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# SEQUENTIAL EQUILIBRIA AND CHEAP TALK IN INFINITE SIGNALING GAMES. PART 2: CHEAP TALK

by

Karl Iorio
and
Alejandro M. Manelli

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# 1 Introduction

Several recent examples (van Damme (1987), Harris (1990), Reny and Robson (1991), Seidmann (1992)) illustrate that well-behaved games with infinite action or type spaces may have no sequential equilibrium. Motivated by the nature of those examples, specially van Damme's, we continue the analysis initiated in Part 1, by considering the addition of cheap talk to infinite signaling games.

In a signaling game, player 1 first learns his private information and then sends a signal. Player 2 observes that signal and responds with an action; the game ends. In the cheap-talk extension of a signaling game, player 1 sends in addition to his signal a payoff irrelevant message (cheap talk) to player 2. A cheap-talk sequential equilibrium is any sequential equilibrium (SE) of the cheap-talk extension game. Infinite signaling games have been extensively applied in economics and finance.[1] There is also a developing literature on signaling games with cheap talk.[2]

It follows from the examples in Part 1 and in the next section of this paper that the non-existence of SE originates with the inability of player 1 to convey information efficiently to player 2. As a result, player 2 cannot coordinate her actions with the private information of player 1. Cheap talk simply and clearly solves the problem by allowing the sender to transmit the extra information. Thus, cheap-talk extension games always have SE. We prove this by showing that the correspondence that assigns to each game its set of cheap-talk SE outcomes is upper hemi-continuous.

Farrell and Gibbons (1989a) argue that cheap talk can be credible, is ubiquitous, and economists and game theorists should give it more attention. We found another reason for adding cheap talk to a signaling game: It solves the non-existence problem without fundamentally altering the nature of the game. We prove that when the signaling space is rich, i.e. it has sufficiently many signals, every cheap-talk SE outcome can be approximated by a sequential $\epsilon$-equilibrium outcome of the game without cheap talk.

We provide two caveats. The approximating SE (of the game without cheap talk) may be contrived; if the signaling space is not rich, for instance when it is finite, the approximation need not occur. In deciding whether to include cheap talk in a model, the analyst must consider the underlying economic situation.

[1] Applications of infinite signaling games (or variants) include Bhattacharya (1978), Leland and Pyle (1977), Milgrom and Roberts (1982), Myers and Majluff (1984), Riley (1979), and Spence (1974). Signaling games have also been used to analyze refinements of the sequential equilibrium concept; for instance, by Banks and Sobel (1987), Cho and Kreps (1987), and Cho and Sobel (1992).

[2] See, for example, Farrell and Gibbons (1989a, b), Matthews, Okuno-Fujiwara and Postlewaite (1989), Seidmann (1990), and Stein (1989).

S3' follows from Lemma 3.

Similarly, since $(\hat{\alpha}, \hat{\eta}, \hat{\beta}) \in SE(\tilde{\Gamma})$, S4 implies

$$\forall x \in \tilde{X}', \int_T U^2(t, \tilde{x}(x), \hat{\eta}(x))\, \beta(x)(dt) \geq \int_T U^2(t, \tilde{x}(x), \eta)\, \beta(x)(dt) \;\; \forall \eta \in M(Y).$$

Once again, S4' follows from Lemma 3. **QED**

**Proof of Theorem 4:** We apply Lemma 2. Let $S = X \times L$ and $v(x, l) = (x, \bar{\eta}(x, l))$. Then, $\tilde{\Gamma} = [(T, \rho), X \times L, Y, U^1, U^2]$. Lemma 2 implies that for any $\hat{\lambda} \in SEO(\Gamma(\Psi^*))$, there exists $(\alpha, \eta, \beta) \in SE(\tilde{\Gamma})$, such that

$\hat{\lambda} = (\hat{\alpha} \bullet \rho) \circ g^{-1}$ where $g(t, (x, l)) = (t, x, \eta((x, l)))$, and

$\tilde{\lambda} = (\hat{\alpha} \bullet \rho) \circ f_{\hat{\eta}}{}^{-1}$ where $fch(t, (x, l)) = (t, (x, l), \hat{\eta}((x, l)))$.

We complete the proof by showing that $\lambda = \tilde{\lambda}_{T \times X \times \Psi}$. Let $A \times B \times C \subseteq T \times X \times \Psi$ be a measurable rectangle. Then

$$\begin{aligned} \hat{\lambda}(A \times B \times C) &= (\hat{\alpha} \bullet \rho) \circ f_{\hat{\eta}}{}^{-1}(A \times B \times C) = (\hat{\alpha} \bullet \rho)(A \times ((B \times L) \cap \eta^{-1}(C))) \\ &= (\hat{\alpha} \bullet \rho) \circ f_{\hat{\eta}}{}^{-1}(A \times B \times L \times C) = \tilde{\lambda}_{T \times X \times \Psi}(A \times B \times C) \end{aligned}$$

**QED**

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The language used for cheap talk deserves consideration. At one extreme there is a language with a unique message. In this case, the cheap-talk game does not differ from the game without cheap talk, and may therefore have no SE. A richer language is required to solve the non-existence problem. We prove that every potential cheap-talk SE outcome can be obtained with any language that distinguishes among player 2's responses. Two examples are the unit interval and the set of all player 2's responses. With the latter, cheap talk may be interpreted as a message that player 1 submits to player 2 suggesting a particular response. This interpretation is justified: We prove that any cheap-talk SE outcome may be realized with strategies in which player 1, in addition to his signal, recommends a response and player 2 for his part follows the recommendations on the equilibrium path. With the former, a particular cheap talk message (an element of the unit interval) has no meaning in itself. It is simply used as a coordination device: cheap talk could also be the color of the dress worn by the player, the temperature set on the thermostat, or the length of the lunch break.

We conclude the introduction with a brief account of the literature. The non-existence of SE has shown common manifestations in different types of infinite games. Infinite games may be approximated by finite games obtained through increasingly finer discretizations of the type and action spaces. In equilibrium, players coordinate their actions in the finite games but are not able to do so in the limit infinite game. Some form of correlation is necessary to reestablish the lost coordination. This is the case in signaling games, and cheap talk is a natural way to achieve the necessary correlation.

Harris (1990) has an example of an infinite stage-game where no equilibrium exists: Players cannot attain the coordination that is possible in the finite version of the game. Harris also proves that a stage randomization device provides the necessary correlation to restore existence. Reny and Robson (1991) provide a minimal (with four players and two stages) non-existence example for the same class of games. It follows from the results in Simon and Zame (1990) that stage-correlated equilibria exist in two-stage games.

Börgers (1991) also anticipates with an example the coordination problem in stage games. In his example, a sequence of equilibrium outcomes of approximating games converges to a limit distribution that cannot be realized by strategies of the limit game. All through the sequence players achieve certain coordination that they loose at the limit. Our examples in Part 1 show the same pattern.

Still a similar situation arises in simultaneous move games of incomplete information. Milgrom and Weber (1985) prove the existence of Bayesian-Nash equilibria under a restriction on the distribution of player types. It is not known whether equilibria exist without that restriction. Simple examples (Milgrom and Weber 1985) show that some coordination possible with finite types may not be so with infinite types. The restriction on types avoids precisely these problems. Relaxing the restriction on each $f_j$ is continuous on the set $\bar{E}_j$ and the collection of these sets is disjoint by construction. Let $X' = \bigcup_j E_j$. By a similar argument, v is a bijection on $X'$.

Write $v(x) = (\tilde{x}(x), \bar{\eta}(x))$. To complete the proof of the Lemma, observe that $x \in X'$ implies $d_x(x, \tilde{x}(x)) < \epsilon$ because, by construction, both $x$ and $\tilde{x}(x)$ are contained in some ball $A_j$ with diameter less than $\delta < \epsilon$. By definition of $d_x$, the function $\tilde{x}(\cdot)$ satisfies (a). (a) and the definition of $\delta$ imply (b). **QED**

We proceed with the proof of the theorem. We will apply Lemma 2. To do so let $S = X'$, and $v$ as in Lemma 3. Thus, there exists $(\hat{\alpha}, \hat{\eta}, \hat{\beta}) \in SE(\tilde{\Gamma})$, supporting the outcome $\tilde{\lambda}$.

Define $g : T \times \tilde{X}' \longrightarrow T \times X \times \Psi$ by setting

$$g(t, x) = (t, v(x)) = (t, \tilde{x}(x), \hat{\eta}(x)).$$

We now show that playing the strategies $(\hat{\alpha}, \hat{\eta})$ results in an outcome $\hat{\lambda}$ with Prohorov distance $p(\tilde{\lambda}, \hat{\lambda}) < \epsilon$). By the definition of outcome,$\hat{\lambda} = (\alpha \bullet \rho) \circ f_{\hat{\eta}}{}^{-1}$.

