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Act-Similarity In Case-Based Decision Theory*

by

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1. Introduction

Motivation

"Case-Based Decision Theory" (CBDT) is a theory of decision making under uncertainty, which suggests that people tend to choose acts which performed well in similar decision situations in the past. More specifically, the theory, in its original version, assumes that a decision maker has "cases" in his/her memory, each of which is a triple (q,a,r), where q is a decision problem, a is the act chosen in it, and r is the result which was obtained. CBDT assumes a utility function over the set of results, u, and a similarity function over the set of problems, s, such that, given a memory (i.e., a set of cases) M and a decision problem p, each act a is evaluated according to the weighted sum

$$(*) \qquad U(a) = U_{p,M}(a) = \sum_{(q,a,r)\in M} s(p,q)u(r).$$

While it stands to reason that past performance of an act would affect the act's evaluation in current problems, it is not necessarily the case that past performance is the only relevant factor in the evaluation process. Specifically, one act's performance may affect the desirability of other, *similar* acts. For instance, suppose that a hypothetical decision maker – to be dubbed "Mrs. Agent" – is looking for a house to buy. One of her options is to purchase a house in a neighborhood she has lived in in the past. Agent hasn't lived in the same house she is now considering, yet it seems unavoidable that her past experience with a close, and probably similar house would color her valuation of the new one. Similarly, consider a decision maker, say, Mr. Agent, who tries to decide whether or not to buy a new product in the supermarket. He has never purchased this product in the past, but he has had the pleasure of consuming similar products by the same producer. Again, we would expect Agent's decision to depend on his experience with similar acts.

Another class of examples illustrating the effects of similarity between acts includes economic problems which involve a continuous parameter. For instance, the decision whether or not to "Offer to sell at price p" for a specific value p, would likely be affected by the results of the same act with different but close values of p. Furthermore, some of the results obtained in Gilboa and

Schmeidler (1993a) and Gilboa and Schmeidler (1993c) for a finite set of acts are likely to have natural extensions to cases with infinitely many acts, provided some notion of similarity over the latter.

Yet another example is Gilboa and Schmeidler (1993a), in which we model a consumer who chooses among products in a repeated problem. In particular, a consumer whose utility function is negative will never be "satisficed;" the more (s)he chooses a product, the more (s)he dislikes it, thus exhibiting change-seeking behavior. In this model, substitution and complementarity effects are modeled by act-similarity. For instance, after attending three (distinct) operas, our consumer may wish to see a movie. Indeed, a fourth opera would also be a "new" act, yet it is "too similar" to the other acts recently chosen.

The need for modeling act-similarity may sometimes be obviated by redefining "acts" and "problems." For instance, when buying a house, Agent's acts may be simply "To Buy" and "Not to Buy," where each possible purchase is modeled as a separate decision problem. However, such a model is hardly very intuitive, especially when many acts are considered simultaneously.

Hence it seems more natural to extend the basic CBDT model and to assume that a decision maker also has a similarity function over acts. Moreover, in many cases the similarity function is most naturally defined on problem-act pairs. For example, "Driving on the left in New York" may be similar to "Driving on the right in London;" "Buying when the price is low" may be more similar to "Selling when the price is high" than to "Selling when the price is low," and so forth.

In short, we would like to have a model in which the similarity function s is defined on such pairs, and – again, given a memory M and a decision problem p, – each act a is evaluated according to the weighted sum

$$(\bullet) \qquad U(a) = U_{p,M}(a) = \sum_{(q,b,r)\in M} s((p,a),(q,b))u(r).$$

In this paper we provide an axiomatization of this decision rule. The axiomatization is somewhat different in nature from those typically found in the literature, as well as from the axiomatization of (*) we present in Gilboa and Schmeidler (1992). We devote the rest of the introduction to explain the conceptual problem involved in the axiomatization and the solution suggested here. Section 2 presents the model and the results. Proofs are to be found in Section 3. Finally, Section 4 concludes with some remarks.

The Problem

We would like to axiomatize the decision rule (•) in terms of (in-principle) observable preferences, thereby rendering it testable and meaningful. That is, we would like to use information about preferences between acts to derive a similarity function between problem-act pairs. However, we here face a problem: the mathematical structure of the decision problem does not relate acts to such pairs in any obvious way. To be precise, any two acts are a-priori related to any problem-act pair in the past to the same extent.

To clarify this point, it may be useful to compare our problem to classical numerical representations of preference relations. For instance, consider von-Neumann and Morgenstern (1944), who use preferences between lotteries to derive a utility function over consequences (together with the expected utility formula). The acts among which their decision maker chooses are naturally tied to the consequences. Furthermore, degenerate lotteries may be identified with consequences. Thus, for example, an ordinal ranking over consequences is implicit in the preference over lotteries. Next consider de Finetti (1937), who derives a subjective probability function over events from preferences between uncertain acts, which are functions from the state space to a utility space. In particular, his structure includes indicator functions, which can be identified with events. Thus, one act may "belong" to a certain event more than another by the very structure of the mathematical constructs. Correspondingly, the likelihood of an event may be measured by the preference for its corresponding indicator function. Similarly, Savage (1954) and Anscombe and Aumann (1963) use similar models, in which acts are differentially related to states and events.

The derivation of CBDT without act similarity follows a similar structure: acts there are identified with their "act profiles," namely with the results to which they led in the past. Thus this model provides a relation between acts and problems: an act is identified with a function from the set of problems in which it was chosen to outcomes. In particular, the similarity between a past problem and a current one may be measured by the preference (in the current problem) for an act which was chosen in this past problem.

By contrast, the model we would like to axiomatize here prohibits the identification of acts with "act profiles." Indeed, since the preference for one act depends on other acts' performance in past cases, it may no longer be assumed that an act's own history is all that matters for its evaluation. Rather, it is the entire history – of this act as well as of others – which may, a-priori, be relevant to the evaluation of each and every act.

The problem posed by this model is not the mere fact that more information pertains to acts' evaluation. The problem is that – as far as an outsider observer can tell from the very structure of the model – it is the *same* information for all acts. When comparing two acts, say, a and c, given a memory M and a decision problem p, there may be nothing in the cases in M which naturally ties them to either a or c. As opposed to the case of (*), in which every act had its own history, now every act may have a claim to any past case. It is the problem-act similarity function which presumably determines which acts – at the new problem p – are more similar to successful ones, and which have to bear the cost of past failures. But when we attempt to axiomatically derive the similarity function, this information is, of course, not yet given.

Thus, on one hand we have the acts, over which preferences may be observed, and on the other – the cases, or problem-act pairs, over which the similarity function is to be defined. If all acts relate to all cases to the same extent, how can the similarity be derived from observations?

One solution to the similarity-derivation problem is to resort to cognitive data, namely to assume that, independently of preferences, we have some problem-act similarity judgments. Such judgments, especially if they can be quantified in an essentially-unique way, can be used to relate current acts to past cases, and thereby to identify acts with various "act profiles" to varying degrees, providing the missing link between acts and cases. Thus one could probably formulate axioms which will dictate that – given cognitive data defining problem-act similarity – preferences are representable by (•) with the same similarity function.

Yet such an approach is vulnerable to standard criticism regarding the validity of cognitive data in economics, as well as the presumed relationship of "similarity" as, say, verbally expressed, and "similarity" as implicitly defined by behavior. In short, although we do not think cognitive data should be dismissed, (and we certainly think it is not "metaphysical nonsense"), we would prefer to have a purely behavioral derivation of the similarity function.

