Discussion Paper #1092

June, 1994

"Property Rights and Efficiency of Public-Good Mechanisms Under Asymmetric Information"

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First version: March, 1994

This version: June, 1994

ABSTRACT

The Coase theorem states that in public good problems, when there are no transaction costs, once property rights are well defined, the efficient outcome will always result. In particular, the assignment of property rights has, by itself, no efficiency implications; different allocations of property rights have merely distributive consequences. Under asymmetric information, however, the situation is completely different. The results of Rob (1989) and Mailath and Postlewaite (1990) indicate that, under asymmetric information, there is no way to overcome the "free-rider" problem. As the number of agents increases, the probability of implementing the efficient outcome decreases to zero. By contrast, in this paper, we identify initial allocations of property rights that are compatible with the operation of an efficient mechanism. We present a mechanism that satisfies the conditions of incentive compatibility and individual rationality (with respect to this initial allocation of property rights) for the individuals and for the firm and that implements the efficient outcome in public good problems under asymmetric information. In this respect, we identify "efficient" property rights structures.

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* I would like to thank Enriqueta Aragonès, V.V. Chari, Peter Klibanoff, Roger Myerson, George Mailath, Steve Matthews, Torben Tranæs, Asher Wolinsky and especially Itzhak Gilboa for their comments and suggestions.
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1. Introduction

Consider the following problem. A certain project is offered to a group of individuals. The project promises the group some benefit but requires it to incur some cost. Alternatively, the proposed project may harm the group but undertaking it promises the group some compensation. The group then has to decide whether or not to implement the project, and in case it does, it has to determine how should the costs or benefits be divided among its individual members. The problem described above is a public good problem. Examples of public good problems span a wide variety including the protection of natural resources, the financing of public radio and television, public schooling and education, public investment in infrastructure, and so on. Other examples are of public "bad" or nuisance problems. For example, suppose that a firm considers operating a pollution generating factory and is willing to compensate those individuals who suffer from the pollution. The neighboring community must reach a decision whether or not to allow for pollution, and in case it does, how to divide the compensation among its individual members.

The Coase theorem asserts that, as long as transaction costs are negligible, well defined property rights will always lead to the efficient resolution of negotiations and in particular, to the efficient resolution of public good problems. The particular assignment of property rights has by itself no efficiency implications; different allocations of property rights have merely distributive consequences. However, a number of results in a variety of related setups have indicated that the Coase theorem may not hold. More specifically, Myerson and Satterthwaite (1983) have shown that under asymmetric information, a buyer and a seller may not reach agreement about the price of a good and may thus, inefficiently, refrain from trading. More related to public good problems, and perhaps more striking, are the results of Rob (1989) and Mailath and Postlewaite (1990) that show that under asymmetric information, the probability of providing a public good tends to zero as the number of individuals increases, although efficiency might require to provide the good with probability 1. Thus, they show that not only is inefficiency possible, it is also inevitable when the number of individuals involved is large.

In this paper, we characterize initial allocations of property rights and an arbitration mechanism that bring to the efficient resolution of public good problems. In this respect, we identify "efficient" assignments of property rights. Our argument relies on two key features. On the relationship between the assignment of property rights and individual rationality constraints, and on the concept of countervailing incentives.

A property right, in law and economics alike, is the right to exclude everyone else from the use of some scarce resource. Typically, property rights are tradable, they can be bought, sold, and exchanged for other property rights. For example, in the pollution problem, the scarce resource is clean air. In case the property rights to clean air belong to the residents living next to the polluting factory, the residents can insist that no pollution will take place. Alternatively, they can demand adequate compensation in case pollution does take place. If, on the other hand, the right to clean air belongs to the firm, it can produce and pollute if it so wishes. It may also concede to reduce its level of pollution if the residents are willing to reimburse it for the associated costs or lost profits. Thus, in the context of public good problems, well defined property rights are actually no more than a complete specification of the outcome in case of a failure to reach agreement. If a member of the negotiating group is dissatisfied with the group's decision, he has the right to be excluded, and get whatever he is entitled to according to the initial assignment of property rights. The initial specification of property rights ensures each member of the group that he will enjoy at least some basic level of utility, regardless of the group's decision. In this respect, the initial assignment of property rights is equivalent to the specification of a reservation value of utility, or an outside option that any agent can exercise regardless of anything else. An agreement amounts in this context to a reallocation of property rights together with some monetary transfers that is agreed upon by everyone. The idea of countervailing incentives is more difficult to explain at this stage. We defer it to a later stage.

The paper follows the following plan: in the next section, we try to demonstrate our point through an informal argument. We provide three examples of possible assignments of property rights and discuss their efficiency properties. In the remainder of the paper, we provide the formal treatment. In section 3, we present the formal set-up of a public good problem and identify the "efficient" property rights structure and mechanism and show that this mechanism is ex-post efficient, incentive compatible, and individually rational with respect to the "efficient" property rights. In section 4, we present a generalization of Rob's (1989) and Mailath and Postlewaite's (1990) result. This general inefficiency result highlights the efficient structure of property rights identified in the previous section. We conclude in section 5 with a brief survey of related literature and a discussion of its relation to our results.

2. The Assignment of Property Rights

For clarity of exposition, we restrict the discussion to a specific example. Let us return to the example of a production facility which, if operated, would generate pollution. Consider the following scenario: in a town inhabited by n residents, a firm is considering whether to operate a factory. The factory can operate at any level, from no production at all to maximal production. The process of production also generates pollution as an inevitable by-product. A higher level of production is associated with a corresponding higher level of pollution. The firm's profits are increasing with the level of production, and the welfare of the residents is decreasing with the level of pollution. Given this background story, we describe different possible assignments of property rights.

EXAMPLE 1: Allocating property rights to the firm

Recall that an assignment of property rights must specify the outcome in case of failure to reach agreement. Allocating property rights to the firm implies that it can operate its factory at any level up to full capacity production and that it has the right to enjoy the associated profits. The town's residents, on the other hand, have the right not to pay anything to the firm. They will, however, suffer from pollution if they choose to do so.

Since the assignment of property rights allows the firm to pollute, if the town's residents want to stop production they have to compensate the firm for its lost profits. In order for the residents to agree to pay their fair share of the associated cost, those who suffer more from the pollution should also be willing to contribute more towards the compensation of the firm. However, under asymmetric information, the degrees of aversion to pollution are not publicly known, and therefore each individual resident has an incentive to understate his aversion to pollution so that he can pay less and "free-ride". As the analysis in Mailath and Postlewaite (1990) shows, as the number of residents increases, the tendency to understate one's aversion to pollution leads to an inability to compensate the firm. The intuition behind this result is that each resident faces the following trade-off: either report the truth and pay accordingly, or lie and pay less, but increase the risk that the residents will not manage to raise enough money to compensate the firm and a higher level of production will take place. As the number of residents increases, the risk that a single resident faces by misreporting his true valuation diminishes. On the other hand, reporting his true valuation still costs him the same. When the number of residents gets large, the latter effect dominates, and eventually all the residents have an incentive to understate their true valuation. Consider a situation in which production is inefficient; that is, the valuations of the people are such that had they been publicly known, a payment scheme that compensates the firm and leaves every resident satisfied could have been devised. For a large enough number of residents, production still takes place. In other words, production results whether it is efficient or not. Moreover, if it so happens that it is inefficient to produce, as the number of residents increases, this inefficiency becomes rampant.[1]

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[1] Mailath and Postlewaite (1990) and Rob (1989) considered a less general problem in which only two options were available to the firm: implement the project or not, or in the terminology of this paper, produce or not. They considered, however, probabilistic decision rules. In this paper, we impose a linear relationship between production and pollution and consider utilities for the residents and profit for the firm which are linear in the level of

EXAMPLE 2: Allocating property rights to the town's residents

When the property rights belong to the town's residents, they can insist that no production would take place without having to reimburse the firm. The firm, on the other hand, has the right not to bear any losses.

Consequently, if the firm still wants to produce, it has to compensate each and every resident. In this example, the town's residents have an incentive to overstate their aversion to pollution so that they can get a higher compensation. As the analysis in Rob (1989) shows, when the number of residents increases, the tendency of the town's residents to overstate their aversion to pollution prevents production, since the firm cannot compensate them and still make a profit. The intuition behind this result is similar to that of the previous example. A resident can either report his true valuation and be compensated accordingly, or lie, report a higher valuation, collect a higher compensation, but increase the risk that the firm will not compensate anyone. As the number of residents increases, reporting the truth still yields the same compensation, but the risk associated with lying decreases. As before, when the number of residents becomes large, the latter effect dominates, and eventually all the residents will report higher valuations. Production will not take place, even in the case that this is an inefficient outcome. That is, the actual valuations of the residents may be such that, had they been publicly known, a payment scheme that sufficiently compensates each resident and still allows the firm to make a positive profit is possible. Yet, no production takes place. Moreover, the larger is the number of residents involved, the higher is the likelihood of this inefficient outcome.

The fact that allocating property rights to the town's residents induces them to report higher valuations, and allocating property rights to the firm induces the residents to report lower valuations, suggests that allocating property rights somewhere in between might provide the residents with the appropriate countervailing incentives to report the truth.[2] Indeed, the efficient mechanism that we propose in this paper will rely on such an assignment of property rights.

However, before we describe the mechanism, we need to specify the initial assignment of property rights. That is, we need to specify the outcome in case either one of the residents, or for that matter, the firm, objects to the use of the mechanism. Consider the following assignment of property rights:

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production. Thus, the level of production in this paper can be interpreted as the probability of production in their papers, respectively.

[2] This intuition is due to Roger Myerson.

