable.

PROOF. Let $A = \{\mu \in M_1 : \mu(B_1) \geq 1, \ldots, \mu(B_k) \geq 1, |\mu| = k\}$ where $k \in \mathbb{N}$ and $B_1, \ldots, B_k$ are rectangles of the form $[a_i, b_i) \times (c_i, d_i]$. As will be seen in Section 4, the collection of subsets of $\mathbb{M}$ of this form is closed under finite intersections and generates $\underline{\underline{M}}$; therefore, by the monotone class theorem we need only show $\theta^{-1}(A) \in \underline{\underline{M}}$. For $k = 0$ it is easy, so assume $k \geq 1$. Let $\bar{b} = \min_{1 \leq i \leq k} b_i$. Consider some $\nu \in \mathbb{M}$ such that $\theta(\nu) \in A$. For this to happen there must be one atom of ν in the strip $[0,\infty) \times (0,\bar{b})$ and the remaining atoms of ν must be contained in rectangles that are translated "up and to the left" from the original B_i's. We now state this rigorously by partitioning the strip. For each $n \in \mathbb{N}_+$ define the strips $C_j(n)$ by $C_j(n) = [0,\infty) \times \left(\frac{j-1}{n}, \frac{j}{n}\right)$ for $j = 1, 2, \ldots, [n\bar{b}]$. (The notation $[x]$ denotes the largest integer less than or equal to x.) For each $n \in \mathbb{N}_+$ and for each $i = 1, 2, \ldots, k$ define the rectangles $B_{ij}(n)$ by $B_{ij}(n) = [0 \vee a_i - \frac{j-1}{n}, b_i - \frac{j}{n}) \times (c_i + \frac{j}{n}, d_i + \frac{j-1}{n}]$ for $j = 1, 2, \ldots, [n\bar{b}]$. These rectangles make sense only for n sufficiently large. We will consider only $n \geq \bar{n}$ where $\bar{n}$ is the smallest positive integer such that $\frac{1}{\bar{n}} < \min_{1 \leq i \leq k} \{(b_i - a_i) \wedge (d_i - c_i)\}$. Note that we can insist on this minimum being positive since rectangles of the form $[a,b) \times (d,d]$ or $[a,a) \times (c,d]$ are empty and therefore can be deleted. For each $n = \bar{n}, \bar{n}+1, \ldots$ and each $j = 1, 2, \ldots, [n\bar{b}]$, define $A_j(n) = \{\mu \in \mathbb{M} : \mu(C_j(n)) \geq 1, \mu(B_{1,j}(n)) \geq 1, \ldots, \mu(B_{k,j}(n)) \geq 1, |\mu| = k+1\}$. Then clearly $\theta^{-1}(A) \subset \bigcup_{j=1}^{[n\bar{b}]} A_j(n)$ and furthermore $\theta^{-1}(A) = \bigcup_{n=\bar{n}}^{\infty} \bigcup_{j=1}^{[n\bar{b}]} A_j(n)$ and thus $\theta^{-1}(A) \in \underline{\underline{M}}$. $\square$

