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A CRITERION FOR EVOLUTIONARY STABILITY IN REPEATED GAMES PLAYED BY FINITE AUTOMATA

by

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1. Introduction

After Aumann (1981) first suggested the use of finite automata to represent the strategies used in repeated games, several papers have further explored that possibility. Rubinstein (1986) focuses on the Repeated Prisoners' Dilemma and incorporates, for the first time, complexity costs into the analysis. Abreu and Rubinstein (1988) extend this framework to general two-person games and show, for the case of the Repeated Prisoners' Dilemma, that the "size" of the Folk Theorem is reduced so that *only* a countable set of payoff points can be achieved in equilibrium. Binmore and Samuelson (1992) consider a modification of the ESS, or MESS, as the equilibrium concept. Their results show that evolutionary stable strategies in the conventional sense (ESS) often fail to exist. They argue that this non existence is due to the fact that for any potential equilibrium automaton, a successful entrant can be constructed by replicating the first automaton and changing those states not used when playing against itself. Under their modified concept (MESS), though, there exist equilibrium automata which achieve the highest payoff possible in the game (utilitarian outcome). Specifically, in the case of the Repeated Prisoners' Dilemma, they show that automata in a polymorphous MESS attain the cooperative payoff. Nevertheless, they express their doubts about how stable this payoff would be if mutations were allowed to overlap, assumption that seems more realistic to a social context. *If mutations were frequent*, they say, *no grounds exist for supposing that utilitarian outcomes will survive*.

Probst (1993) takes on Binmore and Samuelson's challenge and proposes, for the repeated Prisoners' Dilemma game, a "noisy" population in which there always exists a small group of stubborn one-state automata that play "always cooperate" or "always defect". Additionally, he proposes an alternative loosening of the ESS conditions to circumvent Binmore and Samuelson's argument on the reason for the non existence of ESS in the Abreu/Rubinstein automata selection game. Probst considers collection of automata all the elements of which are indistinguishable for the evolutionary process. That is, every element in the collection is of the same complexity, attains the same payoff against itself as the others against themselves and everybody attains that payoff against everybody else in the collection. Then, if every element in the collection is an ESS on its own, the collection is called an *Evolutionary Stable Collection* or ESC. Probst shows that there is a unique set of five three-state automata that satisfies these conditions for ESC. Each of these five automata starts off with defection but attains the cooperative payoff (they are utilitarian in the Binmore and Samuelson's sense) when playing against each other.

Our approach is based on Probst but considers a different type of "noisy" population. We believe that, in a social context, mutations are seldom random. Indeed, we think that new behaviors arise for reasons that have, at least to a certain extent, a logical explanation (or that are not totally irrational) and that respond in some way to the the particular situation of the environment in which they take place. For instance, consider an oligopolistic market in which firms compete in prices (that is, setting a high price is equivalent to cooperate and setting a low price is equivalent to defect). In that situation, one can hardly expect that a new strategy such as: "*I will always set a high price no matter what my competitors do*" will arise for no firm wants to commit suicide. On the other hand, oligopolistic competitors often complain about the behavior of "fringe" firms that go for the short run big profit instead of committing themselves to a long run cooperative strategy. The behavior of these "fringe" firms is to play a short run best response to what they see in the market (setting a low price in the case of price competition) so that they get the highest payoff possible in the short run. This could be consider "myopic" if we think that the firm is going to stay in the market for the long run, but in a situation in which "mutants" constantly enter and are driven out of the population, that behavior is perfectly possible. In that sense, we assume that in any population there is always a small group of people that consistently plays a short run best response to the action taken by the main part of the population. For instance, in the specific context of the Repeated Prisoners' Dilemma, our approach translates into assuming that in any population there is always a small fraction of people that plays "always defect". By doing so, we find that there is one and only one set of two automata that satisfies the conditions for ESC in the Repeated Prisoners' Dilemma. Moreover, the automata in this set turn out to be the ones that appear more often in the literature: *Tit-for-Tat* and *Grim Trigger*. Another interesting application of this idea is to games of common interests. It turns out that for such games there always exist a unique ESC consisting of the one-state automaton that always plays action associated to the Pareto-dominant equilibrium.

We further explore this variation of Probst's idea by extending the analysis to general symmetric games with two players. We prove that, under our modified solution concept, if a game has an evolutionary stable collection, then it is unique. Furthermore, all the automata in the collection are utilitarian in the Binmore and Samuelson's sense, that is, each attains the maximum payoff possible when playing against a replica of itself. We also consider the case of polymorphous populations and show that these two results (uniqueness and efficiency) also hold. Other results refer to the complexity of the equilibrium automata. We show that under our conditions for ESC, the number of strategies of the stage game imposes a uniform bound on the complexity of such equilibrium automata. An even more stringent bound (although not uniform) is determined by the number of different actions that are best response to the strategy induced by the equilibrium automata.

In section 2, we present the formal model and introduce some notation. Section 3 presents the stability conditions. Section 4 contains the main results. In section 5 we extend the analysis by allowing polymorphous populations. In section 6 we discuss the special implications that our approach has in the Repeated Prisoners' Dilemma game. Section 7 studies games of common interest. Section 8 is a summary.

2. The Model

We consider repeated games based on 2-players symmetric stage games of the form $G =< S, U >$ where $S = \{s_1, \ldots, s_n\}$ is the space of actions for each player and $U_i : S \times S \rightarrow \Re$ is the (symmetric) payoff function.

The repeated game $G^{\infty} =< F, \pi >$ is constructed in the usual way based on the stage game G. The payoff functions π_1 and π_2 correspond to the "limit of the means":

$$\pi_i(r_1, r_2) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} U_i(r_1(h_t), r_2(h_t))$$

where h_t denotes the history of the game up to and including time t, and $r_i(h_t)$ is the action taken by player i at time $t+1$

The automaton selection game of Abreu and Rubinstein (1988) is $G^{\#} =< \mathcal{A}, \succ>$. The strategy space is the set $\mathcal{A}$ of finite automata. A finite automaton (with output) or "Moore machine"[1] is just a system that responds to discrete inputs with discrete outputs. Formally, an automaton $a \in \mathcal{A}$ is described by $a = \{Q^a, q_0^a, S, \delta^a, \lambda^a\}$. Q^a is the (finite) set of internal states for automaton a, q_0^a is its initial state, $\lambda^a : Q^a \rightarrow S$ maps each internal state to an action and $\delta^a : Q^a \times S \rightarrow Q^a$ is the transition function that assigns a new state to each internal state depending on the action taken by the opponent. Typically, automata are presented as directed graphs in which each node represents an internal state with the action attached to it and the directed paths represent the transitions. For instance, the two automata below represent the well known strategies "*Tit-*

[1] See Hopcroft and Ullman (1979)

for-Tat" and "*Grim trigger*" in the Repeated Prisoners' Dilemma[2] .

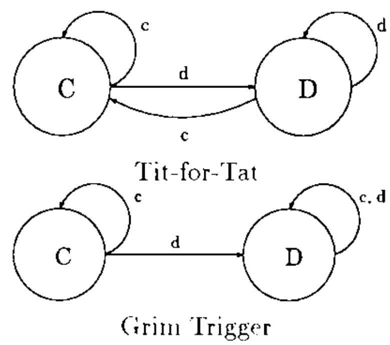


Figure 2.1 *Tit-for-Tat* and *Grim trigger* in the Repeated Prisoners' Dilemma.

The preferences ($\succ$) over automata take into account not only the automata's performance but also their complexity. In this sense, the complexity of an automaton $a \in \mathcal{A}$ is assumed to be its number of internal states and denoted by $|a|$.[3] The preference order over automata a and a' (parameterized by the common opponent a'') satisfy:

$$\begin{aligned} a \succ_1^{a''} a' \iff & \{\pi_1(a,a'') > \pi_1(a',a'')\} \quad \text{or} \\ & \{\pi_1(a,a'') = \pi_1(a',a'') \quad \text{and} \quad |a| < |a'|\}, \end{aligned}$$

with a similar definition for $\succ_2^{a''}$.

The following fact is discussed in Abreu and Rubinstein (1988) and, although it is obvious since we work with automata that have finitely many states, we want to call the attention on it because it will be of importance for the rest of the paper.

Fact 2.1 *Let $a, a' \in \mathcal{A}$. Then, when playing against each other, they get into a finite cycle.*

[2] Our automata are always depicted in such a way that their initial state is always the leftmost state in the picture

[3] See Kalai and Stanford (1988) for some game theoretical properties of this measure of complexity.

We will use $C^a(a') \subset Q^a(a')$ to denote such a cycle, where $Q^a(a') \subset Q^a$ is the set of internal states used by a when playing against a'

The key assumption of our model is that in any population there is always a small group of people that consistently plays a (myopic) best response to the strategy induced by the behavior of the main part of the population. In this sense, the *(stage game) strategy induced by a* is denoted by $\sigma(a) \in \Delta(S)$ and corresponds to the (possibly mixed) strategy for the stage game implicit in $C^a(a)$, that is:

$$\sigma(a, s_i) = \frac{\sum_{q \in C^a(a)} I_{s_i}(\lambda^a(q))}{|C^a(a)|} \qquad s_i \in S$$

and $\sigma(a) = (\sigma(a, s_1), \ldots, \sigma(a, s_n))$. According to the expected utility hypothesis we define $u(s_i, \sigma(a)) = \sum_{j=1}^{n} \sigma(a, s_j) U(s_i, s_j)$. Now, we can define the "*best response*" to a population composed of replicas of automaton a as:

$$\beta(a) = \{ s_i \in S \quad | \quad u(s_i, \sigma(a)) \geq u(s_j, \sigma(a)) \quad \forall s_j \in S \}$$

Therefore, our main assumption is that in a population mainly formed by automata of type a,[4] a small proportion (ϵ) of one state automata always playing some $s_i \in \beta(a)$ completes the environment. Formally, such a population will be denoted by $\mathcal{P}_\epsilon(a, \tilde{s}_i) = \epsilon \tilde{s}_i \dotplus (1-\epsilon)a$, where $0 < \epsilon < \frac{1}{2}$. Thorough the paper, we will use $\tilde{s}_i$ to represent the one state automaton that implements the strategy "*always play s_i*" ($s_i \in S$). Accordingly, the expected payoff of an automaton a' when randomly matched against a population $\mathcal{P}_\epsilon(a, \tilde{s}_i)$ is

$$\pi(a', \mathcal{P}_\epsilon(a, \tilde{s}_i)) = \epsilon \pi(a', \tilde{s}_i) + (1-\epsilon)\pi(a', a).$$

3. The Stability Condition

Probst's (1993) *Payoff Indistinguishable Collection* (PIC) and *Evolutionary Stable Collection* (ESC) are defined for the specific case of the Repeated Prisoners' Dilemma. In this section we generalize these definitions to general symmetric games with two players. Another important difference between Probst's model and ours is the way in which *noise* is added to the population. Probst assumes that every population has the same source of noise exogenously specified, namely the presence of two small group of automata playing "*always cooperate*"

[4] We will see later in the paper what happens if we allow for polymorphous populations.

and "*always defect*" respectively. Our hypothesis, on the other hand, is that this noise is endogenously determined as we assume that the "noisy" players respond, although myopically, to the behavior of the majority of the population.