Given an event $F \subseteq T \times X \times \Psi$, define the set

$$\begin{aligned} G(F) &= g(f_{\hat{\eta}}{}^{-1}(F) \cap (T \times X') \cap supp[\lambda] \\ &= \{g(t, x) \mid x \in X',\ f_{\hat{\eta}}(t, x) \in F,\ g(t, x) \in \text{supp}[\lambda]\}. \end{aligned}$$

Then by definition,

$$\hat{\lambda} = (\alpha \bullet \rho)(f_{\hat{\eta}}{}^{-1}(F)) = \lambda(G(F)).$$

Let $F_\epsilon$ be the set of elements of $T \times X \times \Psi$ within distance $\epsilon$ of F. To show $p(\hat{\lambda}, \lambda) < \epsilon$, it suffices to show that $\tilde{\lambda}(F) \leq \hat{\lambda}(F_\epsilon)$ and $\lambda(F) \leq \tilde{\lambda}(F_\epsilon)$, and to show this it suffices to prove that $G(F) \subseteq F_\epsilon$ and $(F \cap \text{supp}[\hat{\lambda}]) \subseteq G(F_\epsilon)$.

To show $G(F) \subseteq F_\epsilon$, let $g(t, x) \in G(F)$. We will show that $g(t, x) \in F_\epsilon$. By Lemma 3,

$$d(f(t, x), g(t, x)) = d((t, x, \eta(x)), (t, \tilde{x}(x), \eta(x))) < \epsilon.$$

Since $f(t, x) \in F$ by definition of $G(F)$, we have $g(t, x) \in F_\epsilon$ as we were to show.

To prove $(F \cap \text{supp}[\hat{\lambda}]) \subseteq G(F_\epsilon)$, let $g(t, x)$ be an arbitrary element of $F \cap supp[\lambda]$. (Since $\hat{\lambda} = (\hat{\alpha} \bullet \rho) \circ g^{-1} \in F_\epsilon$, and $g$ is continuous on $\text{supp}[\alpha \bullet \rho]$, we can represent any element of $F \cap \text{supp}[\lambda]$ this way (Lemma 3 in Part 1).) As before, $d(f(t, x), g(t, x)) < \epsilon$, so $f(t, x) \in F_\epsilon$, and $g(t, x) \in G(F_\epsilon)$ as desired. This proves $p(\tilde{\lambda}, \hat{\lambda}) < \epsilon$.

We now prove that $\hat{\alpha}$ and $\eta$ satisfy S3' and S4'. Since $(\hat{\alpha}, \hat{\eta}, \hat{\beta}) \in SE(\tilde{\Gamma})$, S3 implies

$$\forall t \in T,\ \int_X U^1(t, \tilde{x}(x), \eta(x))\, \hat{\alpha}(t)(dx) \geq U^1(t, \tilde{x}(x'), \eta(x')),\ \forall x' \in \dot{X}'.$$

types, Cotter (1991) proves the existence of correlated equilibria with public randomization devices.

There is a basic difference, however, between simultaneous move games and signaling games. In Cotter's games the coordination failure arises because of the distribution of types in infinite spaces. The failure is inherent to the structure of the uncertainty. The non-existence in signaling games occurs because of the infinite signaling space, independent of the type space. We proved in Part 1 that finite signaling spaces suffice to guarantee the existence of SE; the coordination problem arises because of decisions made by players.

Crawford and Sobel (1982) provide the original description of cheap talk in signaling games. They characterize the set of Bayesian-Nash equilibria for games in which signals do not affect payoffs.

## 2 Some Examples

**Example 1**, first discussed in Part 1, demonstrates the role of cheap talk in solving the non-existence problem.[3] Consider the signaling game $\Gamma = [(T, \rho), X, Y, U^1, U^2]$, where player 1's private information is $t \in T = \{-1, 1\}$, with probability $\rho(1) = \rho(-1) = 1/2$, his signal $x$ may take any value in $X = [-1, 1]$, and player 2 may respond to a given signal $x$ with any action $y$ in $Y = [-1, 1]$.[4] $U^1(t, x, y) = -x^2 + ty$ is player 1's payoff, and $U^2(t, x, y) = xy$ is player 2's.

Dan, player 1, is a wine lover living in an isolated Nordic town. During some months, he prefers to drink red wine ($t = 1$) and during others, white ($t = -1$). Dan may request either red or white wine from a wine store down south. To do so, he rents air time on a radio station. He may rent time in any amount $r \in [0, 1]$ at a cost $(-r^2)$. The vernacular of the area has two words, 1 and $-1$. Hence, Dan may send any signal $x = r \times 1$ or $x = r \times (-1)$.

Pat, player 2, is in charge of a wine store. She prefers to honor a request if she has received one, and she is indifferent between sending red, white or no wine if she receives no requests. She may only send one type of wine.

Suppose Dan wants to order red wine. He must decide how much air time to rent. Since sending a short request takes virtually no time, any strictly positive time $r$ would suffice to transmit the key word. There is no least expensive way of sending the request; for any $r$, Dan would do better by renting $r/2$. Thus, the only solution to Dan's problem is to rent no space $r = 0$, and arrange with Pat (in advance) that no request means red wine. An analogous argument shows that when Dan prefers white wine, he ought to submit no request and agree with Pat (again in advance.) that no request

[3] The basic idea behind the example is due to Eric van Damme (1987).

[4] To simplify notation we write $\rho(1)$ instead of $\rho(\{1\})$.

Lemma 1 implies conclusion (ii). Using (5) and the remark after Lemma 1, $\hat{\eta} = \eta$ on the supp$[\mu_S]$ and $(\hat{\alpha} \bullet \rho) = \mu$. Therefore, conclusion (i) follows from (i) of Lemma 1. **QED**

**Proof of Theorem 3:** We begin with a lemma providing the function that we will use to convert signals in $X \times \Psi^*$ (of $\Gamma(\Psi^*)$) to signals in X (of $\Gamma$). Let $d$ be the metric on $T \times X \times \Psi$ used to define the Prohorov distance $p$.

**Lemma 3** *If $X$ is rich, there exists a measurable set $X' \subset X$ with closure $\bar{X}'$, and a continuous function $v : \bar{X}' \longrightarrow X \times \Psi$ that is a bijection on $X'$. Fix $\epsilon > 0$ and denote $v(x) = (\tilde{x}(x), \tilde{\eta}(x))$. For $x \notin \bar{X}'$, define $\tilde{x}(x) = x$. Then, we may choose $v$ and $X'$ so that for all $(t, x, \eta) \in T \times X \times \Psi$,*

**(a)** $d((t, x, \eta), (t, \tilde{x}(x), \eta)) < \epsilon$, *and*

**(b)** $| U^i(t, x, \eta) - U^i(t, \tilde{x}(x), \eta) | < \frac{\epsilon}{2}$, *for* $i = 1, 2$.

**Proof:** Choose $\delta$ with $0 < \delta < \epsilon$ so that $\forall x, x' \in X$, $\forall t \in T$, $\forall \eta \in \Psi$, and $i = 1$ or $2$,

$$d((t, x, \eta), (t, \tilde{x}(x), \eta)) < \delta \quad \Rightarrow \quad | U^i(t, x, \eta) - U^i(t, x', \eta) | < \frac{\epsilon}{2}$$

Such $\delta$ exists because the functions $U^i$ are uniformly continuous on the compact set $T \times X \times \Psi$. Define a metric $d_x$ on X by setting

$$d_x(x, x') = \max_{t \in T,\ \eta \in \Psi} d((t, x, \eta), (t, x', \eta)).$$

Using the metric $d_x$ on $X$, let $(A_j)_{j=1}^N$ be a covering of $X$ by $N$ closed balls of positive diameter less than $\delta$. Let $B_1 = A_1$, and for $j = 2, \ldots, N$, define

$$B_j = A_j \setminus \bigcup_{i=1}^{j-1} .$$

Then $X = \bigcup_j B_j$ and each $B_j$ is a measurable set disjoint from the others. Let $M$ be the number of non-empty $B_j$ and reindex these sets so that $B_j$ is non-empty if $j \leq M$. For each $j = 1, \ldots, M$, let $C_j$ be a closed ball of positive diameter less than $\delta/2$ with $C_j \subset B_j$. Such $C_j$ exists because $B_j$ is the intersection of a closed ball and an open set. Since X is rich, there exists a closed set $D_j$, $D_j \subset C_j$ and a continuous surjection $f_j : D_j \longrightarrow \bar{B}_j \times \Psi$. By Theorem 1.4.2 of Parthasarathy (1967), there exists a measurable set $E_j \subset D_j$ such that $f_j$ restricted to $E_j$ is a bijection from $E_j$ onto $B_j \times \Psi$.

We can now define the sets $X'$ and the function $v$ given in the statement of the lemma. We let $\bar{X}' = \bigcup_j \bar{E}_j$ and define $v : \bar{X}' \longrightarrow X \times \Psi$ by setting $v = f_j$ on $\bar{E}_j$. The function $v$ is continuous since means white wine. This is not possible. When Pat receives no requests ($x = 0$), she could respond by sending either type of wine or by mixing (the actions, not the wine), but she cannot perfectly coordinate her actions with Dan's tastes. She ignores them. Whatever Pat decides to do on those occasions will make Dan change his strategy. No SE exists.

Both types of player 1 would like to differentiate themselves but the most inexpensive signal meaning red coincides with the most inexpensive signal meaning white. This prevents costless separation. Paradoxically, having an infinite number of signals implies that there are not enough useful signals (that is, free signals).

