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COMMON BELIEFS AND THE EXISTENCE OF SPECULATIVE TRADE*

by

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ABSTRACT

This paper shows that if rationality is not common knowledge, the no-trade theorem of Milgrom and Stokey (1982) fails to hold. We adopt Monderer and Samet's (1990) notion of common p-belief and show that when traders entertain doubts about the rationality of other traders, and even if these doubts are very small, arbitrarily large volumes of trade as well as rationality may be common p-belief for a large p. Futhermore, rationality and trade may simultaneously be known to arbitrary large (but finite) degree. The underlying intuition of the model is that, in trade situation, every trader may be rational but may believe that the others are not fully rational. Thus, rational traders trade with each other, believing that the other trader might be wrong, while they are right.

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1. Introduction

The information representation literature pioneered by Aumann (1976) has led to the surprising "no-trade theorem" of Milgrom and Stokey (1982). It states that given ex-ante Pareto efficient allocations, the arrival of new information will not induce further trade under the assumption that trade be common knowledge among traders, and hence, it is generally interpreted as a "no-speculation" result. This result is surprising partly because it is in stark disagreement with the general image of the stockbroker as, mostly, a speculant. True, this might be dismissed as laymen's observation; however, economists argue that hedging against risk alone, or trading towards Pareto efficiency, cannot explain the huge volumes of trade we observe in security markets around the world. As an example, we cite Ross (1989) which states "...it is difficult to imagine that the volume of trade in securities markets has very much at all to do with the modest amount of trading required to accomplish the continuous and gradual portfolio rebalancing inherent in our current intertemporal models."

In general, no-trade results rely heavily on the strength of the common knowledge assumption. We say that an event E is *common knowledge* if everybody knows it, everybody knows that everybody knows it, everybody knows that everybody knows that ... , and so forth. While common knowledge of various facts, most notably the model itself, is implicitly assumed in much of economic theory, this concept does not perfectly capture our ordinary, everyday notion of "knowledge". One difficulty it entails is the problem of visualizing high order sentences of the form "...you know that I know that you know that I know E", indeed, the first four levels in this hierarchy of knowledge are sufficient to exhaust most people. Another difficulty, which we regard as being more important, is the difficulty of "knowing". The absolute certainty suggested by knowing, as opposed to believing, seems exaggerated. Especially when the subject of this knowledge may sometimes involve other people's thoughts or future actions. Thus, the motivating argument for this paper is that no-speculation results, while capturing valid intuitions, seem to be too strong. Moreover, and correspondingly, they rely on the assumption of common knowledge, which, again, we regard as too strong. Therefore, it seems desirable to investigate whether an appealing weakening of the common knowledge concept might generate more intuitive results.

This paper attempts to formalize the intuition that people trade because they think they are smarter than others. Specifically, the main claim is that common knowledge of rationality is too strong an assumption, and that when it is relaxed, trade can be explained. Furthermore, rationality may be "almost" common knowledge and still allow for trade to take place.

There are several models of "bounded rationality" which have been proposed as explanations of the no-trade puzzle. Among the many contributions, a number of papers have focused on the axiomatic approach to common-knowledge (e.g. Bacharach 1985; Geanakoplos 1988; Samet 1990). Namely, these papers focus on the axioms on the knowledge operator which characterize no-trade results. For our purposes, it is important to note that one needs to replace partitional information structures with information structures which represent less rational agents in order to avoid no-speculation results, as an example in Rubinstein and Wolinsky (1990) shows. A different approach is proposed by Dow, Madrigal and Werlang (1990); they retain the assumptions about the knowledge operator, that is, they consider partitional information structures, and they generalize Milgrom and Stokey's result as follows: given an ex-ante Pareto efficient allocation, there does not exist any other allocation which is ex-post Pareto dominating the first and is commonly known. Thus, any motivation for trade is eliminated. Their result does not depend on the ex-ante and ex-post partitions, nor on the preferences of the traders which need not be concave (i.e. risk averse), or even increasing. More importantly, it allows traders to have different priors. Their result requires only that markets be complete and that the utility functions be state additive. In fact, they show that this requirement is also necessary. The intuition for their result, and the way their proof works, is that when an event is commonly known (ex-post), this very fact is agreed upon by all traders, and thus can be incorporated into state-contingent trade ex-ante. However, they are able to provide an example where no-speculation fails, i.e., where there is trade, under non-additive probabilities.

The literature thus offers two "bounded rationality" explanations for the no-trade puzzle: one suggests that information structures are non-partitional, and the other -- that beliefs are non-additive. In both cases, the notion of "rationality" and the extent of deviation from it can only be described in the meta-model. That is, a statement such as "trader 1 believes that trader 2 is rational" has no formal content, and perhaps no meaning at all.

By contrast, this paper suggests a model in which rationality of a trader is a well-defined event. We retain the classical assumptions regarding knowledge and beliefs; that is, the information structures are partitional and the priors are additive; furthermore, we retain the common prior assumption. We follow Aumann (1987) in that "rationality" is defined behaviorally and is thus a well defined event over which traders have beliefs. In Aumann (1987) ''every 'state of the world' implies a definite specification of all the parameters which may be the object of uncertainty on the part of any player ... [trader in our model]. In particular, it includes a specification of which action is chosen by each player [trader] ... at each state ω.'' However, for notational convenience we adopt the traditional formulation according to which the strategy choices are external to the states (see remark 6.7). We follow Aumann in that rationality is defined as acting optimally given the trader's information and other traders' actions. An "irrational" trader does not act optimally given his information and other traders' behavior. For simplicity, we assume that an "irrational" trader will choose each possible suboptimal act with equal probability.

In the formal model, relaxing the assumption of common knowledge of rationality implies that there exist states of the world in which traders are indeed "irrational", that is, traders behave suboptimally. However, we introduce these states of the world into the model in order to capture the way in which the traders in the model perceive their world. We do not mean to imply that traders actually act suboptimally, only that they believe that others might do so. (For a discussion of this point, and in general, on the difference between the agent's and the modeler's point of view, see Aumann (1987) and Rubinstein (1991).) Indeed, our analysis focuses on states of the world where all traders are perfectly rational and each of them is aware of his own rationality. Furthermore, they may even be sure of other traders' rationality, and be sure that everybody else is sure of it and so on for any finite degree of knowledge. Yet, it is crucial for our results that the formal model would allow for some positive probability of irrationality. The states at which traders may be irrational are thus not claimed to actually materialize. They are incorporated into the model to reflect traders' beliefs about other traders or about the beliefs of other traders and so forth. An illustration to the way we think about rationality is the following: Suppose that a trader arrives at his office every morning at 9 a.m.. On the way, he goes over the morning news, and when he gets to his office, based on his information, he decides which trades to make. However, this trader may be late to work -- in which case, important trading opportunities may be lost, or a junior trader might buy a certain asset instead of selling it. We think of the trader who arrives at his office on time as being "rational", and of the same trader, when he is late, as being "irrational". The main point is that to explain trade, no trader actually needs to be late. It suffices that some traders suspect that others may suspect that others ... may suspect that a trader has not shown up on time.

Thus, relaxing the common knowledge of rationality assumption does not mean that we wish to dispose of it altogether. It does imply that we cannot use common knowledge as the underlying concept for expressing agreement in the model. Indeed, weaker concepts for expressing mutual agreement of knowledge should be sought. Two such alternative concepts have been proposed in the literature. The first is Rubinstein's (1989) "almost common knowledge" which allows only for a finite hierarchy of knowledge. Rubinstein shows in an example which is the game theoretic formulation of the "coordination attack" problem (see Halpern, 1986) that even for arbitrarily high levels of knowledge, "almost common knowledge" does not approximate common knowledge in the sense that optimizing agents that are in a state of "almost common knowledge" about the game they play cannot behave as if the game is common knowledge; i.e., they cannot play the "natural" Nash equilibria in this game. In a sense, this example shows that the set of Nash equilibria is "discontinuous" at common knowledge: very high levels of knowledge do not yield results which are similar to those implied by common knowledge. Inspired by this example, Monderer and Samet (1989) suggest yet another way of weakening the common knowledge assumption. Instead of truncating the knowledge hierarchy, they replace common knowledge with what they call common belief. This notion weakens "knowledge" to "belief" while retaining an infinite hierarchy of the latter. In this set-up, they obtain an approximation to Aumann's (1976) "agreeing-to-disagree" result as well as continuity of the set of Nash equilibria "at common knowledge". They quantify the degree of belief by a parameter $p \in [0,1]$, and show that if there is common p-belief of the "true" game being played, ϵ-optimizing players can almost mimic the behavior of players to whom the game played is common knowledge and therefore play one of its Nash equilibria.

In this paper we use Monderer and Samet's definition of common p-belief to develop a model of speculative trade in which, under suitable conditions, there is a positive probability of common belief of arbitrarily large volumes of trade. The first step in establishing this result is the observation that, excluding risk management considerations, any speculation or bet originates from a state of conflicting opinions, or more colorfully, "it is a difference of opinion that makes a horse race." Namely, in the context of this paper, starting with a Pareto efficient allocation, traders must have conflicting views regarding the outcome of a certain event in order to induce trade. Following the previous literature, we start with ex-ante Pareto-efficient markets in order to exclude risk management considerations, so that pure speculation is the only possible motivation for trade after the arrival of new information. Nevertheless, disagreement among traders, and even common belief of disagreement, intense as it might be, is not enough to create motivation for trade among completely rational traders. This observation enables us to strengthen Milgrom and Stokey's result, namely: starting with an ex-ante Pareto efficient allocation, a positive probability of a feasible and mutually acceptable trade implies that traders must be indifferent between this trade and the null trade (a similar result is stated in Geanakoplos 1992). Maybe not surprisingly, the intuition for this result is similar to Milgrom and Stokey (1982) and Dow, Madrigal, and Werlang (1990), namely that any ex-post improving trade can be traded ex-ante. Consequently, the second step in constructing our argument is to let traders entertain some doubts concerning the perfect rationality of other traders. The third and last step establishes that common belief of disagreement together with the presence of doubts about the rationality of other traders are sufficient for obtaining common belief of trade of arbitrarily large volumes among not-too-risk-averse traders. It is worth noting that the doubts about the rationality of other traders may be slight enough as to allow for a common p-belief of rationality for p arbitrarily close to 1 together with "almost common knowledge" (as in Rubinstein 1989) of arbitrarily high degree, while still being sufficient for our results. Thus, starting with Pareto efficient allocations, and after the arrival of new information, traders can be "pretty sure" that trade is to take place, and that all other traders are also rational, (although this necessarily means they have conflicting opinions). Moreover, they can be "pretty sure" that everybody else is "pretty sure" about it, and so on.