Definition 3.1 *A set $A \subset \mathcal{A}$ is called Payoff Indistinguishable Collection or PIC if $\forall 0 < \epsilon < 1$, $\forall a, a' \in A$, $\forall s \in \mathcal{J}(a)$ and $\forall s' \in \mathcal{J}(a')$:*

$$\pi(a, \mathcal{P}_\epsilon(a', \tilde{s}')) = \pi(a, \mathcal{P}_\epsilon(a, \tilde{s})) = \pi(a', \mathcal{P}_\epsilon(a', \tilde{s}'))$$
$$\text{and} \qquad |a| = |a'| \qquad [1]$$

An alternative and probably more intuitive characterization of a PIC is given in the following lemma.

Lemma 3.1 *A set $A \subset \mathcal{A}$ is a PIC if and only if $\forall a, a' \in A$,*

$$\begin{aligned}
(i) \quad & \pi(a, a) = \pi(a, a') = \pi(a', a') \\
(ii) \quad & \pi(a, \tilde{s}) = \pi(a, \tilde{s}') = \pi(a', \tilde{s}') \quad \forall s \in \mathcal{J}(a), \quad \forall s' \in \mathcal{J}(a') \\
(iii) \quad & |a| = |a'|
\end{aligned}$$

Proof: *Necessity*: Item (i) follows from a simple continuity argument letting ϵ go to zero in the definition of $\mathcal{P}_\epsilon(a, \tilde{s})$. Then, parts (ii) and (iii) follow immediately from (i) and [1]

Sufficiency: Simply multiply through by $(1 - \epsilon)$ in (i) and by ϵ in (ii) and add them together to obtain [1] ∎

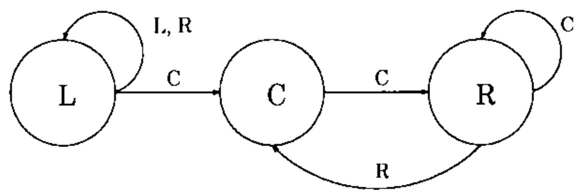
Example 3.1 An example of a PIC is the set $A^* = \{T, G\}$ depicted in Figure 2.2.1, which corresponds to the automata that implement the strategies *Tit-for-Tat* and *Grim Trigger* in the Repeated Prisoner's Dilemma. To see that this set is a PIC, just note that both T and G always cooperate when playing against replicas of themselves or against each other. Additionally, both end up defecting forever when playing against the only myopic best response to any of them, that is, when playing against the automaton that always defects.

Intuitively, it might seem that, if two automata belong to the same PIC, then they must differ only in actions or transitions not used when playing against each other. Although this is true for games with only two strategies, it is not true in general as the following example shows.

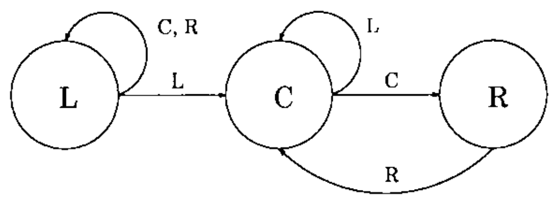
Example 3.2 Consider the repeated game based on the following stage game:

	L	C	R
L	3,3	0,5	0,0
C	5,0	0,0	0,0
R	0,0	0,0	6,6

Consider now the following two automata:[5]



The automaton a



The automaton a'

Figure 3.1 The automata a and a'.

Since $C^{a}(a) = \{L\}$ and $C^{a'}(a') = \{C, R\}$, we have that $\mathcal{B}(a) = \{C\}$ and $\mathcal{B}(a') = \{R\}$. To see that these two automata form a PIC, just verify that $\pi(a, \mathcal{P}_\epsilon(a, \tilde{C})) = \pi(a', \mathcal{P}_\epsilon(a', \tilde{R})) = \pi(a, \mathcal{P}_\epsilon(a', \tilde{R})) = (1-\epsilon)3$. Nevertheless, a and a' induce totally different play paths.

The following definition takes Binmore and Samuelson's (1992) and Probst's (1993) modified versions of Maynard-Smith's (1983) ESS and adapts them to our framework.

Definition 3.2 *Let A^* be a PIC. A^* is an Evolutionary Stable Collection or ESC if $\forall a \in A^*$ and $\forall a' \notin A^*$ $\exists \bar{\epsilon}$ such that $\forall \epsilon \in (0, \bar{\epsilon})$:*

$$
\begin{array}{lll}
 & (i) & \{\pi(a, \mathcal{P}_\epsilon(a, \tilde{s})) > \pi(a', \mathcal{P}_\epsilon(a, \tilde{s})) \quad \forall s \in \mathcal{B}(a)\}, \\
\text{or} & (ii) & \{\pi(a, \mathcal{P}_\epsilon(a, \tilde{s})) \geq \pi(a', \mathcal{P}_\epsilon(a, \tilde{s})) \quad \forall s \in \mathcal{B}(a) \\
 & & \text{and} \quad \pi(a, \mathcal{P}_\epsilon(a', \tilde{s}')) > \pi(a', \mathcal{P}_\epsilon(a', \tilde{s}')) \quad \forall s' \in \mathcal{B}(a')\}, \\
\text{or} & (iii) & \{\pi(a, \mathcal{P}_\epsilon(a, \tilde{s})) \geq \pi(a', \mathcal{P}_\epsilon(a, \tilde{s})) \quad \forall s \in \mathcal{B}(a) \\
 & & \text{and} \quad \pi(a, \mathcal{P}_\epsilon(a', \tilde{s}')) \geq \pi(a', \mathcal{P}_\epsilon(a', \tilde{s}')) \quad \forall s' \in \mathcal{B}(a') \\
 & & \text{and} \quad |a| < |a'|\}.
\end{array}
\qquad [2]
$$

[5] The missing arrows in the two automata below do not need to be specified for this example.

Basically, this definition corresponds to the definition of ESS for automata given by Binmore and Samuelson (1992) in which complexity has been taken into account. Then, it follows Probst (1993) in the sense that potential invaders are restricted to come only from outside the PIC. The two facts that follow show that, although the conditions given above look very similar to the ones given by Binmore and Samuelson, ESC is neither a stronger nor a weaker condition than MESS.

Fact 3.1 *The automaton a being a MESS does not imply that there is an ESC* A^* *such that a belongs to it.*

Indeed, Binmore and Samuelson show that the strategy that they call "*Tat-for-Tit*" constitutes a MESS in the Repeated Prisoners' Dilemma. On the other hand, we will see later that there is only one set of automata that is an ESC in the Repeated Prisoners' Dilemma, and that set does not include the strategy "*Tat-for-Tit*".

Fact 3.2 *The set* A *being an ESC does not imply that* a *is a MESS for any* a *in* A.

Again, the Repeated Prisoners' Dilemma provides an example of it. We will see that the automaton playing "*Tit-for-Tat*" belongs to the only ESC but, as Binmore and Samuelson point out, it is not a MESS.

The following fact indicates that the condition for ESC might be too strong.

Fact 3.3 *There are games with no ESC.*

Consider the following game:

	C	D
C	3,3	4,5
D	5,4	1,1

Figure 3.2 A game without ESC.

Note first that the one state automaton $\tilde{C}$ can not be in any ESC for it is easy to check that the automaton $\tilde{D}$ would invade the population $\mathcal{P}_\epsilon(\tilde{C}, \tilde{D})$

On the other hand, the automaton $\tilde{C}$ can successfully invade any population $\mathcal{P}_\epsilon(a, \tilde{s})$ for any a in any A that is assumed to be an ESC. To see that this is true, we need to use a result that we prove later in the paper and that establishes that any automata in any ESC gets the highest payoff possible when playing against a replica of itself. In this example, that result translates into saying that any automata a that belong to any ESC A must attain the payoff

3. Clearly, this is only possible if a always plays C once in the cycle $C^a(a)$ in which it gets when playing against a replica of itself. If this automaton a reaches such cycle without ever playing D, we have that the automaton $\tilde{C}$ can invade the population formed by the automaton a because:

(i) $\tilde{C}$ does at least as well as a against $\tilde{D} = \beta(a)$.
(ii) $\tilde{C}$ also attains the payoff of 3 when playing against a.
(iii) If $a \neq \tilde{C}$, then $\tilde{C}$ has strictly less internal states than a.

So, before reaching the cycle $C^a(a)$, a must play D at least one time. Let q be this particular state prior to the cycle at which a plays D. Clearly, $\delta^a(q, D) \neq q$ for otherwise a would never reach the cycle $C^a(a)$. Thus, the only possibility left to avoid (ii) above is that $\delta^a(q, C) = q$. But if this is the case, we have that $\pi(\tilde{C}, a) = 4 > 3 = \pi(a, a)$ so that $\tilde{C}$ also invades $\mathcal{P}_\epsilon(a, \tilde{D})$.

Hence, we have seen that $\tilde{C}$ can invade any ESC but $\tilde{C}$ is not evolutionary stable on its own. Therefore, no ESC exists for this game.

4. The Results

We present now our two main results. The first one states that any automaton in any ESC is utilitarian in the following sense:

Definition 4.1 *An automaton $a \in \mathcal{A}$ is said to be utilitarian if $a \in$* $\arg\max_{a \in \mathcal{A}} \pi(a, a)$.

That is, a is an utilitarian automaton if $\pi(a, a) = \pi^*$, where π^* is the maximum payoff achievable when an automaton plays against a replica of itself. Since the game is symmetric, an alternative characterization of π^* is $\pi^* = U(s^*, s^*)$, where $s^* \in \arg\max_{s \in S} U(s, s)$.

Our second result will establish that if an ESC exists, then it is unique.

4.1. Efficiency of ESC

Theorem 4.1.1 *Let $A^* \subset \mathcal{A}$ be an ESC. Then, $\forall a \in A^*$, $\pi(a, a) = \pi^*$.*

Proof: Suppose the contrary, that is, suppose that $\exists a \in A^*$ such that $\pi(a, a) < \pi^*$. From here, we consider two different cases depending on whether a's initial move is in $\beta(a)$ or not. We will show that, in either case, an automaton a' can be constructed in such a way that a' will defeat a, which implies that $\pi(a, a) < \pi^*$ is not possible if a belongs to an ESC.

Case 1. Suppose $\lambda^a(q_0^a) = s_i \notin \beta(a)$. In this case, two things may happen: that a moves to some state in $Q^a(a)$ when its opponent's initial action is different from its initial action (s_i) or that a moves to some state not in $Q^a(a)$.

(1.1) Suppose that $\exists s_j \neq s_i$ such that $\delta^a(q_0^a, s_j) \in Q^a(a)$.

