Discussion Paper No. 1046

AN INTEGRAL REPRESENTATION FOR NON SIMPLE ACTS WITH CERTAINTY EQUIVALENTS

by

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May 1993

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Abstract: in this paper we consider a continuous subjective expected utility model with a connected space of consequences (CSEU, for brevity). This class of models has recently received attention (see Wakker (1989)). Like in Savage (1954), we consider a finitely additive probability measure on a σ-field of events. The aim of the paper is to show that in such a Savagean set-up the appropriate condition to impose on the set of acts to get a CSEU model is to require the existence of certainty equivalents. While quite intuitive, several technical difficulties had to be solved in order to make this argument rigorous, as emerges in the appendix. Nevertheless, the results of the paper are clear and easy to understand.

1. Introduction

In this paper we consider a continuous subjective expected utility model with a connected space of consequences (CSEU, for brevity). This class of models has recently received attention (see Wakker (1989)). Like in Savage (1954), we consider a finitely additive probability measure on a σ-field of events. The aim of the paper is to show that in such a Savagean set-up the appropriate condition to impose on the set of acts to get a CSEU model is to require the existence of certainty equivalents. While quite intuitive, several technical difficulties had to be solved in order to make this argument rigorous, as emerges in the appendix. Nevertheless, the results of the paper are clear and easy to understand.

We now give a more detailed overview of the paper. Let ${}_{c}\mathbf{E}_{c'}$ be a simple act which has **c** as consequence if the true state is in **E**, and **c'** otherwise. A constant act which is indifferent to ${}_{c}\mathbf{E}_{c'}$ is called a certainty equivalent for ${}_{c}\mathbf{E}_{c'}$. Suppose that all simple acts like ${}_{c}\mathbf{E}_{c'}$ have a certainty equivalent. As shown in section 3, this happens if and only if the space of consequences is connected and the von Neumann-Morgestern utility function is continuous. Moreover, it is shown that the existence of certainty equivalents induce a number of interesting topological properties in the space of consequences. In section 4 we show that, under some conditions, the existence of certainty equivalents is equivalent to the existence of an integral representation with a continuous vN-M utility function. Secondly, and most importantly, we obtain a monotone convergence theorem for non simple acts. Besides its technical interest (indeed we are working with just finitely additive measures), this result permits to calculate more easily the value of the integrals involved in Savage's representation theorem for non simple acts. In fact, this value can be calculated as the limit of a sequence of integrals of appropriate simple functions.

2. Preliminary notions

In the state space Ω we consider a σ-field $\mathcal{A}$, whose elements are called events. As a particular case we can have $\mathcal{A}=\mathcal{P}(\Omega)$, the power set of Ω. Let $\mathcal{C}$ be the space of consequences and $\mathcal{F}$ be the set of all functions $\mathbf{f}:\Omega\to\mathcal{C}$ such that $\mathbf{f}^{-1}(\mathbf{E})\in\mathcal{A}$ for all $\mathbf{E}\subseteq\mathcal{C}$. We call acts the elements of $\mathcal{F}$. Of particular importance will be the set of simple acts $\mathcal{S}$, i.e. the acts with finite codomain. Among them, we denote by ${}_{c}\mathbf{E}_{c'}$ the ones that have $\mathbf{c}$ as consequence if $\omega\in\mathbf{E}$ and $\mathbf{c'}$ if $\omega\notin\mathbf{E}$. Moreover, $\hat{\mathbf{c}}$ denotes the constant act which has $\mathbf{c}$ as consequence for all $\omega\in\Omega$.

On $\mathcal{F}$ we consider a preference relation $<$. We assume that $<$ satisfies axioms $\mathbf{P}_1$-$\mathbf{P}_6$ of Savage's theory in the form reported in chapter 14 of Fishburn (1970). Obvious modifications are required since we are considering a σ-field $\mathcal{A}$ and not directly $\mathcal{P}(\Omega)$. The preference relation $<$ induces in a well-known manner two orderings $<^*$ and $<'$ respectively on $\mathcal{C}$ and Ω. In particular from $\mathbf{P}_1$ and $\mathbf{P}_3$ it follows that $<^*$ is a weak order on $\mathcal{C}$. The relations $\approx^*$ and $\lesssim^*$ on $\mathcal{C}$ are defined in the usual manner. If $\mathbf{c',c''}\in\mathcal{C}$, let $(\mathbf{c'},\mathbf{c''})=\{\mathbf{c}\in\mathcal{C}:\mathbf{c'}<^*\mathbf{c}<^*\mathbf{c''}\}$, $(\leftarrow,\mathbf{c'})=\{\mathbf{c}\in\mathcal{C}:\mathbf{c}<^*\mathbf{c'}\}$, $(\mathbf{c'},\rightarrow)=\{\mathbf{c}\in\mathcal{C}:\mathbf{c'}<^*\mathbf{c}\}$. These subsets of $\mathcal{C}$ are called the open intervals. $\mathcal{C}$ can be topologized by taking as a base the collection of all open intervals. $\mathcal{C}/\approx$ denotes the quotient space generated by the equivalence relation $\approx$. Let $[\mathbf{c}]\in\mathcal{C}/\approx$ be the equivalence class with representative $\mathbf{c}\in\mathcal{C}$. We endow $\mathcal{C}/\approx$ with the quotient topology. It is easy to check that this topology has as a base the subsets of the form $([\mathbf{c'}],[\mathbf{c''}])$, $(\leftarrow,[\mathbf{c''}])$ and $([\mathbf{c'}],\rightarrow)$. Equipped with this topology, $\mathcal{C}/\approx$ becomes a completely normal Hausdorff space (cf. Alo' (1971)).

Finally we state Savage's representation theorem for simple acts. In the rest of the paper the term probability measure will always refer to a finitely additive set function.

Proposition 2.1 (Savage). *Let* $<$ *be a preference relation on* $\mathcal{S}$ *obeying* P_1-P_6. *Then there is a probability measure* P *and a real-valued function* $u(\cdot)$ *on* $\mathcal{C}$, *called von Neumann-Morgenstern utility function (vN-M for short), such that:*

(i) for every pair of events A,B *it holds*

$$A<'B \text{ if and only if } P(A)<P(B);$$

(ii) for every $0\leq c\leq P(A)$, *there is an event* B, *with* $B\subset A$, *such that* $P(B)=c$;

(iii) for every pair $c',c''\in\mathcal{C}$ *it holds*

$$c'<^{*}c'' \text{ if and only if } u(c')<u(c'');$$

(iv) $u(\cdot)$ *is unique up to linear positive transformations;*

(v) let $\{A_i\}_{i=1}^{n}$ *and* $\{B_j\}_{j=1}^{m}$ *be two measurable partitions of* Ω *and* $f,g\in\mathcal{S}$ *such that* $f(A_i)=c_i$ *for* $i=1,\dots,n$ *and* $g(B_j)=c'_j$ *for* $j=1,\dots,m$. *It holds*

$$f<g \text{ if and only if } \textstyle\sum_{i=1}^{n}u(f(A_i))P(A_i)<\sum_{j=1}^{m}u(g(B_j))P(B_j).$$

Remark. We conclude with a remark on notation. Let $f\in\mathcal{S}$ and let $X=f(\Omega)$. By definition, X is a finite set. In correspondence with every $x\in X$, let $\omega_x\in f^{-1}(x)$, i.e. ω_x is an element of $f^{-1}(x)$. Set $\Gamma(f)=\{\omega_x:x\in X\}$, i.e. $\Gamma(f)$ is constructed taking a representative from each set $f^{-1}(x)$. As to point (v) in proposition 2.1, we clearly have $\sum_{i=1}^{n}u(f(A_i))P(A_i)=\sum_{\omega\in\Gamma(f)}u(f(\omega))P(f^{-1}(f(\omega)))$. In the following we will always use the r.h.s., and not the l.h.s.. Moreover, for convenience, instead of writing each time $\Gamma(f)$, we will simply write the r.h.s. as $\sum_{\omega\in\Omega}u(f(\omega))P(f^{-1}(f(\omega)))$.

