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# A CHAOTIC MAP ARISING IN THE THEORY OF ENDOGENOUS GROWTH. *

by

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and

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## Abstract

Growth theorists have almost always adopted the assumption of balanced growth in their investigations of development phenomena. In reality countries growth rates oscillate, sometimes wildly, around some average value. The latter is often taken to represent the balanced rate to which the development dynamics spontaneously tends to return after each perturbation. In this paper we try a different interpretation: the growth rate of capital stock in a developing economy oscillates within some bounded interval of feasible values and the balanced growth rate is in fact unstable. These oscillations may be persistent and endogenously determined by the accumulation process itself and they generate a non-trivial, invariant distribution of growth rates. We study a class of two-sector models displaying this feature in the presence of a positive external effect. The qualitative properties of a specific example are analyzed by means of analytical and numerical methods. Our simulations reveal that, while the artificial economy is certainly able to display rather impressive endogenous growth cycles, they occur only when the external effects is unreasonably strong. Similarly to previous tentatives of modelling endogenous oscillations by means a chaotic map, we succeed at the theoretical level but fail short of reproducing some crucial empirical properties of the growth cycles experienced by modern market economies.

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## 1. Introduction

It is a customary practice in the study of growth problems to construct the analytical model around the pivotal assumption that long-run growth occurs at a stable and balanced rate, and that growth rates different from the balanced one may appear, as transient phenomena, only during the initial stages of the development process.

This assumption is shared by the older models in which growth is due to "exogenous" technological change (*e.g.* Solow (1956) or Cass (1965)) and the more recent ones in which growht is "endogenously" determined by the continuous accumulation of some reproducible factor (*e.g.* Lucas (1988), Romer (1990)) †. The task of growth theory is defined as the explanation of the long-run average growth rate. This task is achieved by sweeping aside the continuous fluctuations in growth rates that characterizes actual economies.

This separation between cycles and growth also provides the theoretical underpinnings for the methodology adopted by proponents of Real Business Cycle theory, see *e.g.* Kydland and Prescott (1982) or Hansen (1985). Here the economy is displaced from its constant growth trend by some unpredictable technological shocks, whose effects on the capital accumulation path nevertheless die away rather quickly as the economy returns to its unmodified balanced growth position.

This is rather ironic, as the initial emergence of the real business cycle approach was motivated by the scientific need of connecting the theory of business fluctuations to the theory of economic growth. In the view of its proponents this has been achieved by emphasizing productivity shocks as a main driving force behind the trade cycle: it is the erratic nature of the growth process that provides the impulse for the short run business cycles.

On the other hand, real business cycle supporters seem oblivious to the fatal tendency of their models to go back to a position of complete rest where there are neither cycles nor growth. They handle this lack of internal dynamics by introducing the ancillary hypothesis of an unending stream of technological shocks. These act like the *deus ex machina* of the whole story and provide the real business cycle's transmission mechanism with the continuous impulses that keep it alive. This methodology reduces the claim of having built a joint explanation for cycles and growth to a rather vacuous statement as we have no idea of where growth comes from and if and how the business cycles feed back on it.

† Romer (1986) is an exception: he assumes unbounded and increasing growth rates.

the price of capital (in consumption units) and the real rate of return on capital increase exponentially (the first faster than the second). Neither of this phenomena has been observed in the real world where, if anything, the consumption price of new investments has slightly declined in advanced economies.

Some of these negative properties may depend on the specific functional forms we have chosen and, in particular, on the attribution of the external effect only to the consumption sector. Similar numerical exercises performed on the model described in Section 2.3. show that in that case relative prices remain stationary and that the magnitude of the oscillations of income and consumption can be somewhat reduced.

It remains a general finding, though, that the endogenous oscillations in growth rates are possible only for very large, and in our opinion unrealistic levels of the external effect.

### *4.2 Conclusions.*

We have presented a class of two-sector models in which unbounded accumulation is triggered by the existence of increasing returns in aggregate production. The increasing returns are due to the positive external effect generated by the aggregate stock of capital.

The model predicts that when the external effect is very strong, the growth rate will oscillate forever along cycles of long periodicity or even over a chaotic attractor. We have provided a full characterization of the conditions under which this occurs and illustrated the feature of these endogenous growth cycles by means of numerical simulations.

While the theoretical exercise has certainly been succesful the empirical applicability of the model seems to be still out of reach. Endogenous growth cycles require unreasonably high levels of the external effect when the other model's parameters are tuned to realistic values. Furthermore, the dynamic properties of the aggregate prices and quantities generated by our artificial economy in its chaotic regime fare very poorly when compared to real world data.

The empirical relevance of chaos theory for an understanding of economic growth cycles has yet to be proven.

Indeed the old dicothomy is still the crucial assumption upon which real business cycle theory rests. Without separation between growth and oscillations and, in particular, without the exogeneity of the first to the process generating the second, their model-building technique could not be adopted.

In this paper we point out that there is no reason to maintain this dicothomy, at least at a theoretical level. We show that in models in which unbounded accumulation is due to productive externalities, trend and cycles can be simultaneously generated by the same endogenous economic mechanism.

This is accomplished in a class of stylized two-sector models, without endogenous labor supply and with rather simple forms for the utility and production functions. We show that, as long as the externality is sufficiently strong, it does not matter if it affects the consumption or the capital good sector. In both cases, persistent growth is accompanied by oscillations in growth rates.

Contrary to the examples of endogenous cycles generated by convex versions of the two-sector model (*e.g.* Boldrin (1989)), in the presence of an external factor, endogenous oscillations can be obtained at fairly reasonable values of most parameters.

While the theoretical properties of the class of models presented here are very interesting and suggestive of potentially valuable developments, we are still unable to claim they can provide an explanation of the empirically observed oscillations in the growth rates of most aggregate economic variables. As we have verified by means of many numerical simulations, the models constructed here do not display "realistic" features. Not only they are too stylized to be of serious empirical relevance, but the strenght of the externality required to generate endogenous oscillations implies a behavior of output and consumption which is clearly counterfactual: they both grow too rapidly and the second oscillates much too widely compared to what we observe in reality. It is an open question if more elaborated versions of the same basic models may be able to overcome these limitations.

The rest of the paper is organized as follows. Section 2 introduces the theoretical framework, derive some of its fundamental properties and describes two specific classes of functional forms. In section 3 the simplest among these two models is investigated analytically and its qualitative properties are spelled out. Section 4 briefly summarizes the findings of a number of numerical simulations of the same model and concludes the paper.

$f_\omega(\phi) \in \mathcal{P}$, there exists a continuum of attractive period two cycles. Notice that the periodicity of this attractive cycle is indipendent of the initial condition $z_0 \in \Pi$.

