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An Approach to Equilibrium Selection

by

Akihiko Matsui and Kiminori Matsuyama*

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*Department of Economics, University of Pennsylvania; Department of Economics, Northwestern University, respectively. This is a revision of the paper previously circulated under the same title as CMS-EMS Discussion Paper #970, Northwestern University; Hoover Institution Working Papers E-91-14; CARESS Working Paper #91-36, University of Pennsylvania. We wish to thank R. Boylan, V. Crawford, D. Fudenberg, M. Kandori, J. Swinkels, referees and an associate editor as well as participants of the Conference on Game Theory and Mathematical Economics at Gotemba, Japan, and the workshops at Stanford Business School and the University of Chicago for their comments and discussion.

Abstract

We consider equilibrium selection in 2x2 bimatrix (both symmetric and asymmetric) games with two strict Nash equilibria by embedding it in a dynamic random matching game played by a continuum of anonymous agents. Unlike in the evolutionary game literature, we assume that the players are rational, seeking to maximize the expected discounted payoffs; but they are instead restricted to make a short run commitment when choosing actions. Modelling the friction this way yields the equilibrium dynamics, whose stationary states correspond to the Nash outcomes of the original game. Our selection is based on differential stability properties of the stationary states. It is shown that, as friction becomes arbitrarily small, a strict Nash outcome becomes uniquely absorbing and globally accessible if and only if it satisfies the Harsanyi and Selten (1988) notion of risk-dominance criterion. Our approach thus supplies another support for risk-dominance in addition to those given in the literature.

Keywords: Equilibrium Selection, Random Matching Games, Risk-dominance

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1. Introduction

We approach the problem of equilibrium selection in 2x2 bimatrix (both symmetric and asymmetric) games with two strict Nash equilibria. This class of games, which contains pure coordination games and the battle of the sexes as special cases, is not only important in its own right, but also captures a variety of economic problems in their essentials. Examples include adoption of new technologies (Farrell and Saloner 1985), bank runs (Diamond and Dybvig 1985), choice among alternative currencies as a medium of exchange (Matsuyama, Kiyotaki, and Matsui 1993), geographical distribution of cities (Krugman 1991a, 1991b), Keynesian macroeconomics (Cooper and John 1988), and economic development (Murphy, Shleifer and Vishny 1989; Matsuyama 1991, 1992b). In spite of its central role in game theory and economics, the literature offers very few formal approaches to the problem of equilibrium selection. For instance, the most solution concepts proposed in the literature on refinements of Nash equilibria, such as the strategic stability of Kohlberg and Mertens (1986), have nothing to say about selection among strict Nash equilibria.

Our approach to this problem is to examine the stability of strict Nash equilibria in an explicitly dynamic context. To this end, we consider the society consisting of a continuum of anonymous agents and each agent plays the game repeatedly with an opponent randomly chosen from the population. All players maximize the expected discounted payoffs with one restriction; they need to make a short-run commitment to the action they chose. The opportunity to switch actions arrives stochastically; it follows a Poisson process, which is identical and independent across players. By modelling some friction this way, this dynamic game generates nontrivial equilibrium paths of the behavior patterns in the society, whose stationary states correspond to the Nash equilibrium outcome of the original game.

The stability properties of stationary states, of course, depend on the degree of friction, defined by the expected duration of commitment the players have to make. For example, suppose that the initial behavior patterns are in the neighborhood of a strict Nash equilibrium of the original game, say (L,L) in Figure 1. One can show that, in the presence of large friction, an equilibrium path is unique and the behavior patterns always converge to (L,L). In this sense, any strict Nash is an <u>absorbing</u> state with sufficiently large friction. When friction is small, however, (L,L) may be <u>fragile</u> in that another equilibrium path exists, along which the behavior patterns move away from (L,L) and converge to (R,R). In other words, beliefs that the "band wagon" effects will induce all players to switch from L to R in the future may be consistent, thereby upsetting (L,L). The possibility of such a self-fulfilling prophecy itself should not be a surprise at all, given the multiplicity of Nash equilibria of the original game.[1]

What may be surprising is that the two stationary states, corresponding to the two strict Nash equilibria of the original game, possess different stability properties for a small degree of friction. A seemingly minor (and natural) perturbation of the original game thus help to discriminate between the two strict Nash equilibria of the original game. More specifically, we will show the following results. First, for generic 2x2 bimatrix games, one of the strict Nash equilibria becomes <u>uniquely absorbing</u> (that is, it is the

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[1]It is worth noting that the dynamic nature of the game is not at all responsible for the multiplicity; because of our anonymity assumption, the logic of the folk theorem does not apply here. As a matter of fact, it is straightforward to show that, if the original 2x2 bimatrix game has a unique Nash equilibrium, the equilibirum path of the dynamic game is unique for any initial condition and the behavior pattern converges to the unique Nash equilibrium. The multiplicity of equilibrium paths is thus entirely due to the multiplicity of Nash equilibria of the original 2x2 games.

only absorbing state) and all other states become fragile, as friction becomes arbitrarily small. Second, this uniquely absorbing state has additional stability property, which we call globally accessible; that is, for any initial behavior patterns, there exists an equilibrium path along which the behavior patterns converge to it. We view that these two properties, taken together, make the uniquely absorbing state a natural choice among strict Nash equilibria.[2] Interestingly enough, a strict Nash equilibrium outcome is uniquely absorbing (and globally accessible) if and only if it is risk-dominant in the sense of Harsanyi and Selten (1988). Thus, our selection criterion coincides with their risk-dominance criterion for 2x2 bimatrix games.