Consider a variant of $\Gamma$, in which Dan may also suggests a response to Pat (without renting space). Dan's suggestion (cheap talk,) is not binding and does not affect payoffs. We will call this variation $\Gamma(Y^*) = [(T, \rho), X, Y^*, U^1, U^2]$, the *cheap-talk extension* of $\Gamma$. $Y^*$, a copy of $Y$, is the space of all possible suggestions. The symbol $*$ differentiates a payoff irrelevant message $y^* \in Y^*$ from the response $y \in Y$.[5]

This variant has a SE: Dan sends no written request $x = 0$, but suggests to Pat what to do $y^* = t$ and she follows that suggestion on the equilibrium path. Off the equilibrium path ($x \neq 0$) Pat responds with $y = 1$ to any $x > 0$, and $y = -1$ to any $x < 0$. Since Pat's beliefs do not enter her payoffs, we may use any set of beliefs.

The inability of player 1 to efficiently convey his private information is the main obstacle to the existence of SE: Consider a sequence of increasingly finer discretizations of an infinite game, and a sequence of SE outcomes of those finite games converging to a limit distribution. The limit distribution may fail to be a SE outcome of the limit game for two reasons. First, it may not be feasible: it cannot be realized in the limit game by strategies of the limit game (Example 1 in Part 1). Second, it may be feasible, but not realizable with a continuous strategy of the second player (Examples 2 and 4 in Part 1).

Cheap talk solves both problems: Any limit distribution may be realized by strategies because, through cheap talk, coordination can be obtained in the limit game as well. The limit distribution may be realized with strategies in which player 1 suggests a response and player 2 follows it (Proposition 2). Thus, player 2's strategy will be trivially continuous.

Example 1 captures the essence of the non-existence problem: Cheap-talk extension games always have SE (Corollary 1). We obtain that result from Theorem 1, the upper hemi-continuity of the SE outcome correspondence for cheap-talk extension games.

[5] In the next section, we provide a general definition of cheap-talk extension games that accounts for mixed strategies.

**Proof:** By Parthasarathy Theorem 4.2, page 23, there exists a measurable set $S' \subseteq S$ such that $v : S' \longrightarrow X \times L$ is one to one and onto. We will apply Lemma 1.

Let $(\hat{\varphi}, \hat{\zeta}, \hat{\gamma})$ be a simple SE generating the outcome $\lambda$, and define $\tilde{x}(s) = \text{Proj}_X v(s)$, and $\eta(s) = \text{Proj}_L v(s)$. Both functions so defined are continuous. The function $v$, when restricted to $S'$, is one to one, and therefore has an inverse. We denote by $v_{S'}^{-1} : X \times \Psi^* \longrightarrow S'$ that inverse function. By the Kuratowsky Theorem (Parthasarathy 1967, Corollary I.3.3, page 22), $v_{S'}^{-1}$ is measurable. Define

$$\tilde{w}(x, \eta^*) = v_{S'}^{-1}(x, \eta^*).$$

To define $\mu$, consider any measurable rectangle $A \times B \subseteq T \times S$, and let

$$\mu(A \times B) = \lambda(A \times v(B \cap S')).$$

Since $v$ restricted to $S'$ is one to one, and since $v_{S'}^{-1}$ is measurable, $v(B \cap S')$ is a measurable set. Therefore $\mu$ is a well defined measure on $T \times S$.

We must verify (i)-(iv) of Lemma 1. Let $g(t, s) = (t, v(s)) = (t, \tilde{x}(s), \tilde{\eta}(s))$. $A \subseteq T$ and $D \subseteq X \times \Psi$ be measurable sets. Then, using the definitions,

$$\mu \circ g^{-1}(A \times D) = \mu(g^{-1}(A \times D)) = \mu(A \times v^{-1}(D)) = \lambda(A \times (v^{-1}(D) \cap S').$$

But $v(v^{-1}(D) \cap S') = D$ because $v$ is onto even when restricted to $S'$. Therefore,

$$\mu \circ g^{-1}(A \times D) = \lambda(A \times D),$$

which proves (i).

(ii) is immediate.

By definition of $\tilde{w}$ and $v$, $\tilde{x}(\tilde{w}(x, l)) = \tilde{x}(v^{-1}(x)) = x$, which shows (iii).

Let $h(t, (x, \eta^*)) = (t, \tilde{w}(x, \eta^*))$, and let $A \times B \subseteq T \times S$. Then,

$$(\hat{\varphi} \bullet \rho) \circ h^{-1}(A \times B) = (\varphi \bullet \rho)(A \times \tilde{w}^{-1}(B)).$$

Because $\tilde{w}(X \times \Psi^*) = S'$, and using the definition of $\tilde{w}$,

$$\tilde{w}^{-1}(B) = \tilde{w}^{-1}(B \cap S') = v(B \cap S').$$

Then,

$$(\hat{\varphi} \bullet \rho) \circ h^{-1}(A \times B) = (\varphi \bullet \rho)(A \times v(B \cap S')) = \lambda(A \times v(B \cap S')) = \mu(A \times B),$$

where the second equality follows by 2, and the last one follows by definition of $\mu$. This proves (iv).

Adding cheap talk to a signaling game solves the non-existence problem but raises the question of whether the inclusion of cheap talk fundamentally alters the nature of a game. In $\Gamma$, for instance, separation cannot not be achieved without a cost, but it may be achieved at a very small loss: For a small $\epsilon$, type $t = 1$ signals $\epsilon > x > 0$, and type $t = -1$ signals $-\epsilon < x < 0$. This observation generalizes: When the signaling space X is rich, i.e. it has many signals, every cheap-talk SE outcome can be approximated by a sequential $\epsilon$-equilibrium outcome of the game without cheap talk (Theorem 3).[6]

For finite signaling games, the set of SE outcomes coincides with the set of limits of sequences of sequential $\epsilon$-equilibrium outcomes as $\epsilon \longrightarrow 0$. The example shows this is not true for infinite signaling games. Instead, the set of limits of sequential $\epsilon$-equilibrium outcomes includes the cheap-talk SE outcomes when $X$ is rich.

Games with rich signaling spaces remain fundamentally unchanged by the addition of cheap talk, at least if sequential $\epsilon$-equilibria are considered. We argue with two examples that this conclusion must be taken with care.

**Example 2** shows that, when the signaling space $X$ is not rich, a cheap-talk SE outcome need not be approximated by an $\epsilon$-equilibrium outcome of the game without cheap talk.

Let $\Gamma_2$ differ from $\Gamma$, only in that $X = \{-1, -1/2, 0, 1/2, 1\}$. The cheap-talk extension of this game still has a SE where player 1 sends the signal $x = 0$ with the suggestion $y^* = t$, and player 2 follows the suggestions on the equilibrium path. The corresponding SE outcome cannot be approximated without cheap talk: There are no signals $x$ of arbitrarily low cost that can replace the cheap talk messages; the most inexpensive positive signal is $x = 1/2$ and the most inexpensive negative signal is $x = -1/2$, which costs $1/4$ to player 1.

**Example 3** shows that even when cheap talk can be emulated using the costly signal, doing so may be contrived and unintuitive.

Let $\Gamma_3$ differ from $\Gamma$ only in that $U^2(t, x, y) = -txy$. The cheap-talk extension of this game still has a SE where player 1 sends the signal $x = 0$ with the suggestion $y^* = t$, and player 2 follows that suggestion on the equilibrium path.

Fix $0 < \epsilon < 1$. One sequential $\epsilon$-equilibrium in $\Gamma_3$ has player 1 signaling $x = t\epsilon$ and player 2 responding

$$y(x) = \begin{cases} -1 & \text{if } x \leq -\epsilon \\ 0 & \text{if } -\epsilon < x < \epsilon \\ 1 & \text{if } x \geq \epsilon. \end{cases}$$

[6] The definition of a rich signaling space is provided in Section 5

Alternatively, suppose that $s' \notin \text{supp}[\mu_S]$. It follows from Proposition 1 in Part 1 and (9) that

$$\begin{aligned} U^1(t,\tilde{x}(s),\tilde{\eta}(s)) &= \int_{X\times L} U^1(t,x,\dot{\zeta}(x,l))\,\dot{\varphi}(t)(dx\times dl) \\ &\geq \int_{X\times L} U^1(t,x',\zeta(x',l'))\,\dot{\varphi}(t)(dx\times dl), \forall (x',l') \in X\times L \\ &= U^1(t,\tilde{x}(s'),\zeta(\tilde{x}(s'),\bar{l})) = U^1(t,\tilde{x}(s'),\tilde{\eta}(s')) \end{aligned}$$

where the inequality follows by S3 for $(\varphi,\dot{\zeta},\dot{\gamma})$: the third line follows by defining $x' = \tilde{x}(s')$ and $l' = \bar{l}$, and by definition of $\tilde{\eta}(s')$ when $s' \notin \text{supp}[\mu_S]$. **QED**

**Proof of Proposition 2:** We apply Lemma 1. Let $S = X \times \Psi^*$, $g(t,(x,\eta^*)) = (t,x,\eta^*)$, $\tilde{x}(x,\eta^*) = x$, $\tilde{\eta}(x,\eta^*) = \eta^*$, $\tilde{w}(x,l) = (x,\zeta(x,l))$, and $\mu = \lambda$. With these definitions, $\tilde{\lambda} = \mu \circ f_{\tilde{\eta}}{}^{-1} = \lambda$, since $f_{\tilde{\eta}}{}^{-1}$ is the identity.