For the special case in which $\Lambda$ contains an aperiodic orbit (which will be true for example whenever the hypotheses of Lemma 1 are satisfied) then one could try to apply again Theorem B of Nusse [1987] to conclude that the set of points in $[0, \phi]$ which do not converge to the periodic point of period $p$ is of zero Lebesgue measure †

On the other hand when $f_\omega(\phi) \notin \Pi$ then the asymptotic behavior of the equilibrium growth rate will depend on the specific structure of $\Lambda$. As we have already mentioned, one cannot show that in general $\Lambda$ has measure zero even if it obviously has a Cantor-like structure. If the asymptotic behavior over $\Lambda$ will be periodic or aperiodic can be decided by applying here the same criteria we developed in subsection 3.1.

## 4. Numerical Simulations and Conclusions.

### *4.1 Simulations.*

As pointed out in the Introduction we have tried to parameterize the model using what we consider acceptable values of the parameters and to examine the statistical properties of the aggregate time series so generated. While the detailed results are not reported here to economize on space, some general comments on these findings seem appropriate.

Setting $\theta = 1.2$, $\delta = .75$, $\alpha = .5$ and $\sigma = .5$ this requires a value of $\beta$ around 8. While the first set of parameters are roughly acceptable the level of the external effect $\eta$ implied by the choice of $\beta$ seems to be unreasonably high. This is reflected in the long run behavior of the simulated time series: while they behave reasonably for the first few periods they move on a very steep and exponential growth path soon after that. Not only: the size and the frequency of oscillations in the growth rates of income, capital and consumption become very rapidly of an enormeous magnitude compared to what we observe in the real world data. This is particularly for the consumption and income variables which after detrending display a standard deviation which is almost twenty time larger than observed ones.

Finally the behavior of the associated price sequences also appears unrealistic. As mentioned above in Section 2.3 the relative prices are not stationary in this economy:

† The map is not three times differentiable at two isolated points. It is a tedious but straightforward matter to verify that this does not affect the proof.

## 2. The Model Economies.

### *2.1 The abstract framework.*

The economy is inhabited by a continuum of identical, infinitely lived agents who maximize $\sum_{t=0}^{\infty} u(c_t)\delta^t$ subject to the resource constraints $c_t = F^1(\ell_{1t}, x_{1t}; k_t)$, $x_{t+1} = F^2(\ell_{2t}, x_{2t}; k_t) + (1-\mu)x_t$, $\ell_t = \ell_{1t} + \ell_{2t}$ and $x_t = x_{1t} + x_{2t}$. Here $u(\cdot)$ is a strictly concave and strictly increasing utility function, and $F^1(\cdot,\cdot;k)$, $F^2(\cdot,\cdot;k)$ two concave and increasing production functions.

The notation is standard: $c_t$, $\ell_t$ and $x_t$ denote individual consumption, labor supply and capital stock respectively, $\delta$ and $\mu$ are the discount and depreciation factors. In the two production functions the symbol $k_t$ stands for the average stock of capital in the whole economy. This is assumed to affect the individual production processes as an externality. In equilibrium we will impose that $k_t = x_t$, but it is important to keep a separate notation when discussing the representative agent maximization problem as single individuals have no control on the values of the aggregate capital stock.

One defines the individual production possibility frontier (PPF) as

$$T(x_t, x_{t+1}, \ell_t; k_t) = \max_{\ell_{1t}, x_{1t}} F^1(\ell_{1t}, x_{1t}; k_t)$$

$$\text{subject to}: x_{t+1} \leq F^2(\ell_t - \ell_{1t}, x_t - x_{1t}; k_t) + (1-\mu)x_t.$$

For a given aggregate sequence $\{k_t\}_{t=0}^{\infty}$, we write the individual intertemporal maximization problem in reduced form as

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} V(x_t, x_{t+1}; k_t)\delta^t \tag{P}$$

$$\text{subject to}: (1-\mu)x_t \leq x_{t+1} \leq F^2(1, x_t; k_t) + (1-\mu)x_t$$

where $V(x_t, x_{t+1}; k_t) = u[T(x_t, x_{t+1}, 1; k_t)]$ is the individual return function and the exogenous per-period labor supply $\ell_t$ has been normalized to one.

Equilibria are solutions to $(P)$ that satisfy $x_t = k_t$ for all $t = 0, 1, 2, \ldots$. For the case in which equilibria are interior (i.e. the sequence $\{x_t\}_{t=0}^{\infty}$ satisfies $(1-\mu)x_t < x_{t+1} < F^2(1, x_t; x_t) + (1-\mu)x_t$) they are completely characterized by the following two restrictions

$$V_2(x_t, x_{t+1}; x_t) + \delta V_1(x_{t+1}, x_{t+2}; x_{t+1}) = 0 \tag{EE}$$

that the map $f_\omega$ looks as in Figure 13, with $\phi$ some number less than one. It is obvious that only the subinterval $[0, \phi]$ is now invariant under the action of $f_\omega$.

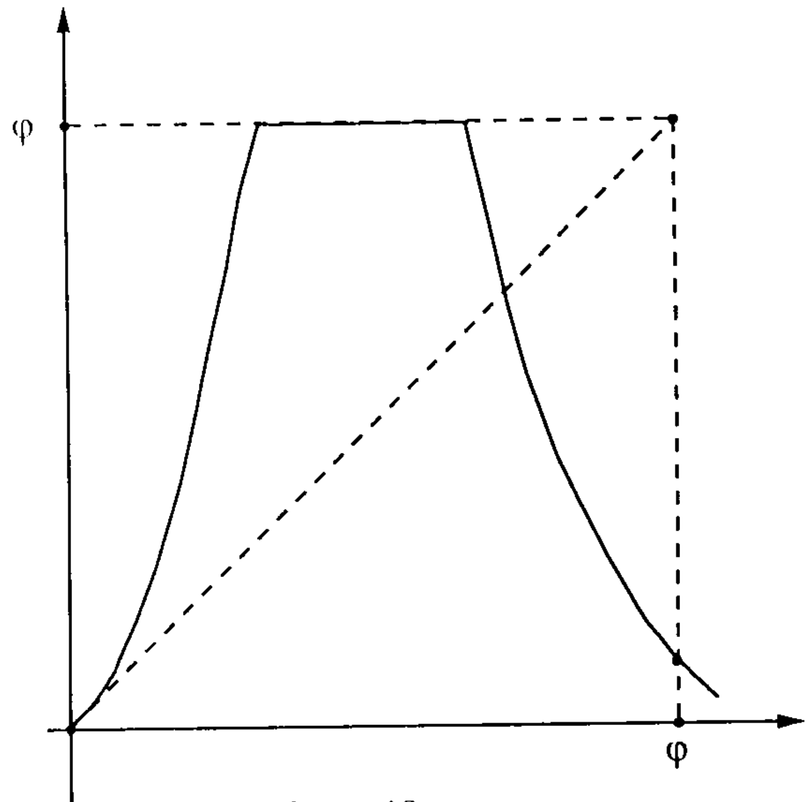


**Figure 13.**

Visual inspection suggests that for values of $\mu$ close to one the behavior of the equilibrium trajectories remain pretty much the same as that discussed in Section 3.1. As $\mu$ increases, unfortunately the structure of the equilibrium set becomes harder to characterize. We will try nevertheless to provide here a short and heuristic discussion.