3. Further axioms and their implications

Besides $\mathbf{P}_1$-$\mathbf{P}_6$ we will make use of two other axioms:

$\mathbf{A}_8$: *let* $\mathbf{c}',\mathbf{c}''\in\mathcal{C}$. *Then for every* $\mathbf{E}\in\mathcal{A}$ *there is a consequence* $\mathbf{c}$ *such that* $\hat{\mathbf{c}}\approx {}_{c'}\mathbf{E}_{c''}$.

$\mathbf{A}_9$: *there is a pair* $\underline{c},\underline{\underline{c}}\in\mathcal{C}$ *such that for all* $\mathbf{c}\in\mathcal{F}$, $\underline{\underline{c}}\lesssim^* c \lesssim^* \underline{c}$.

$\mathbf{A}_8$ is just the certainty equivalent assumption. The next proposition makes its role clear.

Proposition 3.1 *Let* $\mathbf{P}_1$-$\mathbf{P}_6$ *hold. Then the following two statements are equivalent:*

(a) $\mathbf{A}_8$ *holds;*

(b) $\mathcal{C}$ *is a connected space and the vN-M utility function of proposition 2.1 is continuous.*

Proof: Appendix.

The important case of monetary consequences is covered by the next corollary. Let $\mathbf{x},\mathbf{y}\in\mathbb{R}$. A preference relation $<^*$ is called monotone if $\mathbf{x}<^*\mathbf{y}$ whenever $\mathbf{x} < \mathbf{y}$ w.r.t. the natural order in $\mathbb{R}$.

Corollary 3.2. *Let* $\mathcal{C}$ *be an interval of* $\mathbb{R}$ *and* $<$ *a preference relation on* $\mathcal{F}$ *which induces a monotone* $<^*$ *on* $\mathcal{C}$. *Under* $\mathbf{P}_1$-$\mathbf{P}_6$ *the following two statements are equivalent:*

(a) $\mathbf{A}_8$ *holds;*

(b) *the vN-M utility function is continuous.*

Proof: Appendix.

A_8 has a number of consequences on the topological structure of $\mathscr{C}/\approx$. To see them, we introduce some further notation. Let $\pi:\mathscr{C}\rightarrow\mathscr{C}/\approx$ be the projection of $\mathscr{C}$ onto the quotient space $\mathscr{C}/\approx$. By definition π is continuous. Let $<^{\#}$ be the linear order on $\mathscr{C}/\approx$ defined by $[c']<^{\#}[c'']$ if and only if $c'<^{*}c''$, for $c',c''\in\mathscr{C}$. Finally, let $u^{\#}:\mathscr{C}/\approx\rightarrow\mathbb{R}$ be defined by $[c']<^{\#}[c'']$ if and only if $u^{\#}([c'])<u^{\#}([c''])$, for $[c'],[c'']\in\mathscr{C}/\approx$. Clearly $u^{\#}\circ\pi=u$. The existence of $u(\cdot)$ ensures the existence of $u^{\#}(\cdot)$. Recall that a function is closed when the images of closed sets are closed.

Proposition 3.3. *Let* P_1-P_6 *hold. Suppose* A_8 *holds. Then*

(i) The function $u^{\#}:\mathscr{C}/\approx\rightarrow\mathbb{R}$ *is closed.*

(ii) $\mathscr{C}/\approx$ is a locally compact, connected, separable and metrizable space.

(iii) Every closed subspace of $\mathscr{C}/\approx$ is a complete separable metric space, i.e. a Polish space.

Proof: Appendix.

The space $\mathscr{C}/\approx$ is locally compact under A_8. The role of A_9 is to ensure the compactness of $\mathscr{C}/\approx$. This allows a strengthening of proposition 3.1.

Proposition 3.4. *Let* P_1-P_6 *hold. Then the following two statements are equivalent:*

(a) A_8 *and* A_9 *hold;*

(b) $\mathscr{C}$ is a connected space, $\mathscr{C}/\approx$ is a compact space and the vN-M utility function of proposition 2.1 is continuous.

Proof: Appendix.

4. Representation of non simple acts

In this section we first prove that, given $\mathbf{A}_9$, axiom $\mathbf{A}_8$ is a necessary and sufficient condition for the existence of an expected utility representation with a continuous vN-M utility function. Then, and most importantly, we give a monotone convergence theorem for non simple acts. It has some technical interest because we are working with finitely additive measures and order structures. But here it is important because the definition of integral we use in next proposition 4.1 is difficult to apply since it considers a supremum over a large class of simple acts. Instead, the convergence result gives an easier way to figure out the value of these integrals. For, given a non simple function $\mathbf{f}$ and a sequence of simple acts that satisfy the hypoteses of the proposition w.r.t. $\mathbf{f}$, then $\int_{\omega\in\Omega} \mathbf{u}(\mathbf{f}(\omega))\mathbf{dP}$ can be calculated as the limit of the integrals of the simple acts in the sequence. And the integrals of simple functions are easily obtained. Furthermore, lemma 4.4 below will provide a way to construct a sequence of simple acts with the desired properties.

To do all this, we need a further axiom.

$\mathbf{A}_7$: *let* $\mathbf{A}\in\mathcal{A}$. $\left\{\hat{\mathbf{c}}\lesssim\hat{\mathbf{f}}(\omega) \text{ given } \mathbf{A}, \text{ for all } \omega\in\mathbf{A}\right\}$ *implies* $\hat{\mathbf{c}}\lesssim\mathbf{f}$ *given* $\mathbf{A}$, *while* $\left\{\hat{\mathbf{c}}\gtrsim\hat{\mathbf{f}}(\omega) \text{ given } \mathbf{A}, \text{ for all } \omega\in\mathbf{A}\right\}$ *implies* $\hat{\mathbf{c}}\gtrsim\mathbf{f}$ *given* $\mathbf{A}$.

Remark. $\mathbf{A}_7$ is a weakened form of Savage's dominance axiom $\mathbf{P}_7$. This weakening is allowed by $\mathbf{A}_9$. Hence, we have a sort of trade-off between dominance and boundeness requirements. Here we prefer the latter. Axiom $\mathbf{P}_7$ and its variants are discussed at lenght in Schervish and Seidenfeld (1983).