The Context Approach

The solution we propose is the following: assume that we observe preference over a set of acts A at a given problem p and with a given memory M. Further assume that we can also access the decision maker's preferences given a modified memory, in which everything but the outcomes is identical to the given memory M. Given a certain (actual or conceivable) memory, we assume a weak order on the set of acts. However, there is nothing in the structure of an "act" which relates it to past cases. Rather, our axioms are formulated in terms of the memories that would give rise to certain preferences. That is, the axioms will not say anything too exciting about the objects of choice themselves. They will use the structure of memory, or the *context* in which choice is made.

In a sense, we solve the similarity-derivation problem by explicitly modeling the acts-cases relation: the very fact that an act is chosen in a certain context of cases will be brought to the fore, providing the missing link. In other words, since the objects of preference (i.e., acts) are not related to memory (or the "context" of choice), we simply introduce this relation as an additional ingredient of the formal model; we basically assume that we are given a function, which attaches to each possible context (memory) a preference over acts. It follows that, in order to derive a similarity function over past cases, our axioms will not deal only with the objects of choice, but also with the *contexts* of choice.

To clarify this point, let us again compare the structure we employ here with known axiomatizations of numerical representations. For instance, von Neumann and Morgenstern (1944) derive the utility function, to be used in expected utility theory, from preference over lotteries. Indeed, their axioms are formulated in terms of lotteries and exploit their structure, such as the use of compound lotteries. Herstein and Milnor (1953) represent an order over a mixture space, where the axiom of independence uses the mixing operation. Similarly, Savage (1954) derives both utility and probability from preferences over acts, which are functions from states to outcomes. Correspondingly, his axioms make heavy use of this structure, for instance in the "Sure-Thing Principle" (axiom P2), where given acts are "spliced" to define new ones.

By contrast, in this paper the acts – over which preferences are defined – form a completely abstract set, lacking algebraic, topological or any other structure. All we can say about acts directly is that they are ranked by a weak order. On the other hand, the context of decision – the results that were obtained for the problem-act pairs in memory – is a nicely structured set. (In the formal model we assume, for simplicity, that the results are measured in "utiles," and therefore the set of "contexts" is simply $\Re^n$ where n is the number of cases in memory.) Thus most of the mathematical "action" in this paper is not taking place in the set of objects of choice, but in the set of "circumstances" under which decisions are made.

For example, whereas a typical axiom in a more standard model may stipulate that "If x is preferred to y, then $x+z$ is preferred to $y+z$," a typical axiom in our framework states that "If a is preferred to b both in context x and in context y, then a is preferred to b also in context $x+y$." In a sense, the "interesting" axioms are formulated in a space which is a "dual" of the space of acts. Naturally, this approach changes the proof techniques. While separation theorems are as helpful as ever, the proof is quite different from classical ones. (In particular, it will turn out to be the case that there is no distinction between finite and infinite sets of acts.)

From a conceptual viewpoint, in Gilboa and Schmeidler (1992) we also use the "context approach." There we also assume there that an act's ranking depends on memory or on "context." The axioms we use are stated in terms of "act profiles," and they implicitly restrict the circumstances under which certain preference patterns may emerge. For instance, we require that "If x is preferred to y, and z is preferred to w, then $x+z$ is preferred to $y+w$," where all the objects compared are "act profiles." This may be read as, "If a certain act a has yielded the payoff vector x, and a certain act b has yielded the vector y, and in this context a is preferred to b, and if... then..." However, this is only implicit in that model. Mathematically, we assume there that the given data are preferences between the same type of entities which are being mathematically manipulated.

As explained above, we relinquish this more standard approach here since it is not clear how an act should be mapped to the context space. Thus we take an additional step and let the "context approach" be reflected in the formal model as well.

Matsui (1993) also provides an axiomatization of CBDT with problem-act similarity. His model allows the similarity function to depend on past cases' outcomes as well. He uses the classical approach, and assumes preferences over objects which can be algebraically manipulated. However, since acts are not naturally related to past cases in the presence of act similarity, he uses more elaborate entities than "act profiles." There appears to be no simple relationship between Matsui's results and ours. While both papers axiomatize similar functionals (with the exception of the dependence on past cases' results), the entities that characterize these functional are different.

2. The Model

In this section we describe our formal model, axioms and results. We start with the model's primitives, following Gilboa and Schmeidler (1992).

Let P be a nonempty set of *problems*. Let A be a nonempty set of *acts*. For each $p \in P$ there is a nonempty subset $A_p \subseteq A$ of acts available at p. Let R be a set of *outcomes* or *results*. The set of *cases* is $C \equiv P \times A \times R$. A *memory* is a finite subset $M \subseteq C$. W.l.o.g. (without loss of generality) we assume that for every memory M, if $m = (p,a,r), m' = (p',a',r') \in M$ and $m \neq m'$, then $p \neq p'$.

We assume (explicitly) that $R = \Re$ and (implicitly) that it is scaled so that the utility function is the identity. That is, our axioms should be interpreted as if results are measured in "utiles." Needless to say, this model calls for a generalization in which the utility function will also be axiomatically derived, together with the similarity function.

Our axioms will be stated in terms of preferences among acts given a certain memory M and facing a certain problem p. Moreover, they will relate preferences among acts given different possible memories. The following notation will therefore prove useful: given a memory M, denote its projection on $P \times A$ by E. That is,

$$E = E(M) = \left\{ (q,a) \middle| \ \exists\, r \in R, \quad (q,a,r) \in M \right\}$$

In other words, E is the set of problem-act pairs recalled. We will also use the projection of M (or of E) on P, denoted by H. That is,

$$H = H(M) = H(E) = \left\{ q \in P \middle| \exists a \in A, r \in R, \ s.t. \ (q,a,r) \in M \right\}.$$

Thus H is the set of problems recalled.

For every memory M and every problem $p \notin H$ we assume a preference relation over acts, $\geq_{p,M} \subseteq A_p \times A_p$. Our main result derives the numerical representation for a given set E and a given problem $p \notin H$. Let us therefore assume that E and p are given. Every memory M with $E(M) = E$ may be identified with the results it associates with the problem-act pairs, i.e., with a function $x = x(M) \in \Re^E$. An element $x \in \Re^E$ specifies the history of results, or the *context* of the decision problem p. Denoting $n = |E|$, we abuse notation and identify $\Re^E$ with $\Re^n$. (By so doing we implicitly introduce an order over E. However, at no point will it play any role in our results.) Thus a relation $\geq_{p,M}$ over A_p may be thought of as a relation $\geq_x$. Moreover, we will assume that $\geq_x$ is defined for every $x \in \Re^n$. We define $>_x$ and $\approx_x$ to be the asymmetric and symmetric parts of $\geq_x$, as usual.

We will use the following axioms:

A1 Order: For every $x \in \Re^n$, $\geq_x$ is complete and transitive on A_p.

A2 Continuity: For every $\{x_k\}_{k\geq 1} \subseteq \Re^n$ and $x \in \Re^n$, and every $a,b \in A_p$, if $x_k \to x$ (in the standard topology on $\Re^n$) and $a \geq_{x_k} b$ for all $k \geq 1$, then $a \geq_x b$.

A3 Additivity: For every $x,y \in \Re^n$ and every $a,b \in A_p$, if $a \geq_x b$ and $a \geq_y b$, then $a \geq_{x+y} b$.

A4 Reflection: For every $x \in \Re^n$, and every $a,b \in A_p$, if $a \geq_x b$, then $b \geq_{-x} a$.