Denote with $\Pi$ the set of preimages of $\phi$ under finite iterations of the map $f_\omega$, in other words $\Pi$ is defined as the set of all those $z \in [0, \phi]$ for which there exists an $n > 0$ such that $f_\omega^n(z) = \phi$. In particular denote with $\mathcal{P}$ the set of direct preimages of $\phi$, i.e. all those $z \in [0, \phi]$ such that $f_\omega(z) = \phi$. The reader will notice the geometric analogy between this case and the one we just considered for $m(\beta) > 1$, an analogy worth exploiting to shorten the forthcoming discussion. Define

$$\Lambda = [0, \phi] \setminus \Pi.$$

It is clear that most points in $[0, \phi]$ will belong to $\Pi$ but contrary to the previous case one cannot claim this set to be of Lebesgue measure zero.

When the point $\phi$ is mapped into $\Pi$ then all the equilibria with an initial condition $z_0 \in \Pi$ are eventually periodic of some finite period $p$. In the special case in which and

$$\lim_{t\to\infty} \delta^t x_t V_1(x_t, x_{t+1}; x_t) = 0 \tag{TC}$$

There are a number of well known ways in which one can guarantee that the sequences solving $(EE)$ and $(TC)$ are unbounded. We will not spend our time discussing them here and instead assume directly that unbounded growth in the stock of capital occurs (see Boldrin and Rustichini (1994) for details).

When unbounded growth occurs it is useful to transform $(EE)$ into an implicit equation in growth rates, and to study the dynamics induced by the one-parameter family of maps this transformation produces. More precisely: define the growth rate of the capital stock at time $t$ as $\lambda_t = x_{t+1}/x_t$ and let

$$\theta = \max_{x\geq 0} \frac{F^2(1, x; x) + (1-\mu)x}{x}$$

Assume that $F^2$ is such that the latter is a well defined problem with a finite solution. An interior solution to $(EE)$ implies $\lambda_t \in [(1-\mu), \theta]$ for all $t$. Let $x_t = x$, then $(EE)$ generates the one-parameter family of functions $\theta_x : [(1-\mu), \theta] \to [(1-\mu), \theta]$ defined by

$$V_2(x, \lambda_t x; x) + \delta V_1(\lambda_t x, \theta_x(\lambda_t)\lambda_t x; \lambda_t x) = 0 \tag{EE'}$$

One is interested in characterizing the asymptotic behavior of the equilibrium sequences of growth rates $\{\lambda_t\}_{t=0}^{\infty}$ satisfying $(EE')$. In general this is very demanding and cannot be done for arbitrary classes of production and utility functions. Nevertheless it becomes a manegeable task when appropriate choices are made.

In the rest of the paper we will choose particular functional forms for $u$, $F^1$ and $F^2$, such that unbounded accumulation paths are equilibria. Our choice will also guarantee that an analytical study of $(EE')$ and $(TC)$ can be carried out. Our attention will then concentrate on the asymptotic behavior of the orbits of the dynamical sytem induced by $(EE')$ and satisfying $(TC)$.

*2.2 The Basic Example.*

The model we introduce here was first proposed in Boldrin and Rustichini (1994). It is based on the following utility and production functions.

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \quad \text{with } \sigma > 0.$$

(iii) is also satisfied as the set of points in question is the intersection of a collection of closed and bounded subsets of the real line. We have proved in Lemma 2 that the critical point $\hat{z}$ converges to $-\infty$ under repeated iteration of $f_\omega$, hence (iv) is also satisfied. Finally (v) is obviously true. Given that the trajectory of the critical point $\hat{z}$ converges to $-\infty$ there are no asymptotically stable periodic orbits for $f_\omega$ because if such an orbit existed it would attract the trajectory starting at the critical point. Hence the quoted theorem implies our statement. **Q.E.D.**

Summing up we have the following characterization of the equilibrium set $\mathcal{E}$.

**Theorem 1** *Under the assumptions of this subsection the equilibrium set $\mathcal{E}$ defined above in equation* (3.7) *is a closed subset of the unit interval, composed of an (uncountable) number of isolated points. The Lebesgue measure of $\mathcal{E}$ is zero.*

**Proof:** That $\mathcal{E}$ has measure zero follows from Lemma 3. The other properties are trivially verified by construction. **Q.E.D.**

The interpretation of this result is straightforward. In the presence of very strong externalities if the economy follows a perfect foresight path then economic agents must be able to select the initial growth rate $\lambda_0$ from within the equilibrium set $\mathcal{E}$ instead of the whole unit interval. The structure of $\mathcal{E}$ is extremely complicated and furthermore it seems unlikely that points belonging to $\mathcal{E}$ could be selected by chance as the set $\mathcal{E}$ has measure zero. This obviously requirase some incredible computing power on the part of the economic agents, a computing power we very much doubt to possess.

We will now briefly show that a somewhat similar situation arises when a realistic depreciation factor is introduced in the basic model.

### *3.3 Asymptotic Behavior when $\mu < 1$.*

Earlier on we introduced the simplyfying assumption according to which the entire stock of capital depreciates after just one period. This simplified the algebra required to characterize the basic properties of the equilibrium set. We should try to relax this assumption and consider the consequences of setting $0 < \mu < 1$. In general this implies

$$c_t = F^1(\ell_{1t}, x_{1t}; k_t) = k_t^\eta \ell_{1t}^{1-\alpha} x_{1t}^\alpha, \quad \text{with } \alpha \in (0,1) \text{ and } \eta > 0$$

$$i_t = bx_{2t},\ b > 0,\ x_{1t} + x_{2t} = x_t \text{ and } x_{t+1} = (1-\mu)x_t + i_t.$$

After normalizing total labor supply to one, the PPF faced by an individual agent with stock $x_t$, in the presence of an aggregate stock $k_t$ is

$$c_t = T(x_t, x_{t+1}; k_t) = \gamma k_t^\eta (\theta x_t - x_{t+1})^\alpha \tag{2.1}$$

where $0 < \gamma = (1/b)^\alpha < 1$ and $\theta = b + 1 - \mu > 1$ will be assumed. This is enough to guarantee that unbounded accumulation is possible. The analogous to problem $(P)$ is

$$\max_{\{x_t\}_{t=0}^\infty} \sum_{t=0}^{\infty} (1-\sigma)^{-1} \Big(\gamma k_t^\eta (\theta x_t - x_{t+1})^\alpha\Big)^{1-\sigma} \delta^t$$
$$\text{subject to} : (1-\mu)x_t \leq x_{t+1} \leq \theta x_t.$$