(i) follows by definition of $\mu$ and because $g$ is the identity. (ii) follows by definition. (iii) follows because $\tilde{x}(\tilde{w}(x,l)) = \tilde{x}(x,\zeta(x,l)) = x$. To verify (iv), note that, by definition of outcome, $\lambda = (\hat{\varphi} \bullet \rho) \circ h^{-1}$, where $h(t,(x,l)) = (t,x,\zeta(x,l))$. This proves (iv). Lemma 1 then implies that $\tilde{\lambda} = \lambda$ is a SE outcome of $\Gamma(\Psi^*)$. It follows from (5) that $\lambda$ is supported as a simple SE. **QED**

**Proof of Theorem 2:** We apply Lemma 1. Let $S = X$, $\tilde{x}(x) = x$, $\tilde{w}(x,\eta^*) = x$, and $\mu = \lambda_{T\times X}$.

(i), (ii) and (iii) are immediate. To show (iv) holds, let $(\varphi,\zeta,\gamma)$ be a simple SE supporting $\lambda$ in $\Gamma(\Psi^*)$. Then, $(\hat{\varphi} \bullet \rho) = \hat{\lambda}$ by (2). This implies,

$$(\hat{\varphi} \bullet \rho) \circ h^{-1} = \lambda \circ h^{-1},$$

but $h(t,x,\eta^*) = (t,\tilde{w}(x,\eta^*)) = (t,x)$. Thus, $(\hat{\varphi} \bullet \rho) \circ h^{-1} = \lambda_{T\times X} = \mu$. This proves (iv).

With these definitions, $\tilde{\lambda} = \mu \circ f_{\tilde{\eta}}{}^{-1} = \lambda$, by (i). Lemma 1 implies that $\lambda$ is a SE outcome of $\Gamma$. **QED**

The following Lemma will be used to prove Theorems 3 and 4.

**Lemma 2** *Consider a continuous game $\Gamma = [(T,\rho),X,Y,U^1,U^2]$, and its cheap-talk extension $\Gamma(\Psi^*) = [(T,\rho),X,\Psi^*,Y,U^1,U^2]$. Let $S$ be any compact metric space such that there exists a continuous surjective function $v : S \longrightarrow X\times\Psi^*$. Define the game $\tilde{\Gamma} = [(T,\rho),S,Y,\tilde{U}^1,\tilde{U}^2]$, where $\tilde{U}^i(t,x,y) = U^i(t,\ Proj_X v(s),y)$, for $i = 1,\ 2$. Let $\lambda \in SEO(\Gamma(L))$. Then, $\exists(\alpha,\eta,\beta) \in SE(\tilde{\Gamma})$, such that*

**(i)** $\hat{\lambda} = (\hat{\alpha} \bullet \rho) \circ g^{-1}$ *where $g(t,s) = (t,\ Proj_X v(s),\eta(s))$, and*

**(ii)** $\tilde{\lambda} = (\hat{\alpha} \bullet \rho) \circ f_{\hat{\eta}}{}^{-1}$.

To select this response in a sequential $\epsilon$-equilibrium, player 2 must believe that there is equal probability of $t = 1$ or $t = -1$ whenever $||x|| \neq \epsilon$. The cheap-talk equilibrium is simpler and more natural. The analyst must decide whether cheap talk is essential to the economic situation to be modeled.

Adding cheap talk to a game raises several other issues. The use of the set $Y^*$ as the cheap-talk language is justified by two properties (Proposition 2):[7] First, any SE outcome obtained with an arbitrary cheap talk space may also be realized using the set of all player 2's responses. No other SE outcome is possible. Second, any cheap-talk SE outcome can be supported by equilibrium strategies in which player 1 suggests a response $y^*$ to player two, who, in turn, follows that suggestion on the equilibrium path. These properties underlie the interpretation of a cheap-talk message as a recommendation with a precise meaning.

It is not necessary, however, to use a language with an intrinsic meaning. As long as the language distinguishes between player 2's responses, all potential SE outcomes can be realized. This seems to require a language of considerable size. Theorem 4 demonstrates that it is possible to replace the cheap-talk space $Y^*$ with any rich space, for instance the unit interval, and obtain the same results.

Finally, communication may not be possible without the consent of both parties, and player 2 may find it in her best interest to avoid any cheap talk: $\Gamma_2$ has several SE outcomes depending on how the response of player 2 to $x = 0$ is defined. In any of these outcomes, player 2's equilibrium payoff is at least 1/4. In the cheap-talk extension of $\Gamma_2$, the outcome described in Example 2 gives player 2 a payoff of zero. Were this outcome a possibility, Player 2 would clearly prefer to avoid the cheap talk.

# 3 The Cheap-Talk Game

In this section, we define the game and its cheap-talk extension. We also repeat, for completeness and without discussion, some definitions already present in Part 1.

A signaling game is summarized by $\Gamma = [(T, \rho), X, Y, U^1, U^2]$. In this game, player 1 first privately observes his type $t$ from the set $T$ and then sends a signal $x$ from the set $X$. Player 2 observes this signal, infers player 1's probable types, and then responds with an action $y$ from the set $Y$. The game ends and each player $i$ receives payoff $U^i(t, x, y)$. Player 2 has prior beliefs $\rho$ about the possible types $t$ of player 1; $\rho$ is a Borel probability distribution on $T$ that is common knowledge. We denote the support of $\rho$ by supp$[\rho]$ and assume for convenience that supp$[\rho] = T$.

We alter $\Gamma$ to include some payoff irrelevant communication between the players: After observing

[7] In the general case, we use M(Y), the set of all probability distributions on $Y$, to include the mixed responses of player 2.

Since $\tilde{\lambda}_{T \times S} = \mu$, $\bar{\eta} = \eta$ on the support of $\mu_S$, and using the definition of $\tilde{U}^2$,

$$\begin{aligned}
\textstyle\int_{T \times S} \tilde{U}^2(t, s, \hat{\eta}(s))\, \tilde{\lambda}_{T \times S}(dt \times ds) &= \textstyle\int_{T \times S} U^2(t, \tilde{x}(s), \eta(s))\, \mu(dt \times ds) \\
&= \textstyle\int_{T \times X \times \Psi} U^2(t, x, \eta)\, \lambda(dt \times dx \times d\eta) \\
&= \textstyle\int_{T \times X \times L} U^2(t, x, \zeta(x, l))\, (\hat{\varphi} \bullet \rho)(dt \times dx \times dl) \\
&\geq \textstyle\int_{T \times X \times L} U^2(t, x, \zeta'(x, l))\, (\hat{\varphi} \bullet \rho)(dt \times dx \times dl), \quad (7)
\end{aligned}$$

$\forall \zeta'$ measurable. (i) implies the second equality, the fact that $(\hat{\varphi}, \hat{\zeta})$ generates $\lambda$ implies the third one, and S4 for $(\hat{\varphi}, \hat{\zeta}, \hat{\gamma})$ implies the inequality.

Given any continuous $\eta'$, define $\zeta' : X \times L \longrightarrow \Psi$ by

$$\zeta'(x, l) = \eta'(\tilde{w}(x, l)),$$

which is measurable. Then, (iii) implies that

$$\begin{aligned}
&\textstyle\int_{T \times X \times L} U^2(t, x, \zeta'(x, l))(\hat{\varphi} \bullet \rho)(dt \times dx \times d\eta) \\
&\quad= \textstyle\int_{T \times X \times L} U^2(t, \tilde{x}(\tilde{w}(x, l)), \eta'(\tilde{w}(x, l)))\, (\hat{\varphi} \bullet \rho)(dt \times dx \times dl) \\
&\quad= \textstyle\int_{T \times S} U^2(t, \tilde{x}(s), \eta'(s))\, \mu(dt \times ds) \\
&\quad= \textstyle\int_{T \times S} \tilde{U}^2(t, s, \eta''(s))\, \tilde{\lambda}_{T \times S}(dt \times ds) \quad (8)
\end{aligned}$$

where (iv) implies the second equality; the definition of $\tilde{U}^2$ and $\tilde{\lambda}_{T \times S} = \mu$ the last one.

The expressions (7) and (8) imply (6). Then, Proposition 2 in Part 1 asserts that there exist beliefs $\hat{\beta}$ such that $(\hat{\alpha}, \hat{\eta}, \hat{\beta})$ satisfies S2 and S4.