EXAMPLE 3: Allocating γ of the property rights to the firm and 1-γ of the property rights to the residents

Here, the town's residents have the right to insist that the level of production does not exceed γ for some prespecified $\gamma \in [0,1]$ without having to pay for it. That is, they have the right to pay nothing, though they may consequently suffer from a pollution level of γ. The firm, on the other hand, has the right to produce at any level it desires provided that it does not exceed γ. As we shall see later, this allocation of property rights is compatible with the operation of an efficient mechanism for appropriately chosen γ's. We demonstrate what happens in the following two figures:

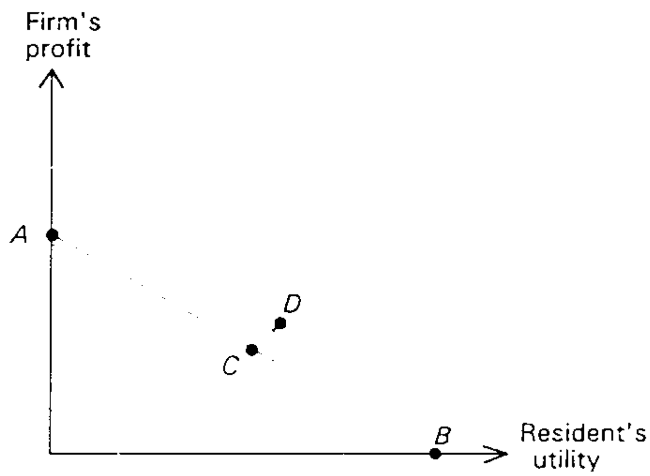

figure 1a

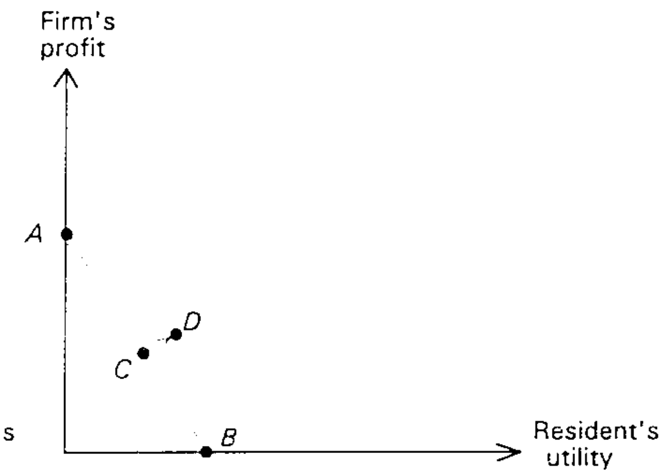

figure 1b

In both figures, the axes denote the utility of the residents and the profit of the firm. For simplicity, we treat all the residents as one in these figures.[3] In both figures, points A, B, and C denote different utility/profit levels which correspond to different property rights specifications. Point A corresponds to example 1 where the property rights are allocated to the firm. This is the structure of property rights that have been studied in Mailath and Postlewaite's (1990) paper. Point B corresponds to example 2 where the property rights are allocated to the town's residents. Rob (1989) analyzes this structure of property rights. Point C corresponds to example 3 where γ of the property rights are allocated to the firm and $1-\gamma$ to the residents. Notice that points A, B, and C specify the utility/profit levels in case of disagreement between the town's residents and the firm. One hopes that with a wisely designed mechanism, these levels of utility/profit can be improved upon. However, notice that *a-priori* we cannot tell whether production will be efficient

[3] One may think of the utility scale as "average utility" or the utility of a "representative agent". However, the analysis that follows does not assume that the agents' utilities are (ex-post) identical.

or not. Either the benefit that the residents derive from no-pollution outweighs the profits of the firm (as in figure 1a where the slope of the line segment connecting points A and B is smaller than 1), or, to the contrary, the profit of the firm surpasses the benefit of no-pollution (as in figure 1b where the slope of AB is larger than 1). Suppose, for example, that the property rights are allocated as in example 1 (point A). If it so happens that the realizations of the valuations of the residents is as in figure 1b, the fact that production will not take place does not violate ex-post efficiency. If, on the other hand, figure 1a applies, Mailath and Postlewaite (1990) show that, for a large n there does not exist an individually rational mechanism that can improve upon the utility/profit levels specified in point A. Rob (1989) proves a similar result when the property rights are allocated as in example 2 (point B). If figure 1a applies, the property rights are efficient. But, if on the other hand, the valuations are as in figure 1b, then for a large n there does not exist an individually rational mechanism that can improve upon the utility/profit levels specified in point B. In section 4, we prove a more general result that shows that under property rights allocations which are associated with utility/profit points that are located anywhere on the line segment connecting points A and B, all mechanisms are bound to be asymptotically inefficient.

By contrast, in this paper we propose a mechanism that operates under an initial allocation of property rights as in example 3 with the corresponding utility/profit levels as in point C. This mechanism succeeds in implementing point D which is *always* ex-post efficient. The mechanism operates as follows: every resident is asked to report his degree of aversion to pollution. If the sum of the residents' reports, that is, the benefit from no-pollution according to the residents' reports exceeds the profits of the firm, no production takes place and the residents reimburse the firm, each according to his report. If on the other hand, the sum of the residents' reports is lower than the profits of the firm, the firm is allowed to produce but it has to compensate the residents, again, each according to his report. As we show later in section 3, there is a way to specify the relationship between the residents' reports and their payments such that truth-telling is an equilibrium of this mechanism. Consequently, the mechanism is able to implement the efficient decision. The intuition which underlies this mechanism is that of providing countervailing incentives to report the truth. When the agents report their valuations, they do not know yet whether they will end up reimbursing the firm or being compensated by it. If the agents knew that they will end up reimbursing the firm they would have reported lower valuations. If, on the other hand, they had reason to believe that the firm will compensate them, they would have exaggerated their reports in hope of collecting a higher compensation. The mechanism that we propose assures us that these two effects exactly cancel each other. Thus, the mechanism creates deliberate ambiguity to induce the agents to report the truth. The fact that the mechanism succeeds in that implies that it is also able to implement the efficient outcome. The idea of creating countervailing incentives to report the truth has already been introduced to mechanism design literature, albeit in a different context, by the work of Cramton, Gibbons, and Klemperer (1987) and Samuelson (1985). We discuss their work in section 5.

3. The Formal Argument

We present the model and discuss the results in terms of the example of the production facility that generates pollution. Yet, our results hold in general for any public good problem. The economy consists of $n+1$ agents, n individuals or residents indexed $\{1,\dots,n\}$ and a firm. A decision about the level of production is to be reached. The firm can operate the factory at any level of production, from full production to no production at all. Production entails pollution, and a higher level of production generates a higher level of pollution. We assume that the relationship between the level of production and the level of pollution is linear and that no production implies no pollution. Given this assumption, assuming further that the level of production is identical to the level of pollution involves no loss of generality. Let the interval $[0,1]$ correspond to the possible levels of production and pollution. Operating the factory at a level of production $\kappa \in [0,1]$ implies that the pollution level endured by the residents is also κ; $\kappa = 1$ corresponds to the maximal level of production and pollution, and $\kappa = 0$ corresponds to no production and no pollution. We assume that all individuals dislike pollution. They may, however, differ with respect to their degree of aversion to pollution. We denote individual i's, $i \in \{1,\dots,n\}$, valuation of no-pollution by v_i. v_i is independently and identically drawn from a distribution F with support $V \subseteq [0,\infty]$ where $\underline{v} \equiv \inf\{v \in V\}$ and $\overline{v} \equiv \sup\{v \in V\}$.[4] Each individual's valuation is known only to himself. The individuals' utilities depend linearly on the level of pollution and on their net payment which might be positive or negative. Thus, an individual i with valuation for no-pollution v_i that receives a transfer of money t_i enjoys a utility of $(1-\kappa)v_i + t_i$ where $\kappa \in [0,1]$ denotes the level of pollution (and production). We assume that the individuals have von Neumann-Morgenstern utilities, that is, they maximize their expected utility.

The firm's profit depends linearly on the level of production (and pollution) and on the sum of its monetary transfers. Thus, the firm's profit equals $\kappa R - \sum_{i=1}^{n} t_i$ where $\kappa \in [0,1]$ denotes the level of production and pollution; $R > 0$ which is publicly known denotes the firm's production profit in case of full production net of its transfers; and t_i, which may be positive or negative, denotes the firm's transfer to individual i. The firm's objective is to maximize its expected profit.

[4] This assumption is made mainly in order to simplify notation. The assumption of independence can be weakened, and the assumption of identical distribution can be dropped.

We employ the following notation: let $v=(v_1,\ldots,v_n)$ denote the vector of individuals' valuations; $\hat{v}=(\hat{v}_1,\ldots,\hat{v}_n)$ denote the vector of individuals' reports; v_{-i} denote the vector $(v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_n)$; and $(\hat{v}_i,v_{-i})$ denote the vector $(v_1,\ldots,v_{i-1},\hat{v}_i,v_{i+1},\ldots,v_n)$. A direct revelation mechanism is described by $n+1$ functions: a decision function $\delta:V^n\rightarrow[0,1]$, where $\delta(\hat{v})$ denotes the level of production as a function of the vector of individuals' reports; and n monetary transfer functions $\tau_i:V^n\rightarrow\Re$ where $\tau_i(\hat{v})$ denotes the monetary transfer made by the firm to individual i as a function of the vector of individuals' reports. Notice that in this formulation the monetary transfer to the firm equals $-\sum_{i=1}^{n}\tau_i(\hat{v})$ and therefore, by definition, the sum of monetary transfers to all the agents in the economy always equals zero. Thus, we restrict our attention to (ex-post) budget balanced mechanisms only. These are "self-reliant" in that they can operate without relying on external credit markets or government subsidies.