Construct an automaton a' as a replica of a but with a new initial state $(q_0^{a'})$ plus some additional states $(q_{s_1}, \ldots, q_{s_n}, q^*, q^-)$ as follows:

$$
\begin{array}{rl}
\text{(i)} & \lambda^{a'}(q_0^{a'}) = s_j \\
\text{(ii)} & \delta^{a'}(q_0^{a'}, s_i) = \delta^a(q_0^a, s_j) \\
\text{(iii)} & \delta^{a'}(q_0^{a'}, s_k) = q_{s_k} \quad \forall s_k \neq s_i,\ s_k \neq s_j \\
\text{(iv)} & \lambda^{a'}(q_{s_k}) \in \mathcal{B}(\hat{s}_k) \\
\text{(v)} & \delta^{a'}(q_{s_k}, s_k) = q_{s_k} \\
\text{(vi)} & \delta^{a'}(q_0^{a'}, s_j) = q^- \\
\text{(vii)} & \lambda^{a'}(q^-) = s^-,\ s^- \neq s_j \\
\text{(viii)} & \delta^{a'}(q^-, s^-) = q^* \\
\text{(ix)} & \lambda^{a'}(q^*) = s^* \\
\text{(x)} & \delta^{a'}(q^*, s^*) = q^* \\
\text{(xi)} & \delta^{a'}(q^-, s_j) = q_{s_j} \\
\text{(xii)} & \lambda^{a'}(q_{s_j}) \in \mathcal{B}(\hat{s}_j) \\
\text{(xiii)} & \delta^{a'}(q_{s_j}, s_j) = q_{s_j}
\end{array}
$$

This new automaton a is much more complex than a, but complexity is less important than performance. What counts is that a' mimics a when playing against a but attains the utilitarian payoff π^* when playing against a replica of itself. Indeed, (i) means that the initial action of a' (s_j) is different than a's initial move (s_i). Then, (ii) guarantees that when a and a' meet, both will transit to state $\delta^a(q_0^a, s_j)$ and from there they will keep playing as if a had met a replica of itself. Item (iii) takes care of the event in which a' meets an automaton that is neither a replica of itself nor the automaton a. This case is relevant only if this automaton that a' meets is in $\mathcal{B}(a)$. Thus, to be in the safe side (just in case a' has indeed met an automaton in $\mathcal{B}(a)$), from $q_0^{a'}$ a' moves to another state (as specified in (iii)) in which a' plays a best response to the action taken by its enigmatic opponent (as (iv) indicates). Finally, (v) guarantees that a' will keep treating its opponent this way as long as its opponent keeps playing the same action. Because of (vi), we have that if a' meets a replica of itself, then it moves to state q^- where, according to (vii), a' finds out whether it is playing against a replica of itself or against a one state automaton that always plays s_j. If it turns out that its opponent is a replica of itself, a' transits to the state q^* (as (viii) indicates) where exerts the "utilitarian" action s^* (item (ix)). Then, (x) establishes that $C^{a'}(a') = \{q^*\}$. Hence, items (vi) through (x) make a' an utilitarian automaton. Finally, items (xi) through (xiii) take care of the case when, once in q^-, a' finds out that is playing against the one state automaton that always plays s_j. Again, just in case s_j is in $\mathcal{B}(a)$, a' always plays a best response to it (as specified by (xii) and (xiii)).

Because of the way a' is constructed, it should be clear that:

(a) $\pi(a', \tilde{s}) \geq \pi(a, \tilde{s}) \quad \forall s \in \mathcal{B}(a)$
(b) $\pi(a', a) = \pi(a, a)$
(c) $\pi(a, a') = \pi(a, a) < \pi^*$
(d) $\pi(a', a') = \pi^*$

Items (a) and (b) imply that $\pi(a', \mathcal{P}_\epsilon(a, \tilde{s})) \geq \pi(a, \mathcal{P}_\epsilon(a, \tilde{s})) \quad \forall s \in \mathcal{B}(a)$. Items (a), (c) and (d) imply that, for ϵ small enough, $\pi(a, \mathcal{P}_\epsilon(a', \tilde{s}')) < \pi(a', \mathcal{P}_\epsilon(a', \tilde{s}')) \quad \forall s' \in \mathcal{B}(a')$. This last inequality clearly implies that a can not belong to an ESC.

Let us now consider the more complicated case in which a transits to some state not in $Q^a(a)$ whenever its opponent initial action is different from a's initial action.

(1.2) Suppose $\forall s_j \neq s_i \quad \delta^a(q_0^a, s_j) \notin Q^a(a)$

Consider the automaton $\dot{a}$ to be an exact replica of a except that $\delta^{\dot{a}}(\delta^{\dot{a}}(q_0^{\dot{a}}, s_j), s_i) \in Q^a(a) \quad \forall s_j \neq s_i$. Note that:

> (a) If two automata $\dot{a}$ meet, they will play against each other exactly as if they where two automata like a since we only changed transitions that occur in one state not visited when a play against a replica of itself.
>
> (b) In a similar way, if matched against an one state automata that always plays an action in $\mathcal{B}(a)$, both $\dot{a}$ and a will play in the same way since $\dot{a}$ and a differ only in transitions that take place when the action taken by the opponent is s_i, which is not in $\mathcal{B}(a)$
>
> (c) Clearly, because of (a), $\mathcal{B}(\dot{a}) = \mathcal{B}(a)$.

Hence, it is clear that $\pi(a, \mathcal{P}_\epsilon(a, \tilde{s})) = \pi(\dot{a}, \mathcal{P}_\epsilon(\dot{a}, \tilde{s})) = \pi(\dot{a}, \mathcal{P}_\epsilon(a, \tilde{s})) \quad \forall s \in \mathcal{B}(a) = \mathcal{B}(\dot{a})$ and $|\dot{a}| = |a|$. Therefore, $\dot{a}$ also belongs to the same ESC as a.

Construct now an invader automaton a' as a replica of $\dot{a}$ but with a new initial state $(q_0^{a'})$ plus some additional states $(q_{s_1}, \ldots, q_{s_n}, q^*, q', q^-)$ as follows:

(i)	$\lambda^{a'}(q_0^{a'}) = s_j \neq s_i$
(ii)	$\delta^{a'}(q_0^{a'}, s_i) = q'$
(iii)	$\lambda^{a'}(q') = s_i$
(iv)	$\delta^{a'}(q', s) = \delta^{\dot{a}}(\delta^{\dot{a}}(q_0^{\dot{a}}, s_j), s_i) \quad \forall s \in S$
(v)	$\delta^{a'}(q_0^{a'}, s_k) = q_{s_k} \quad \forall s_k \neq s_i, \ s_k \neq s_j$
(vi)	$\lambda^{a'}(q_{s_k}) \in \mathcal{B}(\tilde{s}_k)$
(vii)	$\delta^{a'}(q_{s_k}, s_k) = q_{s_k}$
(viii)	$\delta^{a'}(q_0^{a'}, s_j) = q^-$
(ix)	$\lambda^{a'}(q^-) = s^-, \ s^- \neq s_j$
(x)	$\delta^{a'}(q^-, s^-) = q^*$
(xi)	$\lambda^{a'}(q^*) = s^*$
(xii)	$\delta^{a'}(q^*, s^*) = q^*$

(xiii)	$\delta^{a'}(q^-, s_j) = q_{s_j}$
(xiv)	$\lambda^{a'}(q_{s_j}) \in \mathcal{B}(\tilde{s}_j)$
(xv)	$\delta^{a'}(q_{s_j}, s_j) = q_{s_j}$

As before, a' is much more complex than $\dot{a}$, but performs better. In this case, (i) says that $\dot{a}$ and a' have different initial actions (s_i and s_j respectively). Item (ii) states that whenever a' meets a, a' transits to an intermediate state q' in which a' plays s_i which is not in $\mathcal{B}(a)$ (according to (iii)) and then moves to some state in $Q^a(a)$ no matter what its opponent plays. What is important here is that $\dot{a}$ reaches this very same state in $Q^a(a)$ that a' has just reached at the same time (according to (iv)). From here, $\dot{a}$ and a' keep playing as if they were a pair of a automata (or a pair of a automata for that matter). Items (v) through (xv) have the same meaning and implications than items (iii) through (xiii) in the previous case (1.1), that is, they just make sure that a' is an utilitarian automaton and that a' plays a best response to any one state automaton that always plays some action in $\mathcal{B}(\dot{a})$. Hence, as before, we have that:

(a) $\pi(a', \tilde{s}) \geq \pi(\dot{a}, \tilde{s}) \quad \forall s \in \mathcal{B}(\dot{a})$
(b) $\pi(a', \dot{a}) = \pi(\dot{a}, a)$
(c) $\pi(a, a') = \pi(\dot{a}, \dot{a}) < \pi^*$
(d) $\pi(a', a') = \pi^*$

Therefore, $\pi(a', \mathcal{P}_\epsilon(\dot{a}, \tilde{s})) \geq \pi(\dot{a}, \mathcal{P}_\epsilon(\dot{a}, \tilde{s})) \quad \forall s \in \mathcal{B}(\dot{a})$ and, for ϵ small enough, $\pi(\dot{a}, \mathcal{P}_\epsilon(a', \tilde{s}')) < \pi(a', \mathcal{P}_\epsilon(a', \tilde{s}')) \quad \forall s' \in \mathcal{B}(a')$ This last inequality clearly implies that $\dot{a}$ can not belong to an ESC and therefore neither can a.

We turn now to Case 2 in which the initial action of automaton a happens to belong to $\mathcal{B}(a)$.

Case 2. Suppose now that $\lambda^a(q_0^a) = s_i \in \mathcal{B}(a)$. As in Case 1, the automaton a might or might not move to some state in $Q^a(a)$ after its opponent plays some action s_j different from its own initial action s_i.

(2.1) Suppose that $\exists s_j \neq s_i$ such that $\delta^a(q_0^a, s_j) \in Q^a(a)$. In this case, either (i) $\exists q_j^a \in Q^a(a)$ such that $\lambda^a(q_j^a) \neq s_i$ or (ii) $\lambda^a(q) = s_i \quad \forall q \in Q^a(a)$. If (i) holds, construct the automaton a' as a replica of the automaton a but with a different initial state ($q_0^{a'}$) plus some additional states ($q_{s_1}, \ldots, q_{s_n}, q^*, q', q^-$) as follows:

(i)	$\lambda^{a'}(q_0^{a'}) = s_j \neq s_i$
(ii)	$\delta^{a'}(q_0^{a'}, s_i) = \delta^a(q_0^a, s_j)$
(iii)	$\delta^{a'}(q_j^{a'}, s_i) = q'$
(iv)	$\lambda^{a'}(q') \in \mathcal{B}(\tilde{s}_i)$
(v)	$\delta^{a'}(q', s_i) = q'$
(vi)	$\delta^{a'}(q_0^{a'}, s_k) = q_{s_k} \quad \forall s_k \neq s_i,\ s_k \neq s_j$

(vii) $\lambda^{a'}(q_{s_k}) \in \mathcal{B}(\tilde{s}_k)$
(viii) $\delta^{a'}(q_{s_k}, s_k) = q_{s_k}$
(ix) $\delta^{a'}(q_0^{a'}, s_j) = q^-$
(x) $\lambda^{a'}(q^-) = s^-, \; s^- \neq s_j$
(xi) $\delta^{a'}(q^-, s^-) = q^*$
(xii) $\lambda^{a'}(q^*) = s^*$
(xiii) $\delta^{a'}(q^*, s^*) = q^*$
(xiv) $\delta^{a'}(q^-, s_j) = q_{s_j}$
(xv) $\lambda^{a'}(q_{s_j}) \in \mathcal{B}(\tilde{s}_j)$
(xvi) $\delta^{a'}(q_{s_j}, s_j) = q_{s_j}$

According to (i), automaton a and a' play different action sat the beginning. Item (ii) says that if the opponent of a' plays s_i then a' finds out that it is not playing against a replica of itself and then transits to the same state to which automaton a transits when its opponent plays s_j. So, if a and a' meet, after playing different actions at the beginning both will transit to the same state and from there will keep playing as if automata a had met a replica of itself. Item (iii) refers to that particular state q_j^a at which automaton a plays an action different from s_i. This state is important because in it, automaton a' can tell whether it is playing against the automaton a or against $\tilde{s}_i$ which, in this case (Case 2), is in $\mathcal{B}(a)$. Hence, according to (iii), if the opponent of a' plays s_i, a' learns that it is playing against the automaton $\tilde{s}_i \in \mathcal{B}(a)$ and therefore moves to state q' where a' plays a best response to s_i (item (iv)) and keeps doing so as long as its opponent keeps playing s_i (item (v)). If, on the contrary, when a' is at q_j^a its opponent does not play s_i, then a' behaves exactly as automaton a. Items (vi) through (xvi) are like items (iii) through (xiii) in Case 1.1, that is, they just make sure that a' is an utilitarian automaton and that a' plays a best response to any one state automaton that always plays some action in $\mathcal{B}(a)$. Therefore, once again, we have that:

(a) $\pi(a', \tilde{s}) \geq \pi(a, \tilde{s}) \quad \forall s \in \mathcal{B}(a)$
(b) $\pi(a', a) = \pi(a, a)$
(c) $\pi(a, a') = \pi(a, a) < \pi^*$
(d) $\pi(a', a') = \pi^*$

Hence, $\pi(a', \mathcal{P}_\epsilon(a, \tilde{s})) \geq \pi(a, \mathcal{P}_\epsilon(a, \tilde{s})) \quad \forall s \in \mathcal{B}(a)$ and $\pi(a, \mathcal{P}_\epsilon(a', \tilde{s}')) < \pi(a', \mathcal{P}_\epsilon(a', \tilde{s}')) \quad \forall s' \in \mathcal{B}(a')$. Thus, automaton a can not belong to any ESC as it is defeated by an automaton as a'.