PROOF of (3.4). We first show that for each $\mu \in \mathbb{M}$, $P(\mu,\cdot)$ is the probability measure as in (3.3). Given that $X_0 = \mu$, we have $X_1 = \theta(\mu) + \nu_0$ where $\nu_0 \in M_0$; therefore, for X_1 to be in A we need ν_0 to be such that $\nu_0 + \theta(\mu) \in A$. Thus $P(\mu,A) = K(|\mu|, A - \theta(\mu))$. Since $K(n,\cdot)$ is a probability measure for each $n \in \mathbb{N}$ and since for $A_1, A_2, \ldots$ a sequence of sets in $\underline{\underline{M}}$ we have $\bigcup_i A_i - \nu = \bigcup_i (A_i - \nu)$ for any $\nu \in M_1$, it follows that $P(\mu,\cdot)$ is a probability measure. It is now left to show that for each $A \in \underline{\underline{M}}$, $P(\cdot,A)$ is a measurable function. As in the proof of Lemma (3.5), it is only necessary to consider A of the form $A = \{\mu \in \mathbb{M}: \mu(B_1) \geq 1, \ldots, \mu(B_k) \geq 1, |\mu| = k\}$ where $k \in \mathbb{N}$ and $B_1, \ldots, B_k$ are of the form $[a_i,b_i)\times(c_i,d_i]$. It is sufficient to show that for any $r \in \mathbb{R}_+$, $\{\mu \in \mathbb{M}: K(|\mu|, A - \theta(\mu)) \geq r\} \in \underline{\underline{M}}$. For $r = 0$ there is no problem, so let $r > 0$. We now introduce some necessary notation. Let k_0 be the number of rectangles whose boundary is on the line $\{0\}\times(0,\infty)$. Relabel the rectangles so that $B_i \cap \{0\}\times(0,\infty) \neq \emptyset$ for $i = 1,\ldots,k_0$ and $B_j \cap \{0\}\times(0,\infty) = \emptyset$ for $j = k_0+1,\ldots,k$. For $i = 1,\ldots,k_0$ let $C_i = B_i \cap \{0\}\times(0,\infty)$ and let $\hat{B}_i = B_i \setminus C_i$. For the integers $1,2,\ldots,k_0$ let $i_1,i_2,\ldots,i_n$ denote a combination of n of them and let $j_1,j_2,\ldots,j_{k_0-n}$ denote the remaining integers. For any combination $i_1,\ldots,i_n$ let $A_{i_1,\ldots,i_n} = \{\mu \in M_0: \mu(C_{i_1}) \geq 1, \ldots, \mu(C_{i_n}) \geq 1, |\mu| = n\}$ and let $D_{i_1,\ldots,i_n} = \{\mu \in M_1: \mu(\hat{B}_{j_1}) \geq 1, \ldots, \mu(\hat{B}_{j_{k_0-n}}) \geq 1, \mu(B_{k_0+1}) \geq 1, \ldots, \mu(B_k) \geq 1, |\mu| = k-n\}$. Consider some $i_1,\ldots,i_n$ such that $K(k-n+1, A_{i_1,\ldots,i_n}) \geq r$. If some $\mu \in \mathbb{M}$ is such that $\theta(\mu) \in D_{i_1,\ldots,i_n}$, then $\theta(\mu) + \nu \in A$ for any $\nu \in A_{i_1,\ldots,i_n}$, and thus $K(k-n+1, A - \theta(\mu)) \geq r$. Therefore, $\{\mu: K(|\mu|, A - \theta(\mu)) \geq r\} = \bigcup \theta^{-1}(D_{i_1,\ldots,i_n})$ where the union is over all combinations such that $K(k-n+1, A_{i_1,\ldots,i_n}) \geq r$, and if there is no such combination then the set is empty. By Lemma (3.5) we are finished. $\square$

(3.6) PROPOSITION.

$$Q(\mu,A,t) = \begin{cases} P(\mu,A) & \text{if } t \geq m(\mu), \\ 0 & \text{if } t < m(\mu). \end{cases}$$

PROOF. The proof is obvious since $T_{n+1}(\omega) - T_n(\omega) = m(X_n(\omega))$. □

4. EXISTENCE OF AN INVARIANT MEASURE

In order to discuss the limiting properties of the population process Z, we need to know if an invariant measure exists for the imbedded Markov chain X. To prove its existence, the concept of recurrent sets or recurrent chains must be defined. The first step toward this goal is to define a measure on the state space of X.

The procedure for defining a measure on $(\mathbb{M}, \underline{\underline{\mathbb{M}}})$ will be as follows: 1) define a semiring $\underline{\underline{G}}$ of sets of states that will generate $\underline{\underline{\mathbb{M}}}$, 2) define a non-negative real valued set function ϕ with domain $\underline{\underline{G}}$, 3) show ϕ is countably additive on $\underline{\underline{G}}$, and 4) extend ϕ to $\underline{\underline{\mathbb{M}}}$ in the manner of Kolmogorov and Fomin (1960, ch. 7).

Let $\underline{\underline{G}}$ be the set of states of the form $\{\mu \in \mathbb{M}: \mu(B_1) \geq 1,\ldots,\mu(B_k) \geq 1, |\mu| = k\}$ as k varies over $\mathbb{N}$ and $B_1,\ldots,B_k$ (not necessarily distinct) vary over rectangles in $\underline{\underline{\mathbb{E}}}$ of the form $[a,b)\times(c,d]$. Since the rectangles generate $\underline{\underline{\mathbb{E}}}$ it is clear that $\underline{\underline{G}}$ generates $\underline{\underline{\mathbb{M}}}$.