Axiom A1 is probably the most standard of all. It simply required that, given any conceivable context, the decision maker's preference relation over acts is a weak order. Axioms A2-A4 are new in the sense that they are formulated in terms of contexts, rather than in terms of acts. However, at least axioms A2 and A3 cannot fail to remind the reader of standard axioms in the classical approach: A2 requires that preferences would be continuous *in the space of contexts*. A3 states that preferences be additive in this space. That is, that if both contexts x and y suggest that a is preferred to b, then so does the "sum" context $x+y$. The logic of this axiom is that a context may be thought of as "evidence" in favor of one act or another. Thus, if both x and y "lend support" to choosing a over b, then so should the "accumulated evidence" $x+y$.

Needless to say, A3 is one of the main axioms, and carries most of the responsibility for the additive functional we end up with. Naturally, it cannot be any more plausible than the additive functional itself, and there are reasonable examples in which additivity fails. (In particular, such examples are very natural if there is a process of "second-order induction," by which the decision maker refines his/her similarity judgments based on past experience. We discuss this issue in Gilboa and Schmeidler (1993b).) However, the main role of the axiomatization here is to relate the theoretical construct, "problem-act similarity," to observable preferences. Hence we do not attempt to present A3 as a "canon of rationality." While we believe it is a sensible requirement in some cases, we concede it may fail in others.

The "reflection" axiom states, roughly, that "for opposite contexts, one has opposite preferences." That is, if the utilities obtained in the past – given by the context – were to be "reversed," all preferences would also be reversed. For instance, suppose that according to context x, a is preferred to b. Further assume that this preference is mostly due to the fact that an act, which is very similar to a, was extremely successful according to x. Now consider the context $-x$; according to it, the same act – similar to a – was disastrous. It makes sense that in $-x$, b will be preferred to a. As in the case of A3, axiom A4 can also be criticized as too restrictive. However, we find it reasonable as a "first approximation."

Axioms A1-A4 can easily be seen to be necessary for the functional form we would like to derive. By contrast, the next axiom we introduce is not. While the theorem we present is an equivalence theorem, it characterizes a more restricted class of preferences than the decision rule discussed in the introduction, namely those preferences satisfying axiom A5 as well. This axiom should be viewed merely as a technical requirement. It states that preferences are "fine" in the following sense: for any four acts, there is a conceivable context which would distinguish among all four of them.

<u>A5 Discernability:</u> For every distinct $a,b,c,d \in A_p$ there exists $x \in \Re^n$ such that $a >_x b >_x c >_x d$.

(Observe that A5 is trivially satisfied when $|A_p| < 4$.)

Note that, specifically, A5 rules out preferences according to which acts c and d are always "between" a and b. This may be particularly restrictive for some applications. For instance, consider acts which are linearly ordered, say, they are parametrized by quantity. In this case it may well be the case that "Sell

100 shares" is preferred to "Sell 300 shares," or vice versa – but that in both cases, "Sell 200 shares" is ranked in between the two. Yet (in the presence of at least four acts), this is precluded by A5. Therefore there is certainly room to study more general axiom systems. In Section 4 we discuss axiom A5 in more detail, and provide examples to show that axioms A1-A4 alone cannot guarantee the desired result.

Our main result can now be formulated.

Theorem: Let there be given E and p as above. Then the following two statements are equivalent:

(i) $\{\geq_x\}_{x\in\Re^n}$ satisfy A1-A5;

(ii) For every $a \in A_p$ there is a vector $s^a \in \Re^n$ such that:

– for every $x \in \Re^n$ and every $a,b \in A_p$,

$$(**) \qquad a \geq_x b \qquad \text{iff} \qquad \sum_{i=1}^{n} s_i^a x_i \geq \sum_{i=1}^{n} s_i^b x_i ,$$

and

– for every distinct $a,b,c,d \in A_p$, the vectors $(s^a - s^b)$, $(s^a - s^c)$ and $(s^a - s^d)$ are linearly independent.

Furthermore, in this case, if $|A_p| \geq 4$, the vectors $\{s^a\}_{a\in A_p}$ are unique in the following sense: if $\{s^a\}_{a\in A_p}$ and $\{\hat{s}^a\}_{a\in A_p}$ both satisfy (**), then there are a scalar $\alpha > 0$ and a vector $\beta \in \Re^n$ such that for all $a \in A_p$, $\hat{s}^a = \alpha s^a + \beta$.

We remind the reader that $\Re^n$ is used as a proxy for $\Re^E$. Thus the vectors $\{s^a\}_{a\in A_p}$ provided by the theorem can also be thought of as functions from E to $\Re$. Furthermore, these can be viewed as defining similarity on problem-act pairs. Specifically, the theorem implies that under A1-A5, there exists a similarity function

$$s_E : (P \times A)^2 \rightarrow \Re$$

defined by

$$s_E((p,a),(q,b)) = s^a((q,b))$$

for $(q,b) \in E$ and $p \notin H(E)$ (and arbitrarily otherwise), such that the functional

$$U_{p,M}(a) = \sum_{(q,b,r)\in M} s_E((p,a),(q,b))u(r)$$

represents $\geq_{p,M}$ for every p, M with $E(M) = E$ and $p \notin H(M)$.

We now turn to discuss additional properties of the similarity function one may be interested in. First, we observe that the theorem provides vectors $\{s^{ab}\}_{a,b\in A_p}$ such that, for every $x \in \Re^n$ and every $a,b \in A_p$, $a \geq_x b$ iff $s^{ab} \cdot x \geq 0$, where "$\cdot$" denotes vector product, i.e., $x \cdot y = \sum_{i=1}^{n} x_i y_i$. (To see this, set $s^{ab} \equiv s^a - s^b$.) Moreover, these vectors are unique up to multiplication by a positive scalar. Notice that the theorem does not provide any additional information, since the vectors $\{s^a\}_{a\in A_p}$ may be "shifted" by any vector $\beta \in \Re^n$. In particular, no particular importance should be attached to the sign of the similarity vectors. That is, when we consider an act a at a problem p, given E which includes (q,b), it is meaningless to ask whether a at p is similar to b at q to a positive or negative degree. It is only meaningful to ask whether a at p is similar to b at q more or less than a different act c at p is similar to b at q. That is, we are comparing differences of the form $s_E((p,a),(q,b)) - s_E((p,c),(q,b))$.

Similarly, we note that $\{s^{ab}\}_{a,b\in A_p}$ are unique *only* up to a positive scalar. Hence we will not be interested in the actual magnitude of the differences above, rather, only in the ratio between such differences.

Next, we remind ourselves that the similarity function s_E depends on the pairs of problem-act recalled, E. Thus, to obtain the formulation (•), we need to impose an additional condition (or "axiom"), which would imply that the similarity between pairs (p,a) and $(q,b) \in E$ is independent of E.

There are several ways in which one may formulate such a condition. In particular, one may use only the original data, namely the preference orders $\geq_{p,M} \subseteq A_p \times A_p$, and in this language express the fact that the similarity depends only on the pairs compared, rather than on the rest of memory. Alternatively, one may use the language of the similarity function provided by the representation theorem. As we argue in Gilboa and Schmeidler (1992), as long as the theoretical terms used are uniquely defined, there is no theoretical reason to necessarily prefer the more primitive observable data to the derived constructs. Specifically, should one wish to test a certain axiom whose formulation uses the similarity function, one may first measure this function and then check whether it satisfies the required condition. Drawing an analogy to physics, one may formulate an axiom involving the theoretical construct "mass," rather than formulate the axiom directly in terms of "observables." As long as it is known how mass is to be measured, it is perfectly legitimate to use it in further developments of the theory.