Individual i's ex-post utility under a mechanism $(\delta,\tau_1,\ldots,\tau_n)$, or (δ,τ) for short, whose valuation is v_i and whose report is $\hat{v}_i$ equals,

$$u^i_{\delta,\tau}(v_i,\hat{v}_i)=\left(1-\delta(\hat{v})\right)v_i+\tau_i(\hat{v}) \tag{1}$$

Individual i's expected utility in a truth telling equilibrium (his interim utility level) under a mechanism $(\delta,\tau_1,\ldots,\tau_n)$ equals,

$$U^i_{\delta,\tau}(v_i,v_i)=E_{v_{-i}}\left[u^i_{\delta,\tau}(v_i,v_i)\right] \tag{2}$$

$E_w[\cdot]$ denotes the expectation with respect to the random variable w (where $w=v_{-i}$ for individual i and $w=v$ for the firm). The firm's ex-post profit under a mechanism $(\delta,\tau_1,\ldots,\tau_n)$ equals,

$$\pi_{\delta,\tau}(\hat{v})=\delta(\hat{v})R-\sum_{i=1}^{n}\tau_i(\hat{v}) \tag{3}$$

The firm's expected profit in a truth-telling equilibrium (its interim profit) under the mechanism equals,

$$\Pi_{\delta,\tau}=E_v\left[\pi_{\delta,\tau}(v)\right] \tag{4}$$

The following mechanism properties are of interest to us: ex-post efficiency, incentive compatibility, and individual rationality. As is common in the literature, we implicitly assume that if truth-telling is an equilibrium of a mechanism $(\delta,\tau_1,\ldots,\tau_n)$, it will also be played. Hence, we restrict the analysis to the truth-telling equilibrium. We say that a mechanism $(\delta,\tau_1,\ldots,\tau_n)$ is *ex-post efficient* if its decision regarding production at a truth-telling equilibrium (if such exists) is ex-post Pareto efficient; namely, in case the valuations of the individuals exceed the profit of the firm, an ex-post efficient decision is to forbid production, and in case the valuations of the individuals are surpassed by the firm's profit, an ex-post efficient decision requires full production. Thus a mechanism's efficiency depends only on $\delta(\cdot)$. Specifically, a mechanism $(\delta, \tau_1, \dots, \tau_n)$ is ex-post efficient if and only if $\delta(\cdot)$ is given by:

$$\delta(v_1, \dots, v_n) = \begin{cases} 0 & \text{in case } \sum_{i=1}^{n} v_i > R \\ 1 & \text{in case } \sum_{i=1}^{n} v_i < R \end{cases} \tag{5}$$

For a mechanism $(\delta, \tau_1, \dots, \tau_n)$ to induce truthful reporting it must be *incentive compatible* – all individuals should be better off reporting their true valuations (at the interim stage). Formally,

$$\text{(IC)} \qquad U^i_{\delta,\tau}(v_i, v_i) \geq U^i_{\delta,\tau}(v_i, \hat{v}_i), \qquad \forall i, \forall v_i, \hat{v}_i \in V. \tag{6}$$

The following proposition characterizes ex-post efficient and incentive compatible mechanisms. Let $q(\hat{v}_i)$ denote the probability that individual i assigns to the mechanism deciding in favor of no-pollution given his report $\hat{v}_i$ and under the assumption that all other individuals are reporting the truth. That is,

$$q(\hat{v}_i) \equiv P\left(\sum_{j=1}^{n} v_j \geq R \middle| v_i = \hat{v}_i\right). \tag{7}$$

For all $v_i \geq \underline{v}$ denote,

$$Q(v_i) \equiv \int_{\underline{v}}^{v_i} q(v) dF(v) \tag{8}$$

such that $Q(v_i)$ is continuous from the left.[5] Finally, denote $\tau_i^0(v_i) \equiv E_{v_{-i}}\left[\tau_i(v_i, v_{-i}) \middle| \sum_{i=1}^{n} v_i \geq R\right]$ and $\tau_i^1(v_i) \equiv E_{v_{-i}}\left[\tau_i(v_i, v_{-i}) \middle| \sum_{i=1}^{n} v_i < R\right]$.[6] Assume that both $\tau_i^0(v_i)$ and $\tau_i^1(v_i)$ are absolutely continuous on V.[7]

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[5] Notice that when the distribution F has points with positive mass, Q may be discontinuous from the left or from the right. We define Q such that it is continuous from the left because it implies $Q(0) = 0$ and $Q(v) \leq v \quad \forall v \in V$ for any distribution F.

PROPOSITION 1 *Suppose that a mechanism* $(\delta, \tau_1, \ldots, \tau_n)$ *is ex-post efficient. It is incentive compatible if and only if*

$$q(v_i)\tau_i^0(v_i) + (1 - q(v_i))\tau_i^1(v_i) = Q(v_i) - v_i q(v_i) + K \quad \forall i, \forall v_i \in V \tag{9}$$

where K is a constant.

(All proofs are relegated to the appendix.)

The proposition shows that an ex-post efficient and incentive compatible mechanism $(\delta, \tau_1, \ldots, \tau_n)$ must provide individuals with the appropriate countervailing incentives to reveal their true valuations. Namely, an individual i whose valuation is v_i and who reports $\hat{v}_i$ enjoys an expected utility of

$$U^i_{\delta,\tau}(v_i, \hat{v}_i) = q(\hat{v}_i)v_i + Q(\hat{v}_i) - q(\hat{v}_i)\hat{v}_i + K \tag{10}$$

Reporting a high $\hat{v}_i$ increases $q(\hat{v}_i)$ – the probability that the mechanism's decision will bar production and pollution but decreases the individual's expected monetary transfer because $Q(\hat{v}_i) - q(\hat{v}_i)\hat{v}_i$ is decreasing with $\hat{v}_i$. Conversely, reporting a low $\hat{v}_i$ increases the individual's expected monetary transfer but decreases the probability of no-pollution. Thus, the best an individual can do is to report his true valuation. Furthermore, the proposition implies that all ex-post-efficient and incentive compatible mechanisms satisfy the convenient property that individuals' ex-post and interim utilities coincide in the truth telling equilibrium and equal $Q(v_i) + K$ for all $v_i \in V$.

We now turn to define the individual rationality constraints. As we have explained in the introduction, the idea of individual rationality is closely related to that of property rights. Individual rationality constraints guarantee that individuals enjoy a utility level which is at least as high as some basic level of utility which is determined by their property rights. Here, we assume that the initial allocation of property rights is as in example 3. Namely, each individual has the right to insist that the firm does not operate its factory at a production level higher than $\gamma \in [0,1]$, and the firm has the right to operate its factory at any level lower or equal to $\gamma \in [0,1]$. Thus, every individual is guaranteed to enjoy a utility level which is not lower than his utility when he

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[6] The superscripts 0 and 1 serve as a reminder that $\delta = 0$ when $\sum_{i=1}^{n} v_i > R$ and $\delta = 1$ when $\sum_{i=1}^{n} v_i < R$ when a mechanism is ex-post efficient.

[7] A function f is *absolutely continuous* on $[a,b]$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \varepsilon$ for every finite collection $\{(x_i, x_i')\}$ of non overlapping intervals with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$.

suffers from a pollution level of γ but retains his full income. In our formulation, this corresponds to a utility of $(1-\gamma)v_i$. The firm is guaranteed a profit level of at least $\gamma R \geq 0$ (as in point C of figure 1).

A mechanism $(\delta, \tau_1, \ldots, \tau_n)$ satisfies *interim individual rationality* (or, *voluntary participation*) *with respect to an assignment of property rights* γ if the expected utility it promises the residents and the expected profit it promises the firm in the truth-telling equilibrium are at least as high as what they are guaranteed to get by their property rights. Formally,

$$\text{(INTIR} - \gamma) \quad \begin{array}{lll} \text{For the residents} & U^i_{\delta,\tau}(v_i, v_i) \geq (1-\gamma)v_i & \forall i \, \forall v_i \in V \\ \text{For the firm} & \Pi_{\delta,\tau} \geq \gamma R & \end{array} \tag{11}$$

A stronger condition is that of ex-post individual rationality. Namely, a mechanism $(\delta, \tau_1, \ldots, \tau_n)$ satisfies *ex-post individual rationality with respect to an assignment of property rights* γ, if even after the decision of the mechanism is revealed, no agent has reason to object to the mechanism's decision in the truth-telling equilibrium. That is, neither the residents nor the firm are made worse off by the mechanism. Formally,

$$\text{(EXPIR} - \gamma) \quad \begin{array}{ll} \text{For the residents} & u^i_{\delta,\tau}(v_i, v_i) \geq (1-\gamma)v_i \\ \text{For the firm} & \pi_{\delta,\tau}(v) \geq \gamma R \end{array} \quad \forall i \, \forall v \in V^n \tag{12}$$

Requiring that individual rationality be satisfied for the firm as well corresponds to the break-even requirement in Rob (1989) and to ex-post budget balance in Mailath and Postlewaite (1990).[8]

In the sequel, we restrict our attention to the following family of ex-post efficient, incentive compatible and, as we shall see later, individually rational mechanisms $\{(d, t_1^K, \ldots, t_n^K)\}_{K \in \Re}$. By (10), this involves no loss of generality. Let the decision rule d be defined as,

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[8] Although it appears as if the condition of ex-post budget balance is interpreted somewhat differently here, this is not the case. In Mailath and Postlewaite (1990) it is assumed that the public project costs R. In their formulation, budget balance implies that if the project is to be taken, the firm has to be fully reimbursed for incurring this cost. This corresponds to requiring that individual rationality be satisfied for the firm in our formulation. Mailath and Postlewaite do not require the sum of the monetary transfers to equal zero which corresponds to ex-post budget balance in our formulation.

$$d(\hat{v}_1,\ldots,\hat{v}_n)=\begin{cases}0 & \text{in case } \sum_{i=1}^{n}\hat{v}_i \geq R\\ 1 & \text{in case } \sum_{i=1}^{n}\hat{v}_i < R\end{cases} \tag{13}$$

Notice that the definition of d is such that if $(d,t_1^K,\ldots,t_n^K)$ is known to be incentive compatible then it immediately follows that it is also ex-post efficient. The monetary transfer functions t_i^K denote the transfers made by the firm to each individual i. They are identical across individuals and are defined to be,

$$t_i^K(\hat{v}_1,\ldots,\hat{v}_n)=\begin{cases}Q(\hat{v}_i)-\hat{v}_i+K & \text{in case } \sum_{i=1}^{n}\hat{v}_i \geq R\\ Q(\hat{v}_i)+K & \text{in case } \sum_{i=1}^{n}\hat{v}_i < R\end{cases} \tag{14}$$

All the mechanisms $(d,t_1^K,\ldots,t_n^K)$ satisfy (9), and therefore by the proposition they all are incentive compatible. However, it is still instructive to examine the way in which $(d,t_1^K,\ldots,t_n^K)$ administers the monetary transfers. All residents receive a monetary transfer K. Then, if production does not take place, the individuals compensate the firm for its lost profits, each according to his report (notice that $Q(\hat{v}_i)-\hat{v}_i \leq 0 \ \forall \hat{v}_i \in V$); on the other hand, if production does take place, the firm compensates the individuals, each according to his report (notice that $Q(\hat{v}_i) \geq 0 \ \forall \hat{v}_i \in V$). When submitting his report, an individual does not know yet whether he will end up paying the firm or being compensated by it. The mechanism is defined in such a way that when an individual takes into account the sizes of the monetary transfers, the probabilities of paying and being compensated, and the individual's influence on the mechanism's decision, he discovers that revealing his true valuation is his best strategy.