We now introduce the notion of integral we use. By proposition 2.1, for two simple acts $\mathbf{h},\mathbf{h'}$ we have (see the remark on p.3 for notation):

$$h \underset{\sim}{<} h' \quad \text{iff} \quad \sum_{\omega\in\Omega} u(h(\omega))P(h^{-1}(h(\omega))) \le \sum_{\omega\in\Omega} u(h'(\omega))P(h'^{-1}(h'(\omega))).$$

For $f\in\mathcal{F}$, let us define

$$V^-(f)=\sup\left\{\sum_{\omega\in\Omega} u(h(\omega))P(h^{-1}(h(\omega))) \text{ for all } h\in\mathcal{S} \text{ such that } h\underset{\sim}{<}f\right\}$$

$$V^+(f)=\inf\left\{\sum_{\omega\in\Omega} u(h(\omega))P(h^{-1}(h(\omega))) \text{ for all } h\in\mathcal{S} \text{ such that } h\underset{\sim}{>}f\right\}.$$

If $V^-(f)=V^+(f)$, then the common value is denoted by $\int_{\omega\in\Omega} u(f(\omega))dP$. Moreover, set $\mathcal{F}^+=\{f\in\mathcal{F} : V^-(f)=V^+(f)<\infty\}$.

The next result contains the expected utility representation of non simple acts with continuos vN-M utility functions.

Proposition 4.1. *Let* P_1-P_6, A_7 *and* A_9 *hold and let* $f,g\in\mathcal{F}$. *Then the following two statements are equivalent.*

(i) $\mathcal{F}=\mathcal{F}^+$, $\mathcal{C}$ *is connected, and there exists a probability measure* **P** *and a continuos function* $u(\cdot)$ *such that*

$$f<g \text{ if and only if } \int_{\omega\in\Omega} u(f(\omega))dP < \int_{\omega\in\Omega} u(g(\omega))dP.$$

(ii) A_8 *holds.*

Proof: Appendix.

We now give the monotone convergence theorem for non simple acts.

Proposition 4.2. *Suppose* P_1-P_6 *and* A_7-A_9 *hold. Let* $\{g_i\}_{i=1}^{\infty}$ *be a sequence of acts in* $\mathcal{F}$, *and let* $g\in\mathcal{F}$ *such that:*

(i) $g_1(\omega)\underset{\sim}{<}^* \cdots \underset{\sim}{<}^* g_i(\omega)\underset{\sim}{<}^* g_{i+1}(\omega)\underset{\sim}{<}^* \cdots \underset{\sim}{<}^* g(\omega)$ *for all* $\omega\in\Omega$;

(ii) $\lim_{i\to\infty}\{\sup_{\omega\in\Omega}(u(g(\omega))-u(g_i(\omega))\}=0$.

Then $\lim_{i\to\infty}\int u(g_i(\omega))dP=\int u(g(\omega))dP$.

Proof: Appendix.

The proof of this result rests on the next two lemmas. Let **f**,**g** be two simple acts such that $f(\omega)\approx^{*}g(\omega)$ for all $\omega\in\Omega$. By lemma 14.1 of Fishburn (1970) we have $f\approx g$. The following lemma shows that under A_7 the same is true for two non simple acts.

Lemma 4.3. *Let* P_1-P_6, A_7-A_9 *hold and let* **f**,**g** *be two acts. If* $f(\omega)\approx^{*}g(\omega)$ *for all* $\omega\in\Omega$*, then* $f\approx g$.

Proof: Appendix.

Next we give the other lemma needed to prove proposition 4.2.

Lemma 4.4. *Let* P_1-P_6, A_7-A_9 *hold and let* $f\in\mathcal{F}$. *Then there is a sequence* $\{f_i\}_{i=1}^{\infty}$ *of simple acts such that:*

(a) $f_i\lesssim f_j\lesssim f$ *for all* $i<j$;

(b) $f_i(\omega)\lesssim^{*}f(\omega)$ *for all* $\omega\in\Omega$;

(c) $\lim_{i\to\infty}\{\sup_{\omega\in\Omega}(u(f(\omega))-u(f_i(\omega))\}=0$.

Moreover, if a sequence $\{f_i\}_{i=1}^{\infty}$ *of simple acts satisfies (a), (b) and (c), then* $\lim_{i\to\infty}\int u(f_i(\omega))dP=\int u(f(\omega))dP$.

Proof: Appendix.

Appendix

Proof of proposition 3.1. Before begining the proof we recall that every simple act **f** induces in a well known manner a lottery $\mathbf{P}_{\mathbf{f}}$ on $\mathscr{C}$. Similarly $<$ induces a weak order $<''$ on the set $\mathcal{P}$ of all induced lotteries on $\mathscr{C}$. For these notions we refer to ch.14 of Fishburn (1970). Finally, given $c',c''\in\mathscr{C}$, we denote by ${}_{c'}\lambda_{c''}$ the lottery giving **c'** with probability λ and **c"** with probability $1-\lambda$.

The proof is divided in three steps.

(i) In this first step we prove that $\mathbf{A}_8$ implies that $\mathbf{u}(\cdot)$ is continuous. By axiom $\mathbf{P}_5$ there is a pair **c',c"** such that $c'<^{*}c''$. Let us put $u(c'')=0$ and $u(c')=1$. Let $c^{o}\in[c'',c']$. Let $V(u(c^{o}))$ be an open neighborhood of $u(c^{o})$. W.l.o.g., we can suppose $V(u(c^{o}))=(\alpha,\beta)$, with $\alpha<u(c^{o})<\beta$. Suppose $u(c'')<\alpha<\beta<u(c')$. We consider only this case because the other can be handled analogously. By $\mathbf{A}_8$, for any $\lambda\in(0,1)$ there is $c\in(c'',c')$ such that $P_c\approx'' {}_{c''}\lambda_{c'}$. We have, by construction, $u(c)=\lambda$. Hence there exists a pair c_1,c_2 such that $\alpha<u(c_1)<u(c^{o})<u(c_2)<\beta$. Therefore, $u(\Gamma)\subset V(u(c^{o}))$, where $\Gamma=\{c\in\mathscr{C}:c_1<^{*}c<^{*}c_2\}$. This proves that $u(\cdot)$ is continuous at any point $c^{o}\in[c'',c']$. This is enough to prove the desired result.