We here choose to formulate the memory-independence condition in terms of the derived similarity function. However, we can only use terms which are uniquely defined, namely the ratios of differences of the form $s_E((p,a),(q,b)) - s_E((p,c),(q,b))$. Using this language, we can state the following condition:

Memory-Independent Similarity: For every $p,q \in P$, every $a,b,c \in A_p$ and every E^1 and E^2 such that $(q,b) \in E^1, E^2$, and $p \notin H(E^1), H(E^2)$, the following hold:
(i) $s_{E^1}((p,a),(q,b)) - s_{E^1}((p,c),(q,b)) = 0$ iff $s_{E^2}((p,a),(q,b)) - s_{E^2}((p,c),(q,b)) = 0$;
(ii) in case these differences are not zero, then, for every $a',b',c' \in A_p$ and $q' \in P$ with $(q',b') \in E^1, E^2$, the following equality holds:

$$\frac{s_{E^1}((p,a'),(q',b')) - s_{E^1}((p,c'),(q',b'))}{s_{E^1}((p,a),(q,b)) - s_{E^1}((p,c),(q,b))} = \frac{s_{E^2}((p,a'),(q',b')) - s_{E^2}((p,c'),(q',b'))}{s_{E^2}((p,a),(q,b)) - s_{E^2}((p,c),(q,b))}.$$

We will only use this axiom if (i) (and (ii)) of the theorem holds, and if $|A_p| \geq 4$. Under these conditions, the functions s_{E^i} are "essentially unique." To be precise, the expressions on both sides of the equation above do not depend on the specific choice of s_{E^i}.

We note without proof the following result:

Corollary: Assume that $|A_p| \geq 4$, and that the relations

$$\left\{ \geq_{p,M} \mid p \in P \setminus H(M), \quad |M| \geq 3 \right\}$$

satisfy A1-A5 and the Memory-Independent Similarity condition. Then there to exists a single (memory-independent) similarity function

$$s:(P \times A)^2 \to \Re$$

such that

$$(\bullet) \qquad U(a) = U_{p,M}(a) = \sum\nolimits_{(q,b,r)\in M} s((p,a),(q,b))u(r)$$

represents $\geq_{p,M}$ for every p, M.

(The proof is very similar to that of the corresponding theorem – Theorem 2 – in Gilboa and Schmeidler (1992). There the interested reader can also find a "translation" of A6 to the language of observed preferences.[1])

3. Proof

We split the proof into three parts: the equivalence of (i) and (ii) and the uniqueness of the representation.

Part 1: (i) implies (ii)

Throughout this part of the proof, we assume that A1-A4 hold, since they are also necessary for the numerical representation (**). Axiom A5, by contrast, is implied by (**) only in conjunction with the additional condition of linear independence. Since there may be some interest in deriving (**) without linear independence, we will explicitly mention A5 whenever used. Thus, those parts of the proof in which A5 is not mentioned can also be generalized.

The strategy of the proof is as follows: we first show that for every pair of acts $a,b \in A_p$ there is a vector $s^{ab} \in \Re^n$ such that $a \geq_x b$ iff $s^{ab} \cdot x \geq 0$. We then would like to show that the vectors $\{s^{ab}\}_{a,b}$ can be written as differences $s^{ab} = s^a - s^b$ for some $\{s^a\}_a$. This latter part would be first proved by induction for a finite set of acts, and then extended to infinitely many acts.

1 The corresponding axiom in Gilboa and Schmeidler (1992) is named "A5."

We begin with a few auxiliary results. Our first lemma strengthens the additivity axiom to strict preferences:

Lemma 1: For every $x,y \in \Re^n$ and every $a,b \in A_p$, if $a >_x b$ and $a \geq_y b$, then $a >_{x+y} b$.

Proof: Assume, to the contrary, that $b \geq_{x+y} a$. By the reflection axiom, $a \geq_y b$ implies $b \geq_{-y} a$. Using additivity for $(x+y)+(-y)=x$, we obtain $b \geq_x a$, a contradiction. <>

Lemma 2: For every $x \in \Re^n$ and $\lambda > 0$, and every $a,b \in A_p$, if $a \geq_x b$ then $a \geq_{\lambda x} b$.

Proof: Assume that $a \geq_x b$ and let there be given $\lambda > 0$. First consider $\lambda = 1/n$ for some natural number $n > 1$. If $b >_{\lambda x} a$, successive application of Lemma 1 yields $b >_x a$, a contradiction. Hence $a \geq_{\lambda x} b$ holds. By (successive application of) the additivity axiom, we get $a \geq_{\lambda x} b$ for all rational $\lambda > 0$. Continuity completes the proof. <>

Lemma 3: For every $x \in \Re^n$ and $\lambda > 0$, and every $a,b \in A_p$, if $a >_x b$ then $a >_{\lambda x} b$.

Proof: Assume that $a >_x b$ and let there be given $\lambda > 0$. If $b \geq_{\lambda x} a$, then, using Lemma 2 with $\lambda' = 1/\lambda$, we obtain $b \geq_x a$. Hence $a >_{\lambda x} b$ is established. <>

It will be convenient to introduce the following notation: given $a,b \in A_p$, define

$$X^{ab} = \left\{ x \in \Re^n \mid a \geq_x b \right\}$$

and

$$\tilde{X}^{ab} = \left\{ x \in \Re^n \mid a >_x b \right\}.$$

Notice that for all $a,b \in A_p$, by the definition of $>_x$, $\tilde{X}^{ab} \subseteq X^{ab}$ and, using the order axiom as well, $\left(X^{ab}\right)^c = \tilde{X}^{ba}$. Observe also that by the reflection axiom, $X^{ab} = -X^{ba}$.

Lemma 4: For every $a,b \in A_p$, the sets X^{ab}, $\tilde{X}^{ab}$ are convex.

Proof: For X^{ab} convexity follows from the additivity axiom and Lemma 2. For $\tilde{X}^{ab}$ we use Lemmata 1 and 3. <>

Lemma 5: For every $a,b \in A_p$, the set X^{ab} is closed, and $\tilde{X}^{ab}$ is open (in the standard topology on $\mathfrak{R}^n$).

Proof: X^{ab} is closed by the continuity axiom. Since $\tilde{X}^{ab} = \left(X^{ba}\right)^c$, and the latter is closed, $\tilde{X}^{ab}$ is open. <>

Lemma 6: For every $a,b \in A_p$, there exists a vector $s^{ab} \in \mathfrak{R}^n$ such that

$$X^{ab} = \left\{ x \in \mathfrak{R}^n \,|\, s^{ab} \cdot x \geq 0 \right\}.$$

Furthermore, s^{ab} is unique up to a positive multiplicative scalar.

Proof: Let there be given $a,b \in A_p$. By the reflection axiom, $a \geq_0 b$, i.e., $0 \in X^{ab}$. Hence, using Lemmata 4 and 5, X^{ab} is a nonempty, closed and convex subset of $\mathfrak{R}^n$. By the same lemmata, $\left(X^{ab}\right)^c = \tilde{X}^{ba}$ is open and convex. A separating hyperplane argument guarantees the existence of a linear functional S: $\mathfrak{R}^n \to \mathfrak{R}$ and a number $c \in \mathfrak{R}^n$ such that

$$x \in X^{ab} \qquad \text{iff} \qquad S(x) \geq c \qquad\qquad \text{for all } x \in \mathfrak{R}^n.$$

It is easy to verify that $S \equiv 0$ iff $X^{ab} = \mathfrak{R}^n$. In this case the constant c may be any nonpositive number, and, in particular, zero. One may set $s^{ab} = 0$ and note that it is the unique vector satisfying $s^{ab} \cdot x \geq 0$ for all $x \in \mathfrak{R}^n$.