Equilibrium trajectories departing from a given initial condition $x_0$ satisfy:

$$-\alpha\gamma^{1-\sigma} x_t^{\eta(1-\sigma)} (\theta x_t - x_{t+1})^{\alpha(1-\sigma)-1} + \delta\alpha\gamma^{1-\sigma}\theta x_{t+1}^{\eta(1-\sigma)} (\theta x_{t+1} - x_{t+2})^{\alpha(1-\sigma)-1} \tag{EE}$$

and

$$\lim_{t\to\infty} \delta^t x_t c_t^{-\sigma} = 0 \tag{TC}$$

Setting $x_t = x$, $x_{t+1} = \lambda_t x$ and $x_{t+2} = \lambda_{t+1}\lambda_t x$ and simplifying $(EE)$ we obtain the following first order difference equation $\lambda_{t+1} = \tau(\lambda_t)$

$$\lambda_{t+1} = \begin{cases} \theta - (\theta\delta)^\nu \lambda_t^{\beta\nu}(\theta - \lambda_t) & \text{when } \tau(\lambda_t) \geq (1-\mu) \quad (2.2a) \\ (1-\mu) & \text{otherwise} \quad (2.2b) \end{cases}$$

where $\nu = \frac{1}{1-\alpha(1-\sigma)}$ and $\beta = (\eta+\alpha)(1-\sigma) - 1$.

The presence of the floor-value $(2.2b)$ in the definition of the dynamical evolution of $\lambda_t$ is meant to capture the idea that free disposal of the capital stock is really not possible and that $(1-\mu)$ is therefore the minimum admissible value of $\lambda_t$. In order to simplify the analysis we will first assume that $\mu = 1$ holds and concentrate on the portion $(2.2a)$ of $\tau$. In this case the latter will be defined over the whole interval $[0, \theta]$. A positive value for the depreciation rate $\mu$ will be reintroduced later on (Section 3.3) and its implications for the structure of the equilibrium set discussed. The transversality condition can be written as

$$\lim_{t\to\infty} \alpha\theta\gamma^{1-\sigma}\delta^t x_t^{(\eta+\alpha)(1-\sigma)} (\theta - \lambda_t)^{\alpha(1-\sigma)-1} = 0 \tag{2.3}$$

under $g$. Mimicking Collett and Eckmann (1980, p. 99) one can check that in all cases $g$ must have a critical point in $[r, s]$ which is attracted to $z$.

7) Finally consider the case in which $g = \tilde{f}^n$ has a fixed point $g(z) = z$ at which $|g'(z)| = 1$. Only the situation in which $-\infty < z < \infty$ needs to be considered. By 4) there must be a neighborhood $(a, b)$ of $z$ containing no other fixed point of $g$. Again by the same arguments of Collett and Eckmann (1980, p. 100), $g(y) > y$ must hold for $y \in (a, z)$. Let $d$ be the minimum value of $y$ for which $g(y) \geq y$. Observe first that $(d, z) \cap \mathcal{P}^n(1) = \emptyset$ since by definition $x \in (d, z)$ implies $g^k(x) \to z$ as $k \to \infty$ and $x \in \mathcal{P}^n(1)$ implies $g^k(x) \to 0$ as $k \to \infty$. Then either $d = -\infty$ or $g(d) = d$ must hold. The first case contradicts the fact that $\tilde{f}_\omega(y) \to -\infty$ when $y \to \mp\infty$. In the second case $(d, z)$ must contain a critical point of $\tilde{f}_\omega$ as proved in Collett and Eckmann (1980, p. 100). **Q.E.D.**

**Lemma 3** *Under the assumptions of this subsection the equilibrium set $\mathcal{E}$ defined in (3.7) has Lebesgue measure zero.*

**Proof:** Formally we want to show that the set of points $\mathcal{E}$ such that $z \in \{\Re \setminus \mathcal{E}\} \iff \{\tilde{f}_\omega^n(z) \to -\infty\}$ has Lebesgue measure zero when $m(\beta) > 1$. We will make use of the following theorem proved in Nusse (1987).

*Theorem.* Assume that $f : X \to X$ is $C^3$ over the non-trivial interval $X$ and it satisfies the following hypotheses:

(i) there exists at least one aperiodic point for $f$;

(ii) $Sf(x) \leq 0$ for all $x$ such that $f'(x) \neq 0$;

(iii) the set of points whose orbits do not converge to an absorbing boundary point of $X$ is a non-empty compact set;

(iv) the orbits of each critical point of $f$ converge to either some asymptotically stable periodic point or to an absorbing boundary point of $X$;

(v) the fixed points of $f^2$ are isolated.

Then:

a) the set of points whose orbits do not converge to an asymptotically stable periodic point or to an absorbing boundary point of $X$ has Lebesgue measure zero;

b) there exists a positive integer $p$ such that almost every point in $X$ is asymptotically periodic with (not necessarily primitive) period $p$, provided that $f$ is bounded.

We have shown (Lemma 1) that $\tilde{f}_\omega$ has a period-3 cycle for admissible values of $\omega$. This implies the existence of an aperiodic point. $\tilde{f}_\omega$ also satisfies (ii) by construction. Hypothesis

It is straightforward algebra to verify that the map (2.2) is continuous over the interval $[0, \theta]$ if the restriction $\nu\beta > 0$ is satisfied, which we take for granted from now on. This amounts to imposing either aggregate increasing returns in the stock of capital when $0 \leq \sigma < 1$ or a strong *negative* externality ($\eta < -\alpha$) when $\sigma > 1$.

In the special case $\sigma = 1$ (i.e. logarithmic utility function) the dynamics predicted by the model are trivial. The function (2.2) is monotone increasing and maps $(0, \theta]$ into $(-\infty, \theta]$ indepedently of the values taken by the technological parameters $\alpha$ and $\eta$. The two fixed points (corresponding to balanced growth rates) are respectively at $\lambda_1 = \delta\theta$ and $\lambda_2 = \theta$. The first is dynamically unstable while the second is stable but violates the transversality condition. The reasoning of Boldrin and Rustichini (1994) then implies that from every initial condition there is a unique equilibrium path, converging to the balanced growth rate $\lambda_1$.

In the ensuing analysis we will concentrate on the parameter values $0 \leq \sigma < 1$ and $\beta > 0$ as they appear the most relevant from an economic standpoint.

Before proceeding with the study of (2.2) it is opportune to clarify which among the trajectories induced by $\tau$ satisfy $(TC)$. It is readily seen that two different situations may arise.