Harsanyi and Selten offer two justifications of the risk dominance criterion. The one is an axiomatic derivation, based on the following three axioms; i) invariance with respect to isomorphisms, ii) best-reply invariance, and iii) payoff monotonicity (1988, Ch. 3). The other relies on the tracing procedure (Ch. 4). In an attempt to offer a dynamic story of equilibrium selection, Kandori, Mailath and Rob (1993) consider evolutionary models with constant flow of mutations, which generate Markov processes in the behavior patterns.[3] It turns out that the stationary distribution of the Markov processes attaches probability one to the risk dominant outcome in the limit as the rate of mutation goes to zero. In this work, the convergence to a strict Nash equilibrium is studied in the context of repeated play by myopic

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[2]We are perfectly aware that some readers may not find both properties convincing. They do not need to. We offer two distinctive justifications; it suffices if the reader finds one of them convincing.

[3]See also Foster and Young (1990), and for some recent extensions, Ellison (1992), Fudenberg and Harris (1992), Kandori and Rob (1992), Young (1993).

players.[4] Their selection criterion coincides with the risk-dominance criterion, because the risk dominant outcome has a larger basin of attraction. On the other hand, our approach is based on the rational calculations by players.[5] The risk dominant outcome is absorbing because deviating from it implies a payoff loss even under the best possible scenario; the risk dominant outcome is globally accessible for a small degree of friction because, starting from the risk dominated outcome, there exist consistent conjectures with which deviating from it leads to a gain in the expected payoffs. One major strength of our approach over Kandori-Mailath-Rob, other than our rationality assumption, is that the risk dominant outcome is an inescapable one in our model, while the society will escape the risk dominant outcome with probability one in their model. The major weakness of our approach is that the society could escape from the risk-dominated outcome, but it could also stay there forever.

Recently, we become aware of the work by Carlsson and van Damme (1989). They propose to analyze a 2x2 bimatrix game with two strict Nash equilibria as being drawn randomly from the entire class of 2x2 bimatrix games, and the two

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[4]Many studies on evolutionary games also address the question of how a particular equilibrium will emerge in a dynamic context; see, for example, Boylan (1990), Canning (1989), Friedman (1991), Fudenberg and Kreps (1988), Gilboa and Matsui (1991), Matsui (1992), Matsuyama (1992c), Milgrom and Roberts (1990, 1991), Swinkels (1993), and Taylor and Jonker (1978). These studies do not, however, offer an equilibrium selection criterion, since all strict Nash equilibria share the same dynamic properties in their models. It is the presence of constant flow of mutations that enables Kandori, Mailath, and Rob to select the risk-dominant outcome.

[5]Some mention should be made of Kalai and Lehrer (1991). They consider an infinite repetition of a stage game between fixed players. Players have some priors over the opponents' repeated game strategies and try to maximize their expected discounted payoffs. They show that in spite of discrepancy in their initial beliefs, the actual sequence of actions converges to that of Nash equilibrium. Any Nash equilibrium appears as an outcome; their motivation is not to tell a story of equilibrium selection.

players are not completely sure of the game they are playing. Using iterated dominance, Carlsson and van Damme show that the fact that two strict Nash exist is not a part of common knowledge forces the players to coordinate on the risk dominant equilibrium.

We view these alternative approaches for the risk dominance criterion complementary. The two justifications given by Harsanyi and Selten and the one by Carlsson and van Damme are quite convincing when the two players play the game only once in an isolated situation. However, 2x2 bimatrix games with strict Nash equilibria are often used in economics as a model for coordination problems in aggregate economic activities, as indicated in the opening paragraph. The approaches based on dynamic games played by anonymous agents--myopic in the case of Kandori, Mailath and Rob, and rational in our case--, seem more appropriate in this context. Although each may have its own merits and drawbacks, these alternative justifications jointly provide a strong support for the risk-dominance criterion. It is certainly remarkable that such different approaches lead to the same selection criterion.

In section 2, we deal with symmetric games played by the homogeneous population. Asymmetric games played by the heterogeneous population are considered in section 3. Some concluding remarks are given in section 4.

2. Symmetric Games

In this section, we restrict our attention to the symmetric game given in Figure 1. This game has two strict Nash equilibria, (L,L) and (R,R), as well as one mixed strategy equilibrium in which each player chooses L with probability $\mu = (d-c)/\{(a-b)+(d-c)\}$. Instead of analyzing this game in isolation, we envision that this game is played repeatedly in a society with a continuum of identical anonymous players. At every point in time, each player is matched to form a pair with another player, randomly drawn from the population, and they play the game anonymously. All players are highly rational and choose a strategy to maximize the expected discounted payoffs. Because of the anonymity, they are engaged in this maximization without taking into account strategic considerations such as reputation and retaliation.

The key assumption is that no player can switch actions at every point in time. Every player needs to make a commitment to a particular action in the short run. Following Matsuyama (1991, 1992a, b, c), we assume that the opportunity to switch actions arrives randomly; it follows the Poisson process with p being the mean arrival rate. Furthermore, it is assumed that the process is independent across the players and there is no aggregate uncertainty.[6] The strategy distribution in the society as of time t can be thus described as $x_t[L] + (1-x_t)[R]$, where x_t is the fraction of the players that are committed to action L as of time t. We simply call x_t the <u>behavior pattern</u> in the society. Because of the restriction imposed above, x_t changes continuously over time and the rate of change in x_t belongs to $[-px_t, p(1-x_t)]$. Furthermore, any feasible path necessarily satisfies $x_0e^{-pt} \leq x_t \leq 1 - (1-x_0)e^{-pt}$, where the initial condition, x_0, is given exogenously, or "by history."