(ii) In this second step we prove that $\mathbf{A}_8$ implies that $\mathscr{C}$ is connected. It is known that to show this we have to prove that there are no jumps in $\mathscr{C}$ and that every subset of $\mathscr{C}$ which has an upper bound has a supremum (cf. Kelley (1955) p.58 for chains; it easy to check that this is true also for a preorder). The first part is trivial. For the second, let us consider a subset $A\subset\mathscr{C}$. By hypothesis **A** has an upper bound. Let $B=\{c\in\mathscr{C}:c\gtrsim^{*}c_a \text{ for all } c_a\in A\}$. **B** is the subset of all upper bouns of **A**. Pick $c_a\in A$ and $c_b\in B$. By $\mathbf{A}_8$ for every lottery ${}_{c_a}\lambda_{c_b}$ there is a c_l such that $P_{c_\lambda}\approx'' {}_{c_a}\lambda_{c_b}$. On the other hand it is known that for every $c' \in (c_a, c_b)$ there is a unique $\alpha \in (0,1)$ such that $P_{c''} \approx'' {}_{c_a} \alpha_{c_b}$. So $\{c_\lambda : P_{c_\lambda} \approx'' {}_{c_a} \lambda_{c_b}\} = (c_a, c_b)$. Let $\lambda_B = \{\lambda \in [0,1]: c_\lambda \in B\}$. λ_B has an infimum if and only if **B** has an infimum. In fact, suppose that λ_B has an infimum λ' and suppose there is a $c' \in B$ such that $c' <^* c_{\lambda'}$. There is $\alpha \in (0,1)$ such that $P_{c'} \approx'' {}_{c_a} \alpha_{c_b}$. By hypothesis $\alpha \in \lambda_B$ and so $\alpha \geq \lambda'$. But $\alpha \geq \lambda'$ implies that ${}_{c_a} \alpha_{c_b} \gtrsim'' {}_{c_a} \lambda'_{c_b}$ (cf. Fishburn (1970) theorem 8.3) and so $c' \gtrsim^* c_{\lambda'}$, a contradiction. The converse is easy to check. Now, λ_B is a subset of the connected space $[0,1]$. λ_B has a lower bound and so, by Kelley (1955) p.58, it has an infimum. Therefore, **B** also has an infimum.

(iii) In this last step we prove that **(b)** implies **(a)**. Let $\mathscr{C}$ be a connected space and $u(\cdot)$ a continuous function. Let $c_1, c_2 \in \mathscr{C}$ with $c_1 <^* c_2$. Since (c_1, c_2) is a connected subset of $\mathscr{C}$, then $u((c_1,c_2)) = \{u(c): c_1 <^* c <^* c_2\}$ is a connected subset of $\mathbb{R}$. So we can write $u((c_1,c_2)) = (\alpha, \beta)$, with $\alpha = \inf\{u(c): c_1 <^* c <^* c_2\}$ and $\beta = \sup\{u(c): c_1 <^* c <^* c_2\}$. α and β exist because $u(c_1)$ and $u(c_2)$ are, respectively, a lower and upper bound of $\{u(c): c_1 <^* c <^* c_2\}$. We prove that $\alpha = u(c_1)$. Suppose, on the contrary, that $\alpha > u(c_1)$. Let $V(u(c_1))$ be an open neighborhood of $u(c_1)$ such that $x < \alpha$ for all $x \in V(u(c_1))$. Let $V'(c_1)$ be any open neighborhood of c_1 in $\mathscr{C}$. Then for some $c \in V'(c_1)$ we have $c > c_1$. Hence $u(c) > u(c_1)$ and so $u(c) > \alpha$. This implies $u(c) \notin V(u(c_1))$ by the construction of $V(u(c_1))$. Thus $u(V'(c_1))$ is not contained in $V(u(c_1))$. Since $V'(c_1)$ was arbitrary, this contradicts the continuity of $u(\cdot)$. Hence $\alpha = u(c_1)$, as wanted. Similarly, $\beta = u(c_2)$. Now, let $\gamma \in (0,1)$ and let $r = \gamma u(c_1) + (1-\gamma) u(c_2)$. Since we have proved that $u((c_1,c_2)) = (u(c_1), u(c_2))$, there exists $c_r \in (c_1, c_2)$ such that $u(c_r) = r$. Let $V: \mathcal{P} \to \mathbb{R}$ be the linear functional arising in the von Neumann-Morgenstern representation theorem. It is known that $V(P_{c_r}) = \gamma u(c_1) + (1-\gamma) u(c_2) = V(P_{c_1} \gamma P_{c_2})$.

Therefore $P_{c_r} \approx'' P_{c_1} \gamma P_{c_2}$. Now, let ${}_{c_1}A_{c_2}$ be a simple act and suppose that $P(A)=\gamma$. Let $\mathcal{B}$ be the set of all events **B** such that $P(B)=\gamma$. We have proved that there exists a $c_\gamma \in (c_1,c_2)$ such that $P_{c_\gamma} \approx'' P_{c_1} \gamma P_{c_2}$. By definition we have $P_{c_\gamma} \approx'' P_{c_1} \gamma P_{c_2}$ if and only if $\hat{c}_\gamma \approx {}_{c_1}B_{c_2}$ for all events $B \in \mathcal{B}$ (cf. theorem 14.3 of Fishburn (1970), called reduction theorem). In particular, we have $\hat{c}_\gamma \approx {}_{c_1}A_{c_2}$. Since γ was arbitrary, this completes the proof. ■

Proof of corollary 3.2. Since $\overset{*}{<}$ is monotone, it is antisymmetric. Therefore $\mathcal{C}/{\approx}=\mathcal{C}$ and the natural topology of $\mathbb{R}$ coincides with the one induced by $\overset{*}{<}$. Moreover, $u(\cdot)$ is defined on a connected space because $\mathcal{C}$ is an interval. So **(b)** implies **(a)** by proposition 3.1. Finally, **(a)** implies **(b)** by proposition 3.1. ■

Proof of proposition 3.3. (i) We first prove that A_8 implies that $u(\cdot)$ is closed. Let $f,g \in \mathcal{S}$. $P_f \alpha P_g$ is the lottery giving P_f with probability α and P_g with probability $1-\alpha$. P_1-P_6 imply that one of the Herstein and Milnor (1953) axioms holds on $\mathcal{P}$ (cf. Fishburn (1982) theorem 2.1 and Fishburn (1970) pp.203-205). This axiom says that for every $P_{f_1}, P_{f_2}, P_{f_3}$ belonging to $\mathcal{P}$ the sets $\{\alpha : P_{f_1} \alpha P_{f_3} \gtrsim'' P_{f_2}\}$ and $\{\beta : P_{f_1} \beta P_{f_3} \lesssim'' P_{f_2}\}$ are closed in $[0,1]$, and so in $\mathbb{R}$. In particular, this means that $\{\gamma : P_{c_1} \lesssim'' {}_{c_1}\gamma_{c_2} \lesssim'' P_{c_2}\}$ is closed in $[0,1]$ for every $c_1, c_2, c_3 \in \mathcal{C}$ (recall that $P_{c_1} \gamma P_{c_2}$ and ${}_{c_1}\gamma_{c_2}$ describe the same lottery). Let $c', c'' \in \mathcal{C}$ with $c' \overset{*}{<} c''$. Let us put $u(c')=0$ and $u(c'')=1$. We have already shown that A_8 implies that for every lottery ${}_{c'}\gamma_{c''}$ there is a consequence $c \in [c',c'']$ such that $P_c \approx'' {}_{c'}\gamma_{c''}$. Furthermore it is known that, without loss of generality, we can put $u(c)=\gamma$ (cf. Fishburn (1970) ch. 14). It follows that

$\{\gamma:P_{c'}\precsim''_{c'},\gamma_{c''}\precsim''P_{c''}\}=\{u(c):P_{c'}\precsim''P_{c}\precsim''P_{c''}\}=\{u(c):c'\precsim^{*}c\precsim^{*}c''\}=\{u^{\#}(c):c'\precsim^{*}c\precsim^{*}c''\}$.