The formal treatment of the model follows the above line of reasoning. Our first result provides a condition which characterizes the case where there is a positive probability of common belief of disagreement among traders. In addition, this condition allows us to quantify the probability of common belief of disagreement as a function of the underlying information structures. The second result shows that common belief of both disagreement and imperfect rationality are necessary and sufficient for there to exist a price range, in which common belief of arbitrarily large volumes of beneficial trade is possible in the case where traders are risk-neutral. We derive the sufficiency result, namely that common belief of both disagreement and imperfect rationality is sufficient for common belief of arbitrarily large volumes of beneficial trade for sufficiently risk tolerant traders as well. The condition which characterizes the probability of common belief of disagreement in the first result, also ensures the same probability for the second and third results. We also note that the results obtained for common p-belief can be either replicated using Rubinstein's (1989) concept of "almost common knowledge" instead, or, can be obtained simultaneously for both common p-belief and "almost common knowledge".

The paper is organized as follows: In section 2, we start with the fundamental results of the knowledge/belief literature, following Monderer and Samet's paper. In section 3, we present two examples: the first is a simple example of speculative trade, which emphasizes the differences between common knowledge and common belief reasoning; the second example illustrates a case of "almost common knowledge" of trade which is also common belief. Then, in section 4, we develop a model of speculative trade for the simpler case of risk neutral traders. In section 5 we develop the general version of the model, allowing for risk-averse traders. We show that all the results and intuitions of the risk neutral case continue to hold for risk averse traders, once we make the appropriate corrections for risk. Finally, section 6 offers concluding remarks.

2. Set-Up

Let I be a finite set of players and let (Ω,Σ,μ) be a probability space, where Ω is the space of states of the world, Σ is a σ-algebra of events, and μ is a non-atomic probability measure on Σ (to be interpreted as a common prior). For each $i \in I$, Π^i is a finite partition of Ω into measurable sets with positive measure. We use the notation $\Pi^i=\{\Pi_1^i, \Pi_2^i, \ldots ,\Pi_{N_i}^i\}$. For

$\omega\in\Omega$, denote by $\Pi^i(\omega)$ the element of Π^i containing ω. Π^i is interpreted as the information available to agent i; $\Pi^i(\omega)$ is the set of all states which are indistinguishable to i when ω occurs. We denote by $\mathscr{F}^i$ the (σ)-field generated by Π^i. That is, $\mathscr{F}^i$ consists of all unions of elements of Π^i. For $i\in I$, $E\in\Sigma$, $\omega\in\Omega$ and $p\in[0,1]$, we say that *"i believes E with probability at least p at* ω*"*, or simply *"i p-believes E at* ω*"* if $\mu(E|\Pi^i(\omega))\geq p$. Denote by $B_p^i(E)$ the event "i p-believes E." That is,

$$B_p^i(E)=\{\omega:\mu(E\mid\Pi^i(\omega))\geq p\}$$

Notice that this is an event (i.e., it is measurable with respect to Σ). Moreover, it is measurable with respect to $\mathscr{F}^i$.

Definition $E\in\Sigma$ is *evident p-belief* if for each $i\in I$

$$E\subseteq B_p^i(E)$$

Definition An event C is *common p-belief* at ω if there exists an evident p-belief event E such that $\omega\in E$, and for all $i\in I$,

$$E\subseteq B_p^i(C)$$

The intuitive iterative definition of common p-belief is as follows:

Definition For every event C and every $0\leq p\leq 1$ let

$$D_p(C)=\bigcap_{n\geq 1}C_n(p)$$

where $C_0(p)=C$ for any $0\leq p\leq 1$, and for $n\geq 1$, $C_n(p)=\bigcap_{i\in I}B_p^i(C_{n-1}(p))$

The $C_n(p)$'s should be interpreted as follows,

$C_1(p)$: Every agent p-believes C.

$C_2(p)$: Every agent p-believes $C_1(p)$.

$C_3(p)$: Every agent p-believes $C_2(p)$, etc.[1]

Monderer and Samet (1989) showed that the two definitions coincide:

Proposition 2.1 (Monderer and Samet) For every event C and for every $0 \leq p \leq 1$,

(i) $D_p(C)$ is evident p-belief, and $D_p(C) \subseteq B_p^i(C)$ for all $i \in I$.

(ii) C is common p-belief at ω iff $\omega \in D_p(C)$.

The following proposition is a characterization of common p-belief and is a first step towards characterizing common p-belief of trade.

Proposition 2.2 (Neeman, 1993) An event C is common p-belief at $\omega \in \Omega$ if and only if for all $i \in I$ there exists a set $\pi^i = \bigcup_{k \in K^i} \Pi_k^i$ for some $\phi \neq K^i \subseteq \{1, \ldots, N_i\}$ such that $\omega \in \pi^i$ and such that the following two conditions hold,

(i) For all $i \in I$, $\mu(\bigcap_{h \in I} \pi^h | \Pi_k^i) \geq p$ for all $\Pi_k^i \subseteq \pi^i$.

(ii) For all $i \in I$, $\mu(C | \Pi_k^i) \geq p$ for all $\Pi_k^i \subseteq \pi^i$.

Remark 2.3 In all the above definitions and propositions, common 1-belief coincides with common knowledge up to measure 0.

Lastly, we present Rubinstein's (1989) definition of "almost common knowledge:"

Definition An event is C *commonly known of degree n* at ω if $\omega \in C_n(1)$.

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[1] Monderer and Samet use $E^p(C)$ to denote what we denote by $D_p(C)$; we change notation to avoid confusion with the expectation operator.

3. Two Examples

3.1 Common knowledge reasoning as opposed to common p-belief reasoning

This example demonstrates how trade can be sustained under common p-belief (as opposed to common knowledge) and allowing for a small probability of irrationality. Let there be given a probability space (Ω,Σ,μ), and assume we have two traders, 1 and 2. The traders have information structures represented by the following figure and are endowed with ex-ante Pareto efficient allocations which are common knowledge. (The figure shows only the elements of each trader's information structure which are relevant to our argument.)

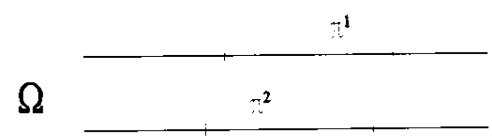

figure 1

Suppose there exists a bet B such that given π^1, trader 1 would like to buy B, and given π^2, trader 2 would like to sell B. Since we start with Pareto efficient allocations, given $\pi^1 \cap \pi^2$, the traders do not want to trade. Furthermore, assume that conditioned on any other element of their information structure, the traders do not want to trade either. Consider the following game: a state of the world is realized and the traders have to decide simultaneously whether they want to trade B. B is traded if and only if one trader decides to buy B and the other trader decides to sell B. Suppose that trader 1 gets a signal $\omega \in \pi^1$. Trader 1 reasons as follows:

> "I know $\omega \in \pi^1$, therefore I want to buy B. However, 2 will sell B if and only if $\omega \in \pi^2$. Therefore, I know that trade will take place if and only if $\omega \in \pi^1 \cap \pi^2$. But in this case, I refuse to trade."

Trader 2 reasons similarly, and indeed, no trade takes place. (The above reasoning resembles Geanakoplos and Polemarchakis (1983) and Sebenius and Geanakoplos (1983).)

However, if we use common belief type reasoning, and introduce imperfect rationality, this argument will not be valid anymore. We use the same set up with some additional assumptions. First, assume that for each trader i there is a positive probability of being irrational. Specifically, we assume: trader i is rational in π^i, the probability that trader i is irrational conditioned on $\pi^j \backslash \pi^i$ is ρ, and $\mu(\pi^1 \cap \pi^2 | \pi^i) = \frac{9}{10}$ for $i=1,2$. Consider the following game: a state of the world is realized. Both traders decide whether they want to trade. They write their decisions on notes which are inserted into sealed envelopes and are sent to a market maker. As before, trade will take place if and only if one trader decides to buy B and the other trader decides to sell B. Consider the following trade B,

$$B(\omega) = \begin{cases} 10 & \omega \in \pi^1 \backslash \pi^2 \\ -10 & \omega \in \pi^2 \backslash \pi^1 \\ 0 & \text{otherwise} \end{cases}$$

For trader 1, $E(B|\pi^1)=1$, and for trader 2, $E(B|\pi^2)=-1$. Consider a state $\omega \in \pi^1 \cap \pi^2$, the conditions we imposed imply that the conditional expectations above are held as common p-belief for $p=\frac{9}{10}$. Therefore, at $\omega \in \pi^1 \cap \pi^2$ we have common (1-p)-belief of rationality together with common p-belief of the subjective valuations of the proposed bet B although these valuations differ. Moreover, this disagreement between traders facilitates trade. We show this for the case of risk neutral traders. Suppose that a price $-\rho < q < \rho$ is announced and that trader 1 decides to buy B on π^1 and trader 2 decides to sell B on π^2. We show that these decisions can indeed be sustained as an equilibrium. Recall that an irrational trader chooses each suboptimal action with equal probability, however, in this case there is only one suboptimal act for each trader. Namely, in this example an irrational trader 1 will choose to sell B, and an irrational trader 2 will choose to buy B. Trader 1 reasons as follows:

> "I believe π^1, therefore, I would like to buy B. As for trader 2, a rational trader 2 will agree to sell B if $\omega \in \pi^1 \cap \pi^2$ because then trader 2 believes that B has a negative expectation. Still, the expectation of B conditional on $\pi^1 \cap \pi^2$ is zero;

however, there is a slight chance that $\omega \in \pi^1 \backslash \pi^2$ and an irrational trader 2 is going to sell B. This is when I'm going to make my profit.''

So, given π^1 and a suggested price $q < \rho$, a rational trader 1 will be happy to buy B for price q although he believes that trader 2 is selling B. Similarly, given π^2, a rational trader 2 will be willing to sell B for any price $q > -\rho$. Or rather, when q is negative, a rational trader 2 is willing to pay $-q$ to get rid of B, although he believes that trader 1 is buying B. Moreover, for $p = \frac{9}{10}$, traders have common p-belief of trade. Thus, common p-belief of disagreement and of rationality leads to common p-belief of trade.

Note that what allows trade to take place is that common p-belief, as opposed to common knowledge, allows traders to disagree. Although a trader believes other traders have different views about the state of affairs, it does not cause him to change his own views. Nonetheless, disagreement between traders alone cannot support trade. If traders are perfectly rational the argument used in the common knowledge case can be repeated to rule out any trade. However, with even the slightest probability of irrationality, disagreement between traders over the value of B allows for the existence of beneficial trade.

3.2 Almost common knowledge of trade

In this example we show that traders can have almost common knowledge (as in Rubinstein 1989) of rationality and trade. We show that traders can have almost common knowledge of level 3, that is, they know that all traders are rational and that trade takes place, they know that others know it, and they know that others know that they know it; however, the hierarchy of knowledge stops here, and this last fact is not known to the traders. The underlying argument of this example can be easily extended to include almost common knowledge of any degree $n \geq 1$.