We characterize the conditions under which the mechanisms $(d,t_1^K,\ldots,t_n^K)$ together with the appropriate allocation of initial property rights satisfy, in addition to ex-post efficiency and incentive compatibility, also individual rationality constraints. However, before presenting the theorem, we establish the following notation. Let $\delta \equiv E[Q(v)]$, and $\eta \equiv \frac{1}{n}E\left[\max\left\{\sum_{i=1}^{n}v_i, R\right\}\right]$.

Notice that both δ and η are determined by the distribution F and that both are non negative. δ measures the expected (ex-ante) utility per resident that is generated by the mechanism $(d,t_1^0,\ldots,t_n^0)$ and η measures the expected (ex-ante) welfare per resident in the economy.

THEOREM 2

(*a*) *The mechanism* $(d,t_1^K,\ldots,t_n^K)$ *satisfies ex-post individual rationality for the residents and interim individual rationality for the firm with respect to an allocation of property rights* γ *if and only if the distribution F and the allocation of property rights* γ *satisfy*

$$\sup_{v\in V}\left\{1-\frac{Q(v)+K}{v}\right\}\leq\gamma\leq\frac{n}{R}(\eta-\delta-K) \tag{15}$$

(*b*) *The mechanism* $(d,t_1^K,\ldots,t_n^K)$ *satisfies ex-post individual rationality with respect to an allocation of property rights* γ *if and only if the distribution F and the allocation of property rights* γ *satisfy*

$$\sup_{v\in V}\left\{1-\frac{Q(v)+K}{v}\right\}\leq\gamma\leq\frac{1}{R}\inf_{v\in V^n}\left\{\pi_{d,t^K}(v)\right\} \tag{16}$$

where $\pi_{d,t^K}(v)=\max\left\{\sum_{i=1}^{n}v_i,R\right\}-\sum_{i=1}^{n}Q(v_i)-nK$.

REMARK 1 The theorem associates with each mechanism $(d,t_1^K,\ldots,t_n^K)$ an interval such that for any allocation of property rights within this interval, the mechanism is ex-post efficient, incentive compatible, and individually rational with respect to this allocation of property rights. Several points should be noted. First, the interval associated with ex-post individual rationality is contained in the interval associated with interim individual rationality Second, both intervals are strictly shifting leftward as K increases because $\sup_{v\in V}\left\{1-\frac{Q(v)+K}{v}\right\}$, $\frac{n}{R}(\eta-\delta-K)$, and $\frac{1}{R}\inf_{v\in V^n}\left\{\pi_{d,t^K}(v)\right\}$ are strictly decreasing in K. Third, if, as seems plausible, we restrict ourselves to allocation of property rights $\gamma\in[0,1]$, we can rule out mechanisms with $K>\eta-\delta$ as unacceptable to the firm at the interim stage and mechanisms with $K<0$ as unacceptable to the residents.

The characterizations above are rather cumbersome, and one may wonder whether such distributions exist at all. In the following two examples we show not only that such distributions exist, but also that they need not be pathological. Specifically, for these examples we choose two of the simplest distributions that are closed under summation (to keep calculations tractable).

EXAMPLE 4 **Interim individual rationality**

Consider an economy with 4 residents. The residents valuations for no-pollution v_i are independently drawn from a Poisson distribution with mean 1. Let $R = 4$. For any value $v = 0, 1, 2, \ldots$

$$Q(v) = \sum_{k=0}^{v-1} q(k) = \begin{cases} 0 & v = 0 \\ .35 & v = 1 \\ .93 & v = 2 \\ 1.73 & v = 3 \\ 2.68 & v = 4 \\ v - .32 & v \geq 5 \end{cases}$$

$\delta = E[Q(v)] = .465$, and $\eta = \frac{1}{4} E\left[\max\left\{\sum_{i=1}^{4} v_i, 4\right\}\right] = 1.195$. The following table shows the property rights intervals that together with a mechanism $(d, t_1^K, \ldots, t_4^K)$, guarantee interim individual rationality.

When K equals	γ has to belong to the interval
0	[.65,.73]
.1	[.55,.63]
.2	[.45,.53]
.3	[.39,.43]
.35	[.36,.38]

In each of these mechanisms, residents' interim and ex-post utilities are given by $Q(v_i) + K$ and the firm's interim profit equals $2.92 - 4K$.

EXAMPLE 5 **Ex-post individual rationality**

Consider an economy with 4 residents. The residents valuations for no-pollution v_i are independently drawn from the following distribution $v = \begin{cases} 0 & \text{with probability } \frac{1}{2} \\ 1 & \text{with probability } \frac{1}{2} \end{cases}$. Let $R = 2$.

$Q(v) = \begin{cases} 0 & v = 0 \\ \frac{1}{2} & v = 1 \end{cases}$. The firm's ex-post profit depends on the residents' reports,

$$\pi_{d,t^K}(v) = \begin{cases} 2-4K & \sum_{i=1}^{4} v_i = 0 \\ \frac{3}{2}-4K & \sum_{i=1}^{4} v_i = 1 \\ 1-4K & \sum_{i=1}^{4} v_i = 2 \\ \frac{3}{2}-4K & \sum_{i=1}^{4} v_i = 3 \\ 2-4K & \sum_{i=1}^{4} v_i = 4 \end{cases}$$

The only mechanism and property rights assignment that satisfy ex-post efficiency, incentive compatibility and ex-post individual rationality with respect to the property rights are the mechanism $(d, t_1^0, \ldots, t_4^0)$ and the property rights assignment $\gamma = \frac{1}{2}$.

The previous theorem characterizes a class of distributions for which there exist efficient mechanisms and property rights allocations. Our next results apply to all the possible distributions. We suggest a more general allocation of property rights and show that there always exist an efficient and incentive compatible mechanism that is individually rational with respect to them. Namely, as before, we assume that the initial allocation of property rights is such that each individual has the right to insist that the firm does not operate its factory at a production level higher than $\gamma \in [0,1]$, and the firm has the right to operate its factory at any level lower or equal to $\gamma \in [0,1]$. However, in contrast to the property rights which were defined before, now, in case of disagreement between the firm and the residents, that is, when the residents and the firm cannot reach agreement about which mechanism to use to settle their claims, the firm is subjected to a tax $T(\kappa)$ which depends on its level of production κ. Formally, the interim and ex-post individual rationality constraints of the residents are unchanged, and the individual rationality constraints for the firm take the following form,

(INTIR – γ, T) $\qquad \Pi_{\delta,t} \geq \max\limits_{0 \leq \kappa \leq \gamma} \{\kappa R - T(\kappa)\}$ $\qquad$ (17)

(EXPIR – γ, T) $\qquad \pi_{\delta,t}(v) \geq \max\limits_{0 \leq \kappa \leq \gamma} \{\kappa R - T(\kappa)\} \qquad \forall v \in V^n$.

Geometrically, this means that point C of figure 1 will now be located below the line connecting points A and B. We emphasize that in case of agreement, that is, when the residents and the firm agree to use an arbitration mechanism, the firm need not pay any taxes.

In order to simplify the following analysis, we assume that F is continuous with support $V = [0, \bar{v}] \subseteq [0, \infty]$ and further restrict our attention to the mechanism $(d, t_1^0, \ldots, t_n^0)$.

THEOREM 3

(a) *If the initial assignment of property rights is such that $\gamma \geq 1-q(0)$ and the tax levied upon the firm in case of disagreement when the firm operates the factory at a level $\kappa \leq \gamma$ equals $T_{INT}(\kappa)=\max\{\kappa R-n(\eta-\delta),0\}$ then The mechanism $(d,t_1^0,\ldots,t_n^0)$ is ex-post individually rational for the residents with respect to γ, and interim individually rational for the firm with respect to γ and the tax schedule T_{INT}.*

(b) *If the initial assignment of property rights is such that $\gamma \geq 1-q(0)$ and the tax levied upon the firm in case of disagreement when the firm operates the factory at a level $\kappa \leq \gamma$ equals $T_{EXP}(\kappa)=\max\left\{\left(\kappa+\frac{Q(\bar{v})}{\bar{v}}-1\right)R,0\right\}$ then The mechanism $(d,t_1^0,\ldots,t_n^0)$ is ex-post individually rational for both the residents and the firm with respect to γ and the tax schedule T_{EXP}. (When V is unbounded, $\frac{Q(\bar{v})}{\bar{v}} \equiv \lim_{v\to\infty}\frac{Q(v)}{v}=1$.)*

As expected, the tax schedule required for interim individual rationality for the firm is lower than the tax schedule required for ex-post individual rationality. Furthermore, unless $q(0)=P\left(\sum_{i=1}^{n-1} v_i \geq R\right)=1$, $\frac{Q(\bar{v})}{\bar{v}}<1$ and the tax is small enough for the firm to enjoy positive after-tax profits in case it chooses to produce without reaching agreement.

Notice that the tax size increases linearly with $\frac{Q(\bar{v})}{\bar{v}}$. In particular, if $\bar{v}$ is small relative to R, the tax need not be large. Moreover, as the number of residents increases, the tax size decreases to zero at a rate of ne^{-nC}. We prove this result for the more general case where the firm's net profit in case of full production, now denoted $R(n)$, depends on n. For simplicity, we assume that $\lim_{n\to\infty}\frac{R(n)}{n}$ exists and denote it r.