Since $\{\gamma:P_{c'}\precsim''_{c'},\gamma_{c''}\precsim''P_{c''}\}$ is closed, it follows that $\{u(c):c'\precsim^{*}c\precsim^{*}c''\}$ also is closed. Therefore, the image of every interval $[c',c'']$ is closed. A similar reasoning shows that the image of every closed interval $(\leftarrow,c]$ and $[c,\rightarrow)$ is closed. Finally, let B be any closed subset of $\mathcal{C}$ and let $\mathfrak{I}$ be the family of closed intervals. The family $\{G\subseteq\mathcal{C}/\approx\ :\ G^{c}\in\mathfrak{I}\}$ is a base for $\mathcal{C}/\approx$. Hence, we have $B=\bigcap_{i\in I}\{B_i\ :\ i\in\mathfrak{I}\}$. Since $u^{\#}(\cdot)$ is one-to-one, we have $u^{\#}(B)=u^{\#}(\bigcap_{i\in I}B_i)=\bigcap_{i\in I}u^{\#}(B_i)$. Since $u^{\#}(B_i)$ is closed for all $i\in I$, we can conclude that the image of each closed set is closed

(ii) By definition π is continuous. So if $\mathcal{C}$ is connected, then also $\mathcal{C}/\approx$ is connected. It is also easy to check that $\mathcal{C}/\approx$ is locally compact. Moreover, by the existence of $u^{\#}(\cdot)$, $\mathcal{C}/\approx$ is separable.

We know that π is closed. Therefore, $u^{\#}(\cdot)$ is a continuous and closed function on $\mathcal{C}/\approx$. Since $u^{\#}(\cdot)$ is one-to-one, we conclude that $\mathcal{C}/\approx$ is homeomorphic to a subset of $\mathbb{R}$. Therefore, since metrizability is a topological invariant, the order topology on $\mathcal{C}/\approx$ can be metrized by means of the metric $d([c],[c'])=|u(c'),u(c'')|$, with $c,c'\in\mathcal{C}$ (see Dugundji (1966) p.186).

(iii) Let C be a closed subspace of $\mathcal{C}/\approx$. Let $\{[c_i]\}_{i=1}^{\infty}$ be a Cauchy sequence in $\mathcal{C}/\approx$. By the definition of the metric $d(\cdot)$, $\{u^{\#}([c_i])\}_{i=1}^{\infty}$ is a Cauchy sequence. Since $u^{\#}(\cdot)$ is one-to-one, $u^{\#}(\cdot)$ is a closed map. Hence, $u^{\#}(C)$ is closed in $\mathbb{R}$, and so is complete. Therefore, there exists $d\in u^{\#}(C)$ such that $\lim_{i\to\infty}u^{\#}([c_i])=d$. Since $u^{\#}(\cdot)$ is one-to-one, there exists $[c]\in\mathcal{C}/\approx$ such that $\lim_{i\to\infty}d([c],\ [c_i])=0$. Hence C is complete. Since $\mathcal{C}/\approx$ is a separable metric space, it is second countable. Hence every subspace of $\mathcal{C}/\approx$ is a second countable metric space, and so a separable metric space. ■

Proof of proposition 3.4. We first prove that **(a)** implies **(b)**. From propositions 3.1 it follows that $\mathscr{C}$ is connected and that the vN-M utility function is continuous. By proposition 3.3, under $\mathbf{A}_8$ the quotient space $\mathscr{C}/\approx$ is connected. By Kelley (1955) p.58 this implies that every subset of $\mathscr{C}/\approx$ with an upper bound has a supremum. If we add $\mathbf{A}_9$, it follows that $(\mathscr{C}/\approx,<^{\#})$ is a complete chain. By Kelley (1955) p.162 the space $\mathscr{C}/\approx$ is compact.

Now we prove that **(b)** implies **(a)**. Since $\mathscr{C}/\approx$ is compact, $(\mathscr{C}/\approx,<^{\#})$ is a complete chain and so it is easy to check that $\mathbf{A}_9$ is satisfied. By proposition 3.1, $\mathbf{A}_8$ also is satisfied ■

Proof of proposition 4.1. Point (i) implies point (ii) by proposition 3.1. We now prove the converse. Suppose $\mathbf{A}_8$ holds. We begin by proving the "only if" part in (i) and that $\mathscr{F}=\mathscr{F}^{+}$. Let $\mathbf{f}\in\mathscr{F}$ and set $\mathbf{B}_f=\{\mathbf{h}\in\mathscr{S} : \mathbf{h}\precsim\mathbf{f}\}$ and $\mathbf{B}^f=\{\mathbf{h}\in\mathscr{S} : \mathbf{h}\succsim\mathbf{f}\}$. By $\mathbf{A}_9$ we have $\underline{\underline{\mathbf{c}}}\precsim^{*}\mathbf{c}\precsim^{*}\underline{\mathbf{c}}$ for all $\mathbf{c}\in\mathscr{C}$. Hence for every $\mathbf{f}\in\mathscr{F}$ and every $\omega\in\Omega$ we have $\underline{\underline{\mathbf{c}}}\precsim^{*}\mathbf{f}(\omega)\precsim^{*}\underline{\mathbf{c}}$. Thus, by $\mathbf{A}_7$, $\hat{\underline{\underline{\mathbf{c}}}}\precsim\mathbf{f}\precsim\hat{\underline{\mathbf{c}}}$. In particular, $\hat{\underline{\underline{\mathbf{c}}}}\precsim\mathbf{h}\precsim\hat{\underline{\mathbf{c}}}$ for all $\mathbf{h}\in\mathscr{S}$. By proposition 2.1 we have $\mathbf{u}(\underline{\underline{\mathbf{c}}})\leq\sum_{\omega\in\Omega}\mathbf{u}(\mathbf{h}(\omega))\mathbf{P}(\mathbf{h}^{-1}(\mathbf{h}(\omega)))\leq\mathbf{u}(\underline{\mathbf{c}})$ for all $\mathbf{h}\in\mathbf{B}_f$ and all $\mathbf{h}\in\mathbf{B}^f$. By the completeness axiom of the real numbers (see e.g. Royden (1988) p.33) this implies that there exist both $\mathbf{sup}\{\sum_{\omega\in\Omega}\mathbf{u}(\mathbf{h}(\omega))\mathbf{P}(\mathbf{h}^{-1}(\mathbf{h}(\omega)))$ for all $\mathbf{h}\in\mathbf{B}_f\}$ and $\mathbf{inf}\{\sum_{\omega\in\Omega}\mathbf{u}(\mathbf{h}(\omega))\mathbf{P}(\mathbf{h}^{-1}(\mathbf{h}(\omega)))$ for all $\mathbf{h}\in\mathbf{B}^f\}$. Moreover, these are finite because $\mathbf{u}(\cdot)$ is bounded. Now, let $\mathbf{f},\mathbf{g}\in\mathscr{F}$ with $\mathbf{f}<\mathbf{g}$, as in (i). By now we know that $\hat{\underline{\underline{\mathbf{c}}}}\precsim\mathbf{f}\precsim\hat{\underline{\mathbf{c}}}$ and $\hat{\underline{\underline{\mathbf{c}}}}\precsim\mathbf{g}\precsim\hat{\underline{\mathbf{c}}}$. From lemma 14.4 of Fishburn (1970). it follows that there is a unique $\lambda^f\in[0,1]$ such that ${}_{\underline{\underline{c}}}\lambda^f_{\underline{c}}\approx'\mathbf{P}_f$ and a unique $\lambda^g\in[0,1]$ such that ${}_{\underline{\underline{c}}}\lambda^g_{\underline{c}}\approx'\mathbf{P}_g$. Clearly $\lambda^f<\lambda^g$. There is a rational number λ such that $\lambda^f<\lambda<\lambda^g$. It is known that ${}_{\underline{\underline{c}}}\lambda^f_{\underline{c}}<'{}_{\underline{\underline{c}}}\lambda_{\underline{c}}<'{}_{\underline{\underline{c}}}\lambda^g_{\underline{c}}$. By point (ii) of proposition 2.1 there is an event $\mathbf{A}\in\mathscr{A}$ with $\mathbf{P(A)}=\lambda$. From the definition of $<''$ it follows $\mathbf{f}<{}_{\underline{\underline{c}}}\mathbf{A}_{\underline{c}}<\mathbf{g}$. By $\mathbf{A}_8$ there is a consequence c such that $\hat{c}\approx_{\underline{\underline{c}}}\mathbf{A}_{\underline{c}}$. Hence $f<\hat{c}<g$. Therefore $h<\hat{c}$ for all $h\in\mathbf{B}_f$, while $\hat{c}\lesssim h$ for some $h\in\mathbf{B}_g$ (indeed $\hat{c}\in\mathbf{B}_g$). This implies that:

$$(*)\quad \sup\{\textstyle\sum_{\omega\in\Omega}u(h(\omega))P(h^{-1}(h(\omega)))\text{ for all }h\in\mathbf{B}_f\} <$$
$$\sup\{\textstyle\sum_{\omega\in\Omega}u(h(\omega))P(h^{-1}(h(\omega)))\text{ for all }h\in\mathbf{B}^f\}.$$

By $(*)$ for every pair $f,g\in\mathcal{F}$ we have $f<g$ if and only if $V^-(f)<V^-(g)$. In a similar way it can be proved that $f<g$ if and only if $V^+(f)<V^+(g)$. Moreover, we have already seen that both $V^+(f)$ and $V^-(f)$ are finite. Therefore, if we show that $V^+(f)=V^-(f)$ for all $f\in\mathcal{F}$, then we prove both the "only if" part and that $\mathcal{F}=\mathcal{F}^+$. Clearly $V^+(h)=V^-(h)=\sum_{\omega\in\Omega}u(h(\omega))P(h^{-1}(h(\omega)))$ for all $h\in\mathcal{S}$. Let us define a functional V on $\mathcal{S}$ by $V(h)=\sum_{\omega\in\Omega}u(h(\omega))P(h^{-1}(h(\omega)))$ for all $h\in\mathcal{S}$.Hence V^+ and V^- are both extension on $\mathcal{F}$ of V. We mantain that these extensions are equal. For, let $\mathcal{C}_r=\{c\in\mathcal{C}\ :\ P_c\approx''_{\underline{\underline{c}}}\lambda_{\underline{c}}$ for some $\lambda\in\mathbb{Q}\cap[0,1]\}$. Let $c,c'\in\mathcal{C}_r$ with $c\overset{*}{<}c'$. By definition there is a pair $\lambda,\lambda'\in\mathbb{Q}\cap[0,1]$, with $\lambda<\lambda'$, such that $P_c\approx''_{\underline{\underline{c}}}\lambda_{\underline{c}}$ and $P_{c'}\approx''_{\underline{\underline{c}}}\lambda'_{\underline{c}}$. Take $\lambda''=(\lambda+\lambda')/2$. Then $P_c<''_{\underline{\underline{c}}}\lambda''_{\underline{c}}<''P_{c'}$; Therefore, by point (ii) of proposition 2.1 and by $\mathbf{A}_8$ there is a $c''\in\mathcal{C}_r$ such that $c\overset{*}{<}c''\overset{*}{<}c'$. Hence $\mathcal{C}_r$ is dense-in-itself. Set $\Gamma=\{V(\hat{c})\ :\ c\in\mathcal{C}_r\}$. Let $c,c'\in\mathcal{C}_r$ with $c\overset{*}{\approx}c'$. Then $u(c)=u(c')$ and so $V(c)=V(c')$. This implies that there is a one-to-one correspondence between Γ and $\mathbb{Q}\cap[0,1]$. So Γ is countable. Moreover, Γ is dense-in-itself. By theorem 1 p.31 of Birkhoff (1948) there is an increasing function $k:\mathbb{R}\to\mathbb{R}$ such that $k(\Gamma)=\mathbb{Q}\cap[k(V(\hat{\underline{\underline{c}}})),k(V(\hat{\underline{c}}))]$. Suppose $V^+\neq V^-$. Then for some $f\in\mathcal{F}$ we have $V^+(f)\neq V^-(f)$. Suppose $V^+(f)<V^-(f)$. Then $k(V^+(f))<k(V^-(f))$. Since $\hat{\underline{\underline{c}}}\lesssim f\lesssim\hat{\underline{c}}$, we have $V^+(f),V^-(f)\in[V(\hat{\underline{\underline{c}}}),V(\hat{\underline{c}})]$. There is a rational number $q\in\mathbb{Q}\cap[k(V(\hat{\underline{\underline{c}}})),k(V(\hat{\underline{c}}))]$ such that $k(V^+(f))<q<k(V^-(f))$. Since $k(\Gamma)=\mathbb{Q}\cap[k(V(\hat{\underline{\underline{c}}})),k(V(\hat{\underline{c}}))]$, there is a $c_q\in\mathcal{C}_r$ such that $k(V(c_q))=q$. Since $k(\cdot)$ is increasing, from $k(V^+(f))<k(V(c_q))<k(V^-(f))$ it follows $V^+(f)<V(c_q)<V^-(f)$. But $V^+(f)<V(c_q)$ implies $f<\hat{c}_q$, while $V(c_q)<V^-(f)$ implies $f>\hat{c}_q$. This contradiction shows that

$V^{+}(f)=V^{-}(f)$ for all $f\in\mathcal{F}$.