It remains to prove S3, which using the definition of $\tilde{U}^1$ is

$$\forall t \in T,\ \forall s \in \text{ supp}[\hat{\alpha}(t)],\ \int_S U^1(t, \tilde{x}(s), \hat{\eta}(s))\, \alpha(t)(ds) \ \geq\ U^1(t, \tilde{x}(s'), \eta(s')),\ \forall s' \in S.$$

Pick any $t \in T$ and $s \in$ supp$[\hat{\alpha}(t)]$. Since $\alpha$ satisfies (4), $(t, s) \in$ supp$[\mu]$. Then, the continuity of $\tilde{x}$ and $\hat{\eta}$ and Lemma 3 in Part 1 imply,

$$(t, \tilde{x}(s), \eta(s)) \in \text{ supp}[\lambda]. \quad (9)$$

Consider any $s' \in S$ and suppose first that $s' \in$ supp$[\mu_S]$. By Lemma 1 in Part 1, there exists $t'$ such that $(t', s') \in$ supp$[\mu_S]$. Once more, the continuity of $\tilde{x}$ and $\eta$, and Lemma 3 in Part 1, imply that $(t', \tilde{x}(s'), \hat{\eta}(s')) \in$ supp$[\hat{\lambda}]$. Then (ii) and (iii) of Proposition 1 in Part 1 imply

$$U^1(t, \tilde{x}(s), \eta(s)) \geq U^1(t, \tilde{x}(s'), \eta(s')).$$

his type, player 1 sends, in addition to a signal x, a message $l$ from the set $L$ of all such messages. Player 2 observes both $x$ and $l$ before responding with an action $y$. The (cheap talk) message $l$ does not affect the payoffs of either player who receives $U^i(t, x, y)$ as in $\Gamma$. We summarize this signaling game with cheap talk by $\Gamma(L) = [(T, \rho), X, L, Y, U^1, U^2]$.

We shall be concerned with continuous games, such as

**Definition 1** $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ *is continuous if* $T$, $X$, *and* $Y$ *are compact metric spaces and* $U^1$ *and* $U^2$ *are continuous. If in addition (the cheap-talk set)* $L$ *is a compact metric space, then* $\Gamma(L) = [(T, \rho), X, L, Y, U^1, U^2]$ *is continuous.*

Fix a continuous game $\Gamma$ for the remainder of the section.

A strategy for a player in $\Gamma(L)$ is a measurable function mapping her information into distributions over her available actions. $\varphi$ represents a generic strategy for player 1, and by $\zeta$ one for player 2. Strategy spaces are therefore,[8]

$$\begin{aligned} \Sigma^1(\Gamma(L)) &= \{\varphi : T \longrightarrow M(X) \times L \mid \varphi \text{ is measurable}\} \\ \Sigma^2(\Gamma(L)) &= \{\zeta : X \times L \longrightarrow M(Y) \mid \zeta \text{ is measurable}\}. \end{aligned}$$

It will be convenient to differentiate strategies in $\Gamma(L)$ from strategies in $\Gamma$: $\alpha$ represents player 1's strategies in $\Gamma$, and $\eta$ player 2's.

After observing a pair $(x, l)$ player 2 forms beliefs about the types of player 1 that may have sent that signal. The definition of sequential equilibrium will require that the beliefs of player 2 be consistent with the strategy of player 1: Let $\zeta \bullet \rho$ denote the joint distribution on $T \times X \times L$ induced by the behavior strategy $\zeta$ when combined with the distribution $\rho$ on $T$. It is defined for all event rectangles $D \times E \subset T \times X \times L$ as

$$(\varphi \bullet \rho)(D \times E) = \int_D \varphi(t)(E)\, \rho(dt).$$

Player 2's beliefs are a function $\gamma : X \times L \longrightarrow M(T)$: $\gamma(x, l)$ is a probability distribution on player 1's types $T$ given that player 1 has sent the signal $x$ and the cheap-talk message $l$. Consistency requires that $\gamma$ be a conditional probability distribution of $t$ given $x$ derived from the joint distribution $\varphi \bullet \rho$.

The set of consistent beliefs-strategy pairs $(\gamma, \varphi)$ is:[9]

$$\Sigma^3 = \{(\gamma, \varphi) \mid \varphi \in \Sigma^1, \;\; \gamma \text{ is measurable and } \gamma \bullet \mu_X = \mu, \;\; \text{where } \mu = \varphi \bullet \rho\}$$

[8] For any metric space $A$, let $M(A)$ denote the set of Borel probability distributions on $A$. Define an *event* of $A$ to be a Borel-measurable subset.

[9] Given two metric spaces $A$ and $B$ and $\mu \in M(A \times B)$, $\mu_A$ and $\mu_B$ denote the marginal distributions on $A$ and $B$.

**(iv)** $\mu = (\hat{\varphi} \bullet \rho) \circ h^{-1}$ *where* $h(t,(x,l)) = (t, \tilde{u}(x,l))$.

*Then,* $\tilde{\lambda} \in SEO(\tilde{\Gamma})$.

**Remark:** $\tilde{\lambda}$ is supported by $(\hat{\alpha}, \eta, \beta)$, where $\alpha$ is a regular conditional distribution of $s$ given $t$ derived from $\mu$, and $\hat{\eta}(s) = \bar{\eta}(s)\ \forall s \in \text{supp}[(\hat{\alpha} \bullet \rho)_S]$

**Proof:** We must find a $(\hat{\alpha}, \eta, \beta) \in SE(\tilde{\Gamma})$ generating the distribution $\tilde{\lambda}$. We first define the strategies $\hat{\alpha}$ and $\hat{\eta}$, and prove that they support the outcome $\tilde{\lambda}$. Then, we apply Proposition 2 in Part 1 to find the beliefs $\hat{\beta}$, and to prove S2 and S4. Finally, using Proposition 1 in Part 1 we verify that S3 holds.

Define $\hat{\alpha}$ to be a version of the regular conditional distribution of $x$ given $t$ derived from $\mu$, that is $\mu = \hat{\alpha} \bullet \mu_S$, with the property that

$$t \in T \text{ and } s \in \text{supp}[\hat{\alpha}(t)] \Rightarrow (t,s) \in \text{supp}[\mu]. \tag{4}$$

Lemma 8 in Part 1 shows that such an $\alpha$ exists. Since $\alpha$ is measurable, $\alpha \in \Sigma^1(\tilde{\Gamma})$. We will use (4) to prove S3 later on.

Pick any $\bar{l} \in L$, and define

$$\eta(s) = \begin{cases} \bar{\eta}(s) & \forall s \in \text{supp}[\mu_S] \\ \zeta(\tilde{x}(s), \bar{l}) & \text{otherwise.} \end{cases} \tag{5}$$

Thus, $\hat{\eta} \in \Sigma^2(\tilde{\Gamma})$ and S1 is satisfied.

We now show that $(\hat{\alpha}, \eta)$ generate the outcome $\tilde{\lambda}$:

$$\tilde{\lambda} = \mu \circ f_{\bar{\eta}}^{-1} = (\hat{\alpha} \bullet \rho) \circ f_{\bar{\eta}}^{-1} = (\alpha \bullet \rho) \circ f_{\eta}^{-1}$$

The first equality follows by definition of $\tilde{\lambda}$. To prove the second one, note that by definition of outcome, $\hat{\lambda} = (\hat{\varphi} \bullet \rho) \circ g_{\dot{\zeta}}^{-1}$. Then, $\lambda_T = \rho$. By (i), $\mu_S = \lambda_T$ and therefore $\mu_S = \rho$. This implies that $(\hat{\alpha} \bullet \rho) = \mu$. Finally, the last equality holds because $\eta = \bar{\eta}$ $[\mu_S]$-almost everywhere.

In three steps, we define a function $\beta^0$ and verify the hypotheses of Proposition 2 in Part 1. First, let $\beta^0$ be any measurable function with

$$\beta^0(s) = \gamma(\tilde{x}(s), l)\ \ \forall s \notin \text{supp}[\mu_S].$$

Second, by S4 of $(\dot{\varphi}, \dot{\zeta}, \dot{\gamma})$, $(\dot{\zeta}(\tilde{x}(s), \bar{l}), \gamma(\tilde{x}(s), l)) \in MBR(\tilde{x}(s), Y, T, U^2)$. Since $\tilde{\lambda}_S = \mu_S$, the definition of $\hat{\eta}$ and $\beta^0$ implies that $\forall s \notin \text{supp}[\tilde{\lambda}_S]$, $(\eta(s), \beta^0(s)) \in MBR(s, Y, T, \tilde{U}^2)$.

Third, we show that $\forall \eta'(\cdot)$ continuous

$$\int_{T \times S} \tilde{U^2}(t, s, \hat{\eta}(s))\, \tilde{\lambda}_{T \times S}(dt \times ds) \ \geq \int_{T \times S} \tilde{U}^2(t, s, \eta'(s))\, \tilde{\lambda}_{T \times S}(dt \times ds). \tag{6}$$

On occasions, it will be more convenient to extend the players' payoff functions $U^i$ from $Y$ to $M(Y)$ by taking expected values: for each $(t, x, \eta) \in T \times X \times M(Y)$, we let

$$U^i(t, x, \eta) = \int_Y U^i(t, x, y)\, \eta(dy). \tag{1}$$

$U^i$ is a continuous function on $T \times X \times Y$ if and only if the extension of $U^i$ to $T \times X \times M(Y)$ is continuous (Lemma 4 in Part 1 proves the non-trivial half of this statement).