When the parameter values are such that $\tau(\lambda) \in [(1-\mu), \theta]$ for all $\lambda \in [(1-\mu), \theta]$, one has $(1-\mu)^t x_0 \leq x_t \leq \theta^t x_0$ for all $t$. In this case, for any initial condition $x_0 \in \Re_+$ the capital accumulation rate remains bounded and one may try to show that all, or at least most, of the trajectories generated by (2.2) satisfy (2.3)

On the other hand, for any $0 \leq \mu \leq 1$ there exist admissible values of the other parameters giving $\tau(\lambda) < (1-\mu)$ for some $\lambda \in [(1-\mu), \theta]$. In these circumstances, for given $x_0$, any feasible $x_1$ such that $\lambda_0 = x_1/x_0$ induces a trajectory of growth rates along which $\tau^t(\lambda_0) \notin [(1-\mu), \theta]$ for some $t$ is not an admissible choice in equilibrium. This is more easily seen in the case $\mu = 1$, where $\lambda_t \notin [0, \theta]$ implies $\lim_{t\to\infty} \lambda_t = \infty$ under the action of $(2.2a)$, which obviously violates $(TC)$. In this case only those values of $x_1$ such that $\lambda_0 = x_1/x_0$ induces an orbit remaining in $[0, \theta]$ for all $t$ are candidate to be selected in equilibrium. As we show later on in section 3.3, this makes the structure of the equilibrium set look particularly intriguing.

In either case, even if $\lambda_t \in [0, \theta]$ for all $t$, an additional joint restriction on parameter values and the "average growth rate" needs to be added to guarantee that (2.3) is satisfied.

Since $\tilde{f}(\tilde{z}) \neq \tilde{z}$, this proves the existence of a period three. $m(\overline{3}) < 1$ holds as a consequence of Proposition 3. **Q.E.D.**

The next step is to show that every stable periodic orbit of the map $\tilde{f}_\omega$ will attract the critical point $\hat{z}$. While tedious, this amounts to verifying that the proof given in Singer (1978) can be replicated for a map defined over the whole real line and with zero Schwartzian derivative at a finite number of isolated points. To save space we will follow the seven steps of the proof as given in Collett and Eckmann (1980, pp. 97-100) and check that they are all satisfied also here.

**Lemma 2** *For all $\omega > 1$ every stable periodic orbit of $\tilde{f}_\omega$ attracts the critical point $\hat{z}$.*

**Proof:** We will proceed by checcking that all the seven properties derived in Collett and Eckmann (1980, pp. 97-100) are satisfied also here.

1) is just a property of Schwartzian derivatives, i.e. if $f$ and $g$ are $C^3$ then $S(f \circ g)(x) = (Sf)(g(x))g'(x)^2 + Sg(x)$.

2) Let $\mathcal{P}^n(1)$ be the set of preimages of 1 under $\tilde{f}^n_\omega$. Then one can verify by direct computation that for all $n \geq 1$, $S\tilde{f}^n < 0$ everywhere on $\Re$ but at $\mathcal{B} = \{\hat{z}, 1, 0\} \cup \mathcal{P}^1(1)$.

3) For all $z \in \{\Re \setminus \mathcal{B}\}$ $|f'|$ has no positive local minimum.

4) The map $\tilde{f}_\omega$ has finitely many points of period $n$ for every integer $n \geq 1$.

*Proof.* Let $\mathrm{Per}(\tilde{f})$ be the set of periodic points. Then it is clear that $\mathcal{P}^n(1) \cap \mathrm{Per}(\tilde{f}) = \emptyset$. Suppose now that for some $n$, $\tilde{f}^n(x) = x$ for infinitely many $x$. Then by the mean value theorem we must have $g'(x_k) = (\tilde{f}^n)'(x_k) = 1$ for infinitely many $x_k \notin \mathcal{P}^n(1)$. Moreover, since the cardinality of $\mathcal{P}^n(1)$ is bounded above by $2^n$, for infinitely many $k$'s $[x_k, x_{k+1}] \cap \mathcal{P}^n(1) = \emptyset$ will hold. Point 3) above then implies that $|g'|$ must vanish on those intervals. But this contradicts the fact that $f$, and hence $\tilde{f}$ and $g$ has finitely many critical points.

5) If $a < b < c$ are consecutive fixed points of $g = \tilde{f}^n$ and if $[a, c]$ contains no critical points of $g$, then $g'(b) > 1$.

6) Let $z \in \Re$ be a stable fixed point for $g = \tilde{f}^n$, and assume $|g'(z)| < 1$. Then the *stable manifold* of $z$ is the set of points converging to $z$ under iteration of $g$ and the *semilocal stable manifold* of $z$ is the connected component of the stable manifold of $z$, which contains $z$. Then by our definition of $\tilde{f}_\omega$, $(r, s)$ is the only possible form that the semilocal stable manifold can assume, where $r$ and $s$ are two finite numbers which are either both fixed points of $g$, or a period-2 cycle for $g$ or one a fixed point of $g$ and the other its preimage

This is

$$\delta\lambda_{\mathrm{AV}}^{\beta+1} < 1 \tag{2.4}$$

Our definition of $\lambda_{\mathrm{AV}}$ goes as follows. Let $\rho$ be a positive measure on the unit interval which is invariant under the iteration of $\tau$, i.e. such that $\rho(\tau^{-1}(A)) = \rho(A)$ for all the Borel subsets $A \subset [0,1]$. Then we have

$$\lambda_{\mathrm{AV}} = \int_0^1 \lambda \cdot \rho(d\lambda)$$

In our applications we will always be considering cases in which there is only one such measure $\rho$ which will be invariant under $\tau$.

Under the parametric restrictions introduced so far the map $\tau$ displays the following properties.

a) It is unimodal over $[0,\theta]$ with a minimum at $\lambda^* = \beta\nu\theta/(1+\beta\nu)$.

b) It has two fixed points at $\lambda_1 = (1/\theta\delta)^{1/\beta}$ and at $\lambda_2 = \theta$.

c) The fixed points are ordered as $\lambda_1 < \lambda_2$, when $\lambda_1$ satisfies condition (2.4) (with $\lambda_{\mathrm{AV}} = \lambda_1$) and $\lambda_2$ does not.

d The latter also implies that $\lambda_2$ is a source whereas $\lambda_1$ can be either a source or a sink.

e) At certain parameter values the equilibrium trajectories $\{\lambda_t\}_{t=0}^{\infty}$ are cyclic or chaotic.

It is also worth noting that when the transversality condition (2.4) is satisfied, the stationary point $\lambda_1$ corresponds to a positive balanced growth rate $g = \lambda_1^{-1} - 1$. Even if property e) may not seem obvious at this point it will become apparent after the next section. In the meanwhile we collect here a few qualitative implications of properties a)-e)

**Proposition 1** *Under the restrictions $\beta > 0$ and $0 \le \sigma < 1$, there exists a more than countable set of equilibrium trajectories for every initial condition $x_0$.*

### *2.3 A More Complicated Example.*

To dispel the impression that the example given above may be special in the class of two-sector economies under consideration, we will introduce here a second specification of $F^1$ and $F^2$ which in fact induces an even more complicated pattern for the equilibrium growth rates.