Nevertheless, this result relies on the assumption (i) that $\exists q_j^a \in Q^a(a)$ such that $\lambda^a(q_j^a) \neq s_i$. There is, though, another possibility. Suppose now that (ii) holds, that is, $\lambda^a(q) = s_i \quad \forall q \in Q^a(a)$. In this case, the invader automaton a' can be easily constructed as a replica of a plus the usual additional states as follows:

(i) $\lambda^{a'}(q_0^{a'}) = s_j \neq s_i$
(ii) $\delta^{a'}(q_0^{a'}, s_i) = \delta^a(q_0^a, s_j)$
(iii) $\delta^{a'}(q_0^{a'}, s_k) = q_{s_k} \quad \forall s_k \neq s_i,\ s_k \neq s_j$
(iv) $\lambda^{a'}(q_{s_k}) \in \mathcal{B}(\tilde{s}_k)$
(v) $\delta^{a'}(q_{s_k}, s_k) = q_{s_k}$
(vi) $\delta^{a'}(q_0^{a'}, s_j) = q^-$
(vii) $\lambda^{a'}(q^-) = s^-,\ s^- \neq s_j$
(viii) $\delta^{a'}(q^-, s^-) = q^*$
(ix) $\lambda^{a'}(q^*) = s^*$
(x) $\delta^{a'}(q^*, s^*) = q^*$
(xi) $\delta^{a'}(q^-, s_j) = q_{s_j}$
(xii) $\lambda^{a'}(q_{s_j}) \in \mathcal{B}(\tilde{s}_j)$
(xiii) $\delta^{a'}(q_{s_j}, s_j) = q_{s_j}$

This automaton a' plays the same role as the one constructed before (when we assumed that (i) hold). The only difference is that, in this case, a' does not need to distinguish between a and $\tilde{s}_i$ because they behave in the same way when playing against a'. Therefore, we have again that $\pi(a', \mathcal{P}_\epsilon(a, \tilde{s})) \geq \pi(a, \mathcal{P}_\epsilon(a, \tilde{s})) \quad \forall s \in \mathcal{B}(a)$ and $\pi(a, \mathcal{P}_\epsilon(a', \tilde{s}')) < \pi(a', \mathcal{P}_\epsilon(a', \tilde{s}')) \quad \forall s' \in \mathcal{B}(a')$ which means that a can not belong to an ESC.

The last case to study is when automaton a transits to some state not in $Q^a(a)$ whenever its opponent's initial action is not the same as a's initial action.

(2.2) Suppose that $\forall s_j \neq s_i \quad \delta^a(q_0^a, s_j) \notin Q^a(a)$.

We will consider here three different cases depending on whether (2.2.1) $Q^a(a) = \{q_0^a\}$, or (2.2.2) $Q^a(a) \neq \{q_0^a\}$ and $\exists s_j \neq s_i$ and $\exists q \in Q^a(a) \backslash q_0^a$ such that $\lambda^a(q) \neq s_j$ or, on the contrary, (2.2.3) $Q^a(a) \neq \{q_0^a\}$ and $\forall s_j \neq s_i$ and $\forall q \in Q^a(a) \backslash q_0^a, \quad \lambda^a(q) = s_j$.

(2.2.1) Suppose that $Q^a(a) = \{q_0^a\}$. Consider the automaton $\dot{a}$ to be an exact replica of a except that $\delta^{\dot{a}}(\delta^{\dot{a}}(q_0^{\dot{a}}, s_j), s_i)) = q_0^a$ for some $s_j \neq s_i$. Since we have only changed a transition not used when a plays against a replica of itself, we have that $\pi(\dot{a}, a) = \pi(a, a) = \pi(\dot{a}, \dot{a})$. Moreover, $\pi(\dot{a}, \tilde{s}) = \pi(a, \tilde{s}) \quad \forall s \in \mathcal{B}(a) = \mathcal{B}(\dot{a})$ for the transition we have changed only occurs when the opponent plays s_i which, in this case, is not in $\mathcal{B}(a)$. Furthermore, $|\dot{a}| = |a|$. Hence, either $\dot{a}$ belongs to the same ESC as a or a is not in an ESC. Consider now the automaton a' to be an exact replica of a plus some additional states as follows:

(i) $\lambda^{a'}(q_0^{a'}) = s_j \neq s_i$
(ii) $\delta^{a'}(q_0^{a'}, s_i) = q'$
(iii) $\lambda^{a'}(q') = s_i$
(iv) $\delta^{a'}(q', s) = q_0^a \quad \forall s \in S$
(v) $\delta^{a'}(q_0^{a'}, s_k) = q_{s_k} \quad \forall s_k \neq s_i,\ s_k \neq s_j$

(vi) $\lambda^{a'}(q_{s_k}) \in \mathcal{B}(\tilde{s}_k)$
(vii) $\delta^{a'}(q_{s_k}, s_k) = q_{s_k}$
(viii) $\delta^{a'}(q_0^{a}, s_j) = q^-$
(ix) $\lambda^{a'}(q^-) = s^-, \ s^- \neq s_j$
(x) $\delta^{a'}(q^-, s^-) = q^*$
(xi) $\lambda^{a'}(q^*) = s^*$
(xii) $\delta^{a'}(q^*, s^*) = q^*$
(xiii) $\delta^{a'}(q^-, s_j) = q_{s_j}$
(xiv) $\lambda^{a'}(q_{s_j}) \in \mathcal{B}(\tilde{s}_j)$
(xv) $\delta^{a'}(q_{s_j}, s_j) = q_{s_j}$

By (i), automaton $\dot{a}$ and a' play different actions at the beginning. Item (ii) says that if the opponent of a' plays s_i the a' finds out that it is not playing against a replica of itself and then transits to the state q' and plays s_i (according to (iii)). By doing that, a' makes $\dot{a}$ go back to $q_0^{\dot{a}}$ and, at the same time, a' also transits to the same state $q_0^{\dot{a}}$ as (iv) indicates. From there on, $\dot{a}$ and a' play against each other as if $\dot{a}$ played against a replica of itself. Items (v) through (xii), as before, just make sure that a' is an utilitarian automaton and that a' plays a best response to any one state automaton that always plays some action in $\mathcal{B}(\dot{a})$. Therefore, we have that:

(a) $\pi(a', \tilde{s}) \geq \pi(\dot{a}, \tilde{s}) \quad \forall s \in \mathcal{B}(\dot{a})$
(b) $\pi(a', \dot{a}) = \pi(\dot{a}, \dot{a})$
(c) $\pi(\dot{a}, a') = \pi(\dot{a}, \dot{a}) < \pi^*$
(d) $\pi(a', a') = \pi^*$

Therefore, $\pi(a', \mathcal{P}_\epsilon(\dot{a}, \tilde{s})) \geq \pi(\dot{a}, \mathcal{P}_\epsilon(\dot{a}, \tilde{s})) \quad \forall s \in \mathcal{B}(\dot{a})$ and, for ϵ small enough, $\pi(\dot{a}, \mathcal{P}_\epsilon(a', \tilde{s}')) < \pi(a', \mathcal{P}_\epsilon(a', \tilde{s}')) \quad \forall s' \in \mathcal{B}(a')$ This last inequality clearly implies that $\dot{a}$ can not belong to an ESC and therefore neither can a.

(2.2.2) Suppose that $\exists s_j \neq s_i$ and $\exists q \in Q^a(a)\backslash q_0^a$ such that $\lambda^a(q) \neq s_j$

Consider the automaton $\dot{a}$ to be an exact replica of automaton a except that $\delta^{\dot{a}}(q_0^{\dot{a}}, s_j) = \delta^{\dot{a}}(q_0^{\dot{a}}, s_i)$ and $\delta^{\dot{a}}(q_j^{\dot{a}}, s_j) = \delta^a(q_0^a, s_j)$ where q_j is (w.l.o.g.) the first state in $Q^a(a)\backslash q_0^a$ in which $\dot{a}$'s action is not s_j. Since we have only changed transitions not used when a plays against a replica of itself, we have that $\pi(\dot{a}, a) = \pi(a, a) = \pi(\dot{a}, \dot{a})$. Also, $\forall s \neq s_j, \quad \pi(\dot{a}, \tilde{s}) = \pi(a, \tilde{s})$ because the changes only matter when the opponent plays s_j. In the case when the opponent is $\tilde{s}_j$, because of the way $\dot{a}$ is constructed, we also have that $\pi(\dot{a}, \tilde{s}_j) = \pi(a, \tilde{s}_j)$. Furthermore, $|\dot{a}| = |a|$. Hence, either $\dot{a}$ belongs to the same ESC as a or a is not in an ESC.

Notice now that $\lambda^{\dot{a}}(q_0^{\dot{a}}) = s_i$ and $\delta^{\dot{a}}(q_0^{\dot{a}}, s_j) = \delta^{\dot{a}}(q_0^{\dot{a}}, s_i) \in Q^{\dot{a}}(a)$. Hence, we can apply the arguments used in (2.1) to conclude that $\dot{a}$ (and, consequently, a) can not belong to an ESC.

(2.2.3) Suppose that $\forall s_j \neq s_i$ and $\forall q \in Q^a(a)\backslash q_0^a, \quad \lambda^a(q) = s_j$

Clearly, the above is possible only if $S = \{s_i, s_j\}$, which implies that $U(s_i, s_i) \neq U(s_j, s_j)$ for otherwise $\pi(a, a) = \pi^*$ and we are assuming the opposite. In this situation, two things might occur: $s_i \in \beta(\tilde{s}_j)$ or $s_i \notin \beta(\tilde{s}_j)$.

(2.2.3.1) Suppose $s_i \in \beta(\tilde{s}_j)$. Consider the automaton $\dot{a}$ to be an exact replica of a but $\delta^{\dot{a}}(q_0^{\dot{a}}, s_j) = q_0^{\dot{a}}$. Clearly, $\pi(\dot{a}, a) = \pi(a, a) = \pi(\dot{a}, \dot{a})$ for we have only changed a transition not used when a plays against a replica of itself. For a similar reason, $\pi(\dot{a}, \tilde{s}_i) = \pi(a, \tilde{s}_i)$. Also, since $s_i \in \beta(\tilde{s}_j)$, $\pi(\dot{a}, \tilde{s}_j) \geq \pi(a, \tilde{s}_j)$ and $|\dot{a}| \leq |a|$. Thus, either a belongs to the same ESC as a or a can not belong to an ESC.