To complete the proof we have to prove the "if" part. Let $f,f',g\in\mathcal{F}$ with $f'\leq f$ and $f\approx g$. By theorem 2.5 of Fishburn (1970) $f'\leq g$. Therefore, from the definitions of V^{+} and V^{-} it follows that $f\approx g$ implies that $V^{+}(f)=V^{+}(g)$ and $V^{-}(f)=V^{-}(g)$. Since we proved that $\mathcal{F}=\mathcal{F}^{+}$, we conclude that $\int_{\omega\in\Omega}u(f(\omega))dP=\int_{\omega\in\Omega}u(g(\omega))dP$. To sum up, we have proved that:

$$\text{if } f<g, \text{ then } \int_{\omega\in\Omega}u(f(\omega))dP < \int_{\omega\in\Omega}u(g(\omega))dP$$
$$\text{if } f\approx g, \text{ then } \int_{\omega\in\Omega}u(f(\omega))dP = \int_{\omega\in\Omega}u(g(\omega))dP.$$

A simple contrapositive is now enough to complete the proof. ■

Proof of lemma 4.3. Let f',g' be two acts such that $\hat{f}'(\omega)\lesssim g'$ for all $\omega\in\Omega$. Since $(\mathcal{C},<^{*})$ is complete, the codomain of f' in $\mathcal{C}$ has a supremum c'. By definition $\hat{f}'(\omega)\lesssim\hat{c}'$ for all $\omega\in\Omega$. By A_7 we have $f'\lesssim\hat{c}'$ and so $f'\lesssim g'$ (a similar reasoning can be found in Schervish and Seidenfeld (1983) p.411). The same holds when $\gtrsim$ takes the place of $\lesssim$. Therefore, a simple consequence respectively of this and of A_7 is: (i) $\hat{f}'(\omega)\approx g'$ for all $\omega\in\Omega$ implies $f'\approx g'$

(ii) $\hat{c}\approx\hat{f}(\omega)$ for all $\omega\in\Omega$ implies $\hat{c}\approx f$.

(Recall that $\hat{f}'(\omega)$ and $\hat{f}(\omega)$ are, respectively, the constant acts on $f'(\omega)$ and $f(\omega)$). Now, let f,g be two non simple acts such that $f(\omega)\approx^{*}g(\omega)$ for all $\omega\in\Omega$. By (ii) $\hat{f}(\omega)\approx g$ for all $\omega\in\Omega$. Thus, by (i), $f\approx g$. ■

Proof of lemma 4.4. The proof is divided in four steps. In (i) there are some preliminaries. In (ii) and (iii) we consider, respectively, $[\mathcal{F}]$ and $\mathcal{F}$. In (iv) we prove the last part.

(i) Let $[f]=\{g\in\mathcal{F}:g(\omega)\approx^{*}f(\omega)\}$. By lemma 4.3, if $g',g''\in[f]$, then $g'\approx g''$. Let $<^{\$}$ be defined on $[\mathcal{F}]=\{[f]:f\in\mathcal{F}\}$ by $[f]<^{\$}[g]$ if and only if $f<g$.

Clearly **[f]** is a function from Ω to $\mathscr{C}/\approx$.

(ii) (the basic idea of the proof comes from proposition 23.12 in Parthasaraty (1977)). In the proof of proposition 3.3 we have seen that $\mathscr{C}/\approx$ can be metrized with the distance $\mathbf{d([c'],[c''])=|u^{\#}([c']),u^{\#}([c''])|}$. Let $\mathbf{[g]}$ be such that $\mathbf{f\in[g]}$ and let $\varepsilon=1$. Since $\mathscr{C}/\approx$ is compact and metrizable, there is a finite open cover $\{A_i\}_{i=1}^{M}$ of $\mathscr{C}/\approx$ such that $\mathbf{diam(A_i)=1}$ for $\mathbf{i=1,\dots,M}$. Let $\mathbf{D_1=A_1,\dots,D_M=A_M\setminus\bigcup_{i=1}^{M-1}A_i}$. $\{D_i\}_{i=1}^{M}$ is a partition of $\mathscr{C}/\approx$ such that $\mathbf{diam(D_i)\le1}$ for $\mathbf{i=1,\dots,M}$. Let $\mathbf{D'_i=[g]^{-1}(D_i)}$ and let $\mathbf{[g_1](\omega)=\sum_{i=1}^{M}\{\inf_{\omega\in D'_i}[g](\omega)\}I_{D'_i}(\omega)}$. Since $\mathscr{C}/\approx$ is compact, $(\mathscr{C}/\approx,<^{*})$ is complete. Therefore $\mathbf{\inf_{\omega\in D'_i}[g](\omega)}$ exists. Moreover, $\mathbf{\inf_{\omega\in D'_i}[g](\omega)\precsim^{*}[g](\omega)}$ for every $\omega\in\mathbf{D'_i}$. Then, from $\mathbf{A_7}$ it follows $\mathbf{[g_1]\precsim^{\$}[g]}$ given $\mathbf{D'_i}$. Applying lemma 14.1 of Fishburn (1970) we obtain $\mathbf{[g_1]\precsim^{\$}[g]}$. Now suppose we have built two partitions $\{{}_mD_i\}_{i=1}^{M'}$ and $\{{}_nD_i\}_{i=1}^{M''}$ of $\mathscr{C}/\approx$ such that $\mathbf{diam({}_mD_i)\le1/m}$ for $\mathbf{i=1,\dots,M'}$ and $\mathbf{diam({}_{m+1}D_i)\le1/m+1}$ for $\mathbf{i=1,\dots,M''}$. Let $\mathbf{{}_mD'_i=[g]^{-1}({}_mD_i)}$ and let $\mathbf{[g_m](\omega)=\sum_{i=1}^{m}\{\inf_{\omega\in {}_mD'_i}[g](\omega)\}I_{{}_mD'_i}(\omega)}$. Clearly $\mathbf{[g_m]\precsim^{\$}[g]}$. Now let $\{C_k\}_{k=1}^{H}$ be the common refinement of $\{{}_mD_i\}_{i=1}^{M'}$ and $\{{}_nD_i\}_{i=1}^{M''}$. We have $\mathbf{diam(C_k)\le1/m+1}$ for $\mathbf{k=1,\dots,H}$. Let $\mathbf{C'_k=[g]^{-1}(C_k)}$ and let $\mathbf{[g_{m+1}](\omega)=\sum_{k=1}^{H}\{\inf_{\omega\in C'_k}[g](\omega)\}I_{C'_k}(\omega)}$. Clearly $\mathbf{[g_{m+1}]\precsim^{\$}[g]}$. Moreover, $\mathbf{[g_{m+1}]}$ and $\mathbf{[g_m]}$ are constant on every $\mathbf{C'_k}$. In particular $\mathbf{[g_{m+1}](\omega)\precsim^{*}[g_m](\omega)}$ for all $\omega\in\mathbf{C'_k}$ and so $\mathbf{[g_{m+1}]\precsim^{\$}[g_m]}$ given $\mathbf{C'_k}$. Applying lemma 14.1 of Fishburn (1970) we obtain $\mathbf{[g_{m+1}]\precsim^{\$}[g_m]}$. So we have obtained an increasing sequence $\mathbf{\{[g_i]\}_{i=1}^{\infty}}$. Let $\mathbf{[g_n]\in\{[g_i]\}_{i=1}^{\infty}}$, with $\{{}_nD_i\}_{i=1}^{M}$ and $\{{}_mD'_i\}_{i=1}^{M}$ the corresponding partition of $\mathscr{C}/\approx$ and Ω. In particular $\mathbf{diam({}_nD_i)\le1/n}$ for $\mathbf{i=1,\dots,M}$. Let $\mathbf{b_{{}_nD_i}=\sup_{\omega\in {}_nD'_i}\{[g](\omega)\}}$ and let $\mathbf{{}_n\bar{D}_i}$ be the closure of $\mathbf{{}_nD_i}$. We have $\mathbf{[[g_n](\omega),b_{{}_nD_i}]={}_n\bar{D}_i}$ for $\omega\in\mathbf{{}_nD'_i}$. Since $\mathbf{diam\,{}_nD_i=diam\,{}_n\bar{D}_i}$, we have $\mathbf{diam\{[[g_n](\omega),b_{{}_nD_i}]\}\le1/n}$. It follows that $\mathbf{d([g_n](\omega),[g](\omega))\le diam\,{}_n\bar{D}_i\le1/m}$ for