**Definition 2** *A* Sequential Equilibrium (SE) *for* $\Gamma(L) = [(T, \rho), X, L, Y, U^1, U^2]$ *is a triple* $(\hat{\varphi}, \hat{\zeta}, \hat{\gamma})$ *satisfying*

**(S1)** $\hat{\varphi} \in \Sigma^1(\Gamma(L))$, $\hat{\zeta} \in \Sigma^2(\Gamma(L))$;

**(S2)** $(\hat{\gamma}, \hat{\varphi}) \in \Sigma^3(\Gamma(L))$;

**(S3)** $\forall t \in T,\ \int_X U^1(t, x, \hat{\zeta}(x, l))\, \hat{\varphi}(t)(dx \times dl) \ \geq\ U^1(t, x', \hat{\zeta}((x', l')))\ \ \forall\, (x', l') \in X \times L$;

**(S4)** $\forall (x, l) \in X \times L,\ \int_T U^2(t, x, \hat{\zeta}(x, l))\, \hat{\gamma}(x, l)(dt) \ \geq\ \int_T U^2(t, x, \eta)\, \hat{\gamma}(x, l)(dt)\ \ \forall\, \eta \in M(Y)$.

*SE($\Gamma(L)$) is the set of SE of* $\Gamma(L)$.

We are concerned with the outcome of signaling games. Any strategy pair in $\Gamma$ generates a distribution on $T \times X \times Y$ and a distribution on $T \times X \times M(Y)$. We call them, respectively, the ***standard outcome*** and the ***outcome*** of playing those strategies. Any strategy pair in a cheap-talk game $\Gamma(L)$ generates a distribution on $T \times X \times M(Y)$, the ***outcome*** of playing those strategies. The outcomes of $\Gamma(L)$ are defined on the payoff-relevant spaces. This makes the comparison of outcomes between $\Gamma(L)$ and $\Gamma$ possible.

**Definition 3** *Let* $\Gamma = [(T, \rho), X, Y, U^1, U^2]$. *The distribution* $\nu$ *on* $T \times X \times Y$ *is a standard outcome of* $\Gamma$ *if there is a strategy pair* $(\hat{\alpha}, \hat{\eta})$ *in* $\Gamma$ *such that* $\nu = \eta \bullet (\alpha \bullet \rho)$. *The distribution* $\lambda$ *on* $T \times X \times M(Y)$ *is an outcome of* $\Gamma$ *if there is a strategy pair* $(\hat{\alpha}, \hat{\eta})$ *in* $\Gamma$ *such that* $\lambda = (\alpha \bullet \rho) \circ f_{\eta}^{-1}$, *where* $f_\eta(t, x) = (t, x, \hat{\eta}(x))$.

*Let* $\Gamma(L) = [(T, \rho), X, L, Y, U^1, U^2]$. *The distribution* $\lambda$ *on* $T \times X \times M(Y)$ *is an outcome of* $\Gamma(L)$ *if there is a strategy pair* $(\hat{\varphi}, \hat{\zeta})$ *in* $\Gamma(L)$ *such that* $\lambda = (\varphi \bullet \rho) \circ g_{\zeta}^{-1}$, *where* $g_\zeta(t, x, l) = (t, x, \zeta((x, l)))$. *We say that the strategy pair* $(\hat{\varphi}, \hat{\zeta})$ *supports (or implements or generates) the outcome* $\lambda$. *SEO($\Gamma(L)$) is the set of outcomes generated by the set of SE of* $\Gamma(L)$.

The standard outcome is the more commonly used notion. In the proofs, specially in Part 1, we need our more refined notion of outcome. *All our results, however, could be stated and interpreted without any changes in terms of standard outcomes.* This assertion is justified by Proposition 1 below.

Example 1, there exists a different SE supporting the same cheap talk outcome in $\Gamma(Y^*)$: Dan sends no request ($x = 0$) and asks for red wine when he wants white and vice versa ($y^* = -t$), and Pat responds by contradicting the cheap-talk suggestions ($\hat{y}(0, y^*) = -y^*$). In this equilibrium a message "I want red ($y^* = 1$)" actually means "I want white": it is the equilibrium that gives its meaning to the cheap-talk messages.

This issue is of particular importance in economic applications where direct communication may be taken as a sign of collusion. It has been argued recently that US airlines communicate their pricing intentions to their rivals by entering letter codes before a particular fare, or by setting fares in advance on their computerized reservation system.

Proposition 2 proved that any cheap-talk SE outcome may be obtained by using player 2's responses as the set of available cheap-talk messages. Any language that has enough symbols (or strings of symbols) to distinguish among player 2's responses would suffice. This is the content of the next theorem.

**Theorem 4** *Consider a continuous game $\Gamma = [(T,\rho), X, Y, U^1, U^2]$ and its cheap-talk extension $\Gamma(\Psi^*) = [(T,\rho), X, \Psi^*, Y, U^1, U^2]$. Let $L$ be any metric space such that there exists a continuous surjective function $\bar{\eta} : L \longrightarrow \Psi^*$. Then, $SEO(\Gamma(\Psi^*)) = SEO(\Gamma(L))$, where $\Gamma(L) = [(T,\rho), X, L, Y, U^1, U^2]$.*

It follows immediately that any rich cheap-talk space $L$, for instance the unit interval, would suffice to obtain all potential cheap-talk SE outcomes.

# 7 Proofs

The proofs of our results are similar in that we use Proposition 2 of Part 1 to obtain the SE beliefs. To avoid repetitions, we first prove an abstract lemma from which our results follow as particular cases.

**Lemma 1** *Let $\Gamma(L) = [(T,\rho), X, L, Y, U^1, U^2]$ be continuous and let $(\varphi, \zeta, \gamma) \in SE(\Gamma(L))$ generate the outcome $\hat{\lambda}$. Suppose there exists a compact metric space $S$, $\mu \in M(T \times S)$, and measurable functions $\tilde{x} : S \longrightarrow X$, $\bar{\eta} : S \longrightarrow \Psi$ and $\tilde{w} : X \times L \longrightarrow S$. Let $\tilde{\Gamma} = [(T,\rho), S, Y, \tilde{U}^1, \tilde{U}^2]$ where $\tilde{U}^i(t,s,y) = U^i(t, \tilde{x}(s), y)$, and define $\tilde{\lambda} = \mu \circ f_{\bar{\eta}}{}^{-1}$, where $f_{\bar{\eta}}(t,s) = (t, s, \bar{\eta}(s))$. Suppose*

**(i)** *$\hat{\lambda} = \mu \circ g^{-1}$, where $g(t,s) = (t, \tilde{x}(s), \eta(s))$.*

**(ii)** *$\tilde{x}(\cdot)$ and $\bar{\eta}(\cdot)$ are continuous on the $supp[\mu_S]$.*

**(iii)** *$\tilde{x}(\tilde{w}(x,l)) = x$, and*

The remainder of this section discusses, for completeness, the relationship between outcome and standard outcome. It is not essential to the results or proofs in this article.

Not all distributions on $T \times X \times M(Y)$ are outcomes, or equivalently, not all distributions can be realized by strategies of the game. (Similarly, not all distributions are standard outcomes.) Although several distributions on $T \times X \times M(Y)$ may generate the same distribution on $T \times X \times Y$; this cannot happen with outcomes. Any given pair of strategies generates a unique outcome and standard outcome. We shall see that there is a continuous, injective mapping between outcomes and standard outcomes. The following simple result is not employed in the paper. We provide it for informative purposes.

**Proposition 1** *Let $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ be continuous, and $\hat{\lambda}$ be an outcome of $\Gamma$, implemented by $\hat{\alpha} \in \Sigma^1(\Gamma)$, and $\hat{\eta} \in \Sigma^2(\Gamma)$. The function $g(\lambda) = \hat{\eta} \bullet (\hat{\alpha} \bullet \rho)$ from the set of outcomes to the set of standard outcomes is an homeomorphism.*

**Proof:** All strategy pairs implementing $\hat{\lambda}$ are equal almost everywhere. Thus, g is a well defined function.

We first show that g is injective. Let $\lambda'$ be another outcome of $\Gamma$, realized by the pair $(\alpha', \eta')$, and suppose that $g(\hat{\lambda}) = g(\lambda')$. Then, their marginal distributions are equal as well:

$$(\hat{\alpha} \bullet \rho) = g(\hat{\lambda})_{T \times X} = g(\lambda')_{T \times X} = (\alpha' \bullet \rho).$$

It follows that $\hat{\eta} = \eta'$ almost everywhere. Therefore, $\hat{\lambda} = (\hat{\alpha} \bullet \rho) \circ f_{\hat{\eta}}^{-1} = (\alpha' \bullet \rho) \circ f_{\eta'}^{-1} = \lambda'$.