(4.1) PROPOSITION. The collection of sets $\underline{\underline{G}}$ is a semiring.

PROOF. To show $\underline{\underline{G}}$ is a semiring, three conditions must be met: a) $\emptyset \in \underline{\underline{G}}$, b) $\underline{\underline{G}}$ is closed under finite intersections, and c) if $A \subset B$ and $A, B \in \underline{\underline{G}}$, then $B\backslash A$ can be represented by a finite union of disjoint sets in $\underline{\underline{G}}$. From the definition of $\underline{\underline{G}}$ it is clear that $\emptyset \in \underline{\underline{G}}$ by letting $k = 0$. To show part (b), let $A_1,A_2 \in \underline{\underline{G}}$ and write $A_1 = \{\mu:\mu(B_1) \geq 1,\ldots,\mu(B_m) \geq 1, |\mu| = m\}$ and $A_2 = \{\mu:\mu(C_1) \geq 1,\ldots,\mu(C_n) \geq 1, |\mu| = n\}$. Let $\pi_1,\pi_2,\ldots,\pi_n$ represent a permutation of the integers $1,2,\ldots,n$. For $A_1 \cap A_2 \neq \emptyset$ we need $m = n$ and $B_i \cap C_{\pi_i} \neq \emptyset$ for all $i = 1,\ldots,m$ for some permutation π. In that case $A_1 \cap A_2 = \{\mu:\mu(B_1 \cap C_{\pi_1}) \geq 1,\ldots,\mu(B_n \cap C_{\pi_n}) \geq 1, |\mu| = n\}$ and since the rectangles of the form $[a,b)\times(c,d]$ are a semiring we have that $A_1 \cap A_2 \in \underline{\underline{G}}$. For proving part (c) let $A_1, A_2 \in \underline{\underline{G}}$ and $A_1 \subset A_2$. We now have that $B_i \subset C_{\pi_i}$ for all $i = 1,\ldots,m$ for some permutation π. Since the rectangles form a semiring, we can find appropriate rectangles such that, for $i = 1,\ldots,m$, we can write $C_{\pi_i} \setminus B_i = \bigcup_{k=1}^{n_i} D_i^k$. It follows that

$$A_1 \setminus A_2 = \bigcup_{k_1=1}^{n_1} \bigcup_{k_2=1}^{n_2} \cdots \bigcup_{k_m=1}^{n_m} \{\mu:\mu(D_1^{k_1}) \geq 1,\ldots,\mu(D_m^{k_m}) \geq 1, |\mu| = m\}$$

and thus $\underline{\underline{G}}$ is a semiring. □

Let $\lambda(\cdot)$ denote the Lebesgue measure defined on $(\mathbb{E}, \underline{\underline{\mathbb{E}}})$. The non-negative, real valued set function ϕ is defined, for $A = \{\mu \in \mathbb{M}:\mu(B_1) \geq 1,\ldots,\mu(B_k) \geq 1, |\mu| = k\}$, by

(4.2) $$\phi(A) = \lambda(B_1)\lambda(B_2)\cdots\lambda(B_k).$$

(4.3) PROPOSITION. The function ϕ is σ-additive on $\underline{\underline{G}}$.