Notice now that $\lambda^{\dot{a}}(q_0^{\dot{a}}) = s_i$ and $\delta^{\dot{a}}(q_0^{\dot{a}}, s_j) = q_0^{\dot{a}} \in Q^{\dot{a}}(\dot{a})$. Hence, we can apply the arguments used in (2.1) to conclude that $\dot{a}$ (and, consequently, a) can not belong to an ESC.

(2.2.3.2) Suppose $s_i \notin \beta(\tilde{s}_j)$. In this case, since $S = \{s_i, s_j\}$, we have that $\{s_j\} = \beta(\tilde{s}_j)$, which implies $\{s_i\} = \beta(\tilde{s}_i)$ for otherwise we would not have $s_i \in \beta(a)$ (main assumption in this Case 2).

Suppose that $U(s_i, s_i) > U(s_j, s_j)$. Clearly, it must then be the case that $\pi(a, \tilde{s}_i) = U(s_i, s_i)$ for otherwise consider the automaton a' to be an exact replica of a plus an additional state q' such that:

$$
\begin{array}{rl}
\text{(i)} & \delta^{a'}(q_0^{a'}, s_i)^2 = q' \\
\text{(ii)} & \lambda(q') = s_i \\
\text{(iii)} & \delta^{a'}(q', s_i) = q'
\end{array}
$$

In this case, $\pi(a', a) = \pi(a, a) = \pi(a', a')$ for we have changed only transitions not used by a against a. For a similar reason, $\pi(a', \tilde{s}_j) = \pi(a, \tilde{s}_j)$. On the other hand, $\pi(a', \tilde{s}_i) = U(s_i, s_i) > \pi(a, \tilde{s}_i)$. Hence, $\pi(a', \mathcal{P}_\epsilon(a, \tilde{s}_i)) > \pi(a, \mathcal{P}_\epsilon(a, \tilde{s}_i))$. Thus, it must be the case that $\pi(a, \tilde{s}_i) = U(s_i, s_i)$. This being the case, though, we have that the automaton $\tilde{s}_i$ defeats a. Indeed, given that $\pi(a, \tilde{s}_i) = U(s_i, s_i)$ is easy to see that $\pi(\tilde{s}_i, a) = U(s_i, s_i) > \pi(a, a)$. Hence, for ϵ small enough, we have that $\forall s \in \beta(a)$, $\pi(\tilde{s}_i, \mathcal{P}_\epsilon(a, \tilde{s})) > \pi(a, \mathcal{P}_\epsilon(a, \tilde{s}))$. Therefore, $\tilde{s}_i$ defeats automaton a, which means that a can not belong to an ESC if $U(s_i, s_i) > U(s_j, s_j)$.

Suppose, on the contrary, that $U(s_i, s_i) < U(s_j, s_j)$. For the same reason as above, $\pi(a, \tilde{s}_j) = U(s_j, s_j)$ for otherwise a replica of a with the differences that follow would defeat a:

$$
\begin{array}{rl}
\text{(i)} & \delta^{a'}(q_0^{a'}, s_j) = q' \\
\text{(ii)} & \lambda^{a'}(q') = s_j \\
\text{(iii)} & \delta^{a'}(q', s_j) = q'
\end{array}
$$

Hence, $\pi(a, \tilde{s}_j) = U(s_j, s_j)$. As before, we have then that, in this case, $\tilde{s}_j$ defeats a for $\pi(\tilde{s}_j, a) = U(s_j, s_j) > \pi(a, a)$ and, therefore, for ϵ small enough, we have that $\pi(\tilde{s}_j, \mathcal{P}_\epsilon(a, \tilde{s})) > \pi(a, \mathcal{P}_\epsilon(a, \tilde{s})) \quad \forall s \in \beta(a)$. Consequently, a can not belong to an ESC in this case either.

We have found that, in any case, for any non utilitarian automaton we can construct an utilitarian automaton that defeats it. Therefore, no ESC can contain a non utilitarian automaton. ∎

The result that follows is of less importance, but indicates that an automaton in an ESC also performs efficiently against any automaton that always plays a myopic best response.

Proposition 4.1.1 *Let $A^* \subset \mathcal{A}$ be an ESC. Then, $\forall a \in A^*$ and $\forall s \in \mathcal{B}(a)$, $\pi(a, \tilde{s}) = \max_{a' \in \mathcal{A}} \pi(a', \tilde{s}) = \max_{s_j \in S} U(s_j, s)$.*

Proof: Suppose that the proposition is not true, that is, suppose that A^* is an ESC but $\exists a \in A^*$ and $\exists s \in \mathcal{B}(a)$ such that $\pi(a, \tilde{s}) < \max_{s_j \in S} U(s_j, s)$. Suppose additionally that a's initial action is to play s, and that it always plays s thereafter if its opponent keeps playing s. It is clear then, by theorem 2.4.1.1, that $s = s^*$ for otherwise a would not belong to an ESC. Hence, in this case, s^* can not be a best response to a. Hence, either a's initial action is different from s or a eventually takes an action different from s when playing against $\tilde{s}$. In either case, let q be that particular state at which a's action is not s. Consider then the automaton a' to be an exact replica of a except that it has an additional state, q^+, and that:

(i) $\delta^{a'}(q, s) = q^+$
(ii) $\lambda^{a'}(q^+) \in \mathcal{B}(s)$
(iii) $\delta^{a'}(q^+, s) = q^+$

Clearly, by the way a' is constructed, we have that $\pi(a', a) = \pi(a, a) = \pi(a', a')$ and $\pi(a', \tilde{s}) > \pi(a, \tilde{s})$. Hence, a' defeats a, which contradicts the assumption that A is an ESC. ∎

4.2. Uniqueness of ESC

Our uniqueness result will refer to games for which there is only one action that yields the utilitarian payoff. The following lemma says that when an equilibrium automaton meets a replica of itself, only this particular action will be used infinitely many times.

Lemma 4.2.1 *Let A^* be an ESC and suppose that that there is a unique $s^* \in S$ such that $U(s^*, s^*) = \pi^*$. Then, $\forall a \in A^*, C^a(a) = \{q^*\}$ and $\lambda^a(q^*) = s^*$.*

Proof: Note first that $\lambda^a(q) = s^* \quad \forall q \in C^a(a)$ for otherwise $\pi(a, a) \neq \pi^*$ and thus, according to theorem 2.4.1.1, a could not belong to an ESC. Next, it must be the case that $C^a(a) = \{q^*\}$ and $\lambda^a(q^*) = s^*$ for if $C^a(a)$ contained more states, we could just "chop off" all those extra states in $C^a(a)$. The result would be an automaton that would behave exactly as a but less complex and, hence, would invade A^*. ∎

The following proposition indicates that the complexity of an equilibrium automaton is determined by the fact that it must attain the utilitarian payoff when playing against a replica of itself and the maximum payoff possible when playing against any of the "myopic" players (as Theorem 2.4.1.1 and Proposition 2.4.1.1 establish). That is, an equilibrium automaton must have just as many internal states as needed to optimally play against a replica of itself and against the myopic players, but no more.

Definition 4.2.1 *$S(a) \subseteq S$ is a subset of S that is minimal with respect to the following two properties:*

(i) $\exists s^* \in S(a)$ such that $U(s^*, s^*) = \pi^*$

(ii) $\forall s_i \in \mathcal{B}(a)$ $\exists s(i) \in S(a)$ such that $s(i) \in \mathcal{B}(\tilde{s}_i)$

Proposition 4.2.1 *Let A^* be an ESC and suppose that that there is a unique $s^* \in S$ such that $U(s^*, s^*) = \pi^*$. Then, $\forall a \in A^*$, $|a| = |S(a)|$.*

Proof: Note first that $|a| \geq |S(a)|$ for otherwise either Theorem 2.4.1.1 or Proposition 2.4.1.1 would not be satisfied by a. Suppose, though, that the proposition is not true, i.e., $|a| > |S(a)|$. In other words, suppose that $\exists a \in A^*$ and $\tilde{q} \in Q^a$ such that $S(a) \subseteq \{\lambda^a(q) \in S \mid q \in Q^a \backslash \tilde{q}\}$. That is, $\tilde{q}$ is an extra state that a carries and that is unnecessary to attain the maximal payoffs against itself and the myopic players.

It should also be clear that $\tilde{q} \in Q^a(x)$ for some $x \in \mathcal{B}(a) \cup a$ for the contrary would mean that a never uses $\tilde{q}$ whatsoever and, therefore, an exact replica of a without this extra state $\tilde{q}$ would invade a population of a's for it would behave exactly as a but would be less complex. From here we would consider two different cases depending on whether $\tilde{q} \in Q^a(a)$ or $\tilde{q} \notin Q^a(a)$.

Case 1. Suppose first that $\tilde{q} \notin Q^a(a)$. In this case, there must exist a $s_i \in \mathcal{B}(a)$ such that $\tilde{q} \in Q^a(\tilde{s}_i)$. We can then construct an invader automaton a' by replicating a, "cutting off" the unused state $\tilde{q}$ (and its successors), and adding a new state q_{s_i} after q^* (remember -Lemma 2.4.2.1- that q^* is the state satisfying: $C^a(a) = \{q^*\}$ and $\lambda^a(q^*) = s^*$) in such a way that:

(i) $\delta^{a'}(q^*, s_i) = q_{s_i}$,

(ii) $\lambda^{a'}(q_{s_i}) \in \mathcal{B}(\tilde{s}_i)$, and

(iii) $\delta^{a'}(q_{s_i}, s_i) = q_{s_i}$.

We will add a new state q_{s_i} as above for each $s_i \in \mathcal{B}(a)$ such that $C^a(s_i)$ comes after $\tilde{q}$ in $Q^a(s_i)$. Clearly, the new automaton a' has one less state (at least) that a and behaves exactly as a when playing against a, itself, or any $s_i \in \mathcal{B}(a) = \mathcal{B}(a')$. Therefore, such an automaton a' would invade a population of a's, contradicting the assumption that a belongs to an ESC.

Case 2. Suppose now that $\tilde{q} \in Q^a(a)$. Two things may happen in this case: either $\lambda^a(\tilde{q}) \neq \lambda^a(q^*)$ or $\lambda^a(\tilde{q}) = \lambda^a(q^*)$.

Case 2.1. Suppose that $\lambda^a(\tilde{q}) \neq \lambda^a(q^*)$. Consider first the automaton a' as an exact replica of a except that $\delta^{a'}(\tilde{q}, \lambda^a(q^*)) = q^*$. Clearly, a and a' are indistinguishable since we have only changed a transition not used by a when playing against itself or against one of the myopic players unless it happens to meet $\tilde{\lambda}^a(q^*)$. The latter would only matter if $\lambda^a(q^*) \in \mathcal{J}(a)$, which would imply that, also in this case, $\pi(a, \tilde{\lambda}^a(q^*)) = \pi(a', \tilde{\lambda}^a(q^*))$ for the transition has been changed in such a way that a' moves to q^* where it plays a best response to $\tilde{\lambda}^a(q^*)$. Therefore, a' must belong also to A^*. Consider now the automaton a'' that is an exact replica of a' with the following modifications:

(i) $\lambda^{a''}(\tilde{q}) = \lambda^a(q^*)$,
(ii) $\delta^{a''}(\tilde{q}, s) = q^* \quad \forall s \in \mathcal{J}(a) = \mathcal{J}(a') = \mathcal{J}(a'')$, and
(iii) $\delta^{a''}(q^*, s_i) \in C^a(\tilde{s}_i) \quad \forall s_i \in \mathcal{J}(a) = \mathcal{J}(a') = \mathcal{J}(a'')$.