When the opportunity to switch arrives, players choose the action which results in the higher expected discounted payoffs, knowing the future path of x as well as their own inability of switching actions continuously. Since the strategy distribution as of time t is $x_t[L] + (1-x_t)[R]$, the value of playing

[6]There are some technical problems concerning the law of large numbers with a continuum of i.i.d. random variables, as first pointed out by Feldman and Gilles (1985) and Judd (1985). Boylan (1992) and Gilboa and Matsui (1992) discuss these issues in the context of random matching games and offer some possible solutions.

action L instead of R as of time t is equal to

$$\{ax_t+c(1-x_t)\} - \{bx_t+d(1-x_t)\} = \{(a-b)+(d-c)\}(x_t-\mu) \ ,$$

and thus players, given the opportunity, commit to play L if $V_t > 0$ and to play R if $V_t < 0$ and are indifferent if $V_t = 0$, where

$$V_t = (p+\theta)\int_0^\infty (x_{t+s}-\mu)e^{-(p+\theta)s}ds \ , \tag{1}$$

with $\theta > 0$ being the discount rate. Therefore, $\{x_t\}_{t=0}^\infty$ is an equilibrium path from x_0 if its right-hand derivative exists and satisfies

$$\frac{d^+x_t}{dt} \in \begin{cases} \{p(1-x_t)\} & \text{if } V_t > 0, \\ [-px_t \ , \ p(1-x_t)] & \text{if } V_t = 0, \\ \{-px_t\} & \text{if } V_t < 0, \end{cases} \tag{2}$$

for all $t \in [0,\infty)$. Equation (2) states that all players currently playing action R (resp. L), if given the opportunity, switch to L (resp. R), when $V_t >$ (resp. $<$) 0.

It is straightforward to show that $x = 0$, μ, and 1 are the only stationary states of the dynamics (1) and (2); that is, $x \in [0,1]$ is a stationary state if and only if it is a Nash equilibrium of the original game. We use (1) and (2) to study the stability of the Nash equilibria.

Since there are generally multiple equilibrium paths from a given initial condition, one need be specific about what the stability means. It is thus necessary to introduce some terminologies.[7]

[7]Alternatively, we could have borrowed a variety of stability concepts in the set-valued differential equations, such as "Absorbent Stable Sets (ASS)" of Gilboa and Samet (1991). We have chosen to avoid introducing such a formality, however, given the simple structure of our dynamics. One can show

Definitions:

i) $x \in [0,1]$ is accessible from $x' \in [0,1]$, if there exists an equilibrium path from x' that reaches or converges to x. $x \in [0,1]$ is globally accessible if it is accessible from any $x' \in [0,1]$.

ii) $x \in [0,1]$ is absorbing if there is a neighborhood of x, U, such that any equilibrium path from U converges to x. $x \in [0,1]$ is fragile if it is not absorbing.

By definition, if an absorbing state, x, is globally accessible, then it is a unique absorbing state in $[0,1]$ and any state in $[0,1]\backslash\{x\}$ is fragile. The definitions do not rule out the possibility that a state may be both fragile and globally accessible, or that a state may be uniquely absorbing but not globally accessible. As will be shown below, however, these situations never exist and a state is uniquely absorbing if and only if it is globally accessible in the dynamics considered in this paper.

Finally, define the degree of friction by $\delta \equiv \theta/p$, the expected duration of the commitment (with the unit of time is normalized so that the discount rate is equal to one).

Lemma 1.

a) $x = 0$ is globally accessible if and only if $(1+\delta)/(2+\delta) \le \mu < 1$,

b) $x = 1$ is globally accessible if and only if $0 < \mu \le 1/(2+\delta)$,

c) $x = 0$ is absorbing if and only if $1/(2+\delta) < \mu < 1$,

d) $x = 1$ is absorbing if and only if $0 < \mu < (1+\delta)/(2+\delta)$.

Proof. See the appendix.

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that any absorbing point, taken as a singleton set, is an ASS.

Lemma 1 implies that there exists at least one and at most two absorbing states. Furthermore, a strict Nash is globally accessible if and only if it is uniquely absorbing.[8] In other words, if $x = 1$ is accessible from $x = 0$, then $x = 0$ is not accessible from $x = 1$, and vice versa. Thus, Lemma 1 can be rephrased as:

Proposition 1.

a) (R,R) is uniquely absorbing and globally accessible if $(1+\delta)/(2+\delta) \leq \mu < 1$; (L,L) is uniquely absorbing and globally accessible if $0 < \mu \leq 1/(2+\delta)$; both (L,L) and (R,R) are absorbing if $1/(2+\delta) < \mu < (1+\delta)/(2+\delta)$.

b) For any $\mu \in (0,1)$, both (L,L) and (R,R) are absorbing for a sufficiently large $\delta > 0$.

c) If $\mu \in (1/2, 1)$, (R,R) is uniquely absorbing and globally accessible for a sufficiently small $\delta > 0$; If $\mu \in (0,1/2)$, (L,L) is uniquely absorbing and globally accessible for a sufficiently small $\delta > 0$.

Figure 2 illustrates Proposition 1. What b) states is that any strict Nash is absorbing in the presence of large friction.[9] More interestingly, unless $\mu = 1/2$, one strict Nash becomes fragile, while the other becomes globally accessible, as friction goes to zero. If $\mu > 1/2$, there is an equilibrium

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[8]The assumption of a positive discount rate is crucial for these results. Note that (1) and (2) give a well defined dynamics, even with a negative discount rate, as long as $1 + \delta > 0$. If we were to allow a negative discount rate, every state in [0,1] would become both fragile and globally accessible, when $(1+\delta)/(2+\delta) \leq \mu \leq 1/(2+\delta)$. (In the terminology of Gilboa and Samet, the entire space, [0,1], becomes an Absorbent Stable Set.)

[9]In the limit as δ goes to infinity, the dynamics (1) and (2) are equivalent to the best response dynamics proposed in Gilboa and Matsui (1991); see also Matsui (1990, 1991) and Matsuyama (1992c). Every strict Nash equilibrium is absorbing in the best response dynamics.

path that traverses from (L,L) to (R,R); that is, even if (L,L) is the initial behavior patterns in this society, there exist consistent beliefs, with which the behavior patterns converge to (R,R) and thereby upsetting (L,L). On the other hand, if the initial behavior patterns are given by (R,R), no consistent beliefs can upset this behavior patterns. In this sense, (R,R) dominates (L,L) if $\mu > 1/2$. Likewise, (L,L) dominates (R,R) if $\mu < 1/2$.