We now show that g and its inverse are continuous. Let $(\alpha^n, \eta^n)$ be a pair of strategies generating the standard outcome $\nu^n$, for $n = 1, 2, \ldots$. We must prove that $\langle \nu^n \rangle \Rightarrow \nu \Leftrightarrow \langle g^{-1}(\nu^n) \rangle \Rightarrow g^{-1}(\nu)$. Let $U(t, x, y)$ be any continuous function on $T \times X \times Y$. Then

$$\begin{aligned} &\int_{T \times X \times Y} U(t, x, y)\, (\eta^n \bullet (\alpha^n \bullet \rho))((dt \times dx \times dy)) \\ &= \int_{T \times X} U(t, x, \eta^n(x))(\alpha^n \bullet \rho)((dt \times dx)) \\ &= \int_{T \times X \times \Psi} U(t, x, \eta)((\alpha^n \bullet \rho) \circ f_{\eta^n}^{-1})(dt \times dx \times d\eta), \end{aligned}$$

where the first equality is obtained integrating over $y$; the second one follows because $(\alpha^n, \eta^n)$ generate the outcome $(\alpha^n \bullet \rho) \circ f_{\eta^n}^{-1}$. Therefore,

$$\int_{T \times X \times Y} U(t, x, y)\nu^n((dt \times dx \times dy)) = \int_{T \times X \times \Psi} U(t, x, \eta) g^{-1}(\nu^n)((dt \times dx \times d\eta)).$$

Since $U(t, x, y)$ is continuous if and only if its extension to $T \times X \times M(Y)$ is continuous (equation (1)), the characterization of weak convergence with continuous functions implies the desired result. **QED**

For instance, any finite space $X$ is not rich. The unit interval is an example of a rich signaling space (Parthasarathy 1967, Theorem I.4.1), but the signaling space $X = [0, 1] \cup \{2\}$ is not rich, because there are not enough signals near the element $x = 2$.

The idea behind Theorem 3 is simple: Let $\lambda$ be a SE outcome of a cheap-talk extension game $\Gamma(\Psi^*)$. By Proposition 2, it can be supported as a simple SE: player 1 suggest a response to player 2 who follows that suggestion on the equilibrium path. When the signaling space X has many signals, it is possible to associate to any pair $(x, \eta^*)$ a unique element $x'$ close to x and, vice versa. Hence, given a SE of $\Gamma(\Psi^*)$ it is possible to construct a sequential $\epsilon$-equilibrium for $\Gamma$ with the same outcome by letting player 1 signal $x'$ instead of $(x, \eta^*)$ and letting player 2 respond to $x'$ with $\eta^*$.

Adding cheap talk to a game with a rich signaling space does not alter it significantly, at least when sequential $\epsilon$-equilibria are considered. We emphasize that the approximating SE may be contrived and un-natural (Example 3), and that the approximation need not hold when the signaling space is not rich (Example 2).

A sequential $\epsilon$-equilibrium differs from a SE in that players may choose $\epsilon$ best responses:

**Definition 8** *A triple $(\tilde{\alpha}, \tilde{\eta}, \tilde{\beta})$ is a sequential $\epsilon$-equilibrium if it satisfies*

**(S1')** $\tilde{\alpha} \in \Sigma^1(\Gamma),\ \tilde{\eta} \in \Sigma^2(\Gamma)$;

**(S2')** $(\tilde{\beta}, \tilde{\alpha}) \in \Sigma^3(\Gamma)$;

**(S3')** $\forall t \in T,\ \int_X U^1(t, x, \tilde{\eta}(x))\, \tilde{\alpha}(t)(dx) \ \geq\ U^1(t, x', \tilde{\eta}(x')) - \epsilon \ \ \forall\, x' \in X$;

**(S4')** $\forall x \in X,\ \int_T U^2(t, x, \tilde{\eta}(x))\, \tilde{\beta}(x)(dt) \ \geq\ \int_T U^2(t, x, \eta)\, \tilde{\beta}(x)(dt) - \epsilon \ \ \forall\, \eta \in M(Y)$.

**Theorem 3** *Let $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ be a continuous game with a rich signaling space $X$. Let $\Gamma(\Psi^*) = [(T, \rho), X, \Psi^*, Y, U^1, U^2]$ be its cheap-talk extension. Given $\lambda \in SEO(\Gamma(\Psi^*))$ and $\epsilon > 0$, there exists $\tilde{\lambda}$ with Prohorov distance $p(\lambda, \tilde{\lambda}) < \epsilon$ such that $\tilde{\lambda}$ is a sequential $\epsilon$-equilibrium of $\Gamma$.* [11]

## 6 The Cheap-Talk Space

Adding cheap talk to a game raises the question of what constitutes cheap talk. We have defined cheap talk as any payoff irrelevant message. This includes simple communication using the vernacular, as well as through any symbol, even without intrinsic meaning; for instance, player 2 observing the color of player 1's tie. The nature of the equilibrium assigns meaning to the different symbols or colors: In

[11] For a definition of the Prohorov distance see Billingsley (1968).

## 4 Existence of SE in Cheap-Talk Games

Consider a continuous signaling game $\Gamma = [(T,\rho), X, Y, U^1, U^2]$ with no SE, for instance the game in Example 1. As previewed in that example, the non-existence problem may be solved by introducing cheap talk in $\Gamma$. Different cheap-talk spaces, however, produce substantial differences in the set of SE outcomes: When only one cheap-talk message is available ( $L$ is a singleton), $\Gamma(L) = [(T,\rho), X, L, Y, U^1, U^2] = \Gamma$, and the non-existence problem prevails. As the language $L$ becomes richer, the possibilities for communication increase and so does the set of SE outcomes. There is a limit to this process. Proposition 2 shows that all potential cheap-talk SE outcomes may be realized by using the set of all player 2's available responses as the language of cheap talk. We formally identify this cheap-talk space.

**Definition 4** *To conserve notation we use $\Psi$ or $\Psi^*$ instead of $M(Y)$, the set of probability distributions on $Y$. We term $\Gamma(\Psi^*) = [(T,\rho), X, \Psi^*, Y, U^1, U^2]$ the* cheap-talk extension of $\Gamma$.

In this game, the cheap-talk space $\Psi^*$ coincides with the set of all possible mixed responses of player 2. An element $\eta \in \Psi$ denotes a generic response by player 2. The symbol $*$ signifies that the variable carrying it does not alter payoffs; it distinguishes between the message $\eta^*$ sent by player 1 and the response $\eta$ of player two. Any cheap-talk message $\eta^*$ in $\Psi^*$ can be naturally interpreted as a suggested response to player 2. This interpretation turns out to be fruitful. Proposition 2 also demonstrates that any SE outcome of $\Gamma(\Psi^*)$can be supported by equilibrium strategies in which player 1 sends a signal and suggests a response to player 2, who follows the suggestions on the equilibrium path. We call this strategies a *simple* SE. Formally,

**Definition 5** *Let $\Gamma(\Psi^*) = [(T,\rho), X, \Psi^*, Y, U^1, U^2]$, and let $(\varphi, \zeta, \gamma)$ be a SE of $\Gamma(\Psi^*)$ with outcome $\hat{\lambda}$. The triple $(\hat{\varphi}, \hat{\zeta}, \hat{\gamma})$ is a* simple *sequential equilibrium if*

**(i)** $\hat{\zeta}(x, \eta^*) = \eta,\ (\eta^* = \eta) \quad \forall (x, \eta^*) \in\ supp[\lambda_{X \times \Psi}]$.

**(ii)** $\hat{\zeta}(x, \eta^*) = \hat{\zeta}(x, \eta^{*\prime}), \quad \forall (x, \eta^*),\ (x, \eta^{*\prime}) \notin\ supp[\lambda_{X \times \Psi}]$.

Condition (i) states that player 2 follows player 1's suggestions on the equilibrium path; (ii) states that player 2's responses do not depend on the suggestions received off the equilibrium path.

We note a fact that we use repeatedly later on: The distribution on $T \times X \times \Psi^*$ generated by player 1's strategy $\hat{\varphi}$ is the SE outcome $\lambda$. By definition of outcome,

$$\hat{\lambda} \quad = \quad (\hat{\varphi} \bullet \rho) \circ g_{\hat{\zeta}}^{-1}, \text{ where } g_{\hat{\zeta}}(t, x, \eta^*) = (t, x, \hat{\zeta}(x, \eta^*))$$

Intuitively, a SE outcome of a cheap talk game can be realized as a simple SE. By hypothesis, it can also be realized by strategies of the game without cheap talk. The strategies in both cases must coincide in general. The proof, which does not follow this reasoning, is provided in Section 7.

Consider now a sequence of games converging to a limit game, and a corresponding sequence of SE outcomes converging to a limit distribution. The limit distribution may not be a SE outcome of the limit game. It follows from Theorem 1 and (3) that it will be a SE outcome of the cheap-talk extension of the limit game:

**Corollary 2** *Let* $\Gamma^n = [(T^n, \rho^n), X^n, Y^n, U^{1n}, U^{2n}]$, $n = 1, 2, \ldots,$ *and* $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ *be continuous games. Let* $\Gamma(\Psi^*) = [(T, \rho), X, \Psi^*, Y, U^1, U^2]$. *Suppose* $\langle \Gamma^n \rangle \longrightarrow \Gamma$, $\hat{\lambda}^n \in SEO(\Gamma^n)$ *and* $\langle \hat{\lambda}^n \rangle \longrightarrow \hat{\lambda}$. *Then,* $\hat{\lambda} \in SEO(\Gamma(\Psi^*))$.

**Remark:** It follows from Corollary 2, and Propositions 4 and 5 in Part 1, that for *strongly monotonic* signaling games, which were defined in Part 1, $SEO(\Gamma) = SEO(\Gamma(\Psi^*))$.