Because of (i) and (ii), a' and a'' are indistinguishable when playing against each other. Moreover, (ii) and (iii) ensures that a'' imitates a (and hence a') when playing against any myopic player. Therefore, a'' must belong to A^* as well. Notice now that the extra state $\tilde{q}$ has been transformed in such a way that when it is active, a'' plays the same action as in q^* and then transits to q^* regardless of the opponent's action. In this situation, this extra state is totally redundant and can be readily eliminated. Consider thus the automaton a''' that is an exact replica of a'' except that it does not have the extra state $\tilde{q}$ and all the transitions that "pointed" to q^* now point to q^* (unless $\tilde{q}$ is the initial state). It is clear that a'' and a''' are totally indistinguishable from a payoff point of view, but a''' is strictly less complex than a'' for it has one less state. Therefore, a''' would invade a population composed of automata in A^*, contradicting the assumption of A^* being an ESC.

Case 2.2. Suppose now that $\lambda^a(\tilde{q}) = \lambda^a(q^*)$. Consider first the automaton a' as an exact replica of a with the following modifications:

(i) $\delta^{a'}(\tilde{q}, s_i) = q^*$for some $s_i \neq \lambda^a(q^*)$.
(ii) $\delta^{a'}(q^*, s_i) \in C^a(s_i)$.

Clearly, a and a' are indistinguishable since we have only changed a transition not used by a when playing against itself or against any of the myopic players unless it happens to meet $\tilde{s}_i$. This would matter if $s_i \in \mathcal{J}(a) = \mathcal{J}(a')$. To be in the safe side, a'' is constructed in such a way that, because of item(ii), $\pi(a, s_i) = \pi(a, s_i)$. Therefore, a' must belong to A^*. Consider next the automaton a'' that is an exact replica of a' except for the following:

(i) $\lambda^{a''}(\tilde{q}) = s_i$,
(ii) $\delta^{a''}(\tilde{q}, s) = q^* \quad \forall s \in \mathcal{J}(a) = \mathcal{J}(a') = \mathcal{J}(a'')$, and
(iii) $\delta^{a''}(q^*, s_j) \in C^a(\tilde{s}_j) \quad \forall s_j \in \mathcal{J}(a) = \mathcal{J}(a') = \mathcal{J}(a'')$.

Because of (i) and (ii), a' and a'' are indistinguishable when playing against each other. Moreover, (ii) and (iii) makes sure that a'' imitates a (and hence a')

when playing against any myopic player. Therefore, a'' must belong to A^* as well. Notice now that the extra state $\tilde{q}$ has been transformed in such a way that when it is active, a'' plays s_i which, by assumption, is different from $\lambda^a(q^*)$. Hence, we can apply the arguments used in Case 2.1 to conclude that A^* can not be an ESC.

We have therefore proved that assuming $|a| > |S(a)|$ leads us to conclude that A^* is not an ESC. Hence, it must be the case that $|a| = |S(a)|$. ∎

The following two corollaries are obvious implications of the previous result. The first one states that if there are two different ESCs, there must be of the same complexity. The second establishes a uniform bound to the complexity of an ESC.

Corollary 4.2.1 *Let A and A' be two ESC and suppose that that there is a unique $s^* \in S$ such that $U(s^*, s^*) = \pi^*$. Then, $\forall a \in A$ and $a' \in A'$, $|a| = |a'|$.*

Corollary 4.2.2 *Let A be an ESC and suppose that that there is a unique $s^* \in S$ such that $U(s^*, s^*) = \pi^*$. Then, $\forall a \in A$, $|a| \leq |S|$.*

Proofs: Obvious.

The lemma that follows is the last result we need before we can state and proof our uniqueness theorem. It indicates that if two ESCs have some elements in common, then the two collections have to be the same.

Lemma 4.2.2 *Let A and A' be two ESC such that $A \cap A' \neq \emptyset$. Then, $A = A'$.*

Proof: The proof is simple. Suppose $\exists a' \in A'$ such that $a' \notin A$ and let $a \in A \cap A'$. Then, A can not be an ESC for a' invades A as it does "as well as" a. ∎

We can now state and prove our second result.

Theorem 4.2.1 *Suppose that there is a unique $s^* \in S$ such that $U(s^*, s^*) = \pi^*$. Then if A^* is an ESC, it is unique.*

Proof: Suppose A and A' are two different ESC and let $a \in A$ and $a' \in A'$. Because of lemma 2.4.2.1, we have that $\lambda^a(q^*) = \lambda^{a'}(q^{*'}) = s^*$, where $\{q^*\} = C^a(a)$ and $\{q^{*'}\} = C^{a'}(a')$. Therefore, $\beta(a) = \beta(a') = \beta(\bar{s}^*)$. Now, if A and A' are two different ESC, we must have that either $C^a(a') \neq C^a(a)$ or $C^{a'}(a) \neq C^{a'}(a')$. Assume (w.l.o.g.) that $C^a(a') \neq C^a(a)$. Let $q_a \in Q^a$ and $q_{a'} \in Q^{a'}$ be the first states in $C^a(a)$ and $C^{a'}(a')$ respectively such that $\lambda^a(q_a) \neq \lambda^{a'}(q_{a'})$. Let $D^a(a) = \{q_a, \ldots, q^*\} \subset C^a(a)$ be the subsequence of states visited by a when plays against a replica of itself that starts with the first states at which a plays a different action than a', and let $D^{a'}(a') = \{q_{a'}, \ldots, q^{*'}\} \subset C^{a'}(a')$ be the analogous subsequence for a'. Construct the automaton b as an exact replica of a except for the following:

(i) $\delta^b(q, \lambda^{a'}(q_{a'})) = \delta^b(q, \lambda^a(q_a)) \quad \forall q \in D^a(a) \backslash q^*$

(ii) If $\lambda^{a'}(q_{a'}) \neq s^*$, $\delta^b(q^*, \lambda^{a'}(q_{a'})) = \delta^b(q_a, \lambda^{a'}(q_{a'}))$

Analogously, we construct b' as an exact replica of a' except for the following:

(i) $\delta^{b'}(q, \lambda^a(q_a)) = \delta^{b'}(q, \lambda^{a'}(q_{a'})) \quad \forall q \in D^{a'}(a') \backslash q^*$

(ii) If $\lambda^a(q_a) \neq s^*$, $\delta^{b'}(q^*, \lambda^a(q_a)) = \delta^{b'}(q_a, \lambda^a(q_a))$

By the way they are constructed, we have that:

(a) $\pi(b, a) = \pi(a, a)$ since we have only changed transitions not used when a plays against a replica of itself.

(b) $\pi(b, \tilde{s}) = \pi(a, \tilde{s})$ because we have only changed transitions that occurs when playing against the automaton that always uses the action $\lambda^{a'}(a')$. Item (ii) makes sure that automaton b plays against this automaton exactly a a would do.

Similarly,

(a) $\pi(b', a') = \pi(a', a')$ since we have only changed transitions not used when a' plays against a replica of itself.

(b) $\pi(b', \tilde{s}) = \pi(a', \tilde{s})$ because we have only changed transitions that occurs when playing against the automaton that always uses the action $\lambda^a(a)$. Item (ii) makes sure that automaton b' plays against this automaton exactly a a' would do.

Hence,

(1) a and b belong to the same ESC, $\mathcal{A}$.

(2) a' and b' belong to the same ESC, $\mathcal{A}'$.

Additionally, note that[6]

(a) $\pi(b, b') = \pi(b', b) = \pi(b, b) = \pi(b', b')$

(b) $\pi(b, \tilde{s}) = \pi(b', \tilde{s}) \quad \forall s \in \mathcal{J}(\tilde{s}^*)$

Hence,

(3) b and b' belong to the same ESC.

[6] It might occur that, still, when b and b' play against each other, they reach to a point at which they use different actions. If this is the case, we can repeat the process (construct two new automata ins the same way we constructed b and b') until that, eventually, they will play like a vs. a or a' vs. a' when playing against each other.

Note that (1), (2), and (3) together imply that $A \cap A' \neq \emptyset$. Henceforth, by lemma 2.4.2.2, we have that $A = A'$ ■

The following example shows that it is necessary to assume that there is only one action ($s*$) in the stage game associated with the utilitarian pair of payoffs in order to have a unique ESC.

Example 4.2.1 Consider the following game:

	B	C	D
B	3,3	0,0	0,5
C	0,0	3,3	0,5
D	5,0	5,0	1,1

Figure 4.2.1 A game with two ESC.

This game is a modification of the Prisoners' Dilemma with the introduction of a new strategy (B) which just "replicates" the role of the strategy C. That is, B vs. B and B vs. D generate the same payoff pairs as C vs. C and C vs. D respectively, and the equivalent holds for D vs B and D vs. C.

In Section 6 we prove that the set composed of the automata that implement the strategies *Tit-for-Tat* and *Grim Trigger* form an ESC of the Repeated Prisoners' Dilemma. Using the same arguments, one can easily prove that the two sets, A and A', below are both ESCs of the game described in Figure 4.2.1.

The set A is composed of all the automata whose basic "skeleton" is the one depicted in figure 4.2.2, that is, automata that are obtained by freely specifying the transitions (arrows) not pictured in Figure 4.2.2. Analogously, the set A' is the set composed of all the automata whose basic "skeleton" appears in Figure 4.2.3.

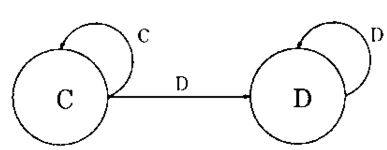


Figure 4.2.2 The *skeleton* of the automata in A.

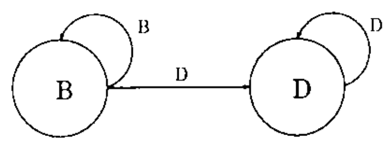


Figure 4.2.3 The *skeleton* of the automata in A'.

5. Polymorphous Populations

We consider now the case in which populations can be composed not only of one type of automaton (plus some best response to it), as it was the case before, but of a mixture of the automata in a PIC. For instance, in the case of the Repeated Prisoners' Dilemma, we have seen that a population composed of automata of the type *Tit-for-Tat* or a population composed of automata of the type *Grim trigger* are both stable in the sense that each can be "invaded" only by automata of the other type and that those automata are "indistinguishable" from a evolutionary point of view. Therefore, nothing prevents any of the automata in an ESC to drift in a population composed of replicas of some other automaton of the same ESC. In this situation, a natural question to ask is: what happens then if a population is composed of a mixture of the automata in the ESC ? We will see in this section that the results obtained in the case of homogeneous populations hold in this more general case as well. For that, we need some additional notation.

Since we will be working with mixtures of automata, the set of reference will be the set of probability distributions over the set of finite automata denoted by $\Delta(\mathcal{A})$ and η will denote a typical element of this set. According to the expected utility hypothesis, we define

$$\pi(\eta, \mu) = \sum_{a_i \in Supp(\eta)} \sum_{a_j \in Supp(\mu)} \eta_{a_i} \mu_{a_j} \pi(a_i, a_j) \quad \forall \eta, \mu \in \Delta(\mathcal{A})$$

Also, as we did in section 2, we define

$$\sigma(a_i, a_j, s) = \frac{\sum_{q \in C^{a_i}(a_j)} I_s(\lambda^{a_i}(q))}{|C^{a_i}(a_j)|}$$

Consequently,

$$\sigma(\eta, s) = \sum_{a_i \in Supp(\eta)} \sum_{a_j \in Supp(\eta)} \eta_{a_i} \eta_{a_j} \sigma(a_i, a_j, s)$$

and $\sigma(\eta) = \{\sigma(\eta, s_1), \ldots, \sigma(\eta, s_n)\}$. Therefore, the expected payoff of using action s_i in a population characterized by η is:

$$u(s_i, \sigma(\eta)) = \sum_{s_j \in S} \sigma(\eta, s_j) U(s_i, s_j)$$

Hence, in this case, the "*best response*" to a population composed of the mixture η is given by:

$$\mathcal{B}(\eta) = \{s_i \in S \quad | \quad u(s_i, \sigma(\eta)) \geq u(s_j, \sigma(\eta)) \quad \forall s_j \in S\}$$

The lemma that follows relate the best response to a mixture with the best response to each automaton in the support of the mixture.