It should be noted that the condition, $\mu > 1/2$, is equivalent to $d - c > a - b$; the deviation loss associated with (R,R) is larger than the deviation loss at (L,L). That is, in the terminology of Harsanyi/Selten, (R,R) risk dominates (L,L). Similarly, (L,L) risk-dominates (R,R) if $\mu < 1/2$. In sum, a Nash equilibrium of the symmetric game given in Figure 1 is a unique absorbing (and globally accessible) state in the presence of sufficiently small friction, if and only if it satisfies the risk-dominant notion of Harsanyi/Selten.

To grasp the intuition behind these results, it is useful to consider a slightly more general game in which the payoff difference of playing L instead of R is given by $\pi(x_t)$, where π is a strictly increasing function and satisfies $\pi(0) < 0$ and $\pi(1) > 0$. (The pairwise random matching game is a special case in which $\pi(x) = x - \mu$.) The outcome (L,L) can be upset when the players have an incentive to deviate for a feasible path from $x = 1$. Because of the monotonicity of π, the incentive to deviate is the strongest if all players are anticipated to switch from L to R in the future, or $x_t = e^{-pt}$. Thus, the condition for $x = 1$ being fragile is

$$V_0 = (p+\theta)\int_0^\infty \pi(e^{-pt})e^{-(p+\theta)t}dt \le 0 \ . \tag{3}$$

As seen from this expression, an increase in the expected duration of the commitment (a small p) has the two opposite effects. On one hand, it reduces the effective discount rate; the players are more concerned about the future when making decisions. On the other hand, it reduces the rate of change in the behavior patterns so that the current strategy distribution becomes more important in calculating the expected discounted payoffs. The strictly positive discount rate, $\theta > 0$, implies that the second effect always dominates the first, since (3) can be rewritten to:

$$V_0 = (1+\delta)\int_0^1 \pi(x)x^\delta dx \leq 0 \ , \tag{4}$$

by letting $x = e^{-pt}$. Condition (4) means that the expected discounted payoff of choosing action L when all other players are anticipated to swicth from L to R is given by the weighted average of π. Note that, as x moves from 1 to 0, the players attach more weight to a higher value of x with a large degree of friction, $\delta = \theta/p$. In the limit as δ goes to infinity, $V_0 = \pi(1) > 0$ so that (4) is violated; or (L,L) becomes absorbing with a sufficiently large friction.

Similarly, starting from x = 0, the incentive to deviate is the strongest when all players are anticipated to switch from R to L in the future, or $x_t = 1 - e^{-pt}$, so that the condition for x = 0 being fragile is given by

$$V_0 = (p+\theta)\int_0^\infty \pi(1-e^{-pt})e^{-(p+\theta)t}dt \geq 0 \ ,$$

or

$$V_0 = (1+\delta)\int_0^1 \pi(x)(1-x)^\delta dx \geq 0 \ . \tag{5}$$

Thus, as x moves from 0 to 1, the players attach more weight to a lower value of x with a large degree of friction, $\delta = \theta/p$. In the limit as δ goes to infinity, $V_0 = \pi(0) < 0$ so that (5) is violated; or (R,R) becomes absorbing with a sufficiently large friction.

The two conditions, (4) and (5), are mutually exclusive for any $\delta > 0$ so that at least one of the two strict Nash outcomes is absorbing. Furthermore, in the limit as δ goes to zero, (4) and (5) become

$$\int_0^1 \pi(x)dx < 0 \ , \tag{6}$$

and

$$\int_0^1 \pi(x)dx > 0 \ , \tag{7}$$

respectively. For the pairwise random matching game, $\pi(x) = x - \mu$ and (6) and (7) are equal to $\mu > 1/2$ and $\mu < 1/2$, respectively. This shows why only one strict Nash outcome remains absorbing as the friction goes to zero for generic games. When the expected duration of the commitments becomes extremely small and the behavior patterns can move between 0 and 1 arbitrarily fast (but are not able to jump between them), all that matters is the average payoff differences. If action L performs better than R on average, then (L,L) is absorbing, while (R,R) is fragile. Note that the uniqueness of the absorbing state in the limit does not depend on the linearity of the payoff differences.

The above discussion also points out the significant difference between the logic behind our result and that of Kandori, Mailath and Rob (1993). Recall that their model is based on the repeated play by myopic players and the constant flow of mutations, so that the stationary distribution of the behavior patterns depends on the size of the basins of attraction. Their selection criterion coincides with the Harsanyi/Selten risk-dominance criterion, because the risk dominant outcome has a larger basin of attraction. On the other hand, we rely on the rational calculations by players. Our selection criterion coincides with the Harsanyi/Selten criterion because deviating from the risk dominant outcome always implies a payoff loss, whereas there exist consistent conjectures with which deviating from the risk dominated outcome leads to a gain in the expected payoffs.[10]

3. Asymmetric Games

In this section, we extend our analysis to the class of asymmetric games given in Figure 3. Again, there are two strict Nash equilibria, (L_1,L_2) and (R_1,R_2), and one mixed strategy Nash equilibrium in which player i plays L_i with probability $\mu_j \equiv (d_j-c_j)/\{(a_j-b_j)+(d_j-c_j)\}$, where $i \neq j$. As in the previous section, we consider the random matching framework, but the players are now divided into two groups of equal size, 1 and 2. Each player from group 1 (player 1) is randomly matched with a player from group 2 (player 2) to play the game under the same restriction with the previous case. Let x_t^i $(i = 1,2)$ denote the fractions of players i who play L_i as of time t. Then, the equilibrium dynamics of the behavior patterns $\{(x_t^1, x_t^2)\}_{t=0}^{\infty}$ are described by

$$\frac{d^+x_t^i}{dt} \in \begin{cases} \{p(1-x_t^i)\} & \text{if } V_t^i > 0, \\ [-px_t^i \ , \ p(1-x_t^i)] & \text{if } V_t^i = 0, \\ \{-px_t^i\} & \text{if } V_t^i < 0, \end{cases} \tag{8}$$

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[10] Matsuyama (1992b) considers a dynamic extension of the coordination game with nonlinear payoff differences and appeals to the equilibrium selection criterion proposed in this paper. Because of the nonlinearity, the selection could be different if the approach by Kandori, Mailath and Rob were adopted.

where

$$V_t^i \equiv (p+\theta)\int_0^\infty (x_{t+s}^j - \mu_i)e^{-(p+\theta)s}ds\ , \qquad (i,\ j,=1,\ 2,\quad i \neq j) \tag{9}$$

as well as the initial condition, (x_0^1, x_0^2).