PROPOSITION 4 *If the distribution $F(\cdot)$ has a bounded support, $\lim_{n\to\infty}\frac{R(n)}{n}=r$, $E[v]\neq r$, and the initial assignment of property rights γ is set equal to $1-q(0)$, then, when the number of residents n is sufficiently large, the maximal tax $T_{EXP}(\gamma)$ is bounded from above by $2rne^{-nC}$ where C is a positive constant.*

REMARK 2 The theorem identifies an efficient assignment of property rights and a mechanism that implements the efficient outcome. Yet, in case of disagreement, as long as the tax $T(\gamma)$ is positive, this assignment of property rights leaves a slack of property rights which are not allocated. The difficulty that this gives rise to is that *in case of disagreement* the firm is required to pay a tax whose proceedings are wasted. This can be avoided by slightly changing the way in which the mechanism operates.

The usual way in which a mechanism operates is that all parties are asked *simultaneously* whether they agree to use the mechanism as an arbitration device. The problem that we have just raised is that in case one of the parties involved (the firm or either one of the residents) objects to the use of the mechanism, the outcome is determined by the property rights. Namely, the firm produces at a level γ and pays a tax $T(\gamma)$ and the residents enjoy a utility of $(1-\gamma)v_i$ each. The tax $T(\gamma)$ that is paid by the firm is not transferred to the residents. If however we adopt a process that obtains the parties' agreement *sequentially*, this problem is avoided, and no resources need be wasted. Consider the following arbitration procedure: first, the firm is asked whether it agrees to abide by the mechanism's decision. If it refuses, it has the right to produce up to capacity γ and it is subjected to the tax specified in theorem 3. The proceedings of the tax are distributed among the residents. If the firm agrees to use the mechanism, it is guaranteed a profit of at least $\gamma R(n) - T(\gamma)$. After obtaining the agreement of the firm, the residents are asked if they are willing to use the mechanism as an arbitration device. If at least one resident voices an objection, the firm is allowed to produce up to capacity γ, but contrary to what happened before, it is not subjected to the tax. The residents' utility in this case is $(1-\gamma)v_i$. If on the other hand, all the residents agree to using the mechanism, the mechanism is implemented. More generally, any procedure that "punishes" the party that objects to using the mechanism by giving it whatever is guaranteed to it by its property rights, and giving the slack or surplus to the other parties will be able to implement the efficient outcome without wasting resources in case of disagreement.

However, a problem that may arise with this procedure is that the firm will have an incentive to bribe residents with low valuations so that it will be able to produce at a level γ and avoid paying the tax. We do not address this problem here.

4. A General Inefficiency Result

The treatment in this section, while more general, resembles the treatment in Mailath and Postlewaite's (1990) appendix. In order to capture the stochastic nature of individuals' valuations as well as any randomness that the mechanism itself induces we embed the economies consisting of n individuals and a firm in a probability space $(\Omega^n, \mathbf{F}^n, \mu^n)$. Individual i's, $i \in \{1,\ldots,n\}$, valuation is given by the random variable $V_i^n : (\Omega^n, \mathbf{F}^n) \to \Re_+$. We assume that the random variables $\{V_i^n\}_{i=1}^n$ are independent for each n. As before, a direct revelation mechanism that determines the level of production is described by $n+1$ functions: a decision function $\delta^n:\Re_+^n \to [0,1]$, which denotes the level of production as a function of individuals' reports[9]; and n monetary transfer functions $\tau_i^n:\Re_+^n \to \Re$ which denote the payment made by the firm to individual i as a function of individuals' reports. In the case that all individuals truthfully report their valuations, the functions δ^n and $\tau^n \equiv (\tau_1^n,\dots,\tau_n^n)$ become random variables $\delta^n(V_1^n(\omega),\dots,V_n^n(\omega)) \in [0,1]$ and $\tau_i^n(V_1^n(\omega),\dots,V_n^n(\omega)) \in \Re$. In the sequel, we refer to δ^n and τ^n both as functions of individuals' valuation and as random variables. For each individual i, let $\mathbf{F}_i^n$ be the smallest σ-algebra with respect to which V_i^n is measurable. Since individual i knows his valuation, he conditions on the realization of V_i^n, that is, $\mathbf{F}_i^n$. Let $\delta_i^n:\Omega^n \to [0,1]$ denote a version of $E[\delta^n|\mathbf{F}_i^n]$ and $E_i[\tau_i^n]:\Omega^n \to \Re$ denote a version of $E[\tau_i^n|\mathbf{F}_i^n]$. Individual i's interim utility from truthfully participating in the mechanism equals $E[(1-\delta^n)V_i^n+\tau_i^n|\mathbf{F}_i^n] = (1-\delta_i^n)V_i^n + E_i[\tau_i^n]$. The firm's interim utility from participating in the mechanism equals $E[\delta^n R(n) - \sum_i \tau_i^n] = E[\delta^n]R(n) - \sum_i E[\tau_i^n]$.

We now reformulate the conditions of efficiency, individual rationality and incentive compatibility in this set-up. Since the conditional probabilities are only unique almost everywhere, efficiency, individual rationality and incentive compatibility hold with probability 1 rather than everywhere.

A mechanism (δ^n, τ^n) is *ex-post efficient* if δ^n is given by:

$$\delta^n(\omega) = \begin{cases} 0 & \text{in case } \sum_{i=1}^{n} V_i^n(\omega) > R(n) \\ 1 & \text{in case } \sum_{i=1}^{n} V_i^n(\omega) < R(n) \end{cases} \qquad \text{for almost all } \omega \in \Omega^n.$$

A mechanism (δ^n, τ^n) satisfies *interim individual rationality with respect to an allocation of property rights* γ if,

$$(\text{INTIR} - \gamma) \quad \begin{array}{l} (1-\delta_i^n)V_i^n + E_i[\tau_i^n] \geq (1-\gamma)V_i^n, \quad \forall i \\ E[\delta^n]R(n) - \sum_i E[\tau_i^n] \geq \gamma R(n), \end{array} \qquad \text{for almost all } \omega \in \Omega^n.$$

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[9] Since the individual utilities and the firm's profits are linear in the level of production, restricting the range of δ^n to $[0,1]$ rather than to the family of distributions over $[0,1]$ is without loss of generality. Both the individuals and the firm are indifferent between the outcome of a distribution over $[0,1]$ and its expectation.

Incentive compatibility is satisfied if,

(IC) $$\exists N \subseteq \Omega^n, \mu^n(N) = 0, \text{ such that } \forall \omega, \omega' \in \Omega^n \setminus N \tag{5.3}$$

$$\left(1-\delta_i^n(\omega)\right)V_i^n(\omega) + E_i\left[\tau_i^n(\omega)\right] \geq \left(1-\delta_i^n(\omega')\right)V_i^n(\omega) + E_i\left[\tau_i^n(\omega')\right].$$

For $0 \leq \beta \leq 1$, define $\Lambda_i^n(\beta) \equiv \inf\left\{x \middle| \mu\left(V_i^n \leq x\right) > \beta\right\}$; Λ_i^n is the inverse distribution function of V_i^n, with a tie-breaking rule.

The next theorem formalizes the following argument. If individuals' valuations are independent then the mechanism determines the level of production "nearly independently" of the valuations of most of them. Thus, only few individuals can be pivotal. The individuals who are not pivotal can avoid paying high taxes in case the mechanism decides on a low level of production by reporting a low valuation (and "free-riding"). Or, alternatively, in case the mechanism decides on a high level of production, these individuals can guarantee themselves a high compensation by reporting a high valuation. In either case, whether it is because the mechanism cannot collect sufficient taxes from the individuals in case of low production, or because the mechanism cannot collect sufficient compensation from the firm in case it decides on a high level of production, inefficiency prevails.

THEOREM 5 *Let* $\left\{\left(\Omega^n, \mathbf{F}^n, \mu^n\right), \left(\delta^n, \tau^n\right)\right\}_n$ *be a sequence of economies and mechanisms that satisfy INTIR -* γ *and IC for all n. If*

A1 $\exists \underline{v}, \bar{v} \geq 0$ *such that* $\forall n, V_i^n : \left(\Omega^n, \mathbf{F}^n\right) \to \left[\underline{v}, \bar{v}\right]$, *and*

A2 $\exists \lambda, \varepsilon > 0$ *such that* $\forall n, \sum_i \Lambda_i^n(\lambda) + n\varepsilon < R(n) < \sum_i \Lambda_i^n(1-\lambda) - n\varepsilon$,

then $E\left[\delta^n\right] \underset{n\to\infty}{\longrightarrow} \gamma$. *That is, the expected level of production converges to* γ *as n tends to infinity.*

We obtain the following corollary about the asymptotic behavior of public good mechanisms,

COROLLARY *Fix* $\gamma \in [0,1]$ *to be an initial allocation of property rights and suppose that*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n E\left[V_i^n\right]}{n} = v,$$

(*a*) *when* $v > r$, *any sequence of mechanisms* (δ^n, τ^n) *satisfying INTIR-*γ *and IC will, asymptotically, implement an inefficient decision with a probability greater or equal to* γ.

(*b*) *when* $v < r$, *any sequence of mechanisms* (δ^n, τ^n) *satisfying INTIR-*γ *and IC will, asymptotically, implement an inefficient decision with a probability greater or equal to* $1-\gamma$.

Notice that ex-post efficiency requires that $E[\delta^n] \underset{n\to\infty}{\to} 0$ when $v > r$, and $E[\delta^n] \underset{n\to\infty}{\to} 1$ when $v < r$. Yet, the previous theorem shows that for any sequence of mechanisms satisfying INTIR - γ and IC, $E[\delta^n] \underset{n\to\infty}{\to} \gamma$. Hence, apart from precluding efficiency, individual rationality actually eliminates any deviation from the status-quo in large economies. In this respect, the previous theorem and corollary demonstrate the crucial importance that the allocation of property rights bears regarding efficiency of public good mechanisms in large economies.

EXAMPLE 6 A mechanism that obtains the bound set by the theorem

Consider the following mechanism,

$$\delta = \begin{cases} 1 & \text{with probability } \gamma \\ 0 & \text{with probability } 1-\gamma \end{cases}$$

$$\tau_i \equiv 0 \quad \forall i.$$

The fact that, asymptotically, this simple mechanism performs as well as any other mechanism, (it satisfies incentive compatibility (trivially) and individual rationality and it obtains the bound set by the corollary), serves as another illustration of the strength of this inefficiency result.