PROOF. Let $\{A_i\}$ be a countable collection of disjoint sets in $\underline{\underline{G}}$ such that $\cup A_i \in \underline{\underline{G}}$. If there are two sets A_i and A_j in the collection such that for $\mu_i \in A_i$ and $\mu_j \in A_j$ we have $|\mu_i| \neq |\mu_j|$, then $\cup A_i \notin \underline{\underline{G}}$; therefore, we can write, for some $m \in \mathbb{N}$, $A_i = \{\mu:\mu(B_1^i) \geq 1,\ldots,\mu(B_m^i) \geq 1, |\mu| = m\}$ for all A_i in the collection. Since $A_i \cap A_j = \emptyset$, we can relabel the rectangles so that $B_1^i \cap B_n^j = \emptyset$ for $n = 1,\ldots,m$ and $i \neq j$. (In other words, there must be one rectangle in A_i that is disjoint with all rectangles in A_j.) Assume this has been done for all sets in the collection. For $m = 1$ it is easy, since $\cup A_i = \{\mu:\mu(\cup B^i) = 1, |\mu| = 1\}$. Consider $m \geq 2$ and write $\cup A_i = \{\mu:\mu(D_1) \geq 1,\ldots,\mu(D_m) \geq 1, |\mu| = m\}$. Clearly we can relabel the rectangles in such a manner so that $\bigcup_i B_k^i \subset D_k$ for $k = 1,\ldots,m$ and still $B_1^i \cap B_n^j = \emptyset$ for $n = 1,\ldots,m$ and $i \neq j$. We shall now show that for $k = 2,\ldots,m$ it must hold that $B_k^1 = B_k^2 = \cdots$. Assume otherwise, that is for some $k = 2,\ldots,m$ there exist integers i and j such that $B_k^i \neq B_k^j$, and let $x \in \mathbb{E}$ be such that $x \in B_k^i$ but $x \notin B_k^j$. Consider $\mu \in M^m$ such that $\mu(B_r^j) \geq 1$ for $r = 1,\ldots,k-1,k+1,\ldots,m$ and $\mu(\{x\}) = 1$. Clearly $\mu \in \bigcup A_i$ since $\mu(D_r) \geq 1$ for $r = 1,\ldots,m$ but there is no n such that $\mu \in A_n$, which is a contradiction. This is true because $\mu(B_1^j) \geq 1$ implies $\mu \notin A_n$ for $n \neq j$, but $\mu(\{x\}) = 1$ implies $\mu \notin A_j$. Thus, $B_k^1 = B_k^2 = \cdots$ for $k = 2,\ldots,m$. We can now write

$$A_i = \{\mu : \mu(B_1^i) \geq 1,\ \mu(B_2) \geq 1,\ldots,\mu(B_m) \geq 1, |\mu| = m\}$$

and

$$\bigcup A_i = \{\mu : \mu(\bigcup B_1^i) \geq 1,\ \mu(B_2) \geq 1,\ldots,\mu(B_m) \geq 1,\ |\mu| = m\},$$

and the result follows. □

The function ϕ is extended to $\underline{\underline{\mathbb{M}}}$ by the formula

$$\phi(A) = \inf_{A \subset \bigcup_n A_n} \sum \phi(A_n) \tag{4.4}$$

where the lower bound is taken over all coverings for which $A_n \in \underline{\underline{G}}$. By Kolmogorov and Fomin (1960, ch. 7), ϕ is a measure on $(\mathbb{M}, \underline{\underline{\mathbb{M}}})$.

We need one more result concerning the measure ϕ.

(4.5) PROPOSITION. The measure ϕ is σ-finite on $(\mathbb{M}, \underline{\underline{\mathbb{M}}})$.

PROOF. Let $B_{r,s} = [r,r+1)\times(s,s+1]$ for any two rational numbers r and s. Let $\bar{r}$ and $\bar{s}$ denote vectors of nonnegative rational numbers of equal dimension; that is, $\bar{r} = (r_1,r_2,\ldots,r_s)$ and $\bar{s} = (s_1,s_2,\ldots,s_k)$. Let $A_{\bar{r},\bar{s}} = \{\mu \in \mathbb{M} : \mu(B_{r_1,s_1}) \geq 1,\ldots,\mu(B_{r_k,s_k}) \geq 1, |\mu| = k\}$. Thus $\phi(A_{\bar{r},\bar{s}}) = 1$ and $\mathbb{M} = \bigcup_{\bar{r},\bar{s}} A_{\bar{r},\bar{s}}$. □

Using the terminology of Orey (1971), the Markov chain $X = \{X_n; n \in \mathbb{N}\}$

with state space $(\mathbb{M}, \underline{\underline{\mathbb{M}}})$ is called ϕ-recurrent if for all $A \in \underline{\underline{\mathbb{M}}}$ with $\phi(A) > 0$ we have $P\{X_n \in A \text{ for some } n \in \mathbb{N}_+ | X_0 = \mu\} = 1$ for each $\mu \in \mathbb{M}$. In order to show X is ϕ-recurrent the following will be assumed.