As before, the set of stationary states of (8) and (9) is $\{(0,0), (\mu_2,\mu_1), (1,1)\}$, which is identical to the set of Nash equilibria of the original game. The definitions of <u>accessible</u>, <u>globally accessible</u>, <u>absorbing</u>, and <u>fragile</u>, can be directly extended into the dynamics on $[0,1]^2$.

To state the properties of (0,0) and (1,1), or equivalently (R_1,R_2) and (L_1,L_2), let us define the following partition of $(0,1)^2 = A(\delta) + B(\delta) + C(\delta)$ (see Figure 4):

$$A(\delta) \equiv \{(\mu_1,\mu_2) \in (0,1)^2\colon\ \mu_2 \geq F_\delta(\mu_1)\ \}\ ,$$

$$B(\delta) \equiv \{(\mu_1,\mu_2) \in (0,1)^2\colon\ 1-\mu_2 \geq F_\delta(1-\mu_1)\ \}\ ,$$

$$C(\delta) \equiv \{(\mu_1,\mu_2) \in (0,1)^2\colon\ 1-F_\delta(1-\mu_1) < \mu_2 < F_\delta(\mu_1)\ \}\ ,$$

where

$$F_\delta(X) \equiv \begin{cases} f_\delta(X), & if\ 0 < X \leq \dfrac{1+\delta}{2+\delta}\ , \\[2ex] f_\delta^{-1}(X), & if\ \dfrac{1+\delta}{2+\delta} \leq X < 1\ , \end{cases}$$

and

$$f_\delta(X) \equiv 1 - (2+\delta)^\delta\left(\frac{X}{1+\delta}\right)^{1+\delta}.$$

Simple algebra shows that $F_\delta(X)$ is strictly decreasing, strictly concave and

$$\lim_{X\to 0}F_\delta(X) = 1,\ \ \lim_{X\to 1}F_\delta(X) = 0,\ \ F_\delta\left(\frac{1+\delta}{2+\delta}\right) = \frac{1+\delta}{2+\delta},\ \ F_\delta'\left(\frac{1+\delta}{2+\delta}\right) = -1.$$

<u>Lemma 2</u>.

a) (0,0) is globally accessible if and only if $(\mu_1,\mu_2) \in A(\delta)$.

b) (1,1) is globally accessible if and only if $(\mu_1,\mu_2) \in B(\delta)$.

c) (0,0) is absorbing if and only if $(\mu_1,\mu_2) \in (0,1)^2 \backslash B(\delta) = A(\delta) + C(\delta)$.

d) (1,1) is absorbing if and only if $(\mu_1,\mu_2) \in (0,1)^2 \backslash A(\delta) = B(\delta) + C(\delta)$.

Proof. See the appendix.

Again, Lemma 2 implies that there is at least one and at most two absorbing state and that a state is uniquely absorbing if and only if it is globally accessible. Thus, one can rephrase it as:

Proposition 2.

a) (R_1,R_2) is uniquely absorbing and globally accessible if $(\mu_1,\mu_2) \in A(\delta)$; (L_1,L_2) is uniquely absorbing and globally accessible if $(\mu_1,\mu_2) \in B(\delta)$; Both (R_1,R_2) and (L_1,L_2) are absorbing if $(\mu_1,\mu_2) \in C(\delta)$.

b) For any $(\mu_1,\mu_2) \in (0,1)^2$, both (R_1,R_2) and (L_1,L_2) are absorbing for a sufficiently large $\delta > 0$.

c) If $\mu_1 + \mu_2 < 1$, (L_1,L_2) is uniquely absorbing and globally accessible for a sufficiently small $\delta > 0$. If $\mu_1 + \mu_2 > 1$, (R_1,R_2) is uniquely absorbing and globally accessible for a sufficiently small $\delta > 0$.

Proof:

a) This follows directly from Lemma 2.

b) Note that $\lim_{\delta\to\infty} F_\delta(X) = 1$ for $0 < X < 1$ monotonically. Thus, $C(\infty) = (0,1)^2$, from which b) follows from a).

c) For any $\delta > 0$, $F_\delta(X) \geq 1 - X$ and $\lim_{\delta\to 0} F_\delta(X) = 1 - X$ for $0 < X < 1$. Therefore, $(\mu_1,\mu_2) \in A(\delta)$ for a sufficiently small $\delta > 0$ if $\mu_1 + \mu_2 > 1$, and $(\mu_1,\mu_2) \in B(\delta)$ for a sufficiently small $\delta > 0$ if $\mu_1 + \mu_2 < 1$. Q.E.D.