Lemma 5.1 $\bigcap_{a \in Supp(\eta)} \mathcal{B}(a) \subset \mathcal{B}(\eta) \subset \bigcup_{a \in Supp(\eta)} \mathcal{B}(a)$

Proof: Clearly, if s is a best response to all the automata in the support of η, it is also a best response to a lineal combination of those automata because of the linear property of the expected utility hypothesis. Reciprocally, if s is a best response to η, there must exist an automaton in the support of η to which s is also a best response. ∎

We present next the definition of a *Polymorphous Evolutionary Stable Collection* (PESC), that is meant to be the appropriate extension of the definition of ESC to include the possibility of a population composed of a mixture of the automata in the collection.

Definition 5.1 *A PIC $A^* \subset \mathcal{A}$ is a Polymorphous Evolutionary Stable Collection or PESC if $\forall \eta \in \Delta(A^*), \forall a \in Supp(\eta)$ and $\forall b \notin A^*, \exists \bar{\epsilon}$ such that $\forall \epsilon \in (0, \bar{\epsilon})$,*

$$\begin{array}{ll}
 & (i)\ \{\pi(a, \mathcal{P}_\epsilon(\eta, \tilde{s})) > \pi(b, \mathcal{P}_\epsilon(\eta, \tilde{s}) \quad \forall s \in \mathcal{B}(\eta)\}, \\
\text{or} & (ii)\ \{\pi(a, \mathcal{P}_\epsilon(\eta, \tilde{s})) \geq \pi(b, \mathcal{P}_\epsilon(\eta, \tilde{s})) \quad \forall s \in \mathcal{B}(\eta) \\
 & \quad \text{and} \quad \forall \mu \in \Delta(\mathcal{A}) \ \text{ such that } \ Supp(\mu) = Supp(\eta) \cup \{b\} \\
 & \quad \exists a' \in Supp(\eta) \quad \text{such that} \quad \forall s' \in \mathcal{B}(\mu) \\
 & \quad \pi(a', \mathcal{P}_\epsilon(\mu, \tilde{s}')) > \pi(b, \mathcal{P}_\epsilon(\mu, \tilde{s}'))\}, \\
\text{or} & (iii)\ \{\pi(a, \mathcal{P}_\epsilon(\eta, \tilde{s})) \geq \pi(b, \mathcal{P}_\epsilon(\eta, \tilde{s})) \quad \forall s \in \mathcal{B}(\eta) \\
 & \quad \text{and} \quad \pi(a', \mathcal{P}_\epsilon(\mu, \tilde{s}')) \geq \pi(b, \mathcal{P}_\epsilon(\mu, \tilde{s}')) \\
 & \quad \forall a' \in Supp(\eta), \forall \mu \text{ as in (ii), and } \forall s' \in \mathcal{B}(b) \\
 & \quad \text{and} \quad \exists a' \in Supp(\eta) \text{ such that } |a'| < |b|\}.
\end{array}$$

Hence, for any mixture η of the elements of the collection A^*, item (i) establishes that any automaton in the mixture must be a best response to the population composed of that mixture together with some myopic best response to it. Item (ii) says that if there is another automaton outside the collection that is a best response to that population as well, then there must exist an automaton in the mixture that destabilizes a hypothetical population composed of the intruder, some of the original inhabitants of the population and some myopic best response to this combination. The idea behind this is that the original mixture can regain its prevalence in the population. Last, item (iii) determines that if it is not possible to drive the intruder out of the population as indicated in (ii), it can be done by means of a less complex automaton in the mixture.

We will show that the results obtained in section 2.4, that is, efficiency and uniqueness of ESC, apply to PESC as well.

Proposition 5.1 *Let A^* be an ESC of $G^\#$ and let G have an strictly dominant strategy. Then, A^* is a PESC of $G^\#$.*

Proof: Let $\eta \in \Delta(A^*)$. Then, $\forall a_i \in Supp(\eta)$ we have that $a_i \in A^*$. Thus, since A^* is an ESC of $G^\#$, we have that $\forall b \notin A^*$ $\pi(a_i, a_j) = \pi(a_j, a_j) \geq \pi(b, a_j)$ $\forall a_j \in Supp(\eta)$. Therefore,

$$\sum_{a_j \in Supp(\eta)} \eta_{a_j} \pi(a_i, a_j) \geq \sum_{a_j \in Supp(\eta)} \eta_{a_j} \pi(b, a_j)$$

and thus,

$$\pi(a_i, \eta) \geq \pi(b, \eta) \qquad [3]$$

Let s^+ be the strategy that is strictly dominant in G. Clearly, $\mathcal{B}(a) = \{s^+\}$ $\forall a \in \mathcal{A}$. Therefore, by proposition 2.4.1.1, $\pi(a_j, \tilde{s}^+) \geq \pi(b, \tilde{s}^+)$ $\forall a_j \in A^*$. In particular,

$$\pi(a_i, \tilde{s}^+) \geq \pi(b, \tilde{s}^+) \qquad [4]$$

Therefore, [3] and [4] together imply that $\forall \epsilon > 0$

$$\pi(a_i, \mathcal{P}_\epsilon(\eta, \tilde{s}^+)) \geq \pi(b, \mathcal{P}_\epsilon(\eta, \tilde{s}^+))$$

If the above inequality always holds as a strict inequality, the proposition is proved. Suppose on the contrary that $\exists \eta \in \Delta(A^*)$, $a_i \in Supp(\eta)$ such that

$$\pi(a_i, \mathcal{P}_\epsilon(\eta, \tilde{s}^+)) = \pi(b, \mathcal{P}_\epsilon(\eta, \tilde{s}^+))$$

or, equivalently,

$$\begin{aligned}\epsilon \pi(a_i, \tilde{s}^+) + (1-\epsilon) \sum_{a_h \in Supp(\eta)} \mu_{a_h} \pi(a_i, a_h) = \\ = \epsilon \pi(b, \tilde{s}^+) + (1-\epsilon) \sum_{a_h \in Supp(\eta)} \mu_{a_h} \pi(b, a_h)\end{aligned} \qquad [5]$$

Now, since $\pi(a_i, \tilde{s}^+) \geq \pi(b, \tilde{s}^+)$ (because of Proposition 2.4.1.1) and $\pi(a_i, a_h) = \pi(a_h, a_h) \geq \pi(b, a_h)$ for otherwise A^* would not be an ESC, we have that [5] implies that

$$\pi(a_h, \mathcal{P}_\epsilon(a_h, \tilde{s}^+)) = \pi(b, \mathcal{P}_\epsilon(a_h, \tilde{s}^+)) \quad \forall a_h \in Supp(\eta)$$

Therefore, by the definition of ESC, either

(a) $\quad \pi(a_h, \mathcal{P}_\epsilon(b, \tilde{s}^+)) > \pi(b, \mathcal{P}_\epsilon(b, \tilde{s}^+))$ or
(b) $\quad \pi(a_h, \mathcal{P}_\epsilon(b, \tilde{s}^+)) = \pi(b, \mathcal{P}_\epsilon(b, \tilde{s}^+))$ and $|a_h| < |b|$.

If (a) holds we have that

$$\epsilon\pi(a_h, \tilde{s}^+) + (1-\epsilon)\pi(a_h, b) > \epsilon\pi(b, \tilde{s}^+) + (1-\epsilon)\pi(b, b)$$

Consider then any $\mu \in \Delta(\mathcal{A})$ satisfying $Supp(\mu) = Supp(\eta) \cup \{b\}$. If (a) holds,

$$\mu_b\epsilon\pi(a_h, \tilde{s}^+) + \mu_b(1-\epsilon)\pi(a_h, b) > \mu_b\epsilon\pi(b, \tilde{s}^+) + \mu_b(1-\epsilon)\pi(b, b) \qquad [6]$$

Also, because of lemma 2.3.1 and because A^* is an ESC, $\pi(a_h, a_j) = \pi(a_j, a_j) \geq \pi(b, a_j)$ and $\pi(a_h, \tilde{s}^+) \geq \pi(b, \tilde{s}^+)$. Therefore, $\forall a_h \in Supp(\eta)$,

$$\mu_{a_j}\epsilon\pi(a_h, \tilde{s}^+) + \mu_{a_j}(1-\epsilon)\pi(a_h, a_j) \geq \mu_{a_j}\epsilon\pi(b, \tilde{s}^+) + \mu_{a_j}(1-\epsilon)\pi(b, a_j) \quad [7]$$

Adding [6] and [7] we get

$$\epsilon\pi(a_h, \tilde{s}^+) + (1-\epsilon)\pi(a_h, \mu) > \epsilon\pi(b, \tilde{s}^+) + (1-\epsilon)\pi(b, \mu)$$

or, equivalently,

$$\pi(a_h, \mathcal{P}_\epsilon(\mu, \tilde{s}^+)) > \pi(b, \mathcal{P}_\epsilon(\mu, \tilde{s}^+))$$

Hence, if (a) holds, we have that A^* is also a PESC of $G^{\#}$.

If, on the contrary, (b) holds we have that, clearly

$$\pi(a_h, \mathcal{P}_\epsilon(\mu, \tilde{s}^+)) \geq \pi(b, \mathcal{P}_\epsilon(\mu, \tilde{s}^+))$$

and $|a_h| < |b|$, which also satisfy the conditions for PESC.

Hence, in either case, we have seen that if A^* is an ESC of $G^{\#}$ and G has a strictly dominant strategy, A^* is also a PESC of $G^{\#}$. ∎

Proposition 5.2 *Let A^* be a PESC. Then, A^* is an ESC.*

Proof: Simply notice that the definition of PESC includes the definition of ESC as a particular case. ∎

The two corollary that follow extend our two main results to polymorphous populations and are easily obtained from the previous results.

Corollary 5.1 *Let $A^* \subset \mathcal{A}$ be a PESC. Then, $\forall a \in A^*$, $\pi(a, a) = \pi^*$.*

Proof: Combine proposition 2.5.2 and theorem 2.4.1.1. ∎

Corollary 5.2 *Suppose that there is a unique $s^* \in S$ such that $U(s^*, s^*) = \pi^*$. Then if A^* is a PESC, it is unique.*

Proof: Combine proposition 2.5.2 and theorem 2.4.2.1. ∎

Therefore, if we consider polymorphous populations, the efficiency and uniqueness results obtained previously are preserved.

6. The Repeated Prisoners' Dilemma

In this section we analyze the Repeated Prisoners' Dilemma based on the game given in Figure 2.6.1 using the approach developed in the previous sections. We will find that the unique ESC in this game contains only two automata that implement the well known *Tit-for-Tat* and *Grim trigger* strategies.