Suppose, then, that S is not identically zero. Since $0 \in X^{ab}$, we have $c \leq 0$. We now wish to show that $c = 0$. Indeed, assume that $c < 0$. In this case we may choose an $x \in \mathfrak{R}^n$ such that $S(x) = c$. Then $x \in X^{ab}$ but $2x \notin X^{ab}$, in contradiction to Lemma 2.

We have therefore established that $c = 0$. The vector s^{ab} defined by S satisfies the desired representation, i.e., $s^{ab} \cdot x \geq 0$ iff $x \in X^{ab}$. Finally, it is straightforward to verify that it is unique up to multiplication by a positive scalar. <>

The scaling of the vectors $\{s^{ab}\}_{ab}$ will play an important role in the sequel. It will therefore be useful to explicitly denote the set of all vectors which identify X^{ab}. For $a,b \in A_p$, denote

$$S^{ab} = \left\{ s^{ab} \in \Re^n \,|\, s^{ab} \cdot x \geq 0 \quad \textit{iff} \quad x \in X^{ab} \right\}.$$

Note that S^{ab} is simply $\left\{ \lambda s^{ab} | \lambda > 0 \right\}$ for any $s^{ab} \in S^{ab}$.

Also observe that $s^{ab} = 0$ for some $s^{ab} \in S^{ab}$ (equivalently, $S^{ab} = \{0\}$) iff $X^{ab} = \Re^n$, which is equivalent to $X^{ba} = \Re^n$ and to $S^{ba} = \{0\}$. That is, for any $a,b \in A_p$, $S^{ab} = \{0\}$ iff $a \approx_x b$ for all $x \in \Re^n$. Since this is an equivalence relation on A_p, we may restrict attention to its equivalence classes. Abusing notation, we will not distinguish between an equivalence class and a representative thereof. Alternatively, we will henceforth simply assume w.l.o.g. that for all $a \neq b$, $S^{ab} \neq \{0\}$.

At this point it may be helpful to review the strategy of the proof in more detail. We have already obtained a numerical representation of sorts: for every $a,b \in A_p$ we can numerically distinguish between vectors $x \in \Re^n$ such that $a \geq_x b$ to those for which $b >_x a$. However, what we wish to obtain is a numerical representation of $\geq_x$ over all of A_p for any given x. To this end, we would like to show that one can find $\{s^a\}_{a \in A}$ such that for every $a,b \in A_p$, $s^{ab} \equiv s^a - s^b \in S^{ab}$. If such vectors $\{s^a\}_{a \in A}$ are found, we can use Lemma 6 to obtain the representation (**). Indeed, in this case we conclude that, for every $x \in \Re^n$ and every $a,b \in A_p$,

$$a \geq_x b \qquad \Leftrightarrow \qquad x \in X^{ab} \qquad \Leftrightarrow \qquad s^{ab} \cdot x \geq 0 \qquad \Leftrightarrow \qquad s^a \cdot x \geq s^b \cdot x.$$

The vectors $\{s^a\}_{a \in A}$, of course, are somewhat arbitrary. Indeed, all that matters for our representation are their pairwise differences $\{s^a - s^b\}_{a,b \in A_p}$. (Hence the uniqueness of the resulting similarity function is only up to a shift by an arbitrary vector.) Correspondingly, we will focus on finding specific vectors $\hat{s}^{ab} \in S^{ab}$ for every $a,b \in A_p$, and then choose $\{s^a\}_{a \in A}$ such that $s^a - s^b = \hat{s}^{ab}$.

However, not every choice of $\hat{s}^{ab} \in S^{ab}$ would allow for such equalities to hold. For instance, it is obviously necessary that the vectors we select satisfy $\hat{s}^{ab} = -\hat{s}^{ba}$; similarly, for every $a,b,c,\in A_p$ we also have to make sure that our vectors satisfy $\hat{s}^{ab} + \hat{s}^{bc} + \hat{s}^{ca} = 0$. Hence, before we continue with the selection of $\{\hat{s}^{ab}\}$, it will be useful (or at least reassuring) to prove that the sets $\{S^{ab}\}$ have some properties corresponding to these necessary conditions. After we establish the following three lemmata, we will turn to define the vectors $\{\hat{s}^{ab}\}$.

Lemma 7: For every $a,b \in A_p$, and every $s^{ab} \in S^{ab}$, $-s^{ab} \in S^{ba}$.

Proof: Let there be given $s^{ab} \in S^{ab}$. By definition, for all $x \in \Re^n$, $a \geq_x b$ iff $s^{ab} \cdot x \geq 0$. Axiom A4 implies that $a \geq_x b$ iff $b \geq_{-x} a$. Thus, for all $x \in \Re^n$, $b \geq_{-x} a$ iff $\left(-s^{ab}\right)(-x) \geq 0$. Since $(-x)$ may be any vector in $\Re^n$, $-s^{ab} \in S^{ba}$ follows. <>

We observe that in the case $|A_p| = 2$, Lemma 7 concludes the proof of (ii): assume that $A_p = \{a,b\}$; choose any vector s^a and any vector $s^{ab} \in S^{ab}$, and define $s^b = s^a - s^{ab}$. Then for every $x \in \Re^n$,

$$a \geq_x b \quad \Leftrightarrow \quad s^{ab} \cdot x \geq 0 \quad \Leftrightarrow \quad s^a \cdot x \geq s^b \cdot x$$

and

$$b \geq_x a \quad \Leftrightarrow \quad -s^{ab} \cdot x \geq 0 \quad \Leftrightarrow \quad s^b \cdot x \geq s^a \cdot x \ .$$

Lemma 8: For every three distinct acts, $a,b,c \in A$, we may select vectors $s^{ab} \in S^{ab}$, $s^{bc} \in S^{bc}$ and $s^{ac} \in S^{ac}$, such that

$$s^{ab} + s^{bc} = s^{ac}.$$

Proof: Let us select some $s^{ab} \in S^{ab}$, $s^{bc} \in S^{bc}$ and $s^{ac} \in S^{ac}$, and define the following LP problem:

$$(P) \qquad Min_{x \in \Re^n} \ s^{ac} \cdot x$$
$$s.t. \qquad s^{ab} \cdot x \geq 0$$
$$\qquad s^{bc} \cdot x \geq 0 \ .$$

Consider also its dual problem,

$$
\begin{aligned}
(D) \quad & Max_{\alpha,\beta} \; 0 \\
& s.t. \quad \alpha s^{ab} + \beta s^{bc} = s^{ac} \\
& \qquad\quad \alpha,\beta \geq 0.
\end{aligned}
$$

Transitivity of $\geq_x$ implies that (P) is bounded: every x which is feasible for (P) satisfies $a \geq_x b$ and $b \geq_x c$. Hence it also satisfies $a \geq_x c$, which implies that $s^{ac} \cdot x \geq 0$. Since (P) is bounded, (D) is feasible. It follows that there are $\alpha,\beta \geq 0$ such that $\alpha s^{ab} + \beta s^{bc} = s^{ac}$. <>

We mention in passing that Lemma 8 concludes the proof of (ii) for the case $|A_p| = 3$. Moreover, it will also play a major role in the induction step for a general A_p. However, for the induction step we would also need another result, guaranteeing that the coefficients α and β above are unique in the cases of interest:

<u>Lemma 9:</u> Assume that $|A_p| \geq 4$ and that A5 holds. Then for every three distinct acts, $a,b,c \in A$, and any $s^{ac} \in S^{ac}$, there are unique vectors $s^{ab} \in S^{ab}$ and $s^{bc} \in S^{bc}$ such that

$$s^{ab} + s^{bc} = s^{ac}.$$

<u>Proof:</u> Assume there is more than one such pair s^{ab}, s^{bc}. Then any two vectors $s^{ab} \in S^{ab}$ and $s^{bc} \in S^{bc}$ are linearly dependent. That is, either $S^{ab} = S^{bc}$ or $S^{ab} = -S^{bc}$. If $S^{ab} = S^{bc}$, there is no $x \in \Re^n$ for which $a >_x c >_x b$. If, on the other hand, $S^{ab} = -S^{bc}$, no x satisfies $a >_x b >_x c$. In both cases, A5 is violated. <>

We are now equipped to deal with the general case. For simplicity, we break the induction step into two: the finite and the transfinite cases.