Let us restrict attention to a linear utility function from the onset. This minimizes algebraic complications and does not alter the qualitative conclusions. Retain notation from the previous subsection, and specify the production functions as

$$c_t = \ell_{1t}^{1-\alpha} x_{1t}^{\alpha}, \text{ for } \alpha \in (0,1) \tag{2.5}$$

containing all the inadmissible initial conditions. The equilibrium set reduces to

$$\mathcal{E} = [0,1] \setminus \left( \cup_{n=0}^{\infty} A_n \right) \tag{3.7}$$

It is straightforward to note that the procedure through which $\mathcal{E}$ is constructed is reminiscent of the deletion algorithm generating the Cantor-Middle-Third set. The only difference here is that the open intervals we are removing are not symmetricaly located around the middle point $1/(1+\omega)$, due to the strong asymmetry of the map $f_\omega$.

In any case this minor difference is not essential and the proof that $\mathcal{E}$ is a closed set may proceed in the standard way (see *e.g.* Devaney (1986)). To show that it is disconnected and it has Lebesgue-measure zero one encounters real difficulties. The slope of $f_\omega$ is zero at $z=1$ and very small nearby. This makes it impossible to show directly that for $\epsilon > 0$ any non negligible interval of the type $[1-\epsilon, 1]$ would contain points that are eventually mapped outside the unit interval by $f_\omega^n$. In spite of the fact that all intervals of that type are mapped into intervals of the type $[0, f(1-\epsilon)]$ and that the slope of $f_\omega$ is very high near zero, the slope of the second iterate of $f_\omega$ becomes as close to zero as one pleases at values of $z \in [1-\epsilon, 1]$ close enough to one.

Hence a complete characterization of the equilibrium set $\mathcal{E}$ cannot rely on existing results and has to derived by other means. This is what we are going to do next.

Begin by extending the function $f_\omega$ over the whole real line

$$\tilde{f}_\omega(z) = \begin{cases} -(z-1)^2 & z > 1 \\ f_\omega(z) & z \in [0,1] \\ f'''(0)\frac{z^3}{3} + f''(0)\frac{z^2}{2} + f(0)z & z < 0 \end{cases}$$

Then $\tilde{f} : \Re \to \Re$ is $C^3$, with $S\tilde{f} \le 0$ and $S\tilde{f} < 0$ if $z \neq 1, \hat{z}$.

**Lemma 1** *There exists an $\overline{\omega} = \nu\overline{\beta}$ such that for all $\omega > \overline{\omega}$ the map $\tilde{f}_\omega$ has a cycle of period three. Also, $m(\overline{\beta}) < 1$.*

**Proof:** Consider the behavior of the third iterate $\tilde{f}_\omega^3$. Denote with $LM = \tilde{f}_\omega^{-2}(\hat{z})$ the left pre-preimage of the critical point and with $Lm = \tilde{f}_\omega^{-1}(\hat{z})$ the left preimage of the critical point. Then $\tilde{f}_\omega^3(LM) = m(\beta)$ and $\tilde{f}_\omega^3(Lm) = \tilde{f}_\omega(m(\beta))$. So $0 < LM < Lm < z^* < 1$. It is easy to see that $m(\beta) \to \infty$ as $\beta \to \infty$. This implies that $\tilde{f}_\omega^3(LM) \to \infty$ and $\tilde{f}_\omega^3(Lm) \to -\infty$ monotonically with $\omega = \nu\beta$. Since $\tilde{f}_\omega$ is continuous and $LM, Lm \in (0,1)$ there will exist an $\overline{\omega}$ such that for all $\omega > \overline{\omega}$ there exists a $\tilde{z}$ satisfying

$$\tilde{f}_\omega^3(\tilde{z}) = \tilde{z}, \quad \text{and} \quad Lm < \tilde{z} < Lm.$$

$$x_{t+1} - (1-\mu)x_t = b\ell_{2t}, \text{ with } b > 0 \text{ and } \ell_{1t} + \ell_{2t} = \ell_t \tag{2.6}$$

Now introduce an external effect in the system by assuming that capital-embodied technological progress acts as a labor-augmenting force. Assuming a constant supply of raw labor time † total labor supply in efficiency units can be written as

$$\ell_t = k_t$$

by choice of an appropriate unit of measurement. The individual agent PPF is then

$$c_t = \gamma\big(bk_t + (1-\mu)x_t - x_{t+1}\big)^{1-\alpha} x_t^\alpha$$

from which the two equilibrium conditions

$$(1-\alpha)(\theta - \lambda_t)^{-\alpha} = \delta\Big((1-\mu)(1-\alpha)(\theta-\lambda_{t+1})^{-\alpha} + \alpha(\theta - \lambda_{t+1})^{1-\alpha}\Big) \tag{2.7}$$

and

$$\lim_{t\to\infty} \gamma\delta^t x_t\Big((1-\mu)(1-\alpha)(\theta-\lambda_t)^{-\alpha} + \alpha(\theta-\lambda_t)^{1-\alpha}\Big) \tag{2.8}$$

are easily derived by exploiting again the substitutions $x_t = x$, $x_{t+1} = \lambda_t x$ and $x_{t+2} = \lambda_{t+1}\lambda_t x$.

Two things are immediately apparent. The transversality condition (2.8) is satisfied as long as the growth rate remains in $[(1-\mu), \theta]$ and $b < \delta^{-1}$ ‡. On the other hand a well defined function $\lambda_{t+1} = \tau(\lambda_t)$ can *never* be derived from (2.7). This implies that, no matter what parameter values one might chose, there are always more than one equilibrium sequences departing from any given initial condition $\lambda_0$. This implies in turn that, for a given stock of capital $x_0$ at the initial time, the set of equilibrium sequences $\{x_t\}_{t=0}^\infty$ has a cardinality which is a countable times that of the continuum. By itself this property guarantees that the trajectories followed by the stock of capital will look rather complicated.

The fact that this growth economy is somewhat more "realistic" than our basic model is also worthy of attention. For example, one can verify that the equilibrium price of the stock of capital $q_t$ and the rate of return on capital $r_t$ (espressed in consumption units) follow respectively

$$q_t = (1-\alpha)\gamma(\theta - \lambda_t)^{-\alpha} \tag{2.9}$$

† Labor-leisure decisions can be introduced by making the algebra more demanding.

‡ This becomes a "curvature-dependent" condition when a nonlinear utility function is used.

**Figure 12.**

*3.2 Asymptotic Behavior when $m(\beta) > 1$.*

As we have already noted the behavior of the equilibrium set may become even more complicated when $m(\beta) > 1$. In this case $f_\omega$ does not map the unit interval into itself, as its maximum is now larger than one. Hence, certain points leave the interval $[0, 1]$ after a finite number of iterations: they are associated with inadmissible sequences of growth rates, along which $\lambda_t \to \infty$. These cannot be equilibrium sequences, as they obviously violate the transversality condition.

We will now argue that in these circumstances the set of equilibria has the structure of a Cantor set, i.e. of a closed, totally disconnected and perfect subset of the unit interval. Such a set has Lebesgue measure zero.