$\omega\in\bar{D}_{ni}$ and $i=1,\dots,M$. Therefore $\sup_{\omega\in\Omega} d([g_n](\omega),[g](\omega))\leq 1/n$ so that $\{[g_i]\}_{i=1}^{\infty}$ converges uniformly to $[g]$, i.e. $\lim_{i\to\infty}\{\sup_{\omega\in\Omega} d([g_i](\omega),[g](\omega))\}=0$. From the definition of the distance $d(\cdot)$ and from $[g_i]\lesssim^{\$}[g]$ for all $i\geq 1$ it follows $\lim_{i\to\infty}\{\sup_{\omega\in\Omega}\{u^{\#}([g](\omega))-u^{\#}([g_i](\omega))\}\}=0$.

(iii) Let f be any act in $[g]$ and f_i any simple act in $[g_i]$. By proposition 4.1 $\{f_i\}_{i=1}^{\infty}$ is an increasing sequence such that $f_i\lesssim f$ for all $i\geq 1$. Observe that the sequence $\{f_i\}_{i=1}^{\infty}$ is not unique since it depends on the choice of f and f_i. In $\mathscr{C}$ can be defined a pseudo-metric $d(c',c'')=|u(c'),u(c'')|$. Clearly $d(c',c'')=d([c'],[c''])$. Therefore we have $\lim_{i\to\infty}\{\sup_{\omega\in\Omega}\{u(f(\omega))-u(f_i(\omega))\}\}=0$.

(iv) Set $d_i=\sup_{\omega\in\Omega}\{u(f(\omega))-u(f_i(\omega))\}$. We can suppose without loss of generality that $(u(f_i(\omega))+d_i)\in[u(\underline{c}),u(\bar{c})]$. Under A_8 we have $u(\mathscr{C})=[u(\underline{c}),u(\bar{c})]$. Hence, for every $\omega\in\Omega$ there is a $c_{\omega}\in\mathscr{C}$ such that $u(c_{\omega})=u(f_i(\omega))+d_i$. Let us define a simple act g on Ω by $g(\omega)=c_{\omega}$ for all $\omega\in\Omega$. By construction, $u(g(\omega))=u(f_i(\omega))+d_i$ for all $\omega\in\Omega$. Furthermore, since by constuction $f_i(\omega)$ and $g(\omega)$ are constant on the same subsets of Ω, we have $g^{-1}(g(\omega))=f_i^{-1}(f_i(\omega))$. Observe that $f(\omega)\lesssim^{*}g(\omega)$ for all $\omega\in\Omega$ because $u(f(\omega))\leq u(g(\omega))$ for all $\omega\in\Omega$. By proposition 4.1 we have $\int u(f(\omega))dP\leq\int u(g(\omega))dP$. But

$$\int u(g(\omega))dP=\sum_{\omega\in\Omega}u(g(\omega))P(g^{-1}(g(\omega)))=$$

$$=\sum_{\omega\in\Omega}(u(f_i(\omega))+d_i)P(g^{-1}(g(\omega)))=\sum_{\omega\in\Omega}(u(f_i(\omega))+d_i)P(f_i^{-1}(f_i(\omega)))=$$

$$=\sum_{\omega\in\Omega}u(f_i(\omega))P(f_i^{-1}(f_i(\omega)))+\sum_{\omega\in\Omega}d_iP(f_i^{-1}(f_i(\omega)))=$$

$$=d_i+\sum_{\omega\in\Omega}u(f_i(\omega))P(f_i^{-1}(f_i(\omega)))=d_i+\int u(f_i(\omega))dP.$$

Therefore, $\{d_i+\int u(f_i(\omega))dP\}\geq\int u(f(\omega))dP$ and so $d_i\geq[\int u(f(\omega))dP-\int u(f_i(\omega))dP]\geq 0$ for all $i\geq 1$. Since $\lim_{i\to\infty}d_i=0$, it follows $\lim_{i\to\infty}\int u(f_i(\omega))dP=\int u(f(\omega))dP$. ∎

Proof of propositon 4.2: without loss of generality we can suppose we have strict inequalities in (i). By lemma 4.4, in correspondence of every element $\mathbf{g}_i$ there is a sequence of simple acts $\{\mathbf{f}^i_j\}$ such that $\lim_{i\to\infty}$ $\int u(f^i_j(\omega))dP=\int u(g^i(\omega))dP$. For j large enough, we have

$$(*)\qquad \int u(g^{i-1}(\omega))dP<\int u(f^i_j(\omega))dP\leq\int u(g^i(\omega))dP.$$

Let $\mathbf{N(i)}$ be the smallest $\mathbf{j}$ for which $(*)$ holds. Then $\{f^i_{N(i)}\}^{\infty}_{i=1}$ is a sequence of simple acts which satisfies (a), (b) and (c) of lemma 4.4. For it is easy to check that $\lim_{i\to\infty}\{\sup_{\omega\in\Omega}(u(g(\omega))-u(f^i_{N(i)}(\omega)))\}=0$. Then:

$$\lim_{i\to\infty} \int u(f^i_j(\omega))dP=\int u(g(\omega))dP.$$

But for all $\omega\in\Omega$ we have:

$$g_1(\omega)<^* f^2_{N(2)}\lesssim^* g_2(\omega)<^* \cdot\ \cdot\ \cdot \lesssim^* g_{i-1}(\omega)<^* f^i_{N(i)}\lesssim^* g_{i+1}(\omega)\lesssim^* \cdot\ \cdot\ \cdot \lesssim^* g.$$

Hence:

$$\int u(g_1(\omega))dP<\int u(f^2_{N(2)})dP\leq\int u(g_2(\omega))dP<\cdot\ \cdot\ \cdot \leq\int u(g_{i-1}(\omega))dP<$$
$$<\int u(f^i_{N(i)})dP\leq\int u(g_{i+1}(\omega))dP\leq\cdot\ \cdot\ \cdot\leq\int u(g)dP.$$

Therefore, $\lim_{i\to\infty} \int u(g_i(\omega))dP=\int u(g(\omega))dP$. ■

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