As before, let there be given a probability space (Ω, Σ, μ), and assume we have two risk neutral traders, 1 and 2. The traders have information structures represented by the following figure and are endowed with ex-ante Pareto efficient allocations which are common knowledge. (The figure shows only the elements of each trader's information structure which are relevant to our argument.)

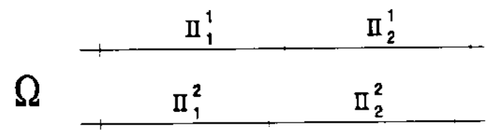

figure 2

Suppose $\mu(\Pi_1^1)=\mu(\Pi_2^1)=\mu(\Pi_2^2)=\mu$; and that $\mu(\Pi_k^2 | \Pi_k^1)=\frac{9}{10}$ for $k=1,2$. Furthermore, suppose that at $\Pi_2^1 \backslash \Pi_2^2$ there is a (conditional) probability $\rho>0$ that trader 2 is irrational. Consider the following proposed trade B,

$$B(\omega)=\begin{cases} 10 & \omega \in \Pi_1^1 \backslash \Pi_1^2 \\ -\frac{10}{9} & \omega \in \Pi_2^1 \cap \Pi_2^2 \\ \frac{20}{\rho} & \omega \in \Pi_2^1 \backslash \Pi_2^2 \\ 0 & \text{otherwise} \end{cases}$$

For trader 2, $E(B | \Pi_1^2)=E(B | \Pi_2^2)=0$. For trader 1, $E(B | \Pi_1^1)=1$, and $E(B | \Pi_2^1)>1$. We show that for any price $0<q<1$, a buy offer by trader 1 at $\Pi_1^1 \cup \Pi_2^1$ and a sell offer by trader 2 at $\Pi_1^2 \cup \Pi_2^2$ can be sustained as an equilibrium. Thus, trade is going to take place at $(\Pi_1^1 \cup \Pi_2^1) \cap (\Pi_1^2 \cup \Pi_2^2)$. Our assumption about the irrationality of trader 2 implies that at $\Pi_2^1 \backslash \Pi_2^2$, trader 2 sells the proposed trade B at any price $0<q<1$ with a probability ρ. (We assume here that at $\Pi_2^1 \backslash \Pi_2^2$, $E(B | \Pi^2(\omega)) \geq 1$. Hence it is optimal not to sell, but an irrational trader will do so.) Note that for any Π_k^i except Π_2^1, the traders buy and sell since they are offered lucrative prices, and they are sure that the other trader will keep his part of the trade because he is rational. For $\omega \in \Pi_2^1$, trader 1 cannot be sure that trade will take place, however, the high value of B, and the possibility that an irrational trader 2 might still sell, make buying his best choice. As for almost common knowledge, note that for any $\omega \in \Pi_1^1 \cap \Pi_1^2$, trader 1 is rational, trader 1 knows that trader 2 is rational, and trader 1 knows that trader 2 knows that he (trader 1) is rational. Similarly, trader 2 is rational, trader 2 knows that trader 1 is rational, and trader 2 knows that trader 1 knows that he is rational. Furthermore, the previous statements can be repeated with "trade" instead of "rationality".

Moreover, one can verify that for any price $0<q<1$, trade is common p-belief at any $\omega\in(\Pi_1^1\cup\Pi_2^1)\cap(\Pi_1^2\cup\Pi_2^2)$ for any $p\leq\frac{9}{10}$.

4. The Results: The case of risk neutral traders.

We formulate the following results for an environment similar to the one used by Milgrom and Stokey (1982). Consider an economy with a set $I=\{1,2\}$ of traders in an uncertain environment represented by a probability space (Ω,Σ,μ). All subsets of and functions on Ω are assumed Σ-measurable. Set inclusion should be interpreted as "μ-almost everywhere". Traders have information structures Π^i, over which they have belief operators $B_p^i(\cdot)$ for some $0\leq p<1$. We restrict our attention to a one dimensional commodity space ("consumption")[2]. Let u^i be trader i's increasing linear (i.e. traders are risk neutral) von Neumann-Morgenstern utility function, $u^i:\mathbb{R}\rightarrow\mathbb{R}$, without loss of generality, we set $u^i(x)=x$ for $x\in\mathbb{R}$. Let $e^i(\omega)$ be trader i's allocation at state ω, $e^i:\Omega\rightarrow\mathbb{R}$, and suppose that the e^i's are ex-ante Pareto efficient, that is, that there does not exist a trade which is ex-ante strictly beneficial to both traders. And, let $B:\Omega\rightarrow\mathbb{R}$ denote a proposed trade or "bet".

As proposition 2.2 suggests, the following assumption is going to play a major role in the results:

A1. For every trader $i\in I$, there exists a set $\pi^i=\bigcup_{k\in K^i}\Pi_k^i$ for some $\phi\neq K^i\subseteq\{1,\ldots,N_i\}$ such that the following two conditions hold,

(i) For all $i\in I$, $\mu(\pi^i|\Pi_k^i)\geq p$ for all $\Pi_k^i\subseteq\pi^i$.

[2] We do not view this set up as conceptually restrictive. It is believed that, while it simplifies the exposition, the results follow in a more general setting as well.

(ii) π^1 and π^2 do not coincide in the following sense: for all $i \in I$ and index subsets $\phi \neq K^{i'} \subseteq K^i$, denote $\pi^{i'} = \bigcup_{k \in K^{i'}} \Pi_k^i$; then $\mu(\pi^{1'} \vartriangle \pi^{2'}) > 0$ (where $\vartriangle$ stands for the symmetric difference between two sets).

In order to create incentives for trade it is necessary that we have a proposed trade B, over which traders have different posterior expectations. The following proposition provides a construction of such a trade by solving a linear programming problem.

Proposition 4.1 Let there be given two sets, $\theta^1 = \bigcup_{k=1}^{n} \Theta_k^1$ and $\theta^2 = \bigcup_{k=1}^{m} \Theta_k^2$, for some $n,m \geq 1$, where $\{\Theta_k^1\}_{k=1}^n$, and $\{\Theta_k^2\}_{k=1}^m$ are each composed of disjoint measurable sets. The condition that θ^1 and θ^2 do not coincide in the sense of *A1* (ii), is necessary and sufficient for there to exist a random variable $B: \Omega \to \mathbf{R}$, and two values $v_1 \neq v_2$ such that $E(B \mid \Theta_k^1) = v_1$ for all $\Theta_k^1 \subseteq \theta^1$, and $E(B \mid \Theta_k^2) = v_2$ for all $\Theta_k^2 \subseteq \theta^2$.

(All proofs are relegated to the appendix.)

The notion of trade involves acts, which, in turn, raise the question of rationality. It may be helpful to formulate an auxiliary result which only deals with beliefs. The following result shows that assumption *A1* applied to the original information partitions characterizes common p-belief of disagreement between traders. We introduce the following notation: let $C^i(B,v)$ denote the event where trader i, given his information, assigns expectation v to the trade B. That is, $C^i(B,v) = \{\omega : E(B \mid \Pi^i(\omega)) = v\}$. Define $C(B,v_1,v_2)$ to be $C^1(B,v_1) \cap C^2(B,v_2)$. $C(B,v_1,v_2)$ is the event where, for all $i \in I$, given his information, each trader i assigns expectation v_i to B.

Theorem 4.2 There exist a trade $B: \Omega \to \mathbf{R}$ and two values $v_1, v_2 \in \mathbf{R}$, $v_1 \neq v_2$, such that $\mu(C(B,v_1,v_2)$ is common p-belief$) > 0$ iff the information partitions satisfy *A1*.

With this result at hand, we move on to establish the existence of common p-belief of trade. Let A be the set of acts available to the traders,

$$A = \{ \text{"}buy\text{"}, \text{"}sell\text{"}, \text{"}refrain\text{"} \}$$

Let a^i be trader i's strategy, $a^i(\omega,B,q) \in A$. That is, a^i is a function attaching to each state ω, and an offered trade B at price q, an act in A. When the offer (B,q) is implied by the context we will also use the notation $a^i(\omega) \in A$ and $a(\omega) = \left(a^i(\omega)\right)_{i \in I}$. Rationality is defined behaviorally as follows: a trader is rational if given any offer (B,q) he chooses the action that maximizes his utility given his information and the other trader's action. Thus, rationality is actually no more than best response of trader i to a^j, and it follows naturally that "rationality" of one trader is defined with respect to the actions of another.[3]

Formally, we say that trader $i \in I$ is rational at $\omega \in \Omega$ if $\omega \in R^i(a)$, where $R^i(a)$ is defined as follows,

$$R^i(a) = \left\{ \omega \mid \ \forall B, q, \quad a^i(\omega,B,q) \in \underset{a^i \in A}{\operatorname{argmax}} \ E\left(u^i(a^i, a^j(\omega), B, q) \mid \Pi^i(\omega)\right) \right\}$$

where, given a proposed trade B and a price q, $u^i(a^i(\omega),a^j(\omega),B,q)$ denotes trader i's utility given $\omega \in \Omega$, the acts (strategies) $a(\omega) = (a^i(\omega),a^j(\omega))$ and that trade takes place if and only if one trader is willing to buy and the other to sell. That is,

$$u^i(a(\omega),B,q) = \begin{cases} u^i(e^i(\omega) + B(\omega) - q) & a^i(\omega) = \text{"}buy\text{"}, \ a^j(\omega) = \text{"}sell\text{"} \\ u^i(e^i(\omega) - B(\omega) + q) & a^i(\omega) = \text{"}sell\text{"}, \ a^j(\omega) = \text{"}buy\text{"} \\ u^i(e^i(\omega)) & \text{otherwise} \end{cases}$$

Remark 4.3 Suppose that there exist a trade B, a price q, and a pair of strategies $a^1(\omega)$, $a^2(\omega)$

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[3] Alternatively, one may define "rationality" independently of other traders, namely, as choosing best responses to one's beliefs. But then one would have to explicitly require that these beliefs coincide with the conditional μ. For simplicity of exposition we stick to the traditional definition of a Nash equilibrium in which beliefs are only implicit.

such that both traders are rational (that is $\mu(R^i(a))=1$ for $i=1,2$) and such that there is a positive probability of trade. Then, neither trader can strictly prefer trade to no-trade.