(4.6) CONDITIONS.

a) $K(1,\{0\}) = 0$ and $K(n,\{0\}) > 0$ for $n \in \mathbb{N}_+\setminus\{1\}$.

b) There exists $n^* \in \mathbb{N}_+$ such that for all $n > n^*$ it follows that $K(n,\{0\}) = 1$.

c) There exists $s^* \in \mathbb{R}_+$ such that for $A = \{\mu \in M_0 : \mu((s^*,\infty)) \geq 1\}$ it follows that $K(n,A) = 0$ for all $n \in \mathbb{N}_+$.

d) Let $M_0^k = \{\mu \in M_0 : |\mu| = k\}$. For each $n = 1,2,\ldots,n^*$ there exists $k_n^* \in \{2,3,\ldots,n^*-n+2\}$ such that for all $k > k_n^*$ it follows that $K(n,M_0^k) = 0$, and thus $K(n, \bigcup_{i=0}^{k_n^*} M_0^i) = 1$.

e) For each $n = 1,2,\ldots,n^*$ consider some $k = 1,\ldots,k_n^*$ and a collection $B_1,B_2,\ldots,B_k$ where, for each i, $B_i \in \underline{\underline{\mathbb{R}}}_+$ and $B_i \subset [0,s^*]$. Let $A = \{\mu \in M_0 : \mu(B_1) \geq 1,\ldots,\mu(B_k) \geq 1, |\mu| = k\}$. Then $K(n,A) > 0$ if and only if $\lambda_0(B_i) > 0$ for each $i = 1,\ldots,k$ where $\lambda_0(\cdot)$ is the Lebesgue measure on $(\mathbb{R}_+, \underline{\underline{\mathbb{R}}}_+)$.

It can be seen from these conditions that many sets of states with positive ϕ measure will never be reached. We therefore define a reduced state space. Let $\hat{\mathbb{E}} = \{(x,y) \in \mathbb{E} : x + y \leq s^*\}$ and let $\hat{\mathbb{M}} = \{\mu \in \mathbb{M} : \mu(\hat{\mathbb{E}}) = \mu(\mathbb{E}) \text{ and } 1 \leq |\mu| \leq n^* + 1\}$. Define $\hat{M}_0$, $\hat{M}_1$, $\hat{M}^k$, $\hat{M}_0^k$, $\hat{\underline{\underline{\mathbb{M}}}}$, and $\hat{\underline{\underline{\mathbb{E}}}}$ in an analogous manner. We are now ready to begin proving the major result of this section, that is, X is ϕ-recurrent on the state space $(\hat{\mathbb{M}}, \hat{\underline{\underline{\mathbb{M}}}})$. We start by proving several lemmas.

(4.7) LEMMA. The function $P(\cdot,\cdot)$ as defined by (3.4) is a transition probability function on $(\hat{\mathbb{M}}, \hat{\underline{\underline{\mathbb{M}}}})$.

PROOF. Considering the proof of (3.4) it is sufficient to show $P(\mu, \hat{\mathbb{M}}) = 1$ for any $\mu \in \hat{\mathbb{M}}$. Condition (4.6e) implies that the probability measure governing life-lengths has no atoms other than zero; therefore with probability one, no two deaths occur at the same epoch. Using this fact and Conditions (4.6c and d), we have that for $n = 2,3,\ldots,n^*+1$

$$K(n, \bigcup_{i=n-1}^{k^*_n} \hat{M}_0^i) = 1 \qquad \text{and} \qquad K(1, \bigcup_{i=1}^{k^*_1} \hat{M}_0^i) = 1.$$

Let $\mu \in \hat{\mathbb{M}}$ with $|\mu| = j$. By (3.4), $P(\mu, \hat{\mathbb{M}}) = K(j,A_j)$ where $A_j = \{\mu_0 \in M_0 : \mu_0((s^*,\infty)) = 0 \text{ and } 1 \le |\mu_0| + j - 1 \le n^* + 1\}$. It is clear that $A_j = \bigcup_{k=1}^{n^*-j+2} \hat{M}_0^k$ and thus $P(\mu, \hat{\mathbb{M}}) = 1$ for any $\mu \in \hat{\mathbb{M}}$. □

(4.8) LEMMA. There exists a positive integer n_1 and a positive real number β_1, independent of ν, such that $P^{n_1}(\nu,\hat{M}^1) \ge \beta_1 > 0$ for all $\nu \in \hat{\mathbb{M}}$.