REMARK 3 To conclude this section, we discuss the relationship between our results and the results obtained by Rob (1989) and Mailath and Postlewaite (1990). Rob (1989) shows that when $E[v] < r$ and $\gamma = 0$, asymptotically, any mechanism is inefficient with probability 1. Mailath and Postlewaite (1990) establish the same result for the case where $E[v] > r$ and $\gamma = 1$. This paper shows that efficiency requires that the assignment of property rights should depend on the number of residents. Moreover, when the number of residents increases the efficient allocation of property rights identified in theorems 2 and 3 requires γ to converge to 1 when $E[v] < r$ and to 0 when $E[v] > r$. Thus, our results show that a simple rule of thumb, namely, assign the property rights according to the relation between $E[v]$ and r will be ex-post efficient in the asymptotic case.

5. Discussion of Related Literature

We devote this section to a brief survey of efficiency and inefficiency results in public good problems. We focus our attention on Bayesian implementation results.[10] The earliest results are due to Clarke (1971) and Groves (1973) who suggested a mechanism that implements efficient provision of a public good in a Bayesian equilibrium as long as the condition of budget balance is not required. Moreover, their mechanism is capable of implementing the efficient outcome using the stronger concept of dominant strategy equilibrium.

A second efficiency result is due to d'Aspremont and Gerard-Varet (1979) and Arrow (1979) (who derived a similar result independently) who suggested a mechanism that is incentive compatible, ex-post efficient and satisfies budget balance. In contrast to the Groves/Clarke mechanism, this mechanism depends crucially on using Bayesian equilibrium as the solution concept. As Green and Laffont (1979) showed, it is impossible to derive this result with dominant strategies.

The first inefficiency result in Bayesian implementation literature is due to Laffont and Maskin (1979) that showed that any budget balanced, incentive compatible, and efficient mechanism (and in particular d'Aspremont and Gerard-Varet's and Arrow's mechanism) is bound to violate individual rationality for some realizations of individuals' valuations. In a more specific setup, Myerson (1981) showed that a revenue maximizing auction is inefficient because the seller (sometimes) retains an item despite the fact that it is worth more to the buyers. Another inefficiency result is due to Myerson and Satterthwaite (1983) that have proven (in the private goods case) the impossibility of attaining ex-post efficiency by means of an incentive compatible, individually rational mechanism. And finally, returning to the public goods case, the results of Rob (1989) and Mailath and Postlewaite (1990) showed that not only is inefficiency inevitable in public goods mechanisms under asymmetric information, but that it happens with a probability that converges to 1 as the number of individuals increases.

The inefficiency results mentioned above show that individual rationality is the main obstacle to efficient Bayesian implementation. As we have argued here, individual rationality is closely related to the initial assignment of property rights. Indeed, two results, that of Cramton, Gibbons and Klemperer (1987) and that of Samuelson (1985) indicate that efficiency may be feasible under certain property rights structures without giving up either budget balance, incentive compatibility, or individual rationality. Cramton, Gibbons and Klemperer (1987) show that a partnership can be efficiently dissolved if the initial individual shares of its partners are

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[10] We follow Fudenberg and Tirole's (1991) chapter on Bayesian games and mechanism design.

approximately the same size. The reason is that the partners have the appropriate countervailing incentives to report their true valuation of the partnership because when a partner reports his valuation or his price, he does not know yet if he will end up buying or selling at this price. Samuelson's (1985) suggestion is similar in this respect. He suggests that in order to get efficiency, the property rights for pollution should not be assigned at all (which is equivalent to having point C of figure 1 located at the origin). Rather, the parties should engage in a bidding game through which the property rights for pollution will be allocated. In contrast to regular auctions, however, each individual pays his bid and receives a transfer of money equal to the average bid. Thus, the bidding game implements the efficient outcome because, again, the negotiating parties have the appropriate countervailing incentives to report the truth. When an individual submits his bid, he does not know yet if he will win the right for pollution and pay, or lose it and be compensated.

The approach of this paper is similar in spirit to that of Cramton, Gibbons and Klemperer (1987) and to Samuelson (1985). In this paper too, the "efficient" structure of property rights identified in theorems 2 and 3 provides the right countervailing incentives to report the truth. Thus, efficiency is retained without relinquishing individually rationality, budget balance, or incentive compatibility.

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Appendix: Proofs

PROOF OF PROPOSITION 1 The proof of this proposition employs a standard technique in mechanism design literature. It follows Myerson (1981) and Myerson and Satterthwaite (1983). We start by showing that (9) implies incentive compatibility. Suppose that $(\delta, \tau_1, \ldots, \tau_n)$ is ex-post efficient. It follows that it is incentive compatible if and only if $q(v_i)(v_i + \tau_i^0(v_i)) + (1 - q(v_i))\tau_i^1(v_i) \geq q(\hat{v}_i)(v_i + \tau_i^0(\hat{v}_i)) + (1 - q(\hat{v}_i))\tau_i^1(\hat{v}_i)$ for all i and $v_i, \hat{v}_i \in V$. Plugging (9) in and simplifying, this holds if and only if $Q(v_i) - Q(\hat{v}_i) \geq (v_i - \hat{v}_i)q(\hat{v}_i)$ for all i and $v_i, \hat{v}_i \in V$. Hence it suffices to show that $\int_{\hat{v}_i}^{v_i} q(s)dF(s) - (v_i - \hat{v}_i)q(\hat{v}_i) \geq 0$, or, $\int_{\hat{v}_i}^{v_i}(q(s) - q(\hat{v}_i))dF(s) \geq 0$, which holds since $q(s)$ is an increasing function.

We now prove the other direction. We claim that the derivative of $q(v_i)\tau_i^0(v_i) + (1 - q(v_i))\tau_i^1(v_i)$ equals $-v_i q'(v_i)$ almost everywhere on V. Incentive compatibility implies that,

$$v_i(q(\hat{v}_i) - q(v_i)) \leq q(v_i)\tau_i^0(v_i) + (1 - q(v_i))\tau_i^1(v_i) - q(\hat{v}_i)\tau_i^0(\hat{v}_i) - (1 - q(\hat{v}_i))\tau_i^1(\hat{v}_i) \leq \hat{v}_i(q(\hat{v}_i) - q(v_i))$$

for all $v_i, \hat{v}_i \in V$. And, when $v_i \neq \hat{v}_i$,

$$\frac{v_i(q(\hat{v}_i) - q(v_i))}{v_i - \hat{v}_i} \leq \frac{q(v_i)\tau_i^0(v_i) + (1 - q(v_i))\tau_i^1(v_i) - q(\hat{v}_i)\tau_i^0(\hat{v}_i) - (1 - q(\hat{v}_i))\tau_i^1(\hat{v}_i)}{v_i - \hat{v}_i} \leq \frac{\hat{v}_i(q(\hat{v}_i) - q(v_i))}{v_i - \hat{v}_i}$$

Letting $\hat{v}_i \to v_i$, this implies $\left[q(v_i)\tau_i^0(v_i) + (1 - q(v_i))\tau_i^1(v_i)\right]' = -v_i q'(v_i) \quad \forall i, \forall v_i \in V$ whenever q is differentiable, which because of monotonicity of q is almost everywhere on V. By assumption $\left[q(v_i)\tau_i^0(v_i) + (1 - q(v_i))\tau_i^1(v_i)\right]'$ is absolutely continuous and therefore by integrating both sides, we obtain, $q(v_i)\tau_i^0(v_i) + (1 - q(v_i))\tau_i^1(v_i) = Q(v_i) - v_i q(v_i) + K \quad \forall i, \forall v_i \in V$ where K is a constant.

QED

PROOF OF THEOREM 2 (a) In the truth-telling equilibrium, the firm's interim expected profit under $(d, t_1^K, \ldots, t_n^K)$ equals $E\left[\max\left\{\sum_{i=1}^{n} v_i, R\right\} - \sum_{i=1}^{n} Q(v_i)\right] - nK = n(\eta - \delta - K)$. Interim individual rationality is satisfied for the firm if and only if $n(\eta - \delta - K) \geq \gamma R$ if and only if $\gamma \leq \frac{n}{R}(\eta - \delta - K)$. Similarly, in the truth-telling equilibrium, an individual with valuation v_i enjoys a utility $Q(v_i)+K$. Ex-post individual rationality is satisfied for the residents if and only if $Q(v_i)+K \geq (1-\gamma)v_i$ for all $v_i \in V$, if and only if $\gamma \geq \sup_{v \in V}\left\{1-\frac{Q(v)+K}{v}\right\}$.

(*b*) In the truth-telling equilibrium, the firm's ex-post expected profit under $(d, t_1^K, \ldots, t_n^K)$ equals $\max\left\{\sum_{i=1}^{n} v_i, R\right\}-\sum_{i=1}^{n} Q(v_i)-nK$. Ex-post individual rationality is satisfied for the firm if and only if $\max\left\{\sum_{i=1}^{n} v_i, R\right\}-\sum_{i=1}^{n} Q(v_i)-nK \geq \gamma R$ for all $v \in V^n$ if and only if $\gamma \leq \frac{1}{R}\inf_{v \in V^n}\left\{\max\left\{\sum_{i=1}^{n} v_i, R\right\}-\sum_{i=1}^{n} Q(v_i)-nK\right\}$. *QED*

PROOF OF THEOREM 3 Ex-post individual rationality is satisfied for the residents if $\gamma \geq \sup_{v \in V}\left\{1-\frac{Q(v)}{v}\right\}$. Since $\frac{Q(v)}{v}$ is increasing in v, the supremum is obtained at $\lim_{v \to 0}\left\{1-\frac{Q(v)}{v}\right\}=1-q(0)$. Interim individual rationality is satisfied for the firm if and only if $n(\eta-\delta) \geq \max_{0 \leq \kappa \leq \gamma}\{\kappa R-T(\kappa)\}$ which is satisfied if $T(\kappa)=T_{INT}(\kappa)\ \forall \kappa \in[0, \gamma]$. Finally, ex-post individual rationality is satisfied if $\max\left\{\sum_{i=1}^{n} v_i, R\right\}-\sum_{i=1}^{n} Q(v_i) \geq \max_{0 \leq \kappa \leq \gamma}\{\kappa R-T(\kappa)\}$ for all $v \in V^n$. The fact that $\frac{Q(v)}{v}$ is increasing in v implies that $\max\left\{\sum_{i=1}^{n} v_i, R\right\}-\sum_{i=1}^{n} Q(v_i) \geq\left(1-\frac{Q(\bar{v})}{\bar{v}}\right) R$ for all $v \in V^n$, and by definition of T_{EXP}, it follows that $\left(1-\frac{Q(\bar{v})}{\bar{v}}\right) R \geq \max_{0 \leq \kappa \leq \gamma}\{\kappa R-T_{EXP}(\kappa)\}$. *QED*

PROOF OF PROPOSITION 4 For any level of production $0 \leq \kappa \leq \gamma$, the required tax $T_{EXP}(\kappa)$ is bounded from above by $(q(\bar{v})-(1-\gamma))R(n)$, which in turn, since $1-q(0)=\gamma$ equals $(q(\bar{v})-q(0))R(n)$. We show that $(q(\bar{v})-q(0))R(n) \xrightarrow[n \to \infty]{} 0$ at a rate ne^{-nC} where $C>0$. The proof is adapted from Durrett (1991) p.59.