The intuition is standard and can be found in Devaney(1986, pages 34-48) where it is applied to the quadratic map. In our setting some extra complications arise because $f_\omega$ is flat near $z = 0$ and the technique adopted by Devaney (which relies on expansiveness) cannot be used.

Let $\omega > \omega^*$ be given and such that $f_\omega((1+\omega)^{-1}) > 1$. Denote with $A_0$ the set of points $z \in [0,1]$ at which $f_\omega(z) > 1$. Then let $A_1 = \{z \in [0,1] \mid f_\omega(z) \in A_0\}$, and inductively define the sequence of intervals

$$A_n = \{z \in [0,1] \mid f_\omega^n(z) \in A_0\} \tag{3.6}$$

and

$$r_t = \alpha\gamma(\theta - \lambda_t)^{1-\alpha} \tag{2.10}$$

which are both stationary processes. This is in sharp contrast with the economy introduced in the previous subsection, where both the price of capital and its rate of return increase as the aggregate capital stock grows (for further details, see below section 4). Unfortunately these more realistic features cannot be fully exploited through numerical simulations, due to the non-invertibility of the implicit function (2.7).

Something can nevertheless be learned about the aymptotic behavior of these economic system. By following the method of Grandmont (1985) one may study the "backward dynamics" generated by the equilibrium condition (2.7). A simple change of variables $\zeta_t = \theta - \lambda_t$ and the adoption of the simplified notation ($a = ((1-\alpha)/\delta)^{1/\alpha}$, $(1-\mu)(1-\alpha) = \tilde{\alpha}$ and $\beta = -1/\alpha$) gives the following map

$$\zeta_t = a\zeta_{t+1}\left(\tilde{\alpha} + \alpha\zeta_{t+1}\right)^{\beta} \tag{2.11}$$

The latter is a unimodal map which at appropriate parameter values displays cycles and chaos. This can be verifed by making use of the same methods we apply in the next section to the first example and will therefore be omitted here †.

## 3. Cyclic and Chaotic Growth Paths.

In light of the previous discussion, this section will concentrate only on the model introduced in subsection 2.2 as the latter lends itself to analytical and numerical investigation. For this and the following two subsections let us also assume that $\mu = 1$.

Our study will be greatly facilitated by the change of variable

$$\lambda_t \longmapsto \frac{\theta - \lambda_t}{\theta} = z_t \tag{3.1}$$

which transforms (2.2$a$) in the map $f : [0, 1] \rightarrow \Re_+$, defined as

$$z_{t+1} = f(z_t) = (\delta\theta)^{\nu}\theta^{\nu\beta}z_t(1 - z_t)^{\nu\beta} \tag{3.2}$$

The latter belongs to a class of unimodal transformations which have been widely studied in the mathematical literature (see *e.g.* Collet and Eckmann (1980), Preston (1982), Devaney (1986)). Because these techniques have become rather standard by now,

† Details are available from the authors

We have also computed the Liapunov exponents of our map at selected parameter values. They follow the already familiar pattern often encountered in unimodal maps. An example of such behavior can be seen in Figures 10-12, which use the same values of $\theta$, $\delta$ and $\nu$ as before and with $\beta$ ranging over selected subintervals of $[5, 9]$.

**Figure 10.**

**Figure 11.**

we will skip the details of the mathematical arguments, which can be recovered in any of the aforementioned references. Indeed a map very similar to (3.2) has already been utilized in an economic context by Matsuyama (1991) †

Equation (3.2) has a stationary state at $z = 0$, corresponding to $\lambda_2 = \theta$, and one at $z^* = 1 - [(\delta\theta)^{1/\beta}\theta]^{-1}$ which corresponds to the previously mentioned balanced growth rate $\lambda_1$. The slope of $f$ at zero is equal to $f'(0) = (\delta\theta^{1+\beta})^\nu$ and at $z^*$ it is equal to $f'(z^*) = 1 - \nu\beta[\theta(\delta\theta)^{1/\beta} - 1]$. It is easily seen that $0 < z^* < 1$ and $f'(0) > 1$ whenever the transversality condition (2.4) is satisfied by the stationary path $\lambda_t = \lambda_1$ for all $t$, while the local stability/instability of $z^*$ under (3.2) depends in a very complicated way on the whole set of model's parameters. This corresponds to our assertion d), according to which $\lambda_1$ can be either a sink or a source.

In the formulation (3.2) the map $f$ depends upon too many parameters, keeping track of which will only cloud the analysis. As we are interested in the impact of the external effect on the growth rates of the capital stock, we will treat $\eta$ (and therefore $\beta$) as our "bifurcation parameter" and take all other as given.

Consider now the function

$$m(\beta) = (\delta\theta)^\nu(\theta\nu\beta)^{\nu\beta}\left(\frac{1}{1+\nu\beta}\right)^{1+\nu\beta}$$

which gives the value of $f$ at its critical point $z = 1/(1+\nu\beta)$. $m(\beta)$ is well defined for $\beta \in [0,\infty)$ and continuously differentiable with $m'(\beta) < 0$ for small values of $\beta$ and $m'(\beta) > 0$ for large values of $\beta$. The meaning of "small" and "large" here, depends on the magnitude of $\theta$ and $\nu$ as $m'(\beta) = m(\beta)\nu[\ln\theta + \ln(\nu\beta) - \ln(1+\nu\beta)]$.

To characterize the attractors of $f_\beta(\cdot)$ we will examine separately the two cases: 1) $m(\beta) \leq 1$, and 2) $m(\beta) > 1$. In the first case $f_\beta$ maps the unit interval into itself, whereas in the second case $f_\beta(z) > 1$ for values of $z \in (0,1)$ and so most orbits of (3.2) will certainly not satisfy the transversality condition.

### *3.1 Asymptotic Behavior when $m(\beta) \leq 1$.*

Simplify notation one more time by setting $A = (\delta\theta)^\nu$ and $\nu\beta = \omega$ to rewrite (3.2) as

$$z_{t+1} = f_\omega(z_t) = A\theta^\omega z_t(1-z_t)^\omega \tag{3.2'}$$

The behavior of the graph of $f_\omega$ as $\omega$ increases is portraied in figure 1, where we have chosen $\theta = 1.6$, $\nu = 2$, $\delta = .5$ and $\beta = 2$, 2.5, 3 and 3.5 respectively.

† He derives a system equivalent to (3.2) from a version of Brock (1975) money-in-the-utility-function model.

Exception made for the case in which $\omega^* = 1$ (when (3.2′) reduces to the well known quadratic equation) we have been unable to derive an analytic representation of the ergodic measure mentioned in proposition 4. Numerical approximations are reproduced in Figures 8 and 9, which have been obtained for the following choices of parameter values: $\theta = 1.2$, $\delta = .75$ $\nu = 2$, $\beta^* = 8$ and $\beta^* = 9$ respectively.

**Figure 8.**

**Figure 9.**

Similar patterns emerge at most other parameter values.