PROOF. Let $\nu \in \hat{M}^k$ and we have $P^{k-1}(\nu,\hat{M}^1) = K(k,\{0\})\cdot K(k-1,\{0\})\cdots K(2,\{0\})$. For any $n \in \mathbb{N}$ it follows that $P^{k-1+n}(\nu,\hat{M}^1) \ge P^{k-1}(\nu,\hat{M}^1)[K(1,\hat{M}^1)]^n$. Let $b(k,n) = P^{k-1}(\nu,\hat{M}^1)[K(1,\hat{M}^1)]^n$ and thus, because of Conditions (4.6a and e), $b(k,n) > 0$. Since $\sup\{|\nu| : \nu \in \hat{\mathbb{M}}\} = n^* + 1$, the lemma is true for $n_1 = n^*$ and $\beta_1 = \min\{b(k,n) : k = 1,2,\ldots,n^*+1 \text{ and } n = n^* - k + 1\}$. □

(4.9) LEMMA. Consider a fixed $k = 1,2,\ldots,n^* + 1$. Let $B_1,B_2,\ldots,B_k$ be rectangles in $\hat{\mathbb{E}}$ of the form $[a_i,b_i)\times(c_i,d_i]$ such that $b_1 > a_1 > b_2 > a_2 > \cdots > b_k > a_k$ and $\lambda(B_i) > 0$ for $i = 1,2,\ldots,k$. Then for $A = \{\mu \in \hat{\mathbb{M}} : \mu(B_1) \ge 1,\ldots,\mu(B_k) \ge 1, |\mu| = k\}$ we have $P^{k+1}(\nu,A) \ge \beta_2 > 0$ for all $\nu \in \hat{M}^1$ for some $\beta_2 \in \mathbb{R}_+$, independent of ν.

PROOF. Let $f_1 = \min\{\frac{d_1 - c_1}{k+1}, \frac{d_2 - c_2}{k},\ldots, \frac{d_k - c_k}{2}\}$,

$$f_2 = \min\{\frac{b_1 - a_1}{k}, \frac{b_2 - a_2}{k-1},\ldots, \frac{b_{k-1} - a_{k-1}}{2}, \frac{b_k - a_k}{2}\},$$

and $f = \min(f_1,f_2)$. Define the following intervals of $\mathbb{R}_+$:

$$D_1 = \left(\frac{a_1+b_1}{2} + \frac{c_1+d_1}{2} - \frac{1}{2}f,\ \frac{a_1+b_1}{2} + \frac{c_1+d_1}{2} + \frac{1}{2}f\right),$$

$$D_1' = \left(\frac{b_1-b_2+a_1-a_2}{2} - \frac{1}{2}f,\ \frac{b_1-b_2+a_1-a_2}{2} + \frac{1}{2}f\right),\ldots,$$

$$D_{k-1} = \left(\frac{a_{k-1}+b_{k-1}}{2} + \frac{c_{k-1}+d_{k-1}}{2} - \frac{1}{2}f,\ \frac{a_{k-1}+b_{k-1}}{2} + \frac{c_{k-1}+d_{k-1}}{2} + \frac{1}{2}f\right),$$

$$D_{k-1}' = \left(\frac{b_{k-1}-b_k+a_{k-1}-a_k}{2} - \frac{1}{2}f,\ \frac{b_{k-1}-b_k+a_{k-1}-a_k}{2} + \frac{1}{2}f\right),$$

$$D_k = \left(\frac{a_k+b_k}{2} + \frac{c_k+d_k}{2} - \frac{1}{2}f,\ \frac{a_k+b_k}{2} + \frac{c_k+d_k}{2} + \frac{1}{2}f\right),$$

$$D_k' = \left(\frac{a_k+b_k}{2} - \frac{1}{2}f,\ \frac{a_k+b_k}{2} + \frac{1}{2}f\right).$$

We first note that since f is the minimum of a finite set of positive numbers, f is positive and therefore $\lambda(D_i) > 0$ and $\lambda(D_i') > 0$ for $i = 1,2,\ldots,k$. Define the following sets of $\hat{M}_0$. For $i = 1,2,\ldots,k$ let $C_i = \{\mu \in \hat{M}_0 : \mu(D_i) \geq 1 \text{ and } \mu(D_i') \geq 1,\ |\mu| = 2\}$. It now follows that for $\nu \in \hat{M}^1$ we have $P^{k+1}(\nu,A) \geq K(1,C_1)\cdot K(2,C_2)\cdots K(k,C_k)K(k+1,\{0\})$, which is positive by Condition (4.6e). □

(4.10) LEMMA. Let $A \in \hat{\underline{\underline{M}}}$ such that $\phi(A) > 0$. Then there exist a positive integer n and a positive real number β, independent of ν, such that $P^n(\nu,A) \geq \beta > 0$ for all $\nu \in \hat{\mathbb{M}}$.