<u>Lemma 10:</u> Assume that axiom A5 holds. Let $A_0 \subseteq A_p$ be finite. Then, for every distinct $a,b \in A_0$, there exists $\hat{s}^{ab} \in S^{ab}$ such that:

(i) $\hat{s}^{ab} = -\hat{s}^{ba}$ for every distinct $a,b \in A_0$;

and

(ii) $\hat{s}^{ab} + \hat{s}^{bc} = \hat{s}^{ac}$ for every distinct $a,b,c \in A_0$.

Proof: We use induction on $k \equiv |A_0|$. For $k = 3$ the proof follows from Lemmata 7-8. Assume, then, that $k \geq 4$ and that the Lemma is true for subsets of A_0 of size $l < k$. Choose $a \in A_0$ and consider $A_1 \equiv A_0 \setminus \{a\}$. Apply the Lemma to obtain $\{\hat{s}^{bc}\}_{b,c \in A_1}$ such that $\hat{s}^{bc} \in S^{bc}$, and such that they satisfy (i) and (ii) on A_1.

For every $b \in A_1$, select a certain $s^{ab} \in S^{ab}$ and denote $s^{ba} \equiv -s^{ab}$. We wish to show that for every such b there exists a constant $\lambda_b > 0$ such that $\hat{s}^{ab} \equiv \lambda_b s^{ab}$ and $\hat{s}^{ba} \equiv -\lambda_b s^{ab}$, together with $\{\hat{s}^{bc}\}_{b,c \in A_1}$, satisfy (i) and (ii) over all A_0.

Consider any distinct $b,c \in A_1$. We argue that the following system (in λ_b and λ_c) has a unique solution:

$$\lambda_b s^{ba} + \lambda_c s^{ac} = \hat{s}^{bc} \qquad ; \qquad \lambda_b, \lambda_c > 0.$$

Indeed, lemma 9 guarantees that a unique solution exists. Yet the coefficient λ_b defined by it may depend on c, and vice versa – λ_c may depend on b. (Note, however, that these coefficients do not depend on the order of b and c, due to property (i) and to the choice of $\{s^{ab}, s^{ba}\}_{b \in A_1}$.) That is, we have found, for any $b,c \in A_1$, unique coefficients $\lambda_b(c) > 0$ and $\lambda_c(b) > 0$ such that

$$\lambda_b(c) s^{ba} + \lambda_c(b) s^{ac} = \hat{s}^{bc}$$

and obviously also

$$\lambda_c(b) s^{ca} + \lambda_b(c) s^{ab} = \hat{s}^{cb}.$$

Our next step is to show that the coefficients above depend only on their subscript. That is, we would like to show that, for every distinct $b,c,d \in A_1$, $\lambda_b(c) = \lambda_b(d)$. Let there be given such $b,c,d \in A_1$. Consider the following three equations:

$$\lambda_b(c) s^{ba} + \lambda_c(b) s^{ac} = \hat{s}^{bc}$$

$\lambda_c(d)s^{ca} + \lambda_d(c)s^{ad} = \hat{s}^{cd}$

$\lambda_d(b)s^{da} + \lambda_b(d)s^{ab} = \hat{s}^{db}$.

Summing them up, we obtain

$$[\lambda_b(c) - \lambda_b(d)]s^{ba} + [\lambda_c(d) - \lambda_c(b)]s^{ca} + [\lambda_d(b) - \lambda_d(c)]s^{da} =$$

$$\hat{s}^{bc} + \hat{s}^{cd} + \hat{s}^{db} = 0.$$

We claim that the vectors s^{ba}, s^{ca} and s^{da} are linearly independent. Indeed, assume they were not. Then s^{da} can be written as a nonnegative linear combination of $\{s^{ab}, s^{ba}, s^{ac}, s^{ca}\}$ (where at most one of the first two and one of the last two vectors have positive coefficients). In this case, $\geq_x$-preferences among $\{a,b,c\}$ dictate $\geq_x$-preference between $\{a,d\}$, and at least one of the 24 permutations of $\{a,b,c,d\}$ is ruled out for all $x \in \Re^n$, in contradiction to A5.

We therefore conclude that s^{ba}, s^{ca} and s^{da} are linearly independent. Hence their coefficients above have to vanish, and, in particular, $\lambda_b(c) = \lambda_b(d)$. Denote this value by λ_b and define $\hat{s}^{ab} = \lambda_b s^{ab}$, and $\hat{s}^{ba} = \lambda_b s^{ba} = -\lambda_b s^{ab}$ for every $b \in A_1$. Since $\lambda_b > 0$, $\hat{s}^{ab} \in S^{ab}$ and $\hat{s}^{ba} \in S^{ba}$. By definition, $\{\hat{s}^{ab}, \hat{s}^{ba}\}_{b \in A_1}$ satisfy property (i). Finally, property (ii) follows from our choice of $\lambda_b = \lambda_b(c)$. This concludes the proof of the Lemma. <>

We now turn to the general case:

<u>Lemma 11:</u> Assume that axiom A5 holds. Then, for every distinct $a,b \in A_p$, there exists $\hat{s}^{ab} \in S^{ab}$ such that:

(i) $\hat{s}^{ab} = -\hat{s}^{ba}$ for every distinct $a,b \in A_p$;

and

(ii) $\hat{s}^{ab} + \hat{s}^{bc} = \hat{s}^{ac}$ for every distinct $a,b,c \in A_p$.

<u>Proof:</u> The proof is by Zorn's lemma, repeating the argument of Lemma 10. Define a set

$$\Psi = \left\{ \left(A, \left\{ s^{ab} \right\}_{a,b \in A} \right) \middle| \begin{array}{l} A \subseteq A_p ; s^{ab} \in S^{ab} \, \forall a,b \in A \\ \left\{ s^{ab} \right\}_{a,b \in A} \; satisfy \; (i),(ii) \end{array} \right\}.$$

Next define a partial order on it, $\geq \subseteq \Psi \times \Psi$, as follows: given $\left(A_1, \left\{s_1^{ab}\right\}_{a,b \in A_1}\right), \left(A_2, \left\{s_2^{ab}\right\}_{a,b \in A_2}\right) \in \Psi$, $\left(A_1, \left\{s_1^{ab}\right\}_{a,b \in A_1}\right) \geq \left(A_2, \left\{s_2^{ab}\right\}_{a,b \in A_2}\right)$ if (a) $A_2 \subseteq A_1$; and (b) $s_2^{ab} = s_1^{ab}$ for all $a,b \in A_2$.