Proof The intuition behind this simple result is that, by definition of rationality, it is as if rational traders "know" when trade is to take place. That is, the traders take into account that trade will actually take place only if their partner agrees. Thus, their conditional expectation calculation is equivalent to the case in which trade is common knowledge. Formally, the conditions imply that there exists a positive probability event T such that,

$$T=\{\omega \mid a^1(\omega,B,q)=\text{"}buy\text{"},\ a^2(\omega,B,q)=\text{"}sell\text{"}\}$$

T is the event where trader 1 buys B at price q, and trader 2 sells it. The rationality of the traders implies that for any $\omega\in\Omega$,

$$E\Big(u^1\big(e^1(\omega)+(B(\omega)-q)\cdot 1_T\big) \mid \Pi^1(\omega)\Big)\geq E\big(u^1(e^1(\omega)) \mid \Pi^1(\omega)\big)$$

aggregating over the different $\Pi^1(\omega)$'s, we get,

$$E\Big(u^1\big(e^1(\omega)+(B(\omega)-q)\cdot 1_T\big)\Big)\geq E\big(u^1(e^1(\omega))\big)$$

Similarly, for trader 2,

$$E\Big(u^2\big(e^2(\omega)-(B(\omega)-q)\cdot 1_T\big)\Big)\geq E\big(u^2(e^2(\omega))\big).$$

Thus, if even one of the traders strictly benefits from trade, we obtain a contradiction to the Pareto optimality of the initial allocations e^i. ■

This remark, which has also been noted by Geanakoplos (1992), seems to strengthen Milgrom and Stokey's result (and even Dow, Madrigal, and Werlang's (1990) result) significantly, since it merely requires the existence of a positive probability of trade, as opposed to common knowledge of trade. It hinges on the assumption that, rational traders "know" (in an informal sense) when their partner will trade, and that rationality is common knowledge at each $\omega\in\Omega$.

The immediate implication of the above is that in our framework we must dispense with common knowledge of rationality if we want to account for trade. Therefore, we assume that traders act suboptimally with a probability $\rho>0$, and use a weaker concept than Nash equilibrium, namely,

Definition $\{a^i(\omega,B,q)\}_{i\in I}$ constitutes a *(1-ρ)-rationality Nash equilibrium* if for all $i\in I$ $\mu(R^i(a))\geq 1\text{-}\rho$. (That is, with probability at least $(1\text{-}2\rho)$, both traders optimize with respect to each other.)

For simplicity we assume that irrational traders choose any suboptimal action with equal probability, and this choice is independent of the suggested trade (B,q). Hence, given an offer (B,q) and $a^j(\omega)$, and if there are two suboptimal acts, say *"buy"* and *"refrain"* then for any $F_k^i=\Pi_k^i\cap(R^i)^c$, $\mu\left(\{\omega\in F_k^i \mid a^i(\omega)=a\}\right)=\frac{1}{2}\mu(F_k^i)$ where a can be either *"buy"* or *"refrain"*. If there is only one suboptimal action a then $\mu\left(\{\omega\in F_k^i \mid a^i(\omega)=a\}\right)=\mu(F_k^i)$. Note that all three actions cannot be suboptimal simultaneously, however, they might all be optimal; for instance, if $a^j(\omega)=$ *"refrain"* for all $\omega\in\Omega$, $\mu(R^i(a^i,a^j))=1$ for all possible a^i. (The suggested intuition for the case where, say, *"sell"* is the only suboptimal action, is that the trader's computer was left with a *"sell"* order which will be implemented for any offer if the trader oversleeps.) However, it should be noted that our results can be easily extended to include all the cases where a positive probability is assigned to each of the suboptimal acts on $(R^i)^c$. We do not assume that a rational trader always knows whether he is rational. This follows from the behavioral definition of rationality; that is, a trader who oversleeps may still be lucky enough as to behave rationally.

We state the following result about trading in markets: there exist a trade B and a $(1\text{-}\rho)$-rationality Nash equilibrium where there exists an interval of prices which support the trade of B, while this trade is common p-belief and where ρ can be arbitrarily close to 0 and p arbitrarily close to 1. Furthermore, it is shown that the volume of trade in this case can be arbitrarily large. We introduce some more notation first: without loss of generality, let trader 1 be the buyer and trader 2 be the seller. Let $T(a,B,q)$ denote the event where both traders are rational, and trade is beneficial to both traders, that is,

$$T(a,B,q) = \left\{ \omega \in R^1 \bigcap R^2 \;\middle|\; \begin{array}{l} \{"buy"\} = \underset{a^1(\omega)\in A}{argmax}\ E\left(u^1(a(\omega),B,q) \mid \Pi^1(\omega)\right) \\ \{"sell"\} = \underset{a^2(\omega)\in A}{argmax}\ E\left(u^2(a(\omega),B,q) \mid \Pi^2(\omega)\right) \end{array} \right\}$$

Note that by definition, "trade" implies rationality of the traders. In particular, when trade will be common p-belief, so will rationality. Also, since *"buy"* and *"sell"* are the only respective optimal actions over $T(a,B,q)$, for all $\omega \in T(a,B,q)$, the definition of trade implies that trading B at price q is strictly Pareto improving.

We remind the reader that the e^i's are assumed ex-ante Pareto efficient.

Theorem 4.4 Let there be given (an arbitrarily small) $\rho > 0$. The following two statements are equivalent:

(i) There exists a proposed trade B, a price q, and strategies $a^1(\omega)$, $a^2(\omega)$ such that:

  (a) the strategies $a^1(\omega)$, $a^2(\omega)$ form a (1-ρ)-rationality Nash equilibrium.

  (b) $\mu(T(a,B,q)$ is common p-belief$) > 0$.

(ii) The information partitions satisfy *A1*.

Furthermore, the volume of trade in this case is arbitrarily large.

Remark 4.5 Trade, as well as rationality, can also be known up to an arbitrarily large degree in a (1-ρ)-rationality Nash equilibrium. Specifically, given information structures that satisfy *A1*, one may subdivide the events over which trade occurs to arbitrarily many events, in such a way that at some states in them the original events are commonly known to any pre-specified degree $n \geq 1$. While n is bounded for any given information structure (say, by $2 \cdot \min_{i\in I}\{N_i\}$), any n may be supported by appropriately defined partitions.

Remark 4.6 While the theorem only guarantees that trade (and rationality) be common p-belief with *some* positive probability, this probability may be arbitrarily close to 1 with an appropriate choice of the information partitions. (Specifically, $\mu(\pi^1 \cap \pi^2)$ may be very close to 1, or, alternatively, one may have trade over the intersection of more than one pair of $\{\pi^1, \pi^2\}$.)

Remark 4.7 The event of "trade" that we construct in the theorem will be independent of the event "trader i is rational" given any element of trader j's partition. That is, to explain trade one need not assume any correlation between the value of the bet and the rationality of the traders.

5. The Results: The general case

In this section we extend the results obtained in section 4 for the case of risk neutral preferences to the general case of risk averse preferences. An analogous argument will show that the results hold for risk seeking preferences as well. We use, basically, the same line of reasoning that we used in section 4. However, as we shall see, incorporating risk aversion into the model causes the prices and volumes under which trade is common belief to change. Under risk aversion, the prices and volumes reflect a "risk-premium" associated with the degree of risk aversion. We show that the results approximate the results of the risk neutral case, as preferences become less risk averse. This is hardly surprising given that we have proved that risk neutral traders have a strict preference for trade. On the other hand, there is some interest in this result since the volume of trade will be limited in this case.[4]

We employ the same framework of section 4, allowing the functions u^i to reflect risk aversion. That is, preferences are represented by strictly increasing, concave, von Neumann-Morgenstern utility functions, u^i:$\mathbf{R} \rightarrow \mathbf{R}$. However, condition *A1* is not sufficient anymore for trade to be common p-belief. A risk averse trader might still refuse to trade because trade increases his exposure to risk; thus, he might refrain from trade even if trade holds a positive expectation for him. Hence, we impose an additional condition which guarantees that the trade that we generate will be uncorrelated with the traders' initial endowments. (Another way of saying this

[4] Compare with Ross (1989) who states ''Surely there can be nothing more embarrassing to an economist than the ability to explain the price in a market while being completely silent on the quantity.''

is that the generated trade is "noise" with respect to the initial endowments.) Thus, the trader's decision of whether to trade or not will be independent of his endowment.

For simplicity, we assume that e^1, e^2, the Pareto efficient initial endowments, are simple functions. We write the canonical representation of e^i as $e^i(\omega)=\sum_{j=1}^{n_i}\alpha_j^i \cdot 1_{\chi_j^i}(\omega)$ where for $i\in I$ the χ_j^i's are disjoint sets with a positive measure and $j\neq h$ implies $\alpha_j^i\neq\alpha_h^i$.

The appropriate version of *A1* will now take the form,

A2. There exists sets $\pi^1=\bigcup_{k=1}^{n}\Pi_k^1$ and $\pi^2=\bigcup_{k=1}^{m}\Pi_k^2$ (where $n\leq N_1$ and $m\leq N_2$) such that the following two conditions hold,

(i) For all $i\in I$, $\mu(\pi^i\,|\,\Pi_k^i)\geq p$ for all $\Pi_k^i\subseteq\pi^i$.

(ii) For $k=1,..,n$ (m) and $j=1,..,n_i$, let $\Theta_{kj}^{\ i}$ be defined as $\Theta_{kj}^{\ i}=\Pi_k^i\cap\chi_j^i$. Then $\theta^1=\bigcup_{\substack{k=1,\ldots,n\\ j=1,\ldots,n_1\\ \mu(\Theta_{k,j}^1)>0}}\Theta_{kj}^1$ and $\theta^2=\bigcup_{\substack{k=1,\ldots,m\\ j=1,\ldots,n_2\\ \mu(\Theta_{k,j}^2)>0}}\Theta_{kj}^2$ do not coincide in the sense of *A1* (ii).

The following theorem is the analog of theorem 4.2. The former differs from the latter in that (i) it only provides sufficiency; (ii) the trade B is also "noise" with respect to the trader's information and initial endowments.

Theorem 5.1 *A2* is sufficient for there to exist a trade $B:\Omega\rightarrow\mathbb{R}$ and two values $v_1,v_2\in\mathbb{R}$, $v_1\neq v_2$, such that for all $i\in I$, $k=1,..,n$ (m), and $j=1,..,n_i$, $E(B\,|\,\{\omega\,|\,e^i(\omega)=\alpha_j^i\}\cap\Pi_k^i)=v_i$ for $i=1,2$ (that is, B is "noise" with respect to the initial endowment e^i and i's information $\Pi^i(\omega)$ at π^i), and such that $\mu(C(B,v_1,v_2)$ is common p-belief$)>0$.

In order to proceed we need a measure of risk aversion which bypasses the difficulty associated with the usual Arrow-Pratt measure of risk aversion, namely, that it is only a partial ordering. The following construction provides a general framework which allows to determine whether a trader with a given utility function can be said to be sufficiently risk tolerant for our purposes. We fix an interval $[-M,M]$. Our discussion is confined to the following family of utility functions which is parametrized by M and ψ,

$$U_{M,\psi}=\{u:[-M,M]\to\mathbb{R} \mid u \textit{ is concave, differentiable, strictly increasing, and } u'(M)\geq\psi\}$$

Note that $U_{\infty,0}$ is the family of general smooth risk averse preferences.