PROOF. Let $A = \{\mu \in \hat{\mathbb{M}} : \mu(B_1) \geq 1,\ldots,\mu(B_k) \geq 1,\ |\mu| = k\}$ for some $k = 1,\ldots,n^*+1$ and for $\{B_i\}$, rectangles in $\hat{\mathbb{E}}$, of the form $[a_i,b_i)\times(c_i,d_i]$ for $i = 1,\ldots,k$. As before, it is enough to prove the lemma for A. Since $\phi(A) > 0$, it follows that $\lambda(B_i) > 0$ for each i. Relabel the rectangles so that $b_1 \geq b_2 \geq \cdots \geq b_k$. If $a_1 > b_2$ let $B_1' = B_1$. If $a_1 \leq b_2$ pick real numbers a_1', a_2', and b_2' such that $a_2' < b_2' < a_1' < b_1$ and $a_2' \geq \max_{j\geq 2} a_j$, $a_1' \geq a_1$, and $b_2' \leq b_2$. Let $B_1' = [a_1',b_1)\times(c_1,d_1]$. Recursively, if B_{i-1}' has been defined, let $B_i' = [a_i', b_i') \times (c_i, d_i]$ if $a_i' > b_{i+1}$, and if $a_i' \leq b_{i+1}$ pick numbers a_i'', a_{i+1}', b_{i+1}' such that $a_{i+1}' < b_{i+1}' < a_i'' < b_i'$ and $a_{i+1}' \geq \max_{j \geq i+1} a_j$, $a_i'' \geq a_i'$, and $b_{i+1}' \leq b_{i+1}$ then let $B_i' = [a_i'', b_i') \times (c_i, d_i]$. Let $A' = \{\mu \in \hat{\mathbb{M}}: \mu(B_1') \geq 1, \ldots, \mu(B_k') \geq 1, |\mu| = k\}$. Since $B_i' \subset B_i$ for each i it follows that $A' \subset A$; therefore, $P^m(\nu, A') \leq P^m(\nu, A)$ for all $m \in \mathbb{N}_+$. Since the set A' satisfies the hypothesis for Lemma (4.9), we are finished by setting $n = n_1 + k + 1$ and $\beta = \beta_1 \cdot \beta_2$ where n_1 and β_1 are as in Lemma (4.8) and β_2 is as in Lemma (4.9). □

(4.11) THEOREM. The Markov chain $X = \{X_n;\ n \in \mathbb{N}\}$ with state space $(\hat{\mathbb{M}}, \underline{\underline{\hat{\mathbb{M}}}})$ is ϕ-recurrent.

PROOF. Consider some $A \in \underline{\underline{\hat{\mathbb{M}}}}$ with $\phi(A) > 0$. Let n and β be such that $P^n(\nu, A) \geq \beta > 0$ for all $\nu \in \hat{\mathbb{M}}$ (Lemma (4.10)). Define a sequence of random variables by

$$Y_j = \begin{cases} 0 & \text{if } X_{jn} \in A^c, \\ 1 & \text{if } X_{jn} \in A \end{cases}$$

for $j \in \mathbb{N}_+$. The sequence $\{Y_j;\ j \in \mathbb{N}_+\}$ forms a Markov chain with state space $\{0,1\}$. Since $P\{Y_{j+1} = 1 | Y_j = 0\} \geq \beta > 0$, it follows that state 0 is not absorbing and thus $P\{Y_j = 1 \text{ for some } j \in \mathbb{N}_+\} = 1$. We now have that X is ϕ-recurrent on $(\hat{\mathbb{M}}, \underline{\underline{\hat{\mathbb{M}}}})$ by observing that

$$\begin{aligned} \{X_m \in A \text{ for some } m \in \mathbb{N}_+\} &\supset \{X_{jn} \in A \text{ for some } j \in \mathbb{N}_+\} \\ &= \{Y_j = 1 \text{ for some } j \in \mathbb{N}_+\}. \end{aligned}$$ □

Since X is ϕ-recurrent on $(\hat{\mathbb{M}}, \underline{\underline{\hat{\mathbb{M}}}})$, there exists a σ-finite measure π on $(\hat{\mathbb{M}}, \underline{\underline{\hat{\mathbb{M}}}})$ such that

(4.12) $$\pi(A) = \int_{\nu \in \mathbb{M}} \pi(d\nu) P(\nu, A) \quad \text{for all } A \in \mathbb{M}.$$

Such a π is called an invariant measure for X. See Orey (1971, pp. 30-31).