	D	C
D	1,1	5,0
C	0,5	3,3

Figure 6.1 The Prisoners' Dilemma

Let $A^* = \{T, G\}$ be the set containing the automaton T, that implements the strategy *Tit-for-Tat*, and the automaton G, that implements the strategy *Grim trigger*, depicted in Figure 2.2.1.

Lemma 6.1 *A^* is a PIC*

Proof: This result is straightforward, just notice that:

$$\pi(G, \mathcal{P}_\epsilon(T, \tilde{D})) = \pi(T, \mathcal{P}_\epsilon(T, \tilde{D})) =$$
$$= \pi(G, \mathcal{P}_\epsilon(G, \tilde{D})) = \pi(T, \mathcal{P}_\epsilon(G, \tilde{D})) = \epsilon + 3(1 - \epsilon)$$ ∎

Proposition 6.1 *A^* is an ESC*

Proof: First, we will prove that no automaton from outside A^* can successfully invade a population consisting mainly of automata of type G. For that, note that

$$\pi(G, \mathcal{P}_\epsilon(G, \tilde{D})) = \epsilon + 3(1-\epsilon)$$

is the highest payoff achievable against $\mathcal{P}_\epsilon(G, \tilde{D})$. Thus, no automaton can do strictly better against $\mathcal{P}_\epsilon(G, \tilde{D})$ than G itself. Therefore, if $a \in \mathcal{A}\backslash A^*$ is a potential invader of $\mathcal{P}_\epsilon(G, \tilde{D})$, it must be the case that:

$$\pi(a, \mathcal{P}_\epsilon(G, \tilde{D})) = \epsilon + 3(1-\epsilon)$$

In order to achieve a payoff of 3 against G, a must necessarily be such that

$$\lambda^a(q_0^a) = C \qquad \text{and} \qquad \lambda^a(\delta^a(q_0^a, C)^n) = C \quad \forall n \geq 1 \tag{[8]}$$

where $\delta^a(q, C)^n = \delta^a(\delta^a(q, C)^{n-1}, C)$ for $n \geq 2$ (similarly for $\delta^a(q, D)^n$). That is, $\delta^a(q, C)^n$ is the active state for automaton a when, being at q, its opponent plays C n consecutive times.

Also, a would need to attain the maximum payoff against $\tilde{D}$, so that it must be the case that

$$\exists \dot{q} \in Q^a \qquad \text{such that} \qquad \lambda^a(\delta^a(\dot{q}, D)^n) = D \quad \forall n \geq 1 \tag{[9]}$$

Now, [8] and [9] together imply that $\pi(a, \mathcal{P}_\epsilon(a, \tilde{D})) = \pi(G, \mathcal{P}_\epsilon(a, \tilde{D}))$ and also that $|a| \geq 2$. Thus, according to (iii) in [2], if a can invade $\mathcal{P}_\epsilon(G, \tilde{D})$, it is necessary that

$$2 \leq |a| \leq |G| = 2 \tag{[10]}$$

It is easy to verify that only two automata satisfy [8], [9] and [10], the one implementing the *Grim Trigger* strategy itself and the one implementing the *Tit-for-Tat* strategy. Therefore, we must conclude that no automaton from outside A^* can enter the population $\mathcal{P}_\epsilon(G, \tilde{D})$.

It remains to prove the analogous for a population consisting mainly on automata of type T, that is, no automaton from outside A^* can successfully invade a population $\mathcal{P}_\epsilon(T, \tilde{D})$. As before, no automaton can do better against $\mathcal{P}_\epsilon(T, \tilde{D})$ than T itself. Therefore, we must look for automata that satisfy $\pi(a, \mathcal{P}_\epsilon(T, \tilde{D})) = \epsilon + 3(1-\epsilon)$. Hence, such automata must satisfy $\pi(a, \tilde{D}) = 1$ and $\pi(a, T) = 3$, which implies

$$\exists \dot{q} \in Q^a \qquad \text{such that} \qquad \lambda^a(\delta^a(\dot{q}, D)^n) = D \quad \forall n \geq 1 \tag{[11]}$$

$$\exists q^* \in Q^a \qquad \text{such that} \qquad \lambda^a(\delta^a(q^*, C)^n) = C \quad \forall n \geq 1 \tag{[12]}$$

Clearly, [11] and [12] imply that $|a| \geq 2$. Additionally, we have that $\pi(T, \mathcal{P}_\epsilon(a, \tilde{D})) = \epsilon + 3(1-\epsilon)$, so that if a can invade $\mathcal{P}_\epsilon(T, \tilde{D})$, it is necessary that

$$2 \leq |a| \leq |T| = 2 \qquad [13]$$

As before, only two automata satisfy [11], [12] and [12], those in A^*. Therefore, we must conclude that no automaton from outside A^* can enter the population $\mathcal{P}_\epsilon(T, \tilde{D})$. ∎

Clearly, since the Prisoners' Dilemma has a strategy that is strictly dominant, the strategies *Tit-for-Tat* and *Grim trigger* also form a PESC. The next proposition formally states this fact.

Proposition 6.2 $A^* = \{T, G\}$ *is the unique PESC of the Repeated Prisoners' Dilemma.*

Proof: Clearly, D is a strictly dominant strategy in the Prisoners' Dilemma. Therefore, according to theorem 2.6.1 and proposition 2.5.1 A^* is a PESC. To see that it is unique, apply corollary 2.5.2. ∎

7. Games of Common Interests

The games of *common interests* where first studied by Aumann and Sorin (1989) and are related to a problem considered one year earlier by Harsanyi and Selten (1988). The problem refers to the question of what equilibrium should be used as the prediction for the outcome of a game if there exist more than one equilibrium. Consider, for example, the following game (from Harsanyi and Selten (1988)):

	s_1	s_2
s_1	9,9	0,8
s_2	8,0	7,7

Figure 7.1 A game with two Nash equilibria

Both (s_1, s_1) and (s_2, s_2) are Nash equilibria of the game, but the pair of payoffs (9,9) associated to (s_1, s_1) strictly Pareto-dominates any other payoff vector. For that reason, one might think of (s_1, s_1) as the "natural" outcome of the game. Nevertheless, (s_2, s_2) also constitutes an equilibrium of the game, the so called *risk-dominant* equilibrium according to Harsanyi and Selten's terminology. The problem in this situation is the following: what assumptions can we make on the way the game is played so that the *Pareto-dominant* equilibrium is selected ?

Aumann and Sorin define a game of *common interests* as a (2-persons) game that has a simple payoff pair that strongly Pareto-dominates all other payoff pairs.[7] They consider repetitions of the game with each player attaches a small but positive probability to the other playing some fixed strategy with bounded recall. They show that this game has an equilibrium in pure strategies with payoffs "close" to the Pareto-dominant vector.

Analyzing this type of games using the approach developed in the previous sections, it turns out that a repeated game based on a game of common interests always has an ESC. Moreover, this ESC consists of an unique automaton that always plays the action associated to the Pareto-dominant equilibrium. The simplicity of this result might be surprising, but it is a rather natural consequence of the the introduction of complexity costs in the evolutionary framework. Indeed, any potential equilibrium automaton should, in the first place, attain the Pareto-dominant payoff for otherwise a successful invader could be constructed (very much like in the proof of Theorem 2.4.1.1) in such a way that attains this Pareto-dominant payoff when playing against itself but imitates the original automaton if faced against it. Therefore, the "noisy" players that respond "myopically" to the behavior of the majority of the population will play this Pareto-dominant strategy all the time. In this situation, it is unnecessarily costly to carry extra states that will not be used in any case.[8]

The following definition and proposition formalizes this discussion.

Definition 7.1 *(Aumann and Sorin) A game G is of common interests if there is a payoff pair $(U_1(s_1^*, s_2^*), U_2(s_1^*, s_2^*))$ that strongly Pareto-dominates all other outcomes, i.e., such that*

$$U_i(s_1^*, s_2^*) > U_i(s_1, s_2) \quad \forall s_1, s_2 \in S \quad \forall i = 1, 2$$

Proposition 7.1 *Let G be a game of common interest and let s^* denote the action associated to the Pareto-dominant equilibrium. Then $G^{\#}$ has a unique ESC consisting of the one-state automaton that always plays the action s^*.*

Proof: Since (s^*, s^*) strictly Pareto-dominates any other pair of payoffs, we have that $\beta(\tilde{s}^*) = \{s^*\}$. Hence, $\pi(s^*, \mathcal{P}(s^*, s^*)) = U(s^*, s^*)$. Let $a \in \mathcal{A}(a \neq \tilde{s}^*)$ be a potential invader. Then a must satisfy $\pi(a, \mathcal{P}(s^*, s^*)) = U(s^*, s^*)$, which implies that $\beta(a) = \{s^*\}$ and hence $\pi(s^*, \mathcal{P}(a, s^*)) = \pi(a, \mathcal{P}(a, s^*))$ and $|\tilde{s}^*| < |a|$. That is, $\tilde{s}^*$ cannot be invade by any automaton thanks to its optimal

[7] There might or might not exist other equilibria of the game.

[8] For a similar reason, the one-state automaton that always plays the action associated to the Pareto-dominant equilibrium is also a MESS (Binmore and Samuelson (1992)). Nevertheless, it is not an ESS (Maynard-Smith (1982))

performance and minimal complexity. Therefore, $\{\bar{s}^*\}$ is an ESC. Moreover, it is unique in virtue of Theorem 2.4.2.1. ∎

8. Conclusions

In this paper we analyze the evolutionary stability of repeated symmetric games in the context of the Abreu and Rubinstein's automaton selection games. We focus on a modification of Probst's (1993) solution concept (ESC) which is a modification of Binmore and Samuelson's (1992) MESS which, in turn, is a modification of the conventional ESS to adapt it to the automata context with complexity costs. We show that, if any population contains always a small fraction of people playing a short run best response to their environment, then there is only one set of automata that is evolutionary stable in the sense that no automaton from outside the set can successfully enter a population composed mainly of such automata. Furthermore, those automata are efficient in the sense that they maximize the payoff that an automaton can obtain when playing against a replica of itself.

The main problem of our approach is that, for some games, there might not exist an ESC. Nevertheless, we think that this could be solved by relaxing the equilibrium conditions. For instance, instead of requiring that *any* mixture of automata in a PIC has to be evolutionary stable, we might require that *some* mixture of the automata in the PIC be stable. By doing that we might (or might not!) solve the problem of the non existence of a solution, but the arguments that we have used to prove that a polymorphous stable population is also efficient and unique would not work anymore. Nevertheless, we think that it is necessary to explore this modification.

Some interesting results are obtained when analyzing typical symmetric games such as the Repeated Prisoners' Dilemma and Games of Common Interests using this solution concept. In the case of the Repeated Prisoners' Dilemma, it turns out that the unique population that is evolutionary stable according to the conditions imposed in this paper is the one composed of the most renowned strategies in the broad literature devoted to this game: *Tit-for-Tat* and *Grim trigger*. For games of Common Interest, we find that only the singleton containing the one-state automaton that always plays the action associated to the Pareto-dominant equilibrium is stable.

We believe that the main attractive of this approach is that simple hypothesis, such as assuming that in any population there is always some people that respond myopically to the behavior of the majority, lead to desirable results such as uniqueness and efficiency. The picture, though, is far from been complete. We think that further research in the direction of relating this approach to some "traditional" or "rationality-based" solution concepts like the ones considered in game theory would be of interest. The goal of such an exercise would be to determine whether the type of evolutionary stability proposed in this paper induces some sort of "rationality" and, if so, of what kind. Furthermore, the solution concept proposed here is a purely static one. The study of a dynamical process that incorporates the ideas discussed in this paper would be, in our opinion, of great interest.

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