Lemma 5.2 Let there be given X, Y, bounded random variables such that $E(Y|X)=0$, and $|X(\omega)|,|Y(\omega)|\leq\frac{M}{2}$ for all $\omega\in\Omega$, together with a constant $\psi>0$, a measurable set T, and another constant $c>0$. Then,

(i) There exists a $\delta>0$ such that for all $u\in U_{M,\psi}$ satisfying $\sup_{\{|x|\leq M\}}|u''(x)|<\delta$,

$E\big(u(X+Y+c)\mid T\big)>E\big(u(X)\mid T\big)$.

(ii) Define $c(u;T)$ to be the solution of the following equation, $E\big(u(X+Y+c)\mid T\big)=E\big(u(X)\mid T\big)$. Then, for all $\epsilon>0$, there exists a $\delta>0$ such that for all $u\in U_{M,\psi}$ satisfying $\sup_{\{|x|\leq M\}}|u''(x)|<\delta$, $c(u;T)<\epsilon$.

Notice that $c(u;T)$ is the compensation required by a trader with a utility function u for accepting a bet Y, given that he considers the information T. Stated otherwise, a trader with an initial allocation $e(\omega)$ (X in the lemma) is sufficiently risk tolerant with respect to e, B, M, and ψ, if the compensation which he requires for accepting a trade B-v (Y in the lemma), v-q ($c(u;T)$ in the lemma) is sufficiently close to 0. Namely, he is willing to trade B for a price q which is sufficiently close to his valuation v. With this at hand, we can prove the analog of theorem 4.4, for the case of risk averse preferences.

Theorem 5.3 Let there be given any positive ρ, M, ψ, and k. Assume that $A2$ holds. Then there exists a $\delta > 0$ such that there exists a trade B with $E(B) \geq k$, a price q, and strategies $a^1(\omega)$, $a^2(\omega)$ such that for all traders with utility functions that satisfy $\sup_{\{|x| \leq M\}} |u''(x)| < \delta$, (that is, sufficiently risk tolerant traders):

(a) the strategies $a^1(\omega)$, $a^2(\omega)$ form a (1-ρ)-rationality Nash equilibrium.

(b) $\mu(T(a,B,q)$ is common p-belief$) > 0$.

As in section 4, a similar result holds for Rubinstein's (1989) notion of almost common knowledge. Trade, as well as rationality, can also be known up to an arbitrarily large degree among sufficiently risk tolerant traders in a (1-ρ)-rationality Nash equilibrium.

6. Concluding Remarks

Remark 6.1 In order to understand the relationship between common p-belief and common knowledge in the context of this paper, we should check what happens as we let p approach 1. Observe that as we let p approach 1, $A1$ is harder to satisfy. That is, for any given information structures that satisfy $A1$ for some $\hat{p} < 1$, there exists a $\hat{p} < \bar{p} < 1$ such that for all $\bar{p} < p$, $A1$ will not hold. Hence, given any information structures, as we let p approach 1, common p-belief of trade disappears. For $p = 1$, we are in the case of common knowledge of trade (and of rationality) and the no trade result of Milgrom and Stokey holds. Similarly, using Rubinstein's notion of "almost common knowledge", given the Π^i's, the maximal possible level of knowledge of trade is bounded by $2 \cdot \min_{i \in I} \{N_i\}$.

Remark 6.2 We wish to emphasize that for any $p < 1$ there exists an information structure such that for any $\rho > 0$: when preferences are fixed, there exist a bet B, a price q, and a small enough $\bar{k}$ such that for any $k \leq \bar{k}$, traders may have common p-belief of trading $k \cdot B$. Alternatively, fix $\bar{k}$ arbitrarily large. Then, for all sufficiently risk tolerant preferences, traders can have common

p-belief of trade of any volume $k \leq \bar{k}$.

Remark 6.3 As shown by the proofs of theorems 4.4 and 5.3, there exists a non-degenerate price interval which supports common p-belief of trade. This suggests introducing a "market-maker" whose role would be to find a random variable B that will induce common p-belief of trade by solving a linear programming problem (as in proposition 4.1). Moreover, when such a trade B exists, v_1-v_2 is unbounded. Therefore the market-maker can charge a bid/ask spread and guarantee himself a positive expected return.

Remark 6.4 The main motivation of this paper was to explain trade in real markets, where trade is a persistent phenomenon. While our model deals only with two periods, the argument made here can be generalized in a dynamic setting to allow for repeated trade. Consider the following: Suppose we have a model with infinitely many periods. Ω is fixed through time and at period 0, an $\omega \in \Omega$ is realized. The information partitions of the traders may depend on time, but at time 0, $\Pi_0^i(\omega)=\Omega$ for all $i \in I$, and traders "learn" over time so that the information partitions at time $t+1$ are refinements of the respective information partitions at time t. At time 0, we start with Pareto efficient allocations. In this setting, we might still get common p-belief of trade at every period. At time t, rationality is defined as behaving optimally on that day. Trader i might want to trade, since he believes that the other traders might not be rational. After trading the allocations may be Pareto optimal with respect to each traders information at time t. But, since rational behavior may be independently determined at each period, the trader cannot be sure about the rationality of other traders and of himself, in the future. Thus, at the following day, new trade might occur precisely for the same reasons. In general, arriving at Pareto efficiency at time t might still allow for further trade at time $t+1$ as the traders' information becomes more refined.

Remark 6.5 The argument of this paper can, of course, be repeated without using the common prior assumption. On the other hand, as Dow, Madrigal and Werlang (1990) show, dispensing with the common prior assumption by itself is not enough to support trade. We should note, however, that using different priors enables us to "close" the model with each trader's prior assigning a probability $\rho > 0$ to other players being irrational, while leaving himself perfectly rational. Thus, it might be argued that using different priors alleviates some of the difficulties with the interpretation of the model, in particular regarding the way we chose to model "irrationality".

Remark 6.6 No-trade results are closely related to (no) agreeing-to-disagree results. Indeed, if risk neutral traders cannot even disagree about the expected value of a certain prospect, why should they trade it, if they are already in possession of a Pareto efficient allocation? In fact, as was shown in Geanakoplos (1988) and Rubinstein and Wolinsky (1990) agreeing-to-disagree results, with the common prior assumption, are stronger than no-trade results in the sense that they continue to hold under weaker requirements on the knowledge operator. As for relaxing these results, Monderer and Samet (1989) show that with common p-belief, the posteriors of an event cannot differ by more than $2(1-p)$.[5] So, for any $p < 1$, players can have common p-belief of disagreement over the values of the posteriors of a certain event. As belief approaches knowledge, that is $p \to 1$, they get the agreeing to disagree result.

Remark 6.7 The argument made here can be repeated in Aumann's (1987) setup. Namely, a state of the world specifies all objects of uncertainty including the traders' actions. The results, in this case, will be similar to the results obtained here. Specifically, instead of postulating the existence of an equilibrium under which trade is common p-belief, in Aumann's (1987) formulation, we can show instead the existence of a prior probability μ such that μ assigns probability one to the event where traders adopt actions $a^1(\omega)$, $a^2(\omega)$, which form a $(1-\rho)$-rationality equilibrium, and under which trade is common p-belief.

Remark 6.8 In a related (but independently developed) paper, Sonsino (1993) also deals with common p-belief of trade. His main results are as following: first, suppose that a proposed trade B is fixed, then if the expected valuations of B are common p-belief then they cannot differ

[5] Neeman (1993) improves this bound to $1-p$.

significantly (the bound on their difference, however, depends on B and can be arbitrarily large). Secondly, it is shown that as p approaches 1, there can be no common p-belief of trade. In his paper "trade" is implicitly assumed to occur whenever the traders have different conditional expectations for some bet B.

Since Sonsino's model does not allow traders to be irrational at any state of the world, his results seem to be in contradiction to remark 4.3. The resolution to this apparent contradiction lies in the implicit notion of rationality: in Sonsino's paper, acts are not formally introduced into the model. Thus, when a bet B is offered, each trader simply computes its expectation, implicitly assuming it would be accepted by the other trader. By contrast, in the model presented above, each trader is explicitly aware of the possibility that the other may refuse to trade, and thus the bet may be called off. That is, he is facing uncertainty both regarding the "objective" state of the world and the other trader's actions.

We believe that in the Milgrom and Stokey framework trade cannot be accounted for if common knowledge of rationality is assumed. Sonsino (1993) implicitly assumes some type of irrationality in that his traders do not fully analyze the model as it is known to the outside observer. In this paper, irrationality is explicit and the model may be assumed to be commonly known. One way or another, remark 4.3 argues that common knowledge of "full" rationality precludes trade.

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Appendix: Proofs

Proof of proposition 4.1

"sufficiency": We formulate the problem as a linear programming problem. We define the sets $\{\Gamma_{i,j}\}_{\substack{i=1,\ldots,n+1\\ j=1,\ldots,m+1}}$ as follows: for $i=1,..,n$ and $j=1,..,m$, $\Gamma_{i,j}=\Theta_i^1\cap\Theta_j^2$; for $i=n+1$, and $j=1,..,m$, $\Gamma_{n+1,j}=\Theta_j^2\backslash\theta^1$; for $j=m+1$, and $i=1,..,n$, $\Gamma_{i,m+1}=\Theta_i^1\backslash\theta^2$; and for $i=n+1$, $j=m+1$, $\Gamma_{n+1,m+1}=(\theta^1)^c\cap(\theta^2)^c$. Observe that for all $i=1,..,n$, $\theta_i^1=\bigcup_{j=1}^{m+1}\Gamma_{i,j}$ and for all $j=1,..,m$, $\theta_j^2=\bigcup_{i=1}^{n+1}\Gamma_{i,j}$.

Consider the following linear programming problem (P),

$$\text{(P)} \qquad \operatorname*{Max}_{\{b_{1,1},\ldots,b_{n+1,m+1},v_1,v_2\}} \; v_1 - v_2$$

$$\text{subject to,} \qquad (1) \qquad \frac{\sum_{j=1}^{m+1}\mu(\Gamma_{i,j})b_{i,j}}{\sum_{j=1}^{m+1}\mu(\Gamma_{i,j})} = v_1, \quad \text{for } i=1,..,n.$$

$$(2) \qquad \frac{\sum_{i=1}^{n+1}\mu(\Gamma_{i,j})b_{i,j}}{\sum_{i=1}^{n+1}\mu(\Gamma_{i,j})} = v_2, \quad \text{for } j=1,..,m.$$

and where the $b_{i,j}$'s are unrestricted. Since $b_{n+1,m+1}$ does not play any part in the following, we set $b_{n+1,m+1}=0$.

Notice that (P) is feasible (set $b_{i,j}=0$ for all i,j, and $v_1=v_2=0$). We adopt the following notation, $\mu(\Gamma_{i,j})$ is denoted by $\gamma_{i,j}$, $\sum_{j=1}^{m+1}\mu(\Gamma_{i,j})$ is denoted by λ_i for $i=1,..,n$, and $\sum_{i=1}^{n+1}\mu(\Gamma_{i,j})$ is denoted by δ_j for $j=1,..,m$. Observe that λ_i, $\delta_j>0$.