Figure 4 illustrates Proposition 2a). It shows that, for a given δ, if the unique mixed strategy equilibrium is close to (L_1,L_2), then (R_1,R_2) is absorbing. That is, for any initial behavior patterns, there is an equilibrium path that converges to (R_1,R_2), and, if any initial behavior patterns are in the neighborhood of (R_1,R_2), any equilibrium path converges to (R_1,R_2). Similarly, for a given δ, if the unique mixed strategy equilibrium is sufficiently close to (R_1,R_2), then (L_1,L_2) is absorbing. If the unique mixed strategy equilibrium belongs to $C(\delta)$, on the other hand, both strict Nash equilibria are absorbing. These regions, $A(\delta)$ and $B(\delta)$ shrink as δ becomes large, and, in the limit as friction goes to infinity, disappear. Thus, as Proposition 2b) states, in the presence of large friction, both strict Nash equilibria become absorbing. Proposition 2c), on the other hand, states that, as friction goes to zero, one strict Nash equilibrium becomes fragile and the other becomes globally accessible. Proposition 2c) also states that (L_1,L_2) becomes absorbing if $\mu_1 + \mu_2 < 1$, which is equivalent to $(1-\mu_1)(1-\mu_2) > \mu_1\mu_2$, or $(a_1-b_1)(a_2-b_2) > (d_1-c_1)(d_2-c_2)$; that is, the product of deviation losses associated with (L_1,L_2) is larger than the product of deviation losses at (R_1,R_2). That is, (L_1,L_2) is absorbing in the presence of small friction if and only if it is risk-dominant in the sense of Harsanyi-Selten.

The basic intuition behind these results is analogous to the case of the symmetric case, but the fact that the behavior pattern is represented in a two dimensional state space introduces one complication, which deserves some emphasis. When the society moves from a neighborhoond of the risk dominated outcome to the risk dominant one, one group of players may not start switching immediately to the risk dominant strategy. For example, suppose that (R_1,R_2) is risk dominant $(\mu_1 + \mu_2 > 1)$, and hence it is uniquely absorbing and globally accessible for a sufficiently small $\delta > 0$. Furthermore, suppose $\mu_1 < 1/2$. Then, as illustrated in Figure 5, an equilibrium path from a neighborhood of the risk dominated outcome, (L_1, L_2), to the risk dominant outcome needs to be designed such that, at an initial stage, group 2 players switch to R_2, while group 1 players continue to play L_1. An equilibrium path to (R_1, R_2) thus cannot be monotone. Group 2 has to act as a leader, in order to make group 1 willing to change. This property has significant implications for coordination problems. For example, such a sequencing may turn out to be critical when designing a successful economic reform program for transitions from socialist economies towards free market economies.

4. Concluding Remarks

We have considered equilibrium selection in 2x2 bimatrix games with two strict Nash equilibria by embedding it in a dynamic random matching framework played by a continuum of anonymous agents. Unlike in the evolutionary game literature, the players are assumed to be rational, seeking to maximize the expected discounted payoffs. They are instead restricted to make a short run commitment when choosing actions. Modelling the friction this way yields the equilibrium dynamics, whose stationary states correspond to the Nash equilibrium outcomes of the original game. Our selection is based on differential stability properties of the stationary states. As friction becomes arbitrarily small, a strict Nash equilibrium outcome becomes <u>uniquely absorbing</u> and <u>globally accessible</u> if and only if it satisfies the Harsanyi and Selten (1988) notion of risk-dominance criterion. Our approach thus supplies another support for risk-dominance in addition to those given in the existing literature.

One weakness of our approach is that, if the initial position of the society is at the risk dominated Nash equilibrium, then playing it forever is a legitimate equilibrium path in our dynamic game. We are not sure whether this weakness is any more serious than the weaknesses of alternative approaches, because our selection is based on the stability properties of equilibrium paths. Nevertheless, it would be nice to be able to eliminate altogether the chance of the risk dominated equilibrium being played forever. One possibility would be to introduce some stochastic shocks on the beliefs, such as sunspots, so that there are always some probability that the society would follow any equilibrium path. Then, eventually, the society would be trapped into the absorbing region of the risk dominant outcome, from which it is unable to escape. Many economists may find an approach based on random shocks due to sunspots combined with the rationality of players more satisfactory than an approach based on the random shocks due to mutations combined with the myopia of players.

Restricting the ability of players to change actions may also be useful for dealing with other problems in game theory. For example, in repeated games, one might expect that such a restriction should be able to narrow down the set of equilibrium payoffs, thereby providing a partial resolution for the folk theorem. A complete characterization of the equilibrium set may be quite difficult because one need to drop the anonymity of players, the assumption that greatly simplifies our analysis. However, any progress in this direction would be highly desirable.

Appendix

Proof of Lemma 1. To prove the "if" part of a) and the "only if" part of d), it suffices to demonstrate that, if $(1+\delta)/(2+\delta) \le \mu < 1$, a feasible path from $x = 1$ to $x = 0$, $x_t = e^{-pt}$ satisfies the equilibrium condition, that is $V_t \le 0$ for all t along this path. This can be checked as follows:

$$V_t = (p+\theta)\int_0^\infty (e^{-p(t+s)} - \mu)e^{-(p+\theta)s}ds = e^{-pt}\left(\frac{1+\delta}{2+\delta}\right) - \mu \le 0 .$$

To prove the "if" part of d) and the "only if" part of a), it suffices to prove that, if $0 < \mu < (1+\delta)/(2+\delta)$, the equilibrium path is unique and converges to $x = 1$ for x_0 sufficiently close to 1. Note that any feasible path from x_0 satisfies $x_t \ge x_0 e^{-pt}$. Therefore, if $\mu(2+\delta)/(1+\delta) < x_0 < 1$,

$$V_0 \ge (p+\theta)\int_0^\infty (x_0 e^{-ps} - \mu)e^{-(p+\theta)s}ds = x_0\left(\frac{1+\delta}{2+\delta}\right) - \mu > 0 .$$

This implies $x_0 \le x_t < 1$, and $V_t > 0$ for all t. Thus, $x_t = 1 - (1-x_0)e^{-pt}$, and $\lim_{t\to\infty} x_t = 1$. This proves a) and d). The proof of b) and c) follows similarly, due to the symmetry. Q.E.D.

Proof of Lemma 2. The proof is divided into three parts.