Distinguish between two cases: (*a*) $Ev>r$, and (*b*) $Ev<r$.

We prove the assertion for case (*a*). Note that $q(\bar{v})-q(0) \leq 1-q(0)$. We show that $1-q(0) \leq e^{-nC}$ for some $C>0$. The fact that F has bounded support implies that for all $\theta \in \Re_+$ the moment generating function of $-v$, $\varphi(\theta) \equiv Ee^{-\theta v}$ is well defined, that is $\varphi(\theta) \equiv Ee^{-\theta v}<\infty$. Applying Chebyshev's inequality to the function $e^{-\theta v}$, we get,

$$e^{-\theta R(n)} P\left(\sum_{i=1}^{n-1} v_i \le R(n)\right) \le \int_0^{R(n)} e^{-\theta \sum_{i=1}^{n-1} v_i} dF\left(\sum_{i=1}^{n-1} v_i\right) \le E e^{-\theta \sum_{i=1}^{n-1} v_i} = \varphi(\theta)^{n-1},$$

or, $$P\left(\sum_{i=1}^{n-1} v_i \le R(n)\right) \le e^{(n-1)\left(\frac{R(n)}{n-1} r\theta + \log\varphi(\theta)\right)}.$$

We will now show that the last exponent is negative for an appropriate choice of θ (and large enough n).

Lemma 1 *If $Ev > r$, there exists a $\theta > 0$ and a large enough N such that for all $n \ge N$, $\frac{R(n)}{n-1}\theta + \log\varphi(\theta) < 0$.*

Proof Denote $f(\theta) = \frac{R(n)}{n-1}\theta + \log\varphi(\theta)$. When $\theta = 0$, $f(\theta) = 0$. We wish to show that the right-hand derivative of f at 0, $\lim_{\substack{h\to 0\\ h\ge 0}} \frac{f(h) - f(0)}{h}$ denoted $D^{+}f(0)$ exists and is negative for large enough n. $\frac{d}{d\theta}\left[\frac{R(n)}{n-1}\theta\right] = \frac{R(n)}{n-1}$ exists and is arbitrarily close to r for large enough n. Hence, it suffices to show that $\exists D^{+} \log\varphi(0) < -r$. Now, $\varphi(0) = 1$. Therefore it suffices to show that $\exists D^{+}\varphi(0) < -r$. However, $\varphi(\theta) = \int_0^{\infty} e^{-\theta v} dF(v)$. It follows from standard arguments (see Durrett (1959) p.59 for details) that for $\theta > 0$, $\exists \varphi'(\theta) = -\int_0^{\infty} v e^{-\theta v} dF(v)$. Hence, $\exists D^{+}\varphi(0) = \lim_{\substack{\theta\to 0\\ \theta\ge 0}} \varphi'(\theta) = -\int_0^{\infty} v dF(v) = -Ev < -r$. □

Since $1 - q(0) = P\left(\sum_{i=1}^{n-1} v_i \le R(n)\right)$, the lemma establishes the existence of a positive constant C and an integer N such that $1 - q(0) \le e^{-nC}$ for all $n \ge N$. Hence, $R(n)\left(q(\bar{v}) - q(0)\right) \le R(n) e^{-nC} \le 2rne^{-nC}$ for large enough n.

The proof of the assertion for case (b) is similar to that of case (a). $q(\bar{v}) - q(0) \le q(\bar{v})$ and we prove $q(\bar{v}) \le e^{-nC}$ for some other $C > 0$. The fact that F has bounded support implies that for all $\theta \in \Re_{+}$ the moment generating function of v, $\psi(\theta) \equiv Ee^{\theta v}$ is well defined. We apply Chebyshev's inequality to the function $e^{\theta v}$,

$$e^{\theta(R(n)-\bar{v})}P\left(\sum_{i=1}^{n-1} v_i \geq R(n)-\bar{v}\right) \leq Ee^{\theta\sum_{i=1}^{n-1} v_i} = \psi(\theta)^{n-1},$$

or, $$P\left(\sum_{i=1}^{n-1} v_i \geq R(n)-\bar{v}\right) \leq e^{(n-1)\left(\log\psi(\theta)-\frac{R(n)-\bar{v}}{n-1}\theta\right)}.$$

LEMMA 2 *If $Ev < r$ there exists a $\theta > 0$ and a large enough N such that for all* $n \geq N$, $\log\psi(\theta) - \dfrac{R(n)-\bar{v}}{n-1}\theta < 0$.

(The proof is similar to the proof of the previous lemma and is omitted.)

Since $q(\bar{v}) = P\left(\sum_{i=1}^{n-1} v_i \geq nr - \bar{v}\right)$, the lemma establishes the existence of a positive constant C and an integer N such that $q(\bar{v}) \leq e^{-nC}$ for all $n \geq N$. Hence, $R(n)(q(\bar{v})-q(0)) \leq R(n)e^{-nC} \leq 2rne^{-nC}$ for large enough n in this case as well, and the proof is complete. *QED*

PROOF OF THEOREM 5 For any number of individuals n, denote $E[\delta^n] \equiv p_n$, and $\mathrm{Var}[\delta^n] \equiv \sigma_n^2$ (Notice that $\sigma_n^2 = \int(\delta^n - p_n)^2 d\mu \leq 1, \forall n$). The superscript n will be suppressed until the end. Define the function $f(\omega) \equiv \dfrac{\delta(\omega)-p_n}{\sigma_n}$. Note that $Ef = 0$ and $Ef^2 = 1$, hence, f is an element of the Hilbert space of square integrable random variables with zero mean and unit variance which is measurable with respect to $\mathbf{F}^n$. f is essentially the level of production δ normalized to have zero mean and unit variance. In this space conditioning on a σ-algebra is a projection operator and conditioning on independent σ-algebras yields orthogonal random variables. Letting $f_i \equiv E[f|\mathbf{F}_i]$, we have that $\left\{\dfrac{f_i}{\|f_i\|}\right\}_{i=1}^{n}$ is an orthonormal set. By Bessel's inequality, $\|f\|^2 \geq \sum_i\left(\int\dfrac{f\cdot f_i}{\|f_i\|}d\mu\right)^2$. That is,

$$1 = \|f\|^2 \geq \sum_i\left(\int\frac{f\cdot f_i}{\|f_i\|}d\mu\right)^2 = \sum_i \|f_i\|^2$$

Hence, no more than $n^{1/2}$ individuals have $\|f_i\|^2 \geq 1/n^{1/2}$.

Define the sets $C_i \equiv \left\{ \omega \middle| |f_i| \geq \frac{1}{n^{1/8}} \right\}$ and $C_i^* \equiv \left\{ \omega \middle| |\delta_i - p_n| \geq \frac{\sigma_n}{n^{1/8}} \right\}$. Notice that these sets coincide up to measure 0. $\|f_i\|^2 = E[f_i^2] \geq \frac{\mu(C_i)}{n^{1/4}}$ so $\mu(C_i^*) \leq n^{1/4}\|f_i\|^2$. Let $I \equiv \left\{ i \middle| \|f_i\|^2 \leq \frac{1}{n^{1/2}} \right\}$ be the set of individuals whose valuations are nearly independent of the mechanism's decision. Since $\int \delta_i \, d\mu = p_n$, for $i \in I$ the probability that individual i's announcement has a significant impact on the level of production is $\mu(C_i^*) \leq \frac{1}{n^{1/4}}$. We prove that $p_n \to \gamma$. It is sufficient to show that each convergent subsequence of $\{p_n\}_n$ converges to γ. Indeed, suppose that $\{p_{n_k}\}_k$ is a subsequence of $\{p_n\}_n$, converging to p. We distinguish three cases: (1) $p < \gamma$; (2) $p > \gamma$; and (3) $p = \gamma$. We show that cases (1) and (2) result in a contradiction and conclude that $\{p_{n_k}\}_k$ has to converge to γ

.