Combining Theorem 1 and Theorem 2 we obtain

**Corollary 3** *Consider the continuous games* $\Gamma^n(\Psi^{n*}) = [(T^n, \rho^n), X^n, \Psi^{n*}, Y^n, U^{1n}, U^{2n}]$, $n = 1, 2, \ldots,$ $\Gamma(\Psi^*) = [(T, \rho), X, \Psi^*, Y, U^1, U^2]$, $\Gamma = [(T, \rho), X, Y, U^1, U^2]$, *and let* $\hat{\lambda}^n \in SEO(\Gamma^n(\Psi^{n*}))$. *Suppose* $\langle \Gamma^n(\Psi^{n*}) \rangle \longrightarrow \Gamma(\Psi^*)$, $\langle \hat{\lambda}^n \rangle \Rightarrow \lambda$, *and that there exists a continuous* $\eta^1 : supp[\lambda_X] \longrightarrow \Psi$ *such that* $\hat{\lambda} = \hat{\lambda}_{T \times X} \circ {f_{\eta^1}}^{-1}$. *Then* $\hat{\lambda} \in SEO(\Gamma)$.

## 5 Does Cheap Talk Significantly Change a Game?

Example 1 illustrates that the addition of cheap talk may increase the set of SE outcomes, and therefore, one is tempted to answer this section's title in the affirmative. When sequential $\epsilon$-equilibria are considered, the answer is not so clear. Theorem 3 states that if the signaling space is sufficiently rich, all cheap-talk SE outcomes can be approximated by a sequential $\epsilon$-equilibrium outcome of the game without cheap talk.

A signaling space is rich if there are sufficiently many signals in any section of the signaling space $X$. Formally,

**Definition 7** *A signaling space* $X$ *is* rich *if for all compact metric spaces* $Z$ *and for all closed balls* $B \subset X$, *there exists a closed set* $A \subset B$ *and a continuous mapping from* $A$ *onto* $Z$.

$$= (\varphi \bullet \rho) \circ g^{-1}, \text{ where } g(t, x, \eta^*) = (t, x, \eta^*)$$
$$= \varphi \bullet \rho \tag{2}$$

where the second line follows by (i), and the third one because $\Psi^*$ and $\Psi$ are two copies of the same space.

**Proposition 2** *Let $\Gamma(L) = [(T, \rho), X, L, Y, U^1, U^2]$ be continuous. Let $(\hat{\varphi}, \hat{\zeta}, \hat{\gamma})$ be a SE of $\Gamma(L)$ with outcome $\hat{\lambda}$. Then, there exists a simple SE for $\Gamma(\Psi^*) = [(T, \rho), X, \Psi^*, Y, U^1, U^2]$ with outcome $\hat{\lambda}$.*

The intuition behind Proposition 2 is as follows. Let $(x, l_1)$ and $(x, l_2)$ be two possible messages on the equilibrium path in $\Gamma(L)$. Player 2's equilibrium strategy prescribes responses $\hat{\zeta}(x, l_1) = \eta^1$ and $\hat{\zeta}(x, l_2) = \eta^2$ respectively. In $\Gamma(\Psi^*)$, player 1 could send a signal and a suggestion, $(x, \eta^{1*})$ instead of $(x, l_1)$, and $(x, \eta^{2*})$ instead of $(x, l_2)$. (Recall that $\eta^{i*} = \eta^i$, $i = 1, 2$; the symbol $*$ only distinguishes a payoff irrelevant suggestion from a response.) Player 2 would respond by following the suggestions.

It is incorrect to assume that player 2 would follow any suggestion: Suppose, for instance, that player 1 suggests to player 2 a response that is strictly dominated (from player 2's point of view). Player 2, of course, will not follow that suggestion. No SE will have that response on the equilibrium path. Thus, player 2's response need not depend on player 1's suggestions off the equilibrium path.

In the proof, which we provide in Section 7, care must be exercised to account for the equilibrium beliefs.

Before proving Theorem 1, the upper hemi-continuity of the SE outcome correspondence for cheap-talk extension games, we must define a notion of convergence of games. We use weak convergence for probability distributions, convergence with respect to the Hausdorff distance for action and type spaces, and continuous convergence for payoff functions. The definition that we now reproduce has been discussed in detail in Part 1.

**Definition 6** *A sequence of continuous games, $\Gamma^n(\Psi^{n*}) = [(T^n, \rho^n), X^n, \Psi^{n*}, Y^n, U^{1n}, U^{2n}]$, converges to a continuous game $\Gamma(\Psi^*) = [(T, \rho), X, \Psi^*, Y, U^1, U^2]$, $\langle \Gamma^n(\Psi^{n*}) \rangle \longrightarrow \Gamma(\Psi^*)$, if*

**(H1)** $\langle X^n \rangle \longrightarrow X$, $\langle Y^n \rangle \longrightarrow Y$, $\langle T^n \rangle \longrightarrow T$, $\langle \rho^n \rangle \Rightarrow \rho$;

**(H2)** $\langle U^{in} \rangle \longrightarrow U^i$ *continuously for $(t, x, y) \in T \times X \times Y$, $i = 1, 2$.*

**Theorem 1** *(Upper hemi-continuity) Consider a sequence of continuous games $\Gamma^n(\Psi^{n*})$, $n = 1, 2, \ldots$ and a continuous game $\Gamma(\Psi^*)$. Suppose $\langle \Gamma^n(\Psi^{n*}) \rangle \longrightarrow \Gamma(\Psi^*)$, $\hat{\lambda}^n \in SEO(\Gamma^n(\Psi^{n*}))$ and $\langle \hat{\lambda}^n \rangle \Rightarrow \hat{\lambda}$. Then, $\hat{\lambda} \in SEO(\Gamma(\Psi^*))$.*

**Proof:** We will apply Theorem 1 of Part 1. Consider alternative but equivalent representations of $\Gamma^n(\Psi^{n*})$, and $\Gamma(\Psi^*)$. Let $\Gamma'^n = [(T^n, \rho^n), X'^n, Y^n, U^{1n}, U^{2n}]$ and $\Gamma' = [(T, \rho), X', Y, U^1, U^2]$ where $X' = X \times \Psi^*$, $X'^n = X \times \Psi^{n*}$. It is immediate that $\langle \Gamma^n(\Psi^{n*}) \rangle \longrightarrow \Gamma(\Psi^*)$. We must verify the hypotheses H3 and H4 of Theorem 1 of Part 1.

By Proposition 2 there is a simple SE $(\hat{\varphi}, \hat{\zeta}, \gamma)$ generating the outcome $\hat{\lambda}^n$ in $\Gamma^n(\Psi^{n*})$. The triple $(\hat{\varphi}, \hat{\zeta}, \hat{\gamma})$ is also a SE of $\Gamma'^n$, and generates an outcome $\lambda'^n$ in that game.

$$
\begin{aligned}
\hat{\lambda}'^n &= (\hat{\varphi}^n \bullet \rho^n) \circ g_{\hat{\zeta}}^{-1} \\
&= (\hat{\varphi}^n \bullet \rho^n) \circ g^{-1}, \text{ where } g(t, (x, \eta^*)) = (t, (x, \eta^*), \eta), \ (\eta^* = \eta) \\
&= \hat{\lambda}^n \circ g^{-1},
\end{aligned}
$$

where the first line follows by definition of outcome, and the last two by the definition of simple SE (ii), and by (2).

Taking limits in the last equation (Hildenbrand 1974, page 51), $\lambda' = \lambda \circ g^{-1}$. This proves H3, and letting $\eta^1(x, \eta^*) = \eta$ (with $\eta^* = \eta$), establishes H4. **QED**

Since SE always exist for finite cheap-talk games,

**Corollary 1** *Every continuous cheap-talk extension game* $\Gamma(\Psi^*) = [(T, \rho), X, \Psi^*, Y, U^1, U^2]$ *has a SE.*

Let $\hat{\lambda}$ be any SE outcome of a signaling game $\Gamma$. Adding cheap-talk to $\Gamma$ will preserve $\lambda$ as a SE outcome: In the modified game player 1 sends always the same cheap-talk message, and player 2's response remains unaltered by the cheap talk. Thus,

$$SEO(\Gamma) \subseteq SEO(\Gamma(\Psi^*)). \tag{3}$$

Example 1 showed that the inclusion may be strict.

For a signaling game $\Gamma$, if $SEO(\Gamma(\Psi^*)) = SEO(\Gamma)$, $\Gamma$ has a SE. Thus, it is important to determine when a SE outcome of $\Gamma(\Psi^*)$ will also be a SE outcome of $\Gamma$.[10]

**Theorem 2** *Let* $\Gamma = [(T, \rho), X, Y, U^1, U^2]$ *be continuous and let* $\lambda$ *be a SE outcome of its cheap-talk extension* $\Gamma(\Psi^*) = [(T, \rho), X, \Psi^*, Y, U^1, U^2]$. *Suppose there exists a strategy pair* $(\alpha, \eta)$ *implementing* $\hat{\lambda}$, *and* $\bar{\eta}$ *is continuous on* $supp[\lambda_X]$. *Then,* $\lambda \in SEO(\Gamma)$.

If a SE outcome $\hat{\lambda}$ of a cheap-talk game can be implemented with a continuous strategy for player 2 in the game without cheap talk, then it is also a SE outcome of the game without cheap talk.

[10] Crawford and Sobel (1982) and Seidmann (1990) present conditions for pure cheap-talk games to be communication-impervious; every SE outcome of the cheap talk extension game is also a SE outcome of the original game.