5. LIMIT THEOREMS

In this section, some expressions for $\lim_{t\to\infty} P_\mu\{Z_t \in A\}$ are given. Since the imbedded Markov chain has an invariant measure on $(\mathbb{M}, \hat{\underline{\mathbb{M}}})$ assuming Conditions (4.6), these conditions are again assumed throughout this section. Let π be the invariant measure associated with X. By Çinlar (1974) the existence of π defined by (4.12) for a ϕ-recurrent imbedded Markov chain is sufficient for the $\lim_{t\to\infty} P_\mu\{Z_t \in A\}$ to exist for $A \in \hat{\underline{\mathbb{M}}}$.

(5.1) THEOREM. Let $A \in \hat{\underline{\mathbb{M}}}$, $m(\cdot)$ as defined by (2.1), and $\hat{\tau}_t(\cdot)$ as defined by (2.8), then $\lim_{t\to\infty} P_\mu\{Z_t \in A\} = c \int_{t\in[0,\infty)} \pi(\hat{\tau}_t(A))dt$, where $c = 1/\int_{\mu\in\hat{\mathbb{M}}} \pi(d\mu)m(\mu)$.

PROOF. From the third equality in the proof of (2.10), we have

$P_\mu\{Z_t \in A\} = \int_{\nu\in\hat{\mathbb{M}}} \int_{s\in[0,t]} R(\mu,d\nu,ds)g_A(\nu,t-s)$, where $g_A(\nu,t) = 1_A(\tau_t(\nu))$ for $t < m(\nu)$ and $g_A(\nu,t) = 0$ for $t \geq m(\nu)$. From Çinlar (1969) we know

$$\lim_{t\to\infty} \int_{\nu\in\mathbb{M}} \int_{s\in[0,t]} R(\mu,d\nu,ds)g_A(\nu,t-s) = c \int_{\nu\in\mathbb{M}} \pi(d\nu) \int_{t\in[0,\infty)} g_A(\nu,t)dt.$$

By (2.8), $c \int_{\nu\in\mathbb{M}} \pi(d\nu) \int_{t\in[0,\infty)} g_A(\nu,t)dt =$

$$c \int_{t\in[0,\infty)} dt \int_{\nu\in\{\mu:m(\mu)>t\}} 1_A(\tau_t(\nu))\pi(d\nu) = c \int_{t\in[0,\infty)} dt \int_{\nu\in\hat{\tau}_t(A)} \pi(d\nu)$$

$$= c \int_{t\in[0,\infty)} \pi(\hat{\tau}_t(A))dt. \qquad \square$$

(5.2) COROLLARY. For any $k = 1,2,\ldots,n^*+1$ and for c defined in (5.1)

$$\lim_{t\to\infty} P_\mu\{Z_t \in \hat{M}^k\} = c \int_{\nu\in\hat{M}^k} \pi(d\nu)m(\nu).$$

PROOF. Since $\hat{M}^k = \hat{\tau}(\hat{M}^k)$,

$$\lim_{t\to\infty} P_\mu\{Z_t \in \hat{M}^k\} = c \int_{\nu\in\hat{\mathbb{M}}} \pi(d\nu) \int_{t\in[0,\infty)} g_{\hat{M}^k}(\nu,t)dt$$

$$= c \int_{\nu \in \hat{\tau}(\hat{M}^k)} \pi(d\nu) \int_{t \in [0, m(\nu))} 1_{\hat{M}^k}(\tau_t(\nu))dt$$

$$= c \int_{\nu \in \hat{M}^k} \pi(d\nu) m(\nu). \qquad \square$$

(5.3) COROLLARY. Let the initial probability measure be identical to the invariant measure; that is, $P\{Z_0 \in A\} = \pi(A)$ for $A \in \hat{\underline{\underline{M}}}$. Also let c be as defined in (5.1), then

$$E[m(Z_0) | Z_0 \in M^k] = 1/c.$$

PROOF. Proof is immediate by (5.1) and (5.2). $\square$