Consider case (1) first. Suppose that $p_{n_k} \to p < \gamma$. We abuse notation and write n instead of n_{k_i}. Let $D_i \equiv \left\{ \omega \middle| V_i(\omega) \leq \Lambda_i\left(1/n^{1/4} \right) \right\}$. The definition of Λ_i implies that $\mu(D_i) \geq \frac{1}{n^{1/4}}$. Let N be the union of the exceptional sets in INTIR-γ and IC for all i. Being the finite union of null sets, N itself is null. If $i \in I$, $\mu(D_i \setminus C_i^*) > 0$, and so there exists an $\omega_i \in \Omega \setminus N$ such that $|\delta_i(\omega_i) - p_n| < \frac{\sigma_n}{n^{1/8}}$ and $V_i(\omega_i) \leq \Lambda_i\left(1/n^{1/4} \right)$. At this point, we use INTIR-γ and IC to bound $E\tau_i$ and then p_n. Consider first $i \in I$, INTIR-γ implies that $-E_i \tau_i(\omega_i) \leq (\gamma - \delta_i(\omega_i))V_i(\omega_i)$ and since $\omega_i \in D_i$, $-E_i \tau_i(\omega_i) \leq (\gamma - \delta_i(\omega_i))\Lambda_i\left(1/n^{1/4} \right)$. Combining with IC yields $-E_i \tau_i(\omega) \leq (\delta_i(\omega_i) - \delta_i(\omega))V_i(\omega) + (\gamma - \delta_i(\omega_i))\Lambda_i\left(1/n^{1/4} \right)$. On $\Omega \setminus C_i^*$, $|\delta_i(\omega) - p_n| < \frac{\sigma_n}{n^{1/8}}$. $\omega_i \notin C_i^*$, therefore $|\delta_i(\omega_i) - \delta_i(\omega)| < \frac{2\sigma_n}{n^{1/8}}$ on $\Omega \setminus C_i^*$. Since $V_i \leq \bar{v}$,

$$-E_i \tau_i(\omega) \leq \frac{2\bar{v}\sigma_n}{n^{1/8}} + \left(\frac{\sigma_n}{n^{1/8}} + \gamma - p_n \right)\Lambda_i\left(1/n^{1/4} \right) \text{ for a.a. } \omega \in \Omega \setminus C_i^*.$$

Hence, $-E\tau_i(\omega) \leq \left(\frac{2\bar{v}\sigma_n}{n^{1/8}} + \left(\frac{\sigma_n}{n^{1/8}} + \gamma - p_n \right)\Lambda_i\left(1/n^{1/4} \right) \right)\mu(\Omega \setminus C_i^*) + \bar{v}\mu(C_i^*)$

$$< \frac{2\bar{v}}{n^{1/8}} + \left(\frac{1}{n^{1/8}} + \gamma - p_n \right)\Lambda_i\left(1/n^{1/4} \right) + \bar{v}/n^{1/4}.$$

If $i \notin I$, $-E\tau_i \leq \bar{v}$. Thus,

$$\leq \sum_{i=1}^{n}\left(\frac{2\overline{v}}{n^{1/8}}+\left(\frac{1}{n^{1/8}}+\gamma-p_n\right)\Lambda_i\left(1/_{n^{1/4}}\right)+\overline{v}/_{n^{1/4}}\right)+\overline{v}n^{1/2}$$

$$\leq\left(\frac{1}{n^{1/8}}+\gamma-p_n\right)\sum_i\Lambda_i\left(1/_{n^{1/4}}\right)+2\overline{v}n^{7/8}+\overline{v}n^{3/4}+\overline{v}n^{1/2}$$

$$\leq(\gamma-p_n)\sum_i\Lambda_i\left(1/_{n^{1/4}}\right)+3\overline{v}n^{7/8}+\overline{v}n^{3/4}+\overline{v}n^{1/2}$$

$$\leq(\gamma-p_n)\sum_i\Lambda_i\left(1/_{n^{1/4}}\right)+5\overline{v}n^{7/8}.$$

Recall that in this case we assume that $p_{n_{k_l}} \to p < \gamma$. Thus, for all sufficiently large n, $p_n < \gamma$. Now, by $A2$ and the firm's INTIR-γ constraint there exist $\varepsilon, \lambda > 0$ such that,

$$(\gamma-p_n)\left(n\varepsilon+\sum_i V_i(\lambda)\right)<(\gamma-p_n)R(n)\leq-\sum_i E\tau_i<(\gamma-p_n)\sum_i\Lambda_i\left(1/_{n^{1/4}}\right)+5\overline{v}n^{7/8}.$$

So, for sufficiently large n (so that $1/_{n^{1/4}} < \lambda$, and $\sum_i V_i(\lambda) > \sum_i \Lambda_i\left(1/_{n^{1/4}}\right)$) it follows that $(\gamma - p_n)n\varepsilon < 5\overline{v}n^{7/8}$. Thus, $\gamma - p_n < \frac{5\overline{v}}{\varepsilon n^{1/8}} \underset{n\to\infty}{\to} 0$, and, hence, $p \geq \gamma$. A contradiction.

We employ a similar method in the second case. Assume that $p_{n_k} \to p > \gamma$. As before, we abuse notation and write n instead of n_{k_l}. Let $D_i' \equiv \left\{\omega \middle| V_i(\omega) \leq \Lambda_i\left(1 - 1/_{n^{1/4}}\right)\right\}$. It follows that $\mu(D_i') \geq 1 - \frac{1}{n^{1/4}}$. Let N be the union of the exceptional sets in INTIR-γ and IC for all i. Being the finite union of null sets, N itself is null. $i \in I$ implies $\mu(D_i' \setminus C_i^*) > 0$, and so there exists an $\omega_i \in \Omega \setminus N$ such that $\left|\delta_i(\omega_i) - p_n\right| < \frac{\sigma_n}{n^{1/8}}$ and $V_i(\omega_i) \leq \Lambda_i\left(1 - 1/_{n^{1/4}}\right)$. As before, at this point we use INTIR-γ and IC to bound $E\tau_i$ and then p_n. Consider first $i \in I$. INTIR-γ implies that $-E_i\tau_i(\omega_i) \leq (\gamma - \delta_i(\omega_i))V_i(\omega_i)$ and since $\omega_i \in D_i'$, $-E_i\tau_i(\omega_i) \leq (\gamma - \delta_i(\omega_i))\Lambda_i\left(1 - 1/_{n^{1/4}}\right)$. Combining with IC yields $-E_i\tau_i(\omega) \leq (\delta_i(\omega_i) - \delta_i(\omega))V_i(\omega) + (\gamma - \delta_i(\omega_i))\Lambda_i\left(1 - 1/_{n^{1/4}}\right)$. On $\Omega \setminus C_i^*$, $\left|\delta_i(\omega) - p_n\right| < \frac{\sigma_n}{n^{1/8}}$. $\omega_i \notin C_i^*$ therefore $\left|\delta_i(\omega_i) - \delta_i(\omega)\right| < \frac{2\sigma_n}{n^{1/8}}$ on $\Omega \setminus C_i^*$. Since $V_i \leq \overline{v}$,

$$-E_i\tau_i(\omega)\leq\frac{2\overline{v}\sigma_n}{n^{1/8}}+\left(\frac{\sigma_n}{n^{1/8}}+\gamma-p_n\right)\Lambda_i\left(1-1/_{n^{1/4}}\right)\text{ for a.a. }\omega\in\Omega\setminus C_i^*.$$

Hence, $-E\tau_i(\omega) \le \left(\frac{2\bar{v}\sigma_n}{n^{1/8}} + \left(\frac{\sigma_n}{n^{1/8}} + \gamma - p_n \right) \Lambda_i \left(1 - \frac{1}{n^{1/4}} \right) \right) \mu(\Omega \setminus C_i^{\bullet}) + \bar{v}\mu(C_i^{\bullet})$

$$< \frac{2\bar{v}}{n^{1/8}} + \left(\frac{1}{n^{1/8}} + \gamma - p_n \right) \Lambda_i \left(1 - \frac{1}{n^{1/4}} \right) + \frac{\bar{v}}{n^{1/4}}.$$

If $i \notin I$, $-E\tau_i \le \bar{v}$. Thus,

$$-\sum_i E\tau_i < \sum_{i \in I} \left(\frac{2\bar{v}}{n^{1/8}} + \left(\frac{1}{n^{1/8}} + \gamma - p_n \right) \Lambda_i \left(1 - \frac{1}{n^{1/4}} \right) + \frac{\bar{v}}{n^{1/4}} \right) + \sum_{i \notin I} \bar{v}$$

$$\le (\gamma - p_n) \sum_i \Lambda_i \left(1 - \frac{1}{n^{1/4}} \right) + 5\bar{v} n^{7/8}.$$

Or, $\quad \sum_i E\tau_i > (p_n - \gamma) \sum_i \Lambda_i \left(1 - \frac{1}{n^{1/4}} \right) - 5\bar{v} n^{7/8}.$

Now, in this case $p_{n_{k_l}} \to p > \gamma$, and therefore, for all sufficiently large n, $p_n > \gamma$. By *A2* and the firm's INTIR-γ constraint there exist $\varepsilon, \lambda > 0$ such that,

$$(p_n - \gamma)\left(\sum_i V_i (1 - \lambda) - n\varepsilon \right) \ge (p_n - \gamma) R(n) \ge \sum_i E\tau_i > (p_n - \gamma) \sum_i \Lambda_i \left(1 - \frac{1}{n^{1/4}} \right) - 5\bar{v} n^{7/8}.$$

For sufficiently large n (such that $1 - \frac{1}{n^{1/4}} > 1 - \lambda$, and $\sum_i \Lambda_i \left(1 - \frac{1}{n^{1/4}} \right) > \sum_i V_i (1 - \lambda)$), it follows that $-n\varepsilon(p_n - \gamma) > -5\bar{v} n^{7/8}$. Thus, $p_n - \gamma < \frac{5\bar{v}}{\varepsilon n^{1/8}} \xrightarrow[n \to \infty]{} 0$. We conclude that $p \le \gamma$. Contradiction.

QED

PROOF OF THE COROLLARY By the previous theorem, for any sequence of mechanisms (δ^n, τ^n) satisfying INTIR-γ and IC, $E[\delta^n] \to \gamma$. The event "the mechanism (δ^n, τ^n) implements the efficient decision" coincides with the set $\left\{ \omega \middle| \sum_i V_i^n > R(n) \text{ and } \delta^n = 0 \text{ or } \sum_i V_i^n < R(n) \text{ and } \delta^n = 1 \right\}$. When $v > r$, by the strong law of large numbers $\lim_{n \to \infty} \frac{\sum_{i=1}^n V_i^n(\omega)}{n} > r$ with probability 1. Since $E[\delta^n] \to \gamma$, it follows that, asymptotically, δ^n will equal 0 with a probability which is at most $1 - \gamma$. Thus, asymptotically, δ^n will implement an inefficient decision with a probability greater or equal to γ. The proof for the case where $v < r$ is analogous. *QED*