We write the dual problem (D) for (P): Let x_i, $i=1,..,n$ be the dual variable corresponding to the i-th constraint of set (1) (in which λ_i appears), and let y_j, $j=1,..,m$ be the dual variable corresponding to the j-th constraint of set (2) (in which δ_j appears).

(D)

$$\underset{\{x_1,..x_n,y_1,..y_m\}}{\text{Min}} \quad 0$$

subject to

(1) $\quad \dfrac{\gamma_{i,j}}{\lambda_i}x_i + \dfrac{\gamma_{i,j}}{\delta_j}y_j = 0, \qquad i=1,..,n;\, j=1,..,m.$

(2) $\quad \dfrac{\gamma_{i,m+1}}{\lambda_i}x_i = 0, \qquad i=1,..,n.$

(3) $\quad \dfrac{\gamma_{n+1,j}}{\delta_j}y_j = 0, \qquad j=1,..,m.$

(4) $\quad \displaystyle\sum_{i=1}^{n} x_i = -1$

(5) $\quad \displaystyle\sum_{j=1}^{m} y_j = 1$

and where the x_i, y_j's are unrestricted.

We wish to show that (P) has a solution with $v_1 \neq v_2$. This is true iff (P) is unbounded, and by using the duality theorem for linear programming, iff its dual (D) is infeasible. We assume the contrary, that is, that there exist $\{x_i\}_{i=1}^{n}$, $\{y_j\}_{j=1}^{m}$ which satisfy all the constraints of (D), and derive a contradiction.

The following facts are direct implications of the dual constraints,

Fact 1 $\quad \gamma_{i,j}>0 \Rightarrow [x_i=0 \Leftrightarrow y_j=0]$ for $i=1,..,n;\, j=1,..,m$ (by (1)).

Fact 2 $\quad \gamma_{i,m+1}>0 \Rightarrow x_i=0$ for $i=1,..,n$ (by (2)).

Fact 3 $\quad \gamma_{n+1,j}>0 \Rightarrow y_j=0$ for $j=1,..,m$ (by (3)).

Since θ^1 and θ^2 do not coincide in the sense of *A1* (ii), $\mu(\theta^1 \triangle \theta^2)>0$. Hence there is an $i \leq n$ such that $\gamma_{i,m+1}>0$ or a $j \leq m$ such that $\gamma_{n+1,j}>0$. Without loss of generality we can relabel the x_i's such that there exists an i^* so that $\gamma_{i,m+1}>0$ for all $i^*<i\leq n+1$; and $\gamma_{i,m+1}=0$ for all

$1 \le i < i^*$; similarly, we relabel the y_j's such that there exists a j^* so that $\gamma_{n+1,j} > 0$ for all $j^* < j \le m+1$; and $\gamma_{n+1,j} = 0$ for all $1 \le j < j^*$. We establish the following results,

Fact 4 For all i, $i^* < i \le n$, $x_i = 0$ (by fact (2)).

Fact 5 For all j, $j^* < j \le m$, $y_j = 0$ (by fact (3)).

(see figure 3)

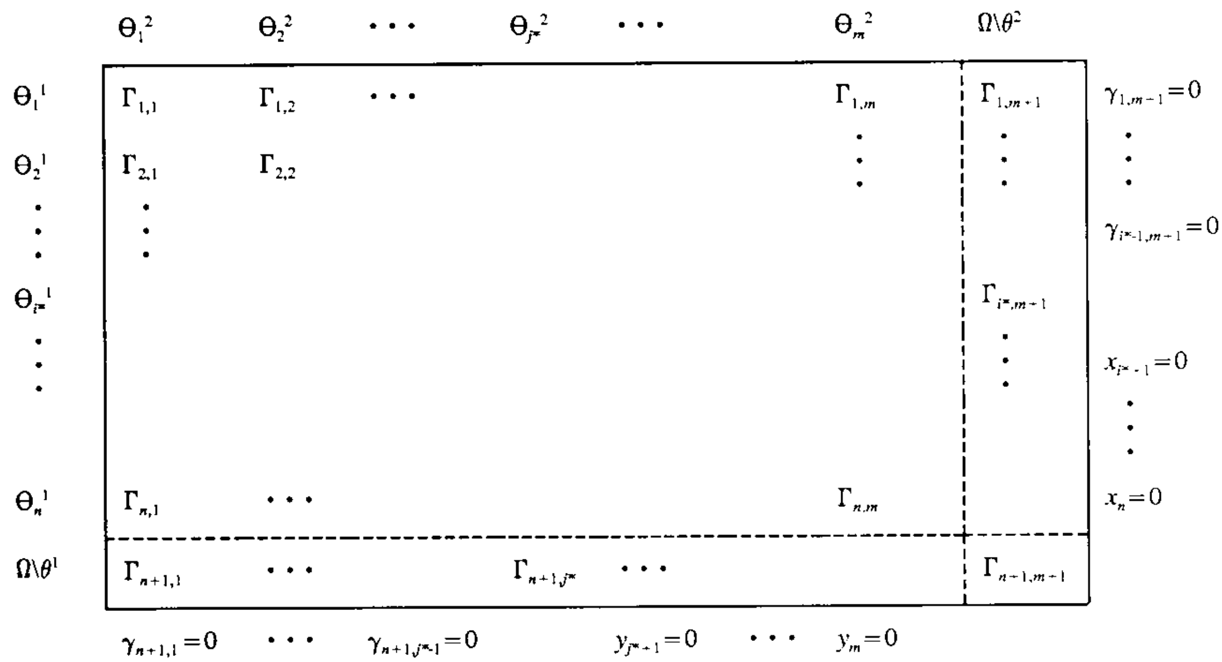

figure 3

Define $\theta^{1\prime} = \bigcup_{x_i \neq 0} \Theta_i^1$, $\theta^{2\prime} = \bigcup_{y_j \neq 0} \Theta_j^2$.

Lemma At least one of the following propositions holds:

(i) $\theta^{1\prime}$ or $\theta^{2\prime}$ are empty.

(ii) $\mu(\theta^{1\prime} \triangle \theta^{2\prime}) = 0$.

Proof We show that not (i) implies (ii). Assume $\theta^{1'}$, $\theta^{2'}$ are non empty, and let $\simeq$ denote equal up to measure 0. $\theta^{1'} = \bigcup_{x_i \neq 0} \Theta_i^1 \simeq \bigcup_{\substack{i \leq n \\ x_i \neq 0 \\ \gamma_{i,j}>0}} \Gamma_{i,j}$. We now show that $\bigcup_{\substack{i \leq n \\ x_i \neq 0 \\ \gamma_{i,j}>0}} \Gamma_{i,j} \simeq \bigcup_{\substack{j \leq m \\ y_j \neq 0 \\ \gamma_{i,j}>0}} \Gamma_{i,j}$. First, consider $i \leq n$, $j \leq m$. Fact (1) ensures that for $i=1,..,n$ and $j=1,..,m$, for $\Gamma_{i,j}$ with $x_i \neq 0$ and $\gamma_{i,j}>0$ also $y_j \neq 0$ and vice versa. Hence, $\Gamma_{i,j}$ appears in both expressions or in neither.

Next, for $i \leq n$, consider the element $\Gamma_{i,m+1}$. Distinguish among three cases: (i) if $i<i^*$, then $\gamma_{i,m+1}=0$, by the choice of i^*, hence $\Gamma_{i,m+1}$ does not appear in either expression; (ii) if $i^*<i \leq n$, then $x_i=0$, by fact (4), hence $\Gamma_{i,m+1}$ does not appear in either expression; (iii) if $i=i^*$, either $\gamma_{i^*,m+1}=0$ or $x_i=0$ by fact (2), and again $\Gamma_{i,m+1}$ does not appear in either expression. A similar argument establishes that for $j \leq m$, $\Gamma_{n+1,j}$ does not appear in either expression.

On the other hand, $\bigcup_{\substack{j \leq m \\ y_j \neq 0 \\ \gamma_{i,j}>0}} \Gamma_{i,j} \simeq \bigcup_{y_j \neq 0} \Theta_j^2 = \theta^{2'}$.

So we have $\theta^{1'} \simeq \theta^{2'}$, or $\mu(\theta^{1'} \triangle \theta^{2'})=0$. //

Since $\mu(\theta^{1'} \triangle \theta^{2'})=0$ contradicts the conditions of the proposition, the above lemma implies that either $\theta^{1'}$ or $\theta^{2'}$ are empty. Therefore, either $x_i=0$ for $i=1,..,n$ or $y_j=0$ for $j=1,..,m$, which violates constraints (4) and (5) of the dual (D) correspondingly. Therefore, we conclude that the dual (D) is infeasible.

"necessity": Otherwise, suppose there exist index subsets $\phi \neq K^{1'} \subseteq \{1,\ldots,n\}$ and $\phi \neq K^{2'} \subseteq \{1,\ldots,m\}$ such that $\mu(\theta^{1'} \triangle \theta^{2'})=0$. Then for any $k^1 \in K^{1'}$, $k^2 \in K^{2'}$, $v_1 = E(B \mid \Theta_{k^1}^1) = E(B \mid \Theta_{k^2}^2) = v_2$, in contradiction to $v_1 \neq v_2$. ■

Proof of theorem 4.2

(⇒): We show the necessity of *A1*.

By proposition 2.2, the fact that $C(B,v_1,v_2)$ is common p-belief implies the existence of sets π^1 and π^2 such that for all $i,j \in I$, $\mu(\pi^j \mid \Pi_k^i) \geq p$ and $\mu(C(B,v_1,v_2) \mid \Pi_k^i) \geq p$ for all $\Pi_k^i \subseteq \pi^i$.

Moreover, π^1 and π^2 satisfy the second part of *A1*, i.e., they do not "coincide". Otherwise, there exist non empty sets $K^{1'} \subseteq K^1$ and $K^{2'} \subseteq K^2$ such that if we denote $\pi^{i'} = \bigcup_{k \in K^{i'}} \Pi_k^i$ for $i \in I$, then

$\mu(\pi^{1'} \Delta \pi^{2'})=0$. But, $\pi^{1'} \cap \pi^{2'} \subseteq B_p^i(C(B,v_1,v_2))$, therefore for $\omega \in \pi^{1'} \cap \pi^{2'}$, $E(B \mid \Pi^1(\omega))=v_1$ and $E(B \mid \Pi^2(\omega))=v_2$. However, since $\mu(\pi^{1'} \Delta \pi^{2'})=0$, $v_1=E(B \mid \bigcup_{k \in K^{1'}} \Pi_k^1)=E(B \mid \bigcup_{k \in K^{2'}} \Pi_k^2)=v_2$, which contradicts $v_1 \neq v_2$.

($\Leftarrow$): We show that *A1* is sufficient.