Given a $\geq$-chain $\left\{\left(A_\gamma, \left\{s_\gamma^{ab}\right\}_{a,b \in A_\gamma}\right)\right\}_{\gamma \in \Gamma}$ in Ψ, consider the ordered pair $\left(A_*, \left\{s_*^{ab}\right\}_{a,b \in A_*}\right)$ defined by

$$A_* = \bigcup_{\gamma \in \Gamma} A_\gamma$$

and

$$s_*^{ab} = s_\gamma^{ab} \qquad \text{for some } \gamma \in \Gamma \text{ such that } a,b \in A_\gamma.$$

To verify that $\left(A_*, \left\{s_*^{ab}\right\}_{a,b \in A_*}\right) \in \Psi$, note that $s^{ab} \in S^{ab}$ for every $a,b \in A_*$, and (i) and (ii) hold since every pair or triple of elements of A_* also belong to A_γ for some $\gamma \in \Gamma$. Hence every $\geq$-chain in Ψ has an upper bound in it.

Furthermore, an element $\left(A, \left\{s^{ab}\right\}_{a,b \in A}\right) \in \Psi$ is $\geq$-maximal only if $A = A_p$: if there exists $a \in A_p \setminus A$, one may define $\left\{s^{ab}\right\}_{b \in A}$ as in Lemma 10, and find a $\geq$-larger element in Ψ for the set of acts $A \cup \{a\}$. However, an element of Ψ corresponding to the set A_p defines the required $\left\{\hat{s}^{ab}\right\}_{a,b \in A_p}$. <>

We finally formulate the main result as follows:

<u>Lemma 12:</u> Assume that axiom A5 holds. Then there are vectors $\left\{s^a\right\}_{a \in A_p}$ such that, for every $a,b \in A_p$ and every $x \in \Re^n$,

$$a \geq_x b \qquad \text{iff} \qquad s^a \cdot x \geq s^b \cdot x.$$

Proof: Choose an act $a \in A_p$ and set s^a to be an arbitrary vector, say, zero. For any other act $b \in A_p$, define $s^b = \hat{s}^{ba}$ where $\{\hat{s}^{ba}\}$ are defined by Lemma 11. Consider any two acts $b, c \in A_p$ and note that

$$s^b \cdot x \geq s^c \cdot x \qquad \Leftrightarrow \qquad \hat{s}^{ba} \cdot x \geq \hat{s}^{ca} \cdot x \qquad \Leftrightarrow \qquad \left(\hat{s}^{ba} - \hat{s}^{ca}\right) \cdot x \geq 0.$$

In view of (i) and (ii) of Lemma 11, this is equivalent to $\hat{s}^{bc} \cdot x \geq 0$. Lemma 11 also states that $\hat{s}^{bc} \in S^{bc}$; using Lemma 6, this implies that $x \in X^{bc}$ and $b \geq_x c$. Conversely, $b \geq_x c$ implies $s^{bc} \cdot x \geq 0$ for all $s^{bc} \in S^{bc}$, and eventually also $s^b \cdot x \geq s^c \cdot x$. <>

To conclude the proof of (ii), we note that, by similar arguments to those employed in the proof of Lemma 10, the vectors $\{\hat{s}^{ab}, \hat{s}^{ac}, \hat{s}^{ad}\}$ defined by Lemma 11, are linearly independent for any distinct $a, b, c, d \in A_p$.

Part 2: (ii) implies (i)

It is straightforward to verify that if $\{\geq_x\}_{x \in \Re^n}$ are representable by $\{s^a\}_{a \in A_p}$ as in (**), they have to satisfy axioms A1-A4. We will therefore only prove that this representation – coupled with the linear independence condition – imply axiom A5.

Assume, then, that $\{\geq_x\}_{x \in \Re^n}$ and $\{s^a\}_{a \in A_p}$ are given, that for every distinct $a, b, c, d \in A_p$, the vectors $\left(s^a - s^b\right)$, $\left(s^a - s^c\right)$ and $\left(s^a - s^d\right)$ are linearly independent but that A5 fails to hold. Specifically, this means that there are four distinct acts $a, b, c, d \in A_p$ such that for no $x \in \Re^n$ is it the case that $a >_x b >_x c >_x d$. In other words, $\{s^a\}_{a \in A_p}$ are such that the following linear system has no solution:

$$\left(s^a - s^b\right) \cdot x = 1$$
$$\left(s^b - s^c\right) \cdot x = 1$$
$$\left(s^c - s^d\right) \cdot x = 1.$$

This implies that these vectors are linearly dependent. That is, there are coefficients α, β, and γ (not all of which are zero), such that

$$\alpha\left(s^a - s^b\right) + \beta\left(s^b - s^c\right) + \gamma\left(s^c - s^d\right) = 0.$$

Notice that

$$\beta\left(s^b - s^c\right) = -\beta\left(s^a - s^b\right) + \beta\left(s^a - s^c\right)$$

and

$$\gamma\left(s^c - s^d\right) = -\gamma\left(s^a - s^c\right) + \gamma\left(s^a - s^d\right).$$

Re-arranging the terms, we get

$$(\alpha - \beta)\left(s^a - s^b\right) + (\beta - \gamma)\left(s^a - s^c\right) + \gamma\left(s^a - s^d\right) = 0.$$

By linear independence, it must be the case that $\alpha - \beta = \beta - \gamma = \gamma = 0$, namely, that $\alpha = \beta = \gamma = 0$, a contradiction. Hence A5 cannot fail to hold if $\left(s^a - s^b\right)$, $\left(s^a - s^c\right)$ and $\left(s^a - s^d\right)$ are linearly independent.<>

Part 3: Uniqueness

Suppose that $|A_p| \geq 4$ and that $\left\{s^a\right\}_{a \in A_p}$ and $\left\{\hat{s}^a\right\}_{a \in A_p}$ both satisfy (**), and we wish to show that there are a scalar $\alpha > 0$ and a vector $\beta \in \Re^n$ such that for all $a \in A_p$, $\hat{s}^a = \alpha s^a + \beta$.

If $s^a = s^b$ for all $a,b \in A_p$, then we also get $\hat{s}^a = \hat{s}^b$ for all $a,b \in A_p$. In this case, setting $\alpha = 0$ and $\beta = \hat{s}^a - s^a$ for some (therefore, all) $a \in A_p$ will do. Assume, then, that for some $a,b \in A_p$, $s^a \neq s^b$, hence also $\hat{s}^a \neq \hat{s}^b$. Since, using the notations and the results in Part 1, $\left(s^a - s^b\right), \left(\hat{s}^a - \hat{s}^b\right) \in S^{ab}$ and $S^{ab} \neq \{0\}$, there exists a unique $\alpha > 0$ such that $\left(\hat{s}^a - \hat{s}^b\right) = \alpha\left(s^a - s^b\right)$. Define $\beta = \hat{s}^a - \alpha s^a$.

We now wish to show that for every $c \in A_p$, $\hat{s}^c = \alpha s^c + \beta$. Let there be given $c \in A_p$. If $s^c = s^a$, then also $\hat{s}^c = \hat{s}^a$. In this case, $\hat{s}^c = \hat{s}^a = \alpha s^a + \beta = \alpha s^c + \beta$. Similarly, if $s^c = s^b$, then also $\hat{s}^c = \hat{s}^b$ and $\hat{s}^c = \hat{s}^b = \alpha s^b + \beta = \alpha s^c + \beta$. We therefore assume that $s^c \neq s^a$ and $s^c \neq s^b$. In this case, there are unique $\gamma, \delta > 0$ such that

$$\left(\hat{s}^c - \hat{s}^a\right) = \gamma\left(s^c - s^a\right)$$

and

$$\left(\hat{s}^b - \hat{s}^c\right) = \delta\left(s^b - s^c\right).$$

Summing up these two with $\left(\hat{s}^a - \hat{s}^b\right) = \alpha\left(s^a - s^b\right)$, we get

$$\alpha\left(s^a - s^b\right) + \gamma\left(s^c - s^a\right) + \delta\left(s^b - s^c\right) = 0.$$

Setting $\left(s^b - s^c\right) = \left(s^b - s^a\right) + \left(s^a - s^c\right)$ and rearranging terms, we get

$$(\alpha - \delta)\left(s^a - s^b\right) + (\delta - \gamma)\left(s^a - s^c\right) = 0.$$

Since $\left(s^a - s^b\right)$ and $\left(s^a - s^c\right)$ are linearly independent (when there are at least four acts), we conclude that $\alpha = \gamma = \delta$. Substituting $\alpha = \gamma$ into $\left(\hat{s}^c - \hat{s}^a\right) = \gamma\left(s^c - s^a\right)$ we get $\hat{s}^c = \alpha\left(s^c - s^a\right) + \hat{s}^a = \alpha s^c + \beta$.<>

This completes the proof of the theorem.