Part 1. Proof that (0,0) is globally accessible if $(\mu_1, \mu_2) \in A(\delta)$:

Without loss of generality, we assume $\mu_1 \le \mu_2$, which can be further divided into the two cases: 1-A) $(1+\delta)/(2+\delta) \le \mu_1 \le \mu_2$, and 1-B) $f_\delta(\mu_1) \le \mu_2$, and $(1+\delta)/(2+\delta) > \mu_1$.

1-A) $(1+\delta)/(2+\delta) \le \mu_1 \le \mu_2$: it suffices to show $(x_t^1, x_t^2) = (x_0^1 e^{-pt}, x_0^2 e^{-pt})$ is an equilibrium path for any $(x_0^1, x_0^2) \in [0,1]^2$, which can be checked as follows: for i, j = 1, 2, $i \ne j$,

$$V_t^i = (p+\theta)\int_0^\infty \{x_0^j e^{-p(t+s)} - \mu_i\} e^{-(p+\theta)s} ds = x_0^j e^{-pt}\left[\frac{1+\delta}{2+\delta}\right] - \mu_i \le 0 .$$

1-B) $f_\delta(\mu_1) \le \mu_2$, and $(1+\delta)/(2+\delta) > \mu_1$: If $x_0^2 \le \mu_1(2+\delta)/(1+\delta)$, one can show $(x_t^1, x_t^2) = (x_0^1 e^{-pt}, x_0^2 e^{-pt})$ is an equilibrium path converging to (0,0), as in 1-A). Suppose $x_0^2 > \mu_1(2+\delta)/(1+\delta)$. We show that a feasible path from (x_0^1, x_0^2) to (0,0), defined by

$$x_t^1 = \begin{cases} 1-(1-x_0^1)\,e^{-pt} & \text{if } t < T, \\ [1-(1-x_0^1)\,e^{-pT}]\,e^{-p(t-T)} & \text{if } t \ge T, \end{cases} \quad \text{and} \quad x_t^2 = x_0^2 e^{-pt}$$

where T satisfies $x_0^2 e^{-pT} = \mu_1\left(\frac{2+\delta}{1+\delta}\right) < 1$, is an equilibrium path. First,

$$V_t^1 = (p+\theta)\int_0^\infty \{x_0^2 e^{-p(t+s)} - \mu_1\} e^{-(p+\theta)s} ds = x_0^2 e^{-pt}\left[\frac{1+\delta}{2+\delta}\right] - \mu_1 ,$$

so that $V_t^1 > 0$ if $t < T$; $= 0$ if $t = T$; < 0 if $t > T$. Second, let y_t be defined by

$$y_t \equiv \begin{cases} 1 & \text{if } t < T, \\ e^{-p(t-T)} & \text{if } t \ge T, \end{cases}$$

Note that y_t is nonincreasing and $y_t \ge x_t^1$ for all t. Therefore,

$$\begin{aligned} V_t^2 &= (p+\theta)\int_0^\infty \{x_{t+s}^1 - \mu_2\} e^{-(p+\theta)s} ds \le (p+\theta)\int_0^\infty \{y_{t+s} - \mu_2\} e^{-(p+\theta)s} ds \\ &\le (p+\theta)\int_0^\infty \{y_s - \mu_2\} e^{-(p+\theta)s} ds = 1 - \mu_2 - \frac{1}{2+\delta} e^{-(p+\theta)T} \\ &= 1 - \mu_2 - \frac{1}{2+\delta}\left[\left[\frac{\mu_1}{x_0^2}\right]\left[\frac{2+\delta}{1+\delta}\right]\right]^{1+\delta} \le f_\delta(\mu_1) - \mu_2 \le 0 . \end{aligned}$$

<u>Part 2</u>. Proof that (1,1) is absorbing if $(\mu_1, \mu_2) \in B(\delta) + C(\delta)$.

Without loss of generality, we assume $\mu_1 \le \mu_2 < f_\delta(\mu_1)$, which implies $\mu_1 < (1+\delta)/(2+\delta)$. First, note that, for any feasible path, if $x_t^2 > \mu_1(2+\delta)/(1+\delta)$, then $x_{t+s}^2 \ge x_t^2 e^{-ps}$, and

$$V_t^1 \ge (p+\theta)\int_0^\infty \{x_t^2 e^{-ps} - \mu_1\} e^{-(p+\theta)s} ds = x_t^2\left[\frac{1+\delta}{2+\delta}\right] - \mu_1 > 0 \ .$$

This implies that, for $x_0^2 > \mu_1(2+\delta)/(1+\delta)$, $V_t^1 > 0$ for all $t < T$, where T satisfies $x_0^2 e^{-pT} = \mu_1\left(\frac{2+\delta}{1+\delta}\right) < 1$. Thus,

$$x_t^1 \ \ge \begin{cases} 1-(1-x_0^1)\,e^{-pt} & \text{if } t < T, \\ [1-(1-x_0^1)\,e^{-pT}]\,e^{-p(t-T)} & \text{if } t \ge T, \end{cases}$$

for all $t > 0$. Since the right hand side is continuous in x_0^1, one can choose x_0^1 sufficiently close to 1 so that, for any $\epsilon_1 > 0$,

$$V_0^2 \ge (p+\theta)\int_0^\infty (y_s - \mu_2) e^{-(p+\theta)s} ds - \epsilon_1 = 1 - \mu_2 - \frac{1}{2+\delta} e^{-(p+\theta)T} - \epsilon_1$$

$$= 1 - \mu_2 - \frac{1}{2+\delta}\left[\left(\frac{\mu_1}{x_0^2}\right)\left(\frac{2+\delta}{1+\delta}\right)\right]^{1+\delta} - \epsilon_1 \ .$$

Therefore, for any $\epsilon_2 > 0$, by choosing x_0^2 sufficiently close to 1,

$$V_0^2 \ge f_\delta(\mu_1) - \mu_2 - \epsilon_1 - \epsilon_2 > 0.$$

This shows that there exists a neighborhood of (1,1) such that V_0^1, $V_0^2 > 0$, thus (1,1) is absorbing.