Let π^1, π^2 satisfy the conditions of *A1*. By proposition 4.1, there exist a bet B and $v_1 \neq v_2$, such that $E(B \mid \Pi_k^1)=v_1$ for all $\Pi_k^1 \subseteq \pi^1$, and $E(B \mid \Pi_k^2)=v_2$ for all $\Pi_k^2 \subseteq \pi^2$. We show that for $\omega \in \pi^1 \cap \pi^2$, $C(B,v_1,v_2)$ is common p-belief. Using proposition 2.2, we need only to demonstrate that for all $i \in I$, $\mu(C(B,v_1,v_2) \mid \Pi_k^i) \geq p$ for all $\Pi_k^i \subseteq \pi^i$. This holds since by proposition 4.1 $\pi^i \subseteq C^i(B,v_i)$ for $i \in I$ which implies $\pi^1 \cap \pi^2 \subseteq C(B,v_1,v_2)$. ■

Proof of theorem 4.4

(i) implies (ii): Suppose that there exists a proposed trade B, a price q, and strategies $a^1(\omega)$, $a^2(\omega)$ such that $T(a,B,q)$ is common p-belief at $\omega \in \Omega$. Recall that by definition of $T(a,B,q)$, trade at $\omega \in T(a,B,q)$ is strictly ex-post Pareto improving. We show that *A1* holds. The fact that $T(a,B,q)$ is common p-belief at $\omega \in \Omega$, implies by proposition 2.2 that there exists two sets π^1, π^2 such that,

(i) For all $i,j \in I$, $\mu(\pi^j \mid \Pi_k^i) \geq p$ for all $\Pi_k^i \subseteq \pi^i$.

(ii) For all $i \in I$, $\mu(T(a,B,q) \mid \Pi_k^i) \geq p$ for all $\Pi_k^i \subseteq \pi^i$.

(i) coincides with *A1* (i). As for *A1* (ii), consider the following: for any $\omega \in T(a,B,q)$, by its definition, *"buy"* is the unique optimizing act on $\Pi^1(\omega)$, and since in addition trader 1 is also rational in $T(a,B,q)$, $a^1(\omega')=$ *"buy"* for all $\omega' \in \Pi^1(\omega)$. Similarly, $a^2(\omega')=$ *"sell"* for all $\omega' \in \Pi^2(\omega)$ such that $\omega \in T(a,B,q)$. Therefore, since for all $\Pi_k^i \subseteq \pi^i$, $\mu(T(a,B,q) \mid \Pi_k^i) \geq p$, for all $\omega \in \Pi_k^1$ such that $\Pi_k^1 \subseteq \pi^1$, $a^1(\omega)=$ *"buy"*, and for all $\omega \in \Pi_k^2$ such that $\Pi_k^2 \subseteq \pi^2$, $a^2(\omega)=$ *"sell"*.

Now, suppose *A1* (ii) fails to hold. That is, there exist index subsets $\phi \neq K^{1'} \subseteq K^1$ and

$\phi \neq K^{2'} \subseteq K^2$ such that $\mu(\pi^{1'} \triangle \pi^{2'})=0$. Notice that at $\omega \in \pi^{1'} \cap \pi^{2'}$, $\pi^{1'} \cap \pi^{2'}$ is common knowledge, and so by the above, at $\omega \in \pi^{1'} \cap \pi^{2'}$ the set $\{\omega | a^1(\omega)= \text{"buy"}, a^2(\omega)= \text{"sell"}\}$ is also common knowledge. So, at $\omega \in \pi^{1'} \cap \pi^{2'}$, we have common knowledge of strictly improving trade among rational traders which contradicts remark 4.3.

(ii) implies (i): The proof proceeds in two steps. In the first step, from $\pi^1 = \bigcup_{k \in K^1} \Pi_k^1$ and $\pi^2 = \bigcup_{k \in K^2} \Pi_k^2$ that satisfy *A1* we generate the sets over which trade will take place. These would take the form $\theta^1 = \bigcup_{k \in K^1} \Theta_k^1$ and $\theta^2 = \bigcup_{k \in K^2} \Theta_k^2$ where $\Theta_k^i \subseteq \Pi_k^i$ and $\pi^1 \cap \pi^2 = \theta^1 \cap \theta^2$. To the θ^i's, we apply proposition 4.1 to get a proposed trade B. In the second step, we suggest a pair of strategies which form a $(1\text{-}\rho)$-rationality equilibrium such that trader 1 will offer to buy B on π^1, trader 2 will offer to sell B on π^2, and over $\pi^1 \cap \pi^2 = \theta^1 \cap \theta^2$ the traders will have common p-belief of trade of arbitrarily large quantities. Trade, however, will actually take place also over $\theta^i \backslash \theta^j$.

Step 1. Suppose $\pi^1 = \bigcup_{k \in K^1} \Pi_k^1$, $\pi^2 = \bigcup_{k \in K^2} \Pi_k^2$ satisfy the conditions in *A1*. Define the sets $\{\Theta_k^1\}_{k \in K^1}$ and $\{\Theta_k^2\}_{k \in K^2}$ as follows: for $i,j \in I$, $i \neq j$, $k \in K^i$ such that $\Pi_k^i \subseteq \pi^j$, set $\Theta_k^i = \Pi_k^i$. Otherwise, for $i,j \in I$, $i \neq j$, $k \in K^i$ such that $\Pi_k^i \not\subseteq \pi^j$, choose $F_k^j \subseteq \Pi_k^i \backslash \pi^j$ with $\mu(F_k^j) > 0$, and $\mu(F^j) = \sum_{k \in K^i} \mu(F_k^j) \leq \rho$. $F^j = \bigcup_{k \in K^i} F_k^j$ will be the event over which trader j is irrational. We define $\Theta_k^i = (\Pi_k^i \cap \pi^j) \cup F_k^j$.

We show that $\theta^1 = \bigcup_{k \in K^1} \Theta_k^1$ and $\theta^2 = \bigcup_{k \in K^2} \Theta_k^2$ satisfy *A1* (ii). Let there be given index subsets $\phi \neq K^{1'} \subseteq K^1$ and $\phi \neq K^{2'} \subseteq K^2$. For $i=1,2$, denote $\pi^{i'} = \bigcup_{k \in K^{i'}} \Pi_k^i$ and $\theta^{i'} = \bigcup_{k \in K^{i'}} \Theta_k^i$. Note that $\mu(\theta^{1'} \triangle \theta^{2'}) = \mu(\theta^{1'} \backslash \theta^{2'}) + \mu(\theta^{2'} \backslash \theta^{1'}) \geq \mu(\theta^{1'} \backslash \pi^{2'}) + \mu(\theta^{2'} \backslash \pi^{1'})$ since $\theta^{i'} \backslash \theta^{j'} \supseteq \theta^{i'} \backslash \pi^{j'}$ for $i \neq j$. We show $\mu(\theta^{1'} \backslash \pi^{2'}) + \mu(\theta^{2'} \backslash \pi^{1'}) > 0$. Distinguish between two cases: (i) Either $\Pi_k^1 \subseteq \pi^2$ for all $k \in K^{1'}$, and $\Pi_h^2 \subseteq \pi^1$ for all $h \in K^{2'}$. In this case $\theta^{1'} = \pi^{1'}$, $\theta^{2'} = \pi^{2'}$ and by *A1* (ii)

$\mu(\theta^{1'}\backslash\pi^{2'})+\mu(\theta^{2'}\backslash\pi^{1'})=\mu(\pi^{1'}\triangle\pi^{2'})>0$. Or, (ii) There exists a $k\in K^{1'}$ such that $\Pi_k^1\not\subset\pi^2$, or an $h\in K^{2'}$ such that $\Pi_h^2\not\subset\pi^1$, in which case $\mu(\theta^{1'}\backslash\pi^{2'})+\mu(\theta^{2'}\backslash\pi^{1'})\geq\max\{\mu(F_k^1),\mu(F_h^2)\}>0$ by the choice of the F_k^{i}'s.

Since $\{\Theta_k^1\}_{k\in K^1}$ and $\{\Theta_k^2\}_{k\in K^2}$ satisfy *A1* (ii), we can apply proposition 4.1 and obtain a trade B such that $E(B|\Theta^1(\omega))=v_1$ for all $\omega\in\pi^1$, $E(B|\Theta^2(\omega))=v_2$ for all $\omega\in\pi^2$, and $v_1\neq v_2$. Without loss of generality, because we identify trader 1 as the buyer and trader 2 as the seller, and because we can add to B a constant, we assume $E(B|\pi^2\backslash\pi^1)\leq v_2<v_1\leq E(B|\pi^1\backslash\pi^2)$. Modify B such that for all $k\in K^i$, $i\neq j$, and $\omega\in(\Pi_k^i\backslash\pi^j)\backslash F_k^j$, $B(\omega)=B(\omega')$ for $\omega'\in F_k^j$. Notice that we extend B such that it equals a constant on any set of the form $\Pi_k^1\cap\Pi_h^2$, so that for $i=1,2$, for all $\omega\notin\pi^2\cup F^2$, $v_1\leq E(B|\Pi^2(\omega)\cap\theta^1)$ (if well defined) and for all $\omega\notin\pi^1\cup F^1$, $E(B|\Pi^1(\omega)\cap\theta^2)\leq v_2$ (if well defined).

Step 2. We construct a pair of strategies $a^1(\omega)$ and $a^2(\omega)$ which form a (1-ρ)-rationality equilibrium and such that $\pi^1\cap\pi^2\subseteq T(a,B,q)$ for $v_2<q<v_1$. Let $a(\omega)$ be defined as follows: for $i=1,2$, for any trade which does not equal the trade B defined above, and for any price q associated with it, let $a^i(\omega)=$ *"refrain"*. Otherwise, for the trade B which is specified above, let

$$a^1(\omega)=\begin{cases}\text{"}buy\text{"} & \omega\in\pi^1\bigcup F^1 \text{ and } q<v_1\\ \text{"}refrain\text{"} & \text{otherwise}\end{cases}$$

$$a^2(\omega)=\begin{cases}\text{"}sell\text{"} & \omega\in\pi^2\bigcup F^2 \text{ and } q>v_2\\ \text{"}refrain\text{"} & \text{otherwise}\end{cases}$$

We show that trader 1 is rational everywhere except F^1, i.e., $R^1(a(\omega))=(F^1)^c$. On π^1, when trader 1 is rational, *"buy"* is the unique optimal action when the trade B is proposed at price $v_2<q<v_1$. On $(R^1)^c$, when trader 1 is irrational, *"buy"* is the unique suboptimal action for $q<v_1$. For any trade B' which is not B above, and for any price q, *"refrain"* is optimal since trader 2 refrains from trade. For the proposed trade B above, a price $v_2<q<v_1$ and for $\omega\in\pi^1$, by construction of B,

$$E\big(B \mid \Pi^1(\omega) \cap \{\omega \mid a^2(\omega,B,q) = \text{"}sell\text{"}\}\big)$$
$$= E\big(B \mid \Pi^1(\omega) \cap (\theta^2 \cup (R^2)^c)\big)$$
$$= v_1.$$

and therefore for any price $v_2 < q < v_1$, *"buy"* is the unique optimal action. For $q \geq v_1$ *"refrain"* is an optimal action, and for $q \leq v_2$ *"buy"* is an optimal action. Otherwise, for $\omega \in \Omega \backslash (\pi^1 \cup F^1)$, either trader 2 refrains from trade, and then *"refrain"* is an optimal action, or, on $\pi^2 \backslash \pi^1$, when $a^2(\omega,B,q) = \text{"}sell\text{"}$, $E\big(B \mid \Pi^1(\omega) \cap \{\omega \mid a^2(\omega,B,q) = \text{"}sell\text{" for some q}\}\big) \leq v_2$ by construction, and since trader 2 sells only for price greater than v_2, *"refrain"* is an optimal action.