4. Remarks

The Role of Axiom A5

As mentioned above, axiom A5 is not necessary for the numerical representation (**). Indeed, when $|A_p| \leq 3$, this representation was obtained without using it. While it is obviously necessary for the linear independence condition in (ii), one may wonder, whether axioms A1-A4 are sufficient for (**) to hold. We answer this question in the negative by two examples.

Example 1:

Let $A_p = \{a,b,c,d\}$ and $n = 3$. Using the notation developed in the proof, we define $\{\geq_x\}_{x \in \Re^3}$ by the sets $\{S^{ab}\}_{a,b \in A_p}$, representatives of which are:

$s^{ab} = (-1,1,0)$; $s^{ac} = (0,-1,1)$; $s^{ad} = (1,0,-1)$;
$s^{bc} = (2,-3,1)$; $s^{cd} = (1,2,-3)$; $s^{bd} = (3,-1,-2)$.

(As in Section 3, we set $s^{uv} = -s^{vu}$ for $u, v \in A_p$.)

We first verify that $\{\geq_x\}_{x \in \Re^3}$ satisfy A1-A4. Axioms A2-A4 hold whenever the relations are defined by some sets $\{S^{uv}\}_{u,v \in A_p}$. Furthermore, completeness is also satisfied whenever $S^{vu} = -S^{uv}$. We should therefore only prove that $\geq_x$ is transitive for all $x \in \Re^3$. However, in view of our proof, it suffices to find, for each triple $T = \{u, v, w\} \subseteq A_p$, coefficients $\lambda_T{}^{uv}, \lambda_T{}^{vw}, \lambda_T{}^{wu} > 0$ such that

$$\lambda_T{}^{uv} s^{uv} + \lambda_T{}^{vw} s^{vw} + \lambda_T{}^{wu} s^{wu} = 0.$$

We have four triples to consider. Denote $T_{-u} = A_p \setminus \{u\}$ for $u \in A_p$. Beginning with T_{-a}, we note that $s^{bc} + s^{cd} + s^{db} = 0$, hence $\lambda_{T_{-a}}{}^{bc} = \lambda_{T_{-a}}{}^{cd} = \lambda_{T_{-a}}{}^{db} = 1$ would do. Next consider T_{-b}. Setting $\lambda_{T_{-b}}{}^{ac} = 2$ and $\lambda_{T_{-b}}{}^{cd} = \lambda_{T_{-b}}{}^{da} = 1$ we obtain the desired equality. Symmetrically, we may choose $\lambda_{T_{-c}}{}^{da} = 2$ and $\lambda_{T_{-c}}{}^{ab} = \lambda_{T_{-c}}{}^{bd} = 1$ for T_{-c}, and for the last triple, T_{-d}, $\lambda_{T_{-c}}{}^{ab} = 2$ and $\lambda_{T_{-c}}{}^{bc} = \lambda_{T_{-c}}{}^{ca} = 1$.

Hence, $\{\geq_x\}_{x \in \Re^3}$ satisfy all of A1-A4. Yet we argue that they cannot be represented as in (**). Indeed, assume such a representation, with vectors $\{s_u\}_{u \in A_p}$ did exist. Define $\hat{s}^{uv} = s^u - s^v$. We know that for every $u, v \in A_p$ there exists a coefficient $\lambda^{uv} > 0$ such that $\hat{s}^{uv} = \lambda^{uv} s^{uv}$. W.l.o.g. we may assume $\lambda^{bc} = 1$. Since s^{bc} and s^{cd} are linearly independent, we also get $\lambda^{cd} = \lambda^{db} = 1$.

Considering the equation $\lambda^{ab} s^{ab} + \lambda^{ca} s^{ca} = s^{cb}$, we notice it has a unique solution with $\lambda^{ab} = 2$. (Where uniqueness follows from linear independence of s^{ab} and s^{ca}.) Hence we have $\hat{s}^{ab} = (-2, 2, 0)$. By a similar token, the equation $\lambda^{da} s^{da} + \lambda^{ab} s^{ab} = s^{db}$ also has a unique solution in which $\lambda^{ab} = 1$. Thus we also have $\hat{s}^{ab} = (-1, 1, 0)$, a contradiction.

Example 2:

Thus we conclude that in the absence of axiom A5, we cannot simply extend the representation given by $\{\hat{s}^{ab}\}_{a,b \in A}$ on a set A to a set $A \cup \{u\}$. Indeed, if we managed to perform this feat the reader would probably become very suspicious: the technique used in the finite and the transfinite induction steps were basically identical. Furthermore, we ended up with a numerical representation of orders on potentially very large sets. Thus, if the induction step may be performed without resorting to A5, one can get such numerical representations without any continuity or separability axiom, which simply does not seem correct.

Indeed, this intuition is reflected in the following example. Let $A_p = [0,1]^2$ and let $\geq_l$ be the lexicographic order on it. For any given $n \geq 1$, define $\{\geq_x\}_{x\in\Re^n}$ as follows:

$$\text{if } \sum_{i=1}^{n} x_i > 0, \quad a >_x b \qquad \text{iff} \qquad a >_l b;$$

$$\text{if } \sum_{i=1}^{n} x_i = 0, \quad a \approx_x b \qquad \text{for all } a,b \in A_p;$$

and

$$\text{if } \sum_{i=1}^{n} x_i < 0, \quad b >_x a \qquad \text{iff} \qquad a >_l b.$$

Thus, for every $x \in \Re^n$, $\geq_x$ is one of (i) $\geq_l$; (ii) $\geq_l^{-1}$; or (iii) the trivial relation (according to which any two acts are equivalent). Hence $\geq_x$ satisfies A1. It can also be verified that A2-A4 are satisfied by $\{\geq_x\}_{x\in\Re^n}$. However, one would not expect to obtain a representation as in (**), since it would imply a numerical representation of $\geq_l$ as well.

We therefore conclude that A5, which is also used in our proof for the finite case, implicitly bounds the cardinality of the set of acts A_p. Specifically, $|A_p| \leq \aleph$ since there cannot be more than a continuum of independent vectors in $\Re^n$.

Finally, we note that if the set of acts contains at least four elements, A5 also restricts the size of memory n: for $n < 3$ and $|A_p| \geq 4$ A5 cannot hold.

Separable Similarity

Our result derives a similarity function over problem-act pairs. While there are numerous examples in which such pairs are the "atoms" of similarity judgments, there are also examples in which one may assume that these judgments are separable; that is, that problem similarity and act similarity are separately assessed, and then aggregated to form a problem-act similarity function. For instance, one such aggregation may be multiplicative, namely

$$s((p,a),(q,b)) = s_P(p,q)s_A(a,b).$$

Given an essentially-unique problem-act similarity function, one may formulate additional axioms on it, which would allow such a decomposition. While we do not follow this track in this paper, we note that classical results on additive and multiplicative separability may be employed here as well.

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