**Figure 1.**

**Proposition 2** *(i) For all values of $\omega > 0$ such that $z^* < \frac{2}{2+\omega}$, the stationary state $z^*$ is globally asymptotically attractive. (ii) When $\omega$ is such that $z^* = \frac{2}{2+\omega}$ a cycle of period two is generated. Let $\omega = \omega_1$ denote such a value. Then all orbits converge to the period two cycle for values of $\omega$ in a right neighborhood of $\omega_1$.*

**Proof:** The proof is standard. One needs only to verify that when (i) is satisfied there are no cycles of period larger or equal to two, and then apply Sarkovskij's theorem. At $\omega = \omega_1$ there is a supercritical flip bifurcation. This is a consequence of standard arguments and the details are therefore omitted (*e.g.* Devaney [1986] or Preston [1982]). **Q.E.D.**

Figure 2 portrays the second iterate of $f_\omega$ for an appropriately chosen value of $\omega_1$. The presence of the cycle of period two is revealed by the triple intersection with the diagonal. Figure 3 does the same thing for the fourth iterate of the map which is plotted for a value of $\omega = \omega_2$ at which a cycle of period four exists.

**Figure 7.**

To procede further along the analythical path one needs to verify that the map (3.2′) has a negative Schwartzian derivative for admissible values of parameters. We recall here that the Schwartzian derivative of a function $f$ is defined as (*e.g.* Devaney [1986])

$$S(f_\omega)(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

One can check that $S(f_\omega)(z) < 0$ for all $z \in [0, 1]$ when $\omega > 1$, which will be assumed from now on.

It is well known that unimodal continuous functions defined on the unit interval and with $S(f_\omega) < 0$ have the property that every stable periodic orbit attracts either the end points of their domain or their critical point (*e.g.* Collet and Eckmann [1980]). The latter result has the following consequence for our model.

**Proposition 4** *For any set of admissible values of the parameters $A$ and $\theta$ there exists a value $\omega^* \geq 1$ such that $f_{\omega^*}$ has exactly one absolutely continuous and invariant measure which is ergodic over* $[0, 1]$.

**Proof:** One needs only to check that, for given $A$ and $\theta$, it is possible to choose $\omega^* \geq 1$ such that

$$A(\omega^*\theta)^{\omega^*} = (1 + \omega^*)^{\omega^*}$$

is true. Then $f_{\omega^*}(\hat{z}) = 1$, $f_{\omega^*}(1) = 0$ and the latter is unstable and such that the origin $0 \notin \{f^n_{\omega^*}(0) \mid n > 0\}^-$. A well known theorem of Misiurewicz (1980) is enough to justify our statement. **Q.E.D.**

**Figure 2.**

**Figure 3.**

that the very complicated pattern depicted in Figures 5-7 appears. Also in this case though one can verify that the period doubling cascade follows the Feigenbaum scenario with $\omega_\infty = 2 \cdot 7.3107\ldots$. Furthermore, as proved below in Proposition 4, an invariant and non-trivial measure can be numerically generated and a non-negative Liapunov exponent does exist after $\beta = 8$.

**Figure 5.**

**Figure 6**

In fact a cascade of period-doubling bifurcations appears as the parameter $\omega$ increases past $\omega_1$. An analytical derivation of the period-doubling values $\omega_n$, $n = 2, 3, \ldots$ is quite cumbersome, to say the least. It is instead possible to generate such behavior numerically, by fixing all model's parameters but $\omega$. Standard renormalization techniques can be used to verify that the period-doubling behavior illustrated below holds for generic values of the parameters.

A simple example can be obtained by choosing $\theta = 1.6$, $\delta = .5$ and $\nu = 2$. Then $\omega = 2\beta$ and the first few period-doubling bifurcations occur at the following points:

$$\omega_1 = 4.46, \quad \omega_2 = 5.308, \quad \omega_3 = 5.504, \quad \omega_4 = 5.5468, \quad \omega_5 = 5.5556,$$
$$\omega_6 = 5.5575, \quad \omega_7 = 5.55796, \quad \omega_8 = 5.55802 \ldots \tag{3.3}$$

Along such cascade an attracting cycle of period $2^n$ is created at $\omega_n$ from a cycle of period $2^{n-1}$ through a supercritical flip bifurcation. These values appear to converge to a limit $\omega_\infty$ in a geometric progression as

$$\omega_n = \omega_\infty - c \cdot \mathcal{F}^n \tag{3.4}$$

We have

$$\omega_\infty = 5.5681 \ldots; \quad c = 7.8749 \ldots; \quad \mathcal{F} = 4.669202 \ldots$$

and therefore $\mathcal{F}$ is the *Feigenbaum constant* (Feigenbaum (1978, 1979). In fact one can easily verify that the map (3.2′) satisfies the universal relation

$$\lim_{n \to \infty} \frac{\omega_n - \omega_{n-1}}{\omega_{n+1} - \omega_n} = 4.669202 \ldots \tag{3.5}$$

Figure 4 reports the bifurcation diagram generated by this choice of parameter values. As it should be expected, the odd numbered cycles appear after the period-doubling process is completed, with a cycle of period three appearing last, as predicted by Sarkovskij's theorem. For the particular configuration of parameter values we have chosen here a period-three cycle exists at $\omega_3 = 2 \cdot 3.08$ and is clearly visible in Figure 4. The existence of windows of stability even well inside the chaotic region is also detectable and will be confirmed later by our calculation of the Liapunov exponents.

**Figure 4.**

More generally one can prove the following

**Proposition 3** *For any choice of $\theta$, $\delta$, $\alpha$ and $\sigma$ satisfying the restrictions introduced in section two there exists a value of $\beta = \beta_3 > 0$ such that (3.2′) exhibits a cycle of period three at $\omega_3 = 2\beta_3$.*

**Proof:** Tedious but relatively straightforward algebra will show that the following is true for a suitable choice of $\omega_3$. Let $a = \min\ \{z \in [0,1] \mid z(1-z)^{\omega} = [(1+\omega)A\theta^{\omega}]^{-1}\}$ denote the smallest preimage of the critical point $\hat{z} = (1+\omega)^{-1}$ and $b = A(\omega\theta)^{\omega}(1+\omega)^{-(1+\omega)}$ the image of $\hat{z}$. Then for given $A < 1$ and $\theta > 1$ satisfying the restrictions listed in section 2 there exists $\omega_3 > 0$ such that

$$f_{\omega_3}([\hat{z}, b]) \supset [a, b]$$

which is known to guarantee the existence of a period-three cycle (see *e.g.* Devaney [1986]).
**Q.E.D.**

Indeed it is interesting to note that the bifurcation behavior of the map $f_\omega$ may be quite more complicated than the one described above. This occurs at those sets of parameter values at which the maximum of $f_\omega$ rises very slowly toward one as $\beta$ increases. For example we have experimented with the following choice: $\theta = 1.2$, $\delta = .75$ and $\nu = 2$ and observed