A similar argument shows that trader 2 is rational everywhere except F^2, i.e., $R^2(a(\omega)) = (F^2)^c$. On π^2, when trader 2 is rational, *"sell"* is the unique optimal action, and on $(R^2)^c$, when trader 2 is irrational, *"sell"* is the unique suboptimal action.

The fact that the traders are rational everywhere except on $(R^i)^c$ and that $\mu((R^i)^c) \leq \rho$ shows that $a(\omega)$ is a $(1\text{-}\rho)$-rationality equilibrium. $T(a,B,q)$ is held as common p-belief at $\omega \in \pi^1 \cap \pi^2$ since π^1, π^2 satisfy *A1* (i) which is sufficient for common p-belief by proposition 2.2.

Furthermore, since both traders are risk neutral, they are willing to trade arbitrarily large quantities of B, i.e., they would be willing to trade $B' = k \cdot B$ for any $k > 0$. Thus, we have common p-belief of trading arbitrarily large quantities of B at a $(1\text{-}\rho)$-rationality Nash equilibrium for any price $v_2 < q < v_1$. ■

Proof of theorem 5.1

Suppose that $\pi^1 = \bigcup_{k=1}^{n} \Pi_k^1$, $\pi^2 = \bigcup_{k=1}^{m} \Pi_k^2$ satisfy condition *A2*. Consider the sets $\{\Pi_k^1 \cap \chi_j^1\}_{k=1,j=1}^{n\ ,\ n_1}$, $\{\Pi_k^2 \cap \chi_h^2\}_{k=1,h=1}^{m\ ,\ n_2}$. Applying proposition 4.1 to these sets generates a trade B and $v_1 \neq v_2$ such that B is "noise" with respect to the initial endowment e^i and i's information structure, i.e., $E(B \mid \Pi_k^1 \cap \chi_j^1) = v_1$ for all $\Pi_k^1 \subseteq \pi^1$, and $\chi_j^1 = \{\omega : e^1(\omega) = \alpha_j^1\}$; and $E(B \mid \Pi_k^2 \cap \chi_h^2) = v_2$ for all $\Pi_k^2 \subseteq \pi^2$, and $\chi_h^2 = \{\omega : e^2(\omega) = \alpha_j^2\}$. The proof continues as in theorem 4.2. ■

Proof of lemma 5.2

(i) We start by observing the fact that any function $u \in U_{\infty,0}$ with a sufficiently small second derivative can be approximated by a linear function.

Fact For all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $u \in U_{\infty,0}$ such that $\sup_{\{|x| \leq M\}} |u''(x)| < \delta$, there exist a linear function $v:\mathbf{R} \to \mathbf{R}$, $v'' \equiv 0$, and $v' \geq u'(M)$ such that $\sup_{\{|x| \leq M\}} |u(x) - v(x)| < \epsilon$.

Proof Observe that for all $|x| \leq M$, $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$, where $\underline{u}$ is a quadratic function with $\underline{u}''$, $\overline{u}$ is a linear function with $\overline{u}' = u'(-M)$ and such that $\underline{u}(-M) = \overline{u}(-M) = u(-M)$. Since $\overline{u}(M) - \underline{u}(M) = 2\delta M^2$, if we fix $v = \overline{u}$, then by setting $\delta = \frac{\epsilon}{2M^2}$, we obtain $\sup_{\{|x| \leq M\}} |u(x) - v(x)| < \epsilon$ and $v' \geq u'(M)$. //

Set $\epsilon < \frac{\psi \cdot c}{2}$. By the above observation, there exists a $\delta > 0$ such that for all $u \in U_{M,\psi}$ satisfying $\sup_{\{|x| \leq M\}} |u''(x)| < \delta$, there exist a linear function v, $v'' \equiv 0$, and $v' \geq u'(M)$ such that $v(x) - \epsilon < u(x) < v(x) + \epsilon$ for all $|x| \leq M$. Therefore,

$$E\big(u(X+Y+c) \mid T\big) > E\big(v(X+Y+c) \mid T\big) - \epsilon$$

$$\geq E\big(v(X) \mid T\big) + \psi \cdot c - \epsilon \qquad \text{(because of linearity of } v \text{ and } v' \geq \psi)$$

$$> E\big(v(X) \mid T\big) + \epsilon \qquad \text{(by choice of } \epsilon)$$

$$\geq E\big(u(X) \mid T\big).$$

(ii) Fix an $\epsilon > 0$, by (i) there exists a $\delta > 0$ such that for all $u \in U_{M,\psi}$ satisfying $\sup_{\{|x| \leq M\}} |u''(x)| < \delta$, $E\big(u(X+Y+\epsilon) \mid T\big) > E\big(u(X) \mid T\big)$. However, replacing ϵ with $c^{\circ}(u;T)$, causes the last inequality to become an equality, therefore, we deduce $c^{o}(u;T) < \epsilon$. ■

Proof of theorem 5.3

The proof follows the proof of theorem 4.4 while making the necessary corrections for risk.

Step 1. From $\pi^1 = \bigcup_{k \in K^1} \Pi_k^1$ and $\pi^2 = \bigcup_{k \in K^2} \Pi_k^2$ that satisfy *A2* we generate $\theta^1 = \bigcup_{k \in K^1} \Theta_k^1$ and $\theta^2 = \bigcup_{k \in K^2} \Theta_k^2$ as in the proof of theorem 4.4. To $\left\{\Theta_k^1 \cap \chi_j^1\right\}_{\substack{k \in K^1 \\ j=1,\dots,n_1}}$, $\left\{\Theta_k^2 \cap \chi_h^2\right\}_{\substack{k \in K^2 \\ h=1,\dots,n_2}}$, we apply proposition 4.1 to get a proposed trade B which satisfies all the conditions in theorem 4.4 and in addition, $E(B \mid \Theta^1(\omega) \cap \chi_j^1) = v_1$ for all $\omega \in \pi^1$ and χ_j^1, $E(B \mid \Theta^2(\omega) \cap \chi_h^2) = v_2$ for all $\omega \in \pi^2$ and χ_h^2. Without loss of generality, multiply B to get $E(B) \geq k$.

Step 2. We construct a pair of strategies $a^1(\omega)$ and $a^2(\omega)$ that form a $(1-\rho)$-rationality equilibrium. For $i=1,2$, let the $a^i(\omega)$'s be defined as follows: for any trade which is not B above, and for any price q, let $a^i(\omega) =$ *"refrain"*. Otherwise, for the trade B specified above, consider the sets F_k^i which are defined as in the proof of theorem 4.4., and let $a^1(\omega,B,q)$ and $a^2(\omega,B,q)$ be defined as follows,

$$a^1(\omega) = \begin{cases} \text{"}\mathit{buy}\text{"} & \omega \in \pi^1 \bigcup F^1 \text{ and } q < v_1 - c^1(\omega,B) \\ \text{"}\mathit{refrain}\text{"} & \text{otherwise} \end{cases}$$

$$a^2(\omega) = \begin{cases} \text{"}\mathit{sell}\text{"} & \omega \in \pi^2 \bigcup F^2 \text{ and } q > v_2 + c^2(\omega,B) \\ \text{"}\mathit{refrain}\text{"} & \text{otherwise} \end{cases}$$

where $c^1(\omega,B)$ is chosen such that $E\big(u^1(e^1(\omega)+B-c^1(\omega,B)) \mid \Pi^1(\omega)\big) = E\big(u^1(e^1(\omega)) \mid \Pi^1(\omega)\big)$ for all

$\omega \in \pi^1$, and $c^2(\omega,B)$ is chosen such that $E\big(u^2(e^2(\omega)-B+c^2(\omega,B)) \mid \Pi^2(\omega)\big)=E\big(u^2(e^2(\omega)) \mid \Pi^2(\omega)\big)$ for all $\omega \in \pi^2$.

As in the proof of theorem 4.4, we show that trader 1 is rational everywhere except F^1. On π^1, when trader 1 is rational and is sufficiently risk tolerant, *"buy"* is the unique optimal action when the trade B is proposed at price $v_2+c^2(\omega,B)<q<v_1-c^1(\omega,B)$. On F^1, when trader 1 is irrational, *"buy"* is the unique suboptimal action. For any trade B' which is not B above, and for any price q, *"refrain"* is optimal since trader 2 refrains from trade. For the proposed trade B above, by lemma 5.3 there exist a $\delta_1>0$ such that for all traders with utilities satisfying $\sup_{\{|x|\le M\}} |u''(x)|<\delta_1$, $\frac{v_1-v_2}{2}<v_1-c^1(\omega,B)$. Therefore for the price $q=\frac{v_1-v_2}{2}$, *"buy"* is the unique optimal action for a sufficiently risk tolerant trader 1. For a price $q \ge v_1-c^1(\omega,B)$, *"refrain"* is an optimal action. Otherwise, for $\omega \in \Omega \backslash (\pi^1 \cup F^1)$, either trader 2 refrains from trade, and then *"refrain"* is an optimal action, or, on $\pi^2 \backslash \pi^1$, when $a^2(\omega,B,q)=$ *"sell"*, $E\big(B \mid \Pi^1(\omega) \cap \{\omega \mid a^2(\omega,B,q)=\text{"}sell\text{" for some q}\}\big) \le v_2$ by construction, and since trader 2 sells only for price greater than v_2, *"refrain"* is an optimal action.

A similar argument shows that trader 2 is rational everywhere except F^2. On π^2, when trader 2 is rational and is sufficiently risk tolerant, *"sell"* is the unique optimal action, and on F^2, when trader 2 is irrational, *"sell"* is the unique suboptimal action.

The fact that the traders are rational everywhere except on $F^i=(R^i)^c$ and that $\mu((R^i)^c) \le \rho$ shows that $a(\omega)$ is a $(1-\rho)$-rationality equilibrium for sufficiently risk tolerant traders. $T(a,B,\frac{v_1-v_2}{2})$ is held as common p-belief at $\omega \in \pi^1 \cap \pi^2$ between sufficiently risk tolerant traders since π^1, π^2 satisfy *A1* (i) which is sufficient for common p-belief by proposition 2.2. ■