Part 3.

From Part 1 and Part 2, a) and d) follow immediately. b) and c) can be proved similarly, due to the symmetry. Q.E.D.

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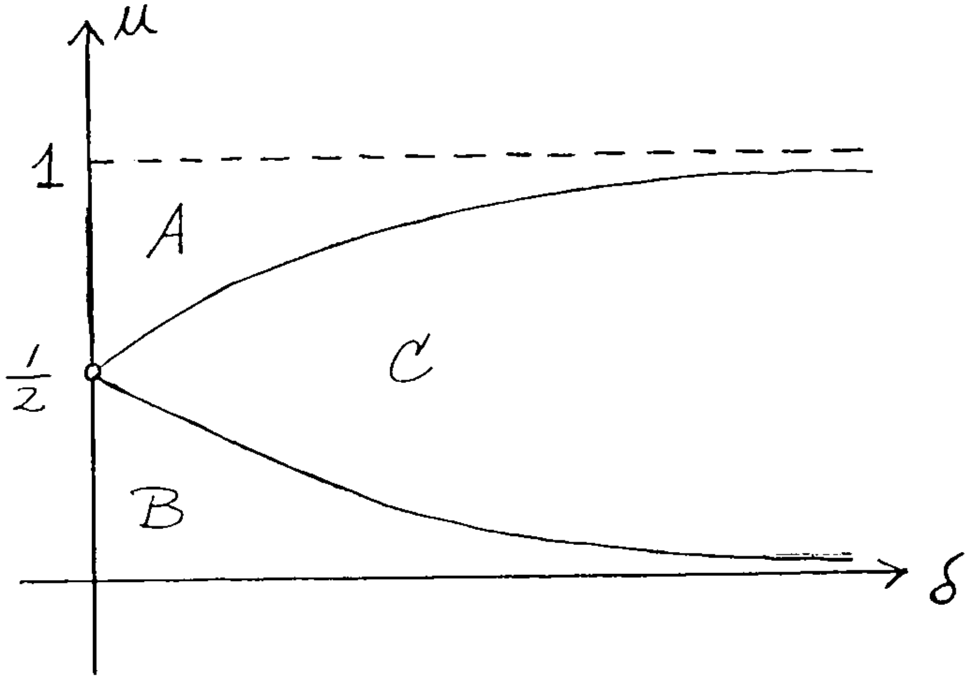
Figure 1:

$a > b$

$c < d$

	L	R
L	a, a	c, b
R	b, c	d, d

Figure 2:

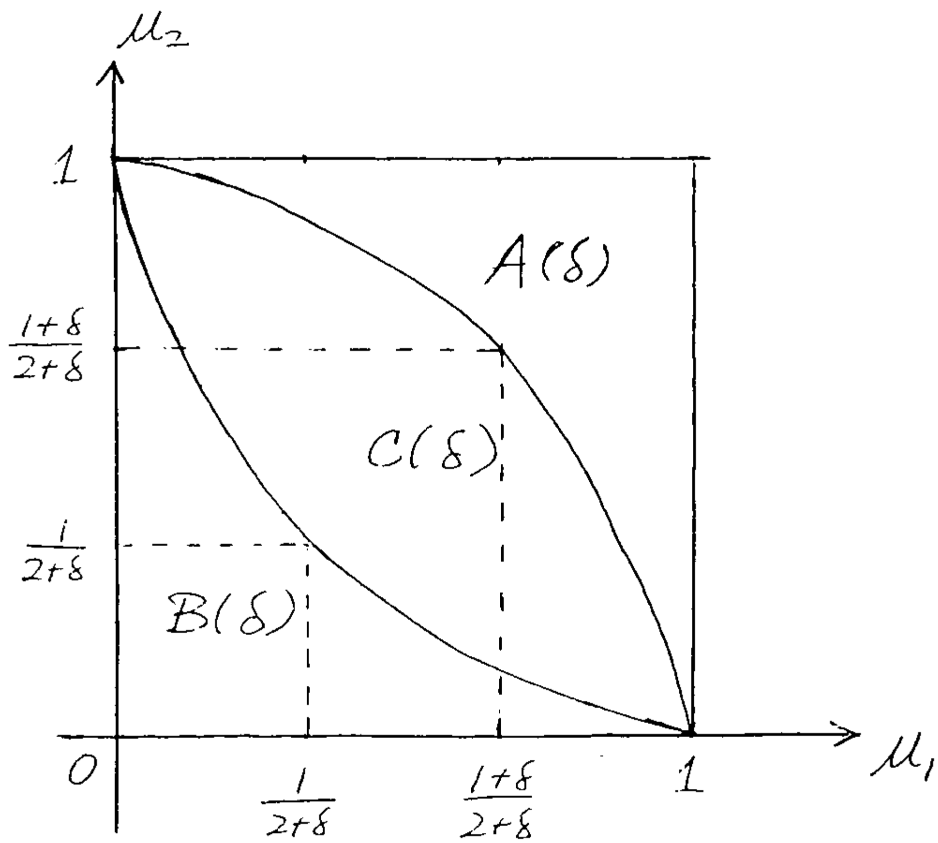


A: (R,R) is uniquely absorbing and globally accessible.
B: (L,L) is uniquely absorbing and globally accessible.
C; Both (R,R) and (L,L) are absorbing.

Figure 3:

	L_2	R_2
L_1	a_1, a_2	c_1, b_2
R_1	b_1, c_2	d_1, d_2

Figure 4:



$A(\delta)$: (R_1, R_2) is uniquely absorbing and globally accessible.

$B(\delta)$: (L_1, L_2) is uniquely absorbing and globally accessible.

$C(\delta)$: Both (R_1, R_2) and (L_1, L_2) are absorbing.

Figure 5: