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A DYNAMIC MODEL OF MULTIPARTY COMPETITION*

by

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ABSTRACT

We construct a dynamic voting model of three-party competition in order to capture the following facts: voters base their decision on past economic performance of the parties; parties and candidates have different objectives; finally, a candidate while in office can only have a small effect on the economy. The properties that characterize the electoral system are the following: each voter has a single vote to cast and there is a single-winner elected under plurality rule. Given the decision rule of the voters we have sincere voting and, because our voters do not consider the possibility of abstention, all votes are to be cast.

We show the existence of equilibrium and the compatibility of the different objectives of parties and candidates. Our model may explain the emergence of ideologies and shows that in multicandidate elections held under the plurality system, Hotelling's principle of minimum differentiation is no longer satisfied.

1. INTRODUCTION

Most of the literature on electoral competition focuses on the study of static models. Parties decide strategically which policy to advocate in order to win the election. Voters decide which party they like best and the model typically only attempts to predict the elections' outcome. While this type of models capture some of the important features of party competition, they ignore some others. In reality, elections are repeated over time. The dynamic nature of the process changes both the circumstances under which elections are held and the voters' preferences. Finally, it introduces a distinction between a candidate and her party as separate decision makers who may have different goals. This paper presents a simple model of three-party competition, focusing on these aspects. For simplicity and clarity of exposition we ignore many of the standard aspects of parties competition which are widely studied in the literature.

We start with the obvious observation that elections are not isolated. During pre-election periods newspapers do not only discuss platforms that parties present but also comments and complaints on the policies implemented by the incumbent and by other parties in the past. All this information affects the results of each election, since parties and candidates as well as voters make use of it. Therefore, a more accurate model should capture these effects and the natural way to do it is to think of elections as repeated over time.

If elections take place over time, we have to deal with the fact that candidates and parties have different "life" horizon. While candidates typically cannot be reelected forever, parties can. Thus it is natural to make a distinction between candidates and parties with respect to their objective functions. Since candidates cannot be reelected over and over again, they do not care about the long run. It is assumed that their objective while in office is to maximize their popularity. Specifically, we assume that the strategy of the candidate in office is vote maximization: she maximizes the proportion of votes that her party will get in next election. This can be taken as a measure of her popularity and of the number of copies of her autobiography which will be sold. Parties, on the other hand, care about the future and they try to maximize the number of times they will be in power over time. We can think of candidates as players in a one period game and of parties – as players in an infinite period one. We will see that the different objectives of parties and candidates are compatible. Furthermore, their behavior can be interpreted as the formation of different ideologies for different parties.

While it is customary to assume that the party in power can choose any policy in a given space, it is rarely the case that they can do so independently of the past. When parties take office they have to deal with legislation, volume of deficit, agreements and other issues that are given by their predecessors' policies. Thus a policy in our model does not determine what the economy will be like but rather in which direction it will move. For simplicity, we only consider two different types of policies $\{0,1\}$ to be interpreted as either a positive or a negative effect on a given economic variable. We assume that the implementation of any policy affects the economic variables in a continuous way. Hence, given a position of the economic variables, the performance of a party in office for one period can decide the direction in which the economy is going to move but cannot position the economy in a point of the state space far away from the original one. For simplicity we assume that the state of the economy is an aggregate (or average) of all previously implemented policies and will be represented by a point in $[0,1]$. As an example, policies of type 0 can be thought of as "to cut taxes" or "protectionist policies" or "decrease public expenditure" and policies of type 1 as "to raise taxes" or "nonprotectionist policies" or "increase public expenditure" respectively. The economic configuration at each time is given by the frequencies with which these policies have been chosen in the past. Since economic policies appear to be one of the most important factors for parties and voters alike we will be referring mostly to them throughout the paper; however, the model can also apply to any other kind of policy. For instance, to increase or decrease aid to fledgling democracies.

Voters' preferences are defined on the space of states of the economy, represented by the interval $[0,1]$. If a voter were a dictator, he would choose a policy, 0 or 1, taking into account the position of the economy at that time in the interval $[0,1]$. Each voter is assumed to have an ideal point in the interval $[0,1]$. The relative frequency at which he would choose each policy would be given by his ideal point. The fact that voters would switch between the two policies may be interpreted as if after choosing the same policy several times, they "get tired" of it. That is, the negative implications of and the problems associated with a given policy become more salient in the voters' minds. Thus, voters would try to minimize the cumulative dissatisfaction with policies.

Since voters delegate the choice of a policy to parties, the acts available to the voters are the parties. Similarly to the hypothetical case of a "dictator" voter, who would express dissatisfaction with policies, real voters express dissatisfaction with their available acts, i.e., the parties. In other words, since it is much more common to observe people expressing dissatisfaction than satisfaction with government policies, we assume that voters vote *against* rather than *for* parties. There are several psychological explanations for this phenomenon. For instance, people tend to recall negative experiences (such as unemployment or high inflation) more than positive ones (which are often the absence of such problems). Alternatively, one may argue that people tend to associate success with their own deeds, but to blame failures on others (i.e. the party in power). Be it as it may, our model assumes that it is dissatisfaction, rather than satisfaction that drives voters' choices. Furthermore, recalling that they care about the cumulative "state" of the economy, we conclude that voters get "more tired" of parties the longer they are in office. So the record that voters keep of the parties will reflect how much they disliked them rather than how much they liked them and this record is updated every time that a new party takes office.

To describe voters' preferences we therefore need a model of cumulative experiences. In "Case-Based Consumer Theory" (CBCT) Gilboa and Schmeidler (1993) suggest a theory of repeated choice where the decision makers' utility is cumulative. Applied to our case, elections take place at periods $t = 0,1,2,...$ At each time t, voters face a new decision problem represented by election and a memory M^t. Before each election voters update their memory by including the results of last election, given by the last party in office and the policy it chose to implement. Therefore, the cases in the memory of the voters are simply described by the winner of the election (p) and the policy she implemented at that time (x). We can write the decision rule of the voters as maximizing:

$$U^t(p) = \sum_{(p,x)\in M^t} u(x)$$

Following one interpretation of CBCT, the memory of past performance of the parties directly affects the nature and "utility" of present experiences. Therefore, even if voters know how much they like or dislike a policy implemented by one of the parties at a certain time, $u(x)$, this instantaneous utility function does not summarize all the relevant information. According to this interpretation, tastes, and thus decisions, are intrinsically history-dependent. Hence, the function $u(.)$ should be thought of as some derivative of the "real" utility, $U(.)$, which in turn, is the aggregate of $u(.)$ values. This interpretation reflects the fact that the evaluation of parties by the voters is constantly changing over time as new parties are taking office. Since, as we have argued, voters get more tired of parties the longer they are in power, we define $u(.)$ as the negative of the distance between the voter's ideal point and the policy chosen by the party.

The properties that characterize the electoral system in the present model are the following: each voter has a single vote to cast and there is a single-winner elected under plurality rule. Given the decision rule of the voters we have sincere voting and, because our voters do not consider the possibility of abstention, all votes are to be cast. There is a continuum of voters with ideal points uniformly distributed on the interval $[0,1]$. The number of parties is exogenous and the main results are given for a three party competition. Parties do not have ideal points to begin with. These are endogenously determined by maximizing their objective functions. Each party presents one candidate to each election and candidates cannot be reelected. We can think of candidates as players in a one period game and of parties -- as players in an infinite period one. Parties and candidates know the decision rule of the voters. Therefore there is no need for platforms before elections, since voters have already made their decision. The winner of the election has to choose a policy to implement in the set $\{0,1\}$. Since candidates cannot be reelected they maximize the proportion of votes stagewise. Parties, on the other hand, care about the future and maximize the limit frequencies of the number of times they win.

In the first version of the model we assume that candidates are loyal in the following sense: when they are indifferent between policies of type 0 and 1 they will choose the policy that the party has chosen in the past (this will turn out to be well-defined). In this case the solution shows that no party mixes policies of different types, i.e., if a party starts choosing a policy of type 1 it will continue choosing this type of policy for ever (Theorem 1). Then we drop the assumption of loyalty and in the long run we have a similar result (Theorem 2), i.e., in the first periods one of the parties may mix different types of policies but at some point it decides on one of them and continues with the chosen one for ever. Since we assume that parties have no ideal point to begin with, this result can be interpreted as suggesting that ideologies may emerge from the actions of the candidates when maximizing the popularity of the party in the short run.

We define the economic configuration at time t, e^t, as the ratio between the number of times that type 1 policies have been chosen in the past and the total number of elections. In both cases we have that the economic configuration tends to stabilize at one half This result reflects the fact that parties tend to satisfy the median voter in some sense. On the other hand, our result also reflects Cox's (1987) suggestion that in multiparty competition under plurality system, candidates have no incentives to minimize their differences. To be precise, on the aggregate level there is some support to Hotelling's result. If we consider the aggregate economic variable when each party is in power, it tends to one half for all parties. However, this is a result of aggregation over time, taking into account the cumulative effect of past elections. By contrast, when we focus on each party's decision variable, i.e., whether to lead the economy to the "left" or "right" end of the state space we find the opposite of Hotelling's result, i.e., that parties are pushed to one of the extreme policies ("always right" or "always left"). In our model, it is in the interest of each party to have no other party choosing the same type of policy it has decided to implement. The two solutions described above yield the same results for the long run: one of the parties chooses one type of policy and wins one half of the time and the other two parties choose policies of the other type and each wins one fourth of the time.

In the solutions described above we have considered candidates as the players of the game at each stage. If parties instead of candidates were to decide on policies, their objective would be to maximize the number of times that the party wins, i.e., they would take into account that they are going to participate in all elections and so they would prefer to sacrifice some of the votes in a given period in order to increase the total number of times in office. We show that if candidates are stagewise vote-maximizers and loyal, their choices constitute a Nash equilibrium path in the infinite-stage game. On the one hand this proves the existence of equilibrium for three-party competition with uniform distribution of voters in a plurality system where parties maximize votes. On the other hand, considering different objectives for candidates and parties, they appear to be compatible.

The rest of the paper is organized as follows. In section 2 we compare our model to other existing models in the literature. Section 3 describes the model. In Section 4 we present the results. Section 5 includes some concluding remarks.

2. COMPARISON TO THE LITERATURE.

There exist some theories of voting which suggest that voters base their decision on past performance of the parties. The Reward-Punishment theory proposed by Key (1966) is based on the assumption that voters only care about the effects of the policies that parties choose and they are looking at past performance when deciding how much confidence to give to each party. Downs (1957) proposed a theory according to which parties' past performance is the cheapest way for voters to predict future performance. In this model, voters care about the policy that a party implements on top of its effects. He assumes that political parties must be consistent over time in the policies they advocate and implement. Our interpretation of the voters behavior is different than Downs' but we find that consistency over time in the policies implemented is a result of optimal choice of the parties. Fiorina (1981) builds a dynamic model for two parties that combines features of both theories and examines it at the empirical level. He assumes that voters base their decision not only on past performance of the parties but also on past promises and hypothetical choices of policies. He shows that most of the assumptions of his theory are supported by the data. Our model is much simpler than Fiorina's. Past promises or hypothetical choices are not considered by the voters in their evaluation of the parties. For simplicity we also assume that platforms have no effect on the evaluation of the parties.

Other variations of retrospective voting have been suggested in analyzing how voters ought to behave if they wish to get their representatives to pursue their interests. The solution is an optimal decision rule for the voters given that they know the objective function of the parties. Ferejohn (1986) and Austen-Smith and Banks (1989) are two examples. In our model, we assume that voters use very little information to make their decision. They do not know the objectives that define parties' behavior and they use a very simple rule to evaluate parties based on past performance.

Kramer (1977) presents a dynamic model of multidimensional competition. In each period two political parties compete for votes by advocating particular policies, and whichever party wins takes office and puts the policies it advocated into effect. In the next period, a new election is held and another policy implemented. As this process is repeated over time, it generates a sequence of implemented policies. He shows that over time, these sequences converge on the minmax set, which he interprets as the generalization of the median for multidimensional policy spaces. The main assumptions are the following: parties maximize votes myopically; the preferences of the voters are constant over time and are defined on the policy space; at any time the challenger can choose any policy in the policy space while the incumbent must defend the same policy.

Here we propose a different dynamic model of competition in one dimension which we analyze for the case of three-party competition. In our model parties maximize votes, as in Kramer's, but we relax the myopic behavior and compare it to intertemporal strategies. The preferences of the voters change over time and are defined on the space of consequences of the policies (i.e., the space of states of the economy). We restrict the set of alternatives of the parties and assume that the policy chosen by a party while in office can only produce small effects on an aggregate state variable. Given these assumptions, we show the existence of equilibrium and the compatibility of short run and long run objectives of the parties. The results we obtain can be interpreted as explaining the emergence of ideologies and we show that in multicandidate elections held under the plurality system, Hotelling's principle of minimum differentiation is no longer satisfied.

The classical (Hotelling) results of the spatial voting model with plurality rule and vote maximization follow from the model of spatial economic competition analyzed by Eaton and Lipsey (1975). They show that there is no equilibrium for the case of three-party competition. Palfrey (1984) defines a perturbation of the original game and finds an equilibrium in a spatial model of three party competition. He also shows that the principle of minimum differentiation does not hold by assuming certain asymmetries among the parties. In our model we assume that parties are identical. Yet, in equilibrium, i.e., as a result of competition, they try to differentiate themselves from each other.

3. THE MODEL

The preferences of the voters are defined on the space of states of the economy represented by the interval $[0,1]$. There is a continuum of voters. Each voter is characterized by his ideal point and they are distributed uniformly on $[0,1]$. Their instantaneous utility functions are single peaked: $u_i(x) = -|x_i - x|$ where x_i represents the ideal point of voter i. It can be interpreted as follows: if voter i were a dictator, he would choose policy 1 a proportion x_i of the times and policy 0 a proportion $(1-x_i)$ of the times. At each time $t = 0,1,2,\ldots$ an election takes place. At time t all voters have memory $(M^t)_i = M^t$ of past elections. An element of memory, which we call a case, is represented by $m^t = (p^t, x^t)$ where p^t represents the party that won the election at time t and x^t the type of policy it implemented. The memory of voters at time $t+1$ is $M^{t+1} = M^t \cup \{(p^t, x^t)\}$ and $M^0 = \emptyset$. For voter i at time t, each party p is ranked by aggregation of all cases in the memory M^t in which p won the election, i.e., $m^k = (p^k, x^k) \in M^t$ such that $p^k = p$ for $k < t$ and it is given by:

$$U_i^t(p) = \sum_{(p,x^k)\in M^t} u_i(x^k)$$

When there are no cases in memory to calculate the utility, its default value is zero. Voters give their vote to the party that gives them the highest utility level. If a voter is indifferent among different parties, he will choose each of them with equal probability. We will assume that the law of large numbers hold, i.e., that if a proportion μ of voters are indifferent among parties in a set P_o, each party in P_o gets a proportion $\frac{\mu}{|P_o|}$ of the votes[1] .

At each time t the party that obtains the largest proportion of votes wins the election and has to choose a policy to implement. Formally, let μ_p^t be the proportion of votes for party p at time t; party p^* wins election t only if $\mu_{p^*}^t \geq \mu_p^t$ for all $p \in \{a,b,c\}$. Ties are broken by fair lotteries. The policies are $\{0,1\}$ to be interpreted as a small change in one of the two possible directions. Let $k_1^p = k_1^p(t)$ be the number of times that party p was in power and chose policy 1 up to time t. Similarly, we define $k_0^p = k_0^p(t)$ and $k^p = k^p(t) = k_0^p + k_1^p$. Thus, for every $t, t = k_0^a + k_1^a + k_0^b + k_1^b + k_0^c + k_1^c$ and the economic configuration at time t is given by $e^t = \frac{k_1^a + k_1^b + k_1^c}{t}$.

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1 See Judd (1985) for the mathematical subtleties involved.

It will simplify notation to define the infinite period game for the parties and embed the candidates' game in it. That is, instead of defining the candidates' game as a game with infinitely many stagewise vote-maximizer players we will model them as agents of the parties, and reflect their utility maximization by an appropriate choice of strategies for the parties.

Let us formally define the parties game to be the following infinite-stage game. The set of players is $\{a,b,c\}$. For every $t \geq 0$ nature chooses a "winner" from $\{a,b,c\}$ as follows: if there exists a party p such that $\mu_p^t > \mu_{p'}^t$ for all $p' \neq p$ then p is the winner with probability one. Otherwise, nature chooses one of the maximizers of μ_p^t with equal probability. Then, for the same $t \geq 0$, the winner chooses an element of $\{0,1\}$, $U_i^t(p)$ is updated and the game proceeds to stage $t+1$. (The voters' choices are incorporated into the rules of the game as defining μ_p^t.)

Let H^t be the set of histories of the game at time t and let h^t denote an element of this set. That is, h^t is a sequence of t elements of the set $H = \{(p,x): p \in \{a,b,c\} \; and \; x \in \{0,1\}\}$. We use $\circ$ for concatenation. Thus, $h^{t+1} = h^t \circ (p^t, x^t)$ where p^t is the party that wins election t and x^t is the policy it implements (h^0 is the empty sequence). The proportion of votes for party p at time t is a function of the history of the game at time t, formally $\mu_p^t : H^t \to [0,1]$.

A strategy for party p is defined by $x^p = (x_0, x_1, x_2, \ldots, x_t, \ldots)$ where $x_t : H^t \to \{0,1\}$ is a function that assigns to each history a policy $x \in \{0,1\}$. We consider the following two sets of strategies: the set of vote maximizing strategies

$$X^p = \left\{ (x_0, x_1, x_2, \ldots, x_t, \ldots) : \mu_p^{t+1}\left(h^t \circ \left(p, x_t(h^t)\right)\right) \geq \mu_p^{t+1}\left(h^t \circ (p,x)\right) for\, all\, x \in \{0,1\} \, and\, all\, t \geq 0 \right\}$$

and its subset of loyal strategies

$$XL^p = \left\{ (x_0, x_1, x_2, \ldots, x_t, \ldots) \in X^p : \begin{array}{c} x_t(h^t) = x \; if \; \mu_p^{t+1}\left(h^t \circ (p,0)\right) = \mu_p^{t+1}\left(h^t \circ (p,1)\right) \\ and \; x_j(h^j) = x, for\, every\, j < t \; such\; that \; p^j = p \end{array} \right\}$$

4. RESULTS.

First, we analyze the behavior of the candidates as autonomous agents in the game. At each election t we have a different candidate for each party. The candidate of the winning party has to choose a policy. We assume that at each time t, the candidate that wins the election, implicitly assumed to know the decision rule of the voters, chooses a policy in the set $\{0,1\}$ to maximize the proportion of votes that she can obtain in the next election (stagewise vote-maximizer). In addition, we have an assumption of loyalty: if a candidate is indifferent between the two policies (in terms of vote maximization) and if all previous candidates of her party happened to have chosen the same policy, so will she. (Notice that this assumption does not restrict a candidate choice if she is the first to win on her party's behalf or if past winners happened to choose different policies.) These agents' preferences are equivalent to assuming that the party chooses a strategy x^p in the set XL^p.

Theorem 1: Competition of loyal candidates.

If for all $p \in P, x^p \in XL^p$ we have the following results up to any permutation of parties and/or of policies:

$$I.\ For\ all\ t = 4k, k = 1,2,...$$
$$(i)\ k_0^a(t) = k^a(t) = k,\ k_0^b(t) = k^b(t) = k\ and\ k_1^c(t) = k^c(t) = 2k$$
$$(ii)\ e^t = \frac{1}{2}$$
$$(iii)\ For\ all\ i,\ U_i^t(a) = -kx_i,\ U_i^t(b) = -kx_i\ and\ U_i^t(c) = -2k(1-x_i)$$
$$II. \lim_{t\to\infty} \frac{k^a(t)}{t} = \lim_{t\to\infty} \frac{k^b(t)}{t} = \frac{1}{4}\ and\ \lim_{t\to\infty} \frac{k^c(t)}{t} = \frac{1}{2}$$

(All proofs are relegated to an appendix.)

Part I of the theorem states that under the above assumptions parties will always choose the same type of policy. In this case, we have that for all $t = 4k$ parties a and b, which have been choosing policies of type 0, win one fourth of the time and party c, which has been choosing policies of type 1, wins one half of the time. The economic configuration at each time $t = 4k$, given by the frequencies at which the policies have been chosen in the past, is $e^t = 1/2$ which means that every four periods the economy is balanced.

Part II concludes that in the long run the limit frequencies of the time in office for the parties are as follows: party c, who has been choosing only policies of type 1, wins one half of the time and parties a and b, who have chosen only policies of type 0, win one fourth of the time each.

Next we drop the loyalty assumption. We still assume that at each time t, the candidate that wins the election, implicitly assumed to know the decision rule of the voters, chooses a policy in the set $\{0,1\}$ to maximize the proportion of votes that she can obtain in the next election (stagewise vote-maximizer). Formally, the party chooses a strategy $x^p \in X^p$.

Theorem 2: Competition of candidates.

If for all $p \in P$, $x^p \in X^p$ then we have the follwing results up to any permutation of parties and/or of policies:

I. For all $t = 4k$, $k = 1,2,\dots$ *there exist* k_1 *and* k_2 *with* $k = k_1 + k_2$ *and* $\min\{k_1, k_2\} \leq 3$ *such that:*

(i) $k_0^a(t) = k_1, k_1^a(t) = k_2, k_0^b(t) = k^b(t) = k_1 + 2k_2$ *and* $k_1^c(t) = k^c(t) = 2k_1 + k_2$

(ii) $e^t = \dfrac{1}{2}$

(iii) *For all* i, $U_i^t(a) = -k_1 x_i - k_2(1 - x_i)$,

$$U_i^t(b) = -(k_1 + 2k_2)x_i \text{ and}$$

$$U_i^t(a) = -(2k_1 + k_2)(1 - x_i)$$

$$II.\ \lim_{t\to\infty} \frac{k^a(t)}{t} = \lim_{t\to\infty} \frac{k^b(t)}{t} = \frac{1}{4} \text{ and } \lim_{t\to\infty} \frac{k^c(t)}{t} = \frac{1}{2}$$

Furthermore, for all $p \in \{a,b,c\}$, *if* $k_0^p(t) > 0$ *and* $k_1^p(t) > 0$ *for some* t, *then* $\lim_{t\to\infty} \dfrac{k^p(t)}{t} = \dfrac{1}{4}$.

Theorem 2 states that two of the parties will always choose the same type of policy, regardless of how the third party chooses to mix the policies. That is, two of the parties are behaving as if they were loyal, while the third one is "almost" loyal: it will choose the same policy whenever in power, except for at most three times. The economic configuration at each time $t = 4k$ is $e^t = 1/2$ which means that every four periods the economy is balanced.

In the long run the limit frequencies of the periods in which the parties are in office are as follows: If party a, who started mixing, ends up choosing a policy of type 0, then party c, who has been choosing only policies of type 1 will win one half of the time and parties a and b, who have chosen mostly policies of type 0 will win one fourth of the time each. If party a ends up choosing a policy of type 1 then party b, who has been choosing only policies of type 0 will win one half of the time and parties a and c, who have chosen mostly policies of type 1, will win one fourth of the time each. At any rate, the long-run frequencies are as especified in Theorem 1; however, in case one party mixes the two types of policies, it cannot be the one which wins one half of the times.

Finally, we consider the case of competition of parties. As explained in the introduction, stagewise maximization characterizes the objective of the candidates. Since candidates cannot be reelected, they only care about their popularity one period ahead. In contrast, parties participate in elections at all periods. Therefore the objective of the parties can be characterized by the maximization of the limit frequency of the number of times that the party is in power. If candidates are stagewise vote-maximizers and loyal, as assumed in the first solution, we also have that their choices constitute a Nash equilibrium path in the parties game. The payoff function of party p is assumed to be $\Pi^p = \limsup_{t\to\infty} \frac{k^p(t)}{t}$ where $k^p(t)$ is the number of times that party p has been in office up to time t.

Theorem 3: Competition of parties.

For every x^a, x^b and x^c such that $x^p \in XL^p, \left(x^a, x^b, x^c\right)$ is a Nash equilibrium of the parties game.

Remark:

If x^a, x^b and x^c are such that $x^p \in X^p, \left(x^a, x^b, x^c\right)$ need not be a Nash equilibrium of the parties game. (A counterexample is provided in the appendix.)

5. CONCLUDING REMARKS

The interesting features of the model are the following: it is a dynamic model of three-party competition; it shows the existence of equilibrium and the compatibility of the different objectives of parties and candidates; it explains the emergence of ideologies and shows that in multicandidate elections held under the plurality system, where policies are restricted to be the effects on an aggregate state variable, Hotelling's principle of minimum differentiation is no longer satisfied.

We start by assuming that all parties are identical, that they do not have preferences over policies; rather, they are vote maximizers. In equilibrium parties and candidates behave as if they had ideal points, i.e., each party chooses always the same policy. If, instead, we assume that parties have non-identical preferences over policies and their objective is not only to win elections but also to implement their most preferred policies one can show that the result will not change. Therefore, the fact that parties always choose the same policy is compatible with (at least) two theories: (i) the parties are only interested in vote maximization, and ideologies "emerge" from strategic considerations; and (ii) parties do have ideologies to begin with, and these determine their initial choices, since vote maximization leaves them indifferent between the two policies; however, in later stages vote maximization and ideological considerations coincide.

The two solutions described in the paper for the "candidates' game" show the same results for the long run: one of the parties chooses one type of policy and wins one half of the time and the other two parties choose policies of the other type and each wins one fourth of the time. By comparison, consider the two-party case in our model. It is easy to see that as a result of two-party competition none of the parties can win more than one half of the times in the limit. Hence our results may also be interpreted as if there were "no room" for a third party. On the one hand, the two parties that end up choosing the same type of policy have no incentives to form a coalition, since by appearing as a single party they can only win one half of the times which is exactly what they do in aggregate. On the other hand, if we start with two parties, none of them seems to gain anything by dividing into two different parties.

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APPENDIX

To simplify notation we will write $\mu_p^t(x)$ and μ_p^t instead of $\mu_p^t\left(h^{t-1}\circ(p,x)\right)$ and $\mu_p^t\left(h^t\right)$ repectively.

Proof of Theorem 1:

Part I: first we prove (iii) by induction on k:

At $t=0$, we have $U_i^0(a)=U_i^0(b)=U_i^0(c)=0$, therefore $\mu_p^0=\frac{1}{3}$ for all p which implies a tie among the three parties. If party a wins the first election $\mu_a^1(0)=\mu_a^1(1)=0$, i.e., party a is indifferent between policies 0 and 1.

At $t=1$, if $\left(p^0,x^0\right)=(a,0)$ then $U_i^1(a)=-x_i$ and $U_i^1(b)=U_i^1(c)=0$. Therefore $\mu_a^1=0$ and $\mu_b^1=\mu_c^1=\frac{1}{2}$ implies a tie between parties b and c. If party b wins the second election $\mu_b^2(0)=\mu_b^2(1)=0$, i.e., party b is indifferent between policies 0 and 1.

At $t=2$, if $\left(p^0,x^0\right)=(a,0)$ and $\left(p^1,x^1\right)=(b,0)$ then $U_i^2(a)=U_i^2(b)=-x_i$ and $U_i^2(c)=0$. Therefore $\mu_a^2=\mu_b^2=0$ and $\mu_c^2=1$ implies that party c wins the third election and $\mu_c^3(0)=\frac{1}{3}<\frac{1}{2}=\mu_c^3(1)$, i.e., party c chooses policy 1.

At $t=3$, given that $\left(p^0,x^0\right)=(a,0),\left(p^1,x^1\right)=(b,0)$ and $\left(p^2,x^2\right)=(c,1)$, we have $U_i^3(a)=U_i^3(b)=-x_i$ and $U_i^3(c)=-\left(1-x_i\right)$. Therefore $\mu_a^3=\mu_b^3=\frac{1}{4}$ and $\mu_c^3=\frac{1}{2}$ implies that party c wins the fourth election and $\mu_c^4(0)=0<\frac{1}{3}=\mu_c^4(1)$, i.e., party c chooses policy 1.

In addition to $[(a,0),(b,0),(c,1),(c,1)]$, other possible results for this period are: $[(a,0),(b,1),(c,0),(b,1)]$ and $[(a,0),(b,1),(c,1),(a,0)]$ up to any permutation of parties or policies. At $t=4$, *i.e.* $k=1$, for all possible results we have $U_i^4(a)=U_i^4(b)=-x_i$ and $U_i^4(c)=-2\left(1-x_i\right)$ up to any permutation of parties or policies.

Now, suppose that the result is true for $k > 1$. Then, at $t = 4k$ we have $U_i^{4k}(a) = U_i^{4k}(b) = -kx_i$ and $U_i^{4k}(c) = -2k(1 - x_i)$. This implies that $\mu_p^{4k} = \frac{1}{3}$ for all p, i.e., a tie among the three parties. Here we have two cases depending on who breaks the tie.

<u>Case 1</u>.- If party a (or b) wins the election at time $t = 4k$ we have $\mu_a^{4k+1}(0) = \mu_a^{4k+1}(1) = 0$. Thus, party a will choose policy 0, because it is the policy it has already chosen in the past.

At $t = 4k + 1$ we have:

$$U_i^{4k+1}(a) = -(k+1)x_i, U_i^{4k+1}(b) = -kx_i \text{ and } U_i^{4k+1}(c) = -2k(1 - x_i).$$

Therefore $\mu_a^{4k+1} = 0, \mu_b^{4k+1} = \frac{2}{3}$ and $\mu_c^{4k+1} = \frac{1}{3}$ which implies that party b wins this election and $\mu_b^{4k+2}(0) = \frac{k}{3k+1} > \frac{k-1}{6k-2} = \mu_b^{4k+2}(1)$, i.e., party b chooses policy 0.

At $t = 4k + 2$ we have:

$$U_i^{4k+2}(a) = U_i^{4k+2}(b) = -(k+1)x_i \text{ and } U_i^{4k+2}(c) = -2k(1 - x_i).$$

Therefore $\mu_a^{4k+2} = \mu_b^{4k+2} = \frac{k}{3k+1} < \frac{k+1}{3k+1} = \mu_c^{4k+2}$ which implies that party c wins this election and $\mu_c^{4k+3}(0) = \frac{1}{3} < \frac{k+1}{3k+2} = \mu_c^{4k+3}(1)$, i.e., party c chooses policy 1.

At $t = 4k + 3$ we have:

$$U_i^{4k+3}(a) = U_i^{4k+3}(b) = -(k+1)x_i \text{ and } U_i^{4k+3}(c) = -(2k+1)(1 - x_i).$$

Therefore $\mu_a^{4k+3} = \mu_b^{4k+3} = \frac{2k+1}{6k+4} < \frac{k+1}{3k+2} = \mu_c^{4k+3}$ which implies that party c wins this election and $\mu_c^{4k+4}(0) = \frac{k}{3k+1} < \frac{1}{3} = \mu_c^{4k+4}(1)$, i.e., party c chooses policy 1.

At $t = 4(k+1)$ we have: $U_i^{4(k+1)}(a) = U_i^{4(k+1)}(b) = -(k+1)x_i$ and $U_i^{4(k+1)}(c) = -2(k+1)(1 - x_i)$.

<u>Case 2</u>.- If party c wins the election at time $t = 4k$ we have:

$$\mu_c^{4k+1}(0) = \frac{k-1}{3k-1} < \frac{k}{3k+1} = \mu_c^{4k+1}(1),$$

i.e., party c will choose policy 1.

At $t = 4k + 1$ we have:

$$U_i^{4k+1}(a) = U_i^{4k+1}(b) = -kx_i \text{ and } U_i^{4k+1}(c) = -(2k+1)(1-x_i).$$

Therefore $\mu_a^{4k+1} = \mu_b^{4k+1} = \dfrac{2k+1}{6k+2}$ and $\mu_c^{4k+1} = \dfrac{k}{3k+1}$ which implies that party a (or b) wins this election and $\mu_a^{4k+2}(0) = \mu_a^{4k+2}(1) = 0$. Thus, party a chooses policy 0 because it is the policy it has already chosen in the past.

At $t = 4k + 2$ we have:

$$U_i^{4k+2}(a) = -(k+1)x_i, U_i^{4k+2}(b) = -kx_i \text{ and } U_i^{4k+2}(c) = -(2k+1)(1-x_i).$$

Therefore $\mu_a^{4k+2} = 0, \mu_b^{4k+2} = \dfrac{2k+1}{3k+1} > \dfrac{k}{3k+1} = \mu_c^{4k+2}$ which implies that party b wins this election and $\mu_b^{4k+3}(0) = \dfrac{2k+1}{6k+4} > \dfrac{1}{6} = \mu_b^{4k+3}(1)$, i.e., party b chooses policy 0.

At $t = 4k + 3$ we have:

$$U_i^{4k+3}(a) = U_i^{4k+3}(b) = -(k+1)x_i \text{ and } U_i^{4k+3}(c) = -(2k+1)(1-x_i).$$

Therefore $\mu_a^{4k+3} = \mu_b^{4k+3} = \dfrac{2k+1}{6k+4} < \dfrac{k+1}{3k+2} = \mu_c^{4k+3}$ which implies that party c wins this election and $\mu_c^{4k+4}(0) = \dfrac{k}{3k+1} < \dfrac{1}{3} = \mu_c^{4k+4}(1)$, i.e., party c chooses policy 1.

At $t = 4(k+1)$ we have: $U_i^{4(k+1)}(a) = U_i^{4(k+1)}(b) = -(k+1)x_i$ and $U_i^{4(k+1)}(c) = -2(k+1)(1-x_i)$.

(i) and (ii) follow directly from (iii). Part II follows from part I.♠

Proof of Theorem 2:

Part I: first we prove (iii) by induction on k. Assume that at $t = 4k$ we have $k_1 = k$ and $k_2 = 0$.

Then $U_i^{4k}(a) = U_i^{4k}(b) = -kx_i$ and $U_i^{4k}(c) = -2k(1-x_i)$. Furthermore, suppose that party a wins the election at $t = 4k$ and chooses policy 1. (If party a chooses policy 0, at $t = 4(k+1)$ we have $k_1 = k+1$ and $k_2 = 0$, from the previous proof.)

At $t = 4k + 1$ we have:

$$U_i^{4k+1}(a) = -kx_i - (1-x_i), U_i^{4k+1}(b) = -kx_i \text{ and } U_i^{4k+1}(c) = -2k(1-x_i).$$

Therefore $\mu_a^{4k+1} = 0, \mu_b^{4k+1} = \frac{2}{3}$ and $\mu_c^{4k+1} = \frac{1}{3}$ which implies that party b wins this election and $\mu_b^{4k+2}(0) = \frac{1}{2} > \frac{2k-1}{6k-2} = \mu_b^{4k+2}(1)$, i.e., party b chooses policy 0.

At $t = 4k+2$ we have:

$$U_i^{4k+2}(a) = -kx_i - (1-x_i), U_i^{4k+2}(b) = -(k+1)x_i \text{ and } U_i^{4k+2}(c) = -2k(1-x_i).$$

Therefore $\mu_a^{4k+2} = \frac{k-1}{6k-2}, \mu_b^{4k+2} = \frac{1}{2}$ and $\mu_c^{4k+2} = \frac{k}{3k-1}$ which implies that party b wins this election and $\mu_b^{4k+3}(0) = \frac{1}{3} > 0 = \mu_b^{4k+3}(1)$, i.e., party b chooses policy 0.

At $t = 4k+3$ we have:

$$U_i^{4k+3}(a) = -kx_i - (1-x_i), U_i^{4k+3}(b) = -(k+2)x_i \text{ and } U_i^{4k+3}(c) = -2k(1-x_i).$$

Therefore $\mu_a^{4k+3} = \frac{3k-2}{9k-3}, \mu_b^{4k+3} = \frac{1}{3}$ and $\mu_c^{4k+3} = \frac{k}{3k-1}$ which implies that party c wins this election and $\mu_c^{4k+4}(0) = \frac{k-1}{3k-1} < \frac{1}{3} = \mu_c^{4k+4}(1)$, i.e., party c chooses policy 1.

At $t = 4(k+1)$ we have:

$$U_i^{4(k+1)}(a) = -kx_i - (1-x_i), U_i^{4(k+1)}(b) = -(k+2)x_i \text{ and } U_i^{4k+4}(c) = -(2k+1)(1-x_i)$$

Now suppose that at $t = 4k$ party c has won the election. From the previous proof we know that it chooses policy 1.

At $t = 4k+1$ we have:

$$U_i^{4k+1}(a) = U_i^{4k+1}(b) = -kx_i \text{ and } U_i^{4k+1}(c) = -(2k+1)(1-x_i).$$

Therefore $\mu_a^{4k+1} = \mu_b^{4k+1} = \frac{2k+1}{6k+2}$ and $\mu_c^{4k+1} = \frac{k}{3k+1}$ which implies that party a (or b) wins this election and $\mu_a^{4k+2}(0) = \mu_a^{4k+2}(1) = 0$. Suppose party a chooses policy 1. (The case in which party a chooses policy 0 is analized in the previous proof.)

At $t = 4k+2$ we have:

$$U_i^{4k+2}(a) = -kx_i - (1-x_i), U_i^{4k+2}(b) = -kx_i \text{ and } U_i^{4k+2}(c) = -(2k+1)(1-x_i).$$

Therefore $\mu_a^{4k+2}=0, \mu_b^{4k+2}=\frac{2k+1}{3k+1}>\frac{k}{3k+1}=\mu_c^{4k+2}$ which implies that party b wins this election and $\mu_b^{4k+3}(0)=\frac{1}{2}>\frac{1}{3}=\mu_b^{4k+3}(1)$, i.e., party b chooses policy 0.

At $t=4k+3$ we have:

$$U_i^{4k+3}(a)=-kx_i-(1-x_i), U_i^{4k+3}(b)=-(k+1)x_i \text{ and } U_i^{4k+3}(c)=-(2k+1)(1-x_i).$$

Therefore $\mu_a^{4k+3}=\frac{1}{6}, \mu_b^{4k+3}=\frac{1}{2}$ and $\mu_c^{4k+3}=\frac{1}{3}$ which implies that party b wins this election and $\mu_b^{4k+4}(0)=\frac{1}{3}>0=\mu_b^{4k+4}(1)$, i.e., party b chooses policy 0.

At $t=4(k+1)$ we have:

$$U_i^{4(k+1)}(a)=-kx_i-(1-x_i), U_i^{4(k+1)}(b)=-(k+2)x_i \text{ and } U_i^{4k+4}(c)=-(2k+1)(1-x_i)$$

which proves the result for $k_1=k$ and $k_2=1$. Similarly it can be proven for $k_1=1$ and $k_2=k$.

Now suppose that it is true for $k_1>1$ and $k_2>1$ and $k_1+k_2=k$.

At $t=4k$ we have:

$$U_i^{4k}(a)=-k_1x_i-k_2(1-x_i), U_i^{4k}(b)=-(k_1+2k_2)x_i \text{ and } U_i^{4k}(c)=-(2k_1+k_2)(1-x_i).$$

<u>Case 1</u>.- Suppose that party a wins the election at $t=4k$. Then

$$\mu_a^{4k+1}(0)=\frac{3k_1k_2-2k_1-k_2}{(3k_1+1)(3k_2-1)}, \mu_a^{4k+1}(1)=\frac{3k_1k_2-k_1-2k_2}{(3k_1-1)(3k_2+1)},$$

$$\mu_a^{4k+1}(0)>\mu_a^{4k+1}(1) \text{ iff } \left\{k_1>k_2 \text{ and } k_1>\frac{9k_2+1}{9k_2-9}\right\} \text{or} \left\{k_1<k_2 \text{ and } k_1<\frac{9k_2+1}{9k_2-9}\right\}$$
$$\mu_a^{4k+1}(0)<\mu_a^{4k+1}(1) \text{ iff } \left\{k_1>k_2 \text{ and } k_1<\frac{9k_2+1}{9k_2-9}\right\} \text{or} \left\{k_1<k_2 \text{ and } k_1>\frac{9k_2+1}{9k_2-9}\right\}$$
$$\mu_a^{4k+1}(0)=\mu_a^{4k+1}(1) \text{ iff } k_1=k_2$$

<u>Case 1.1</u>.- If at $t=4k$ party a chooses policy 0. Then at $t=4k+1$ we have:

$$U_i^{4k+1}(a)=-(k_1+1)x_i-k_2(1-x_i), U_i^{4k+1}(b)=-(k_1+2k_2)x_i \text{ and } U_i^{4k+1}(c)=-(2k_1+k_2)(1-x_i).$$

Therefore $\mu_a^{4k+1} = \dfrac{3k_1k_2 - 2k_1 - k_2}{(3k_1+1)(3k_2-1)}, \mu_b^{4k+1} = \dfrac{k_2}{3k_2-1}, \mu_c^{4k+1} = \dfrac{k_1+1}{3k_1+1}$ and

$$\mu_a^{4k+1} < \mu_c^{4k+1} \le \mu_b^{4k+1} \text{ } iff \text{ } k_2 \le \frac{k_1+1}{2}.$$

<u>Case 1.1.1</u>.- If at $t = 4k+1$ party b wins then $\mu_b^{4k+2}(0) = \dfrac{1}{3} > \dfrac{k_2-1}{3k_2-2} = \mu_b^{4k+2}(1)$, i.e., party b chooses policy 0.

At $t = 4k+2$ we have:

$U_i^{4k+2}(a) = -(k_1+1)x_i - k_2(1-x_i), U_i^{4k+2}(b) = -(k_1+2k_2+1)x_i$ and $U_i^{4k+2}(c) = -(2k_1+k_2)(1-x_i)$.

Therefore $\mu_a^{4k+2} = \dfrac{3k_1-1}{9k_1+3}, \mu_b^{4k+2} = \dfrac{1}{3}$ and $\mu_c^{4k+2} = \dfrac{k_1+1}{3k_1+1}$ which implies that party c wins this election and $\mu_c^{4k+3}(0) = \dfrac{1}{3} < \dfrac{k_1+1}{3k_1+2} = \mu_c^{4k+3}(1)$ i.e., party c chooses policy 1.

At $t = 4k+3$ we have:

$$U_i^{4k+3}(a) = -(k_1+1)x_i - k_2(1-x_i), U_i^{4k+3}(b) = -(k_1+2k_2+1)x_i \text{ and}$$
$$U_i^{4k+3}(c) = -(2k_1+k_2+1)(1-x_i).$$

Therefore $\mu_a^{4k+3} = \dfrac{3k_1+1}{9k_1+6}, \mu_b^{4k+3} = \dfrac{1}{3}$ and $\mu_c^{4k+3} = \dfrac{k_1+1}{3k_1+2}$ which implies that party c wins this election and $\mu_c^{4k+4}(0) = \dfrac{k_1-2}{3k_1-1} < \dfrac{1}{3} = \mu_c^{4k+4}(1)$, i.e., party c chooses policy 1.

<u>Case 1.1.2</u>.- If at $t = 4k+1$ party c wins then $\mu_c^{4k+2}(0) = \dfrac{1}{3} < \dfrac{k_1+1}{3k_1+2} = \mu_c^{4k+2}(1)$, i.e., party c chooses policy 1.

At $t = 4k+2$ we have:

$U_i^{4k+2}(a) = -(k_1+1)x_i - k_2(1-x_i), U_i^{4k+2}(b) = -(k_1+2k_2)x_i$ and $U_i^{4k+2}(c) = -(2k_1+k_2+1)(1-x_i)$

Therefore $\mu_a^{4k+2} = \dfrac{3k_1k_2 - 2k_1 + k_2 - 1}{(3k_1+2)(3k_2-1)}, \mu_b^{4k+2} = \dfrac{k_2}{3k_2-1}$ and $\mu_c^{4k+2} = \dfrac{k_1+1}{3k_1+2}$ which implies:

$$\mu_a^{4k+2} < \mu_c^{4k+2} \le \mu_b^{4k+2} \text{ } iff \text{ } k_2 \le k_1 + 1.$$

Case 1.1.2.1.- If at $t=4k+2$ party b wins then $\mu_b^{4k+3}(0)=\frac{1}{3}>\frac{k_2-1}{3k_2-2}=\mu_b^{4k+3}(1)$, i.e., party b chooses policy 0.

At $t=4k+3$ we have:

$$U_i^{4k+3}(a)=-(k_1+1)x_i-k_2(1-x_i), U_i^{4k+3}(b)=-(k_1+2k_2+1)x_i \text{ and}$$
$$U_i^{4k+3}(c)=-(2k_1+k_2+1)(1-x_i).$$

Therefore $\mu_a^{4k+3}=\frac{3k_1+1}{9k_1+6}, \mu_b^{4k+3}=\frac{1}{3}$ and $\mu_c^{4k+3}=\frac{k_1+1}{3k_1+2}$ which implies that party c wins this election and $\mu_c^{4k+4}(0)=\frac{k_1}{3k_1+1}<\frac{1}{3}=\mu_c^{4k+4}(1)$, i.e., party c chooses policy 1.

Case 1.1.2.2.- If at $t=4k+2$ party c wins then $\mu_c^{4k+3}(0)=\frac{k_1}{3k_1+1}<\frac{1}{3}=\mu_c^{4k+3}(1)$, i.e., party c chooses policy 1.

At $t=4k+3$ we have:

$U_i^{4k+3}(a)=-(k_1+1)x_i-k_2(1-x_i)$, $U_i^{4k+3}(b)=-(k_1+2k_2)x_i$, and $U_i^{4k+3}(c)=-(2k_1+k_2+2)(1-x_i)$.

Therefore $\mu_a^{4k+3}=\frac{3k_2-2}{9k_2-3}, \mu_b^{4k+3}=\frac{k_2}{3k_2-1}$ and $\mu_c^{4k+3}=\frac{1}{3}$ which implies that party b wins this election and $\mu_b^{4k+4}(0)=\frac{1}{3}>\frac{k_2-1}{3k_2-2}=\mu_b^{4k+4}(1)$, i.e., party b chooses policy 0.

At $t=4(k+1)$, for all possible results of Case 1.1. we have:

$$U_i^{4(k+1)}(a)=-(k_1+1)x_i-k_2(1-x_i), U_i^{4(k+1)}(b)=-((k_1+1)+2k_2)x_i \text{ and}$$

$$U_i^{4(k+1)}(c)=-(2(k_1+1)+k_2)(1-x_i).$$

Case 1.2.- If at $t=4k$ party a chooses policy 1 then at $t=4k+1$ we have:

$$U_i^{4k+1}(a)=-k_1x_i-(k_2+1)(1-x_i), U_i^{4k+1}(b)=-(k_1+2k_2)x_i \text{ and } U_i^{4k+1}(c)=-(2k_1+k_2)(1-x_i).$$

Therefore $\mu_a^{4k+1}=\frac{3k_1k_2-k_1-2k_2}{(3k_1-1)(3k_2+1)}, \mu_b^{4k+1}=\frac{k_2+1}{3k_2+1}$ and $\mu_c^{4k+1}=\frac{k_1}{3k_1-1}$. Hence:

$$\mu_a^{4k+1}<\mu_c^{4k+1}\leq\mu_b^{4k+1} \text{ iff } k_1\geq\frac{k_2+1}{2}.$$

By a similar argument we can prove that at $t = 4(k+1)$ we have:

$$U_i^{4(k+1)}(a) = -k_1 x_i - (k_2+1)(1-x_i),\ U_i^{4(k+1)}(b) = -(k_1 + 2(k_2+1))x_i \text{ and}$$

$$U_i^{4(k+1)}(c) = -(2k_1 + (k_2+1))(1-x_i).$$

<u>Case 2</u>.- Suppose that party b wins the election. Then $\mu_b^{4k+1}(0) = \frac{k_2}{3k_2+1} > \frac{k_2-1}{3k_2-1} = \mu_b^{4k+1}(1)$, i.e., party b chooses policy 0.

At $t = 4k+1$ we have:

$$U_i^{4k+1}(a) = -k_1 x_i - k_2(1-x_i),\ U_i^{4k+1}(b) = -(k_1 + 2k_2 + 1)x_i \text{ and } U_i^{4k+1}(c) = -(2k_1 + k_2)(1-x_i).$$

Therefore $\mu_a^{4k+1} = \frac{3k_2+2}{9k_2+3}, \mu_b^{4k+1} = \frac{k_2}{3k_2+1}$ and $\mu_c^{4k+1} = \frac{1}{3}$ which implies that party a wins $\mu_a^{4k+2}(0) = \frac{3k_1-1}{9k_1+3}, \mu_a^{4k+2}(1) = \frac{3k_1k_2 + k_1 - 2k_2 - 1}{(3k_1-1)(3k_2+1)}$ and

$$\mu_a^{4k+2}(0) \geq \mu_a^{4k+2}(1) \textit{ iff } k_2 \leq \frac{9(k_1)^2 - 6k_1 + 5}{9k_1 - 9}.$$

<u>Case 2.1</u>.- If at $t = 4k+1$ party a chooses policy 0 then at $t = 4k+2$ we have:

$$U_i^{4k+2}(a) = -(k_1+1)x_i - k_2(1-x_i),\ U_i^{4k+2}(b) = -(k_1 + 2k_2 + 1)x_i \text{ and } U_i^{4k+2}(c) = -(2k_1 + k_2)(1-x_i).$$

Therefore $\mu_a^{4k+2} = \frac{3k_1-1}{9k_1+3}, \mu_b^{4k+2} = \frac{1}{3}$ and $\mu_c^{4k+2} = \frac{k_1+1}{3k_1+1}$ which implies that party c wins and $\mu_c^{4k+3}(0) = \frac{1}{3} < \frac{k_1+1}{3k_1+2} = \mu_c^{4k+3}(1)$, i.e., party c chooses policy 1.

At $t = 4k+3$ we have:

$$U_i^{4k+3}(a) = -(k_1+1)x_i - k_2(1-x_i),\ U_i^{4k+3}(b) = -(k_1 + 2k_2 + 1)x_i \text{ and}$$
$$U_i^{4k+3}(c) = -(2k_1 + k_2 + 1)(1-x_i).$$

Therefore $\mu_a^{4k+3} = \frac{3k_1+1}{9k_1+6}, \mu_b^{4k+3} = \frac{1}{3}$ and $\mu_c^{4k+3} = \frac{k_1+1}{3k_1+2}$ which implies that party c wins and $\mu_c^{4k+4}(0) = \frac{k_1}{3k_1+1} < \frac{1}{3} = \mu_c^{4k+4}(1)$, i.e., party c chooses policy 1.

At $t=4(k+1)$ we have:

$$U_i^{4(k+1)}(a)=-(k_1+1)x_i-k_2(1-x_i),\ U_i^{4(k+1)}(b)=-((k_1+1)+2k_2)x_i \text{ and}$$

$$U_i^{4(k+1)}(c)=-(2(k_1+1)+k_2)(1-x_i).$$

Case 2.2.- If at $t=4k+1$ party a chooses policy 1 then at $t=4k+2$ we have:

$$U_i^{4k+2}(a)=-k_1x_i-k_2(k_2+1)(1-x_i),\ U_i^{4k+2}(b)=-(k_1+2k_2+1)x_i \text{ and}$$
$$U_i^{4k+2}(c)=-(2k_1+k_2)(1-x_i).$$

Therefore $\mu_a^{4k+2}=\dfrac{3k_1k_2+k_1-2k_2-1}{(3k_1-1)(3k_2+2)}$, $\mu_b^{4k+2}=\dfrac{k_2+1}{3k_2+2}$ and $\mu_c^{4k+2}=\dfrac{k_1}{3k_1-1}$ which implies that party c wins and $\mu_c^{4k+3}(0)=\dfrac{k_1+1}{3k_1-2}<\dfrac{1}{3}=\mu_c^{4k+3}(1)$, i.e., party c chooses policy 1.

At $t=4k+3$ we have:

$$U_i^{4k+3}(a)=-k_1x_i-k_2(k_2+1)(1-x_i),\ U_i^{4k+3}(b)=-(k_1+2k_2+1)x_i \text{ and}$$
$$U_i^{4k+3}(c)=-(2k_1+k_2+1)(1-x_i).$$

Therefore $\mu_a^{4k+3}=\dfrac{3k_1+1}{9k_1+6}$, $\mu_b^{4k+3}=\dfrac{k_2+1}{3k_2+2}$ and $\mu_c^{4k+3}=\dfrac{1}{3}$ which implies that party b wins and $\mu_b^{4k+4}(0)=\dfrac{1}{3}>\dfrac{k_2}{3k_2+1}=\mu_b^{4k+4}(1)$, i.e., party b chooses policy 0.

At $t=4(k+1)$ we have:

$$U_i^{4(k+1)}(a)=-k_1x_i-(k_2+1)(1-x_i),\ U_i^{4(k+1)}(b)=-(k_1+2(k_2+1))x_i \text{ and}$$

$$U_i^{4(k+1)}(c)=-(2k_1+(k_2+1))(1-x_i).$$

Case 3.- Suppose that party c wins the election. Then, using an argument symmetric to case 2 we obtain the same results.

By combining the different cases we conclude that if a party mixes the two policies (because, given that candidates are stagewise vote-maximizers, they are indifferent between 0 and 1), eventually it will decide for one of them (candidates are no longer indifferent) and will continue choosing this same policy forever. This result is represented in Figure 1. For k_1 and k_2 with values in region I, party a maximizes the proportion of votes for next election by choosing policy 1. For k_1 and k_2 with values in region II, party a maximizes the proportion of votes for next election by choosing policy 0. Party a is indifferent between policies 0 and 1 if $k_1 = k_2$ or $k_j = 0 \; or \; 1 \; for \; j = 1 \; or \; 2$. Finally, for k_1 and k_2 with values outside of these regions, party a maximizes the proportion of votes for next election by choosing policy 0 or 1 depending on the results of previous elections. In all regions parties b and c maximize the proportion of votes for next election by choosing always the same policy, 0 and 1 respectively.

(i) and (ii) follow directly from (iii). Part II follows from part I.

Lemma 1: (i) If $k_0^a(t) < k_0^b(t)$ then $k_1^a(t) \geq k_1^b(t)$.

(ii) If $k_0^a(t) = k_0^b(t)$ then $\left|k_1^a(t) - k_1^b(t)\right| \leq 1$.

Proof: First we prove part (i). Suppose it is not true, i.e., at time t.we have $k_0^a(t) < k_0^b(t)$ and $k_1^a(t) < k_1^b(t)$. Consider $t_b < t$, the last time that party b won an election. Since $t_b < t$ we must have $k_0^a(t_b) \leq k_0^a(t)$ and $k_1^a(t_b) \leq k_1^a(t)$. If at t_b party b chose policy 0 then $k_0^b(t_b) = k_0^b(t) - 1$ and $k_1^b(t_b) = k_1^b(t)$. Therefore $k_0^a(t_b) \leq k_0^b(t_b)$ and $k_1^a(t_b) < k_1^b(t_b)$ which implies that at t_b party b could not win. If at t_b party b chose policy 1 then $k_0^b(t_b) = k_0^b(t)$ and $k_1^b(t_b) = k_1^b(t) - 1$. Therefore $k_0^a(t_b) < k_0^b(t_b)$ and $k_1^a(t_b) \leq k_1^b(t_b)$ which implies that at t_b party b could not win. Part (ii) follows from a similar argument.

Lemma 2: $\dfrac{k^p(t)}{t} \leq \dfrac{1}{2} + \dfrac{1}{t}$ for all $p \in \{a,b,c\}$ and all $t > 0$.

Proof: Define $X^t(p,q) = \{x_i \in [0,1] : U_i^t(p) = U_i^t(q)\}$ and let $x^t(p,q)$ be an element of $X^t(p,q)$. To prove the Lemma we consider different cases depending on the values of $k_0^p(t)$ and $k_1^p(t)$ for all p (to simplify notation we will drop the time index). By the previous Lemma, it suffices to study the following cases:

$$(i)\, k_0^b > k_0^a > k_0^c \; and \; k_1^b < k_1^a < k_1^c$$
$$(ii)\, k_0^b > k_0^a > k_0^c \; and \; k_1^b = k_1^a < k_1^c$$
$$(iii)\, k_0^b > k_0^a > k_0^c \; and \; k_1^b < k_1^a = k_1^c$$
$$(iv)\, k_0^b = k_0^a > k_0^c \; and \; k_1^b = k_1^a < k_1^c$$
$$(v)\, k_0^b = k_0^a > k_0^c \; and \; k_1^b < k_1^a = k_1^c$$
$$(vi)\, k_0^b = k_0^a = k_0^c \; and \; k_1^b = k_1^a = k_1^c$$
$$(vii)\, k_0^b = k_0^a = k_0^c \; and \; k_1^b = k_1^a < k_1^c$$
$$(viii)\, k_0^b > k_0^a > k_0^c \; and \; k_1^b < k_1^a = k_1^c$$

Suppose the contrary, i.e., $\frac{k^p(t)}{t} > \frac{1}{2} + \frac{1}{t}$ for some p. Next we find a contradiction for each party in each case:

(i) Suppose that parties' performance has been such that $k_0^a + k_1^a > k_0^b + k_1^b + k_0^c + k_1^c + 1$. Since $k_0^a < k_0^b$ and $k_1^a < k_1^c$ we have a contradiction.

Suppose that parties' performance has been such that $k_0^b + k_1^b > k_0^a + k_1^a + k_0^c + k_1^c + 1$. It is necessary that, given the performances of parties a and c, party b wins an election with either $k_0^b - 1$ or $k_1^b - 1$. If party b chose policy 1 and case (i) applies, we have $\mu_b = \min\{x(a,b), x(b,c)\}$. For party b to win we need $x(a,b) \geq \frac{1}{3}$ for $\left(k_0^b, k_1^b - 1\right)$. This condition implies $2k_1^a + k_0^a - k_1^b + 2 \geq k_0^b + k_1^b$ and contradicts the initial assumption. If party b chose policy 0 and we have $k_0^b - 1 > k_0^a$ then case (i) applies and we have $\mu_b = \min\{x(a,b), x(b,c)\}$. For party b to win we need $x(a,b) \geq \frac{1}{3}$ for $\left(k_0^b - 1, k_1^b\right)$. This condition $2k_1^a + k_0^a - k_1^b + 1 \geq k_0^b + k_1^b$ contradicts the initial assumption. If party b chose policy 0 and we have $k_0^b - 1 = k_0^a$ then case (iii) applies and we have $\mu_b = x(b,c)$. For party b to win we need $x(b,c) \geq \frac{1}{2}$ for $\left(k_0^b - 1, k_1^b\right)$. This condition $k_0^c + k_1^c + 1 \geq k_0^b + k_1^b$ contradicts the initial assumption.

Suppose that parties' performance has been such that $k_0^c + k_1^c > k_0^a + k_1^a + k_0^b + k_1^b + 1$. Since in this case party c is symmetric to party b the last argument applies here.

(ii) For parties a and b the contradiction follows directly.

Suppose that parties' performance has been such that $k_0^c + k_1^c > k_0^a + k_1^a + k_0^b + k_1^b + 1$. It is necessary that, given the performances of parties a and b, party c wins an election with either $k_0^c - 1$ or $k_1^c - 1$. If party c chose policy 0 then case (ii) applies and we have $\mu_c = 1 - x(a,c)$. For party c to win we must have $k_0^a = k_0^b$ and $1 - x(a,c) \geq \frac{1}{3}$ for $\left(k_0^c - 1, k_1^c\right)$. This condition $2k_0^a + k_1^a - k_0^c + 2 \geq k_0^c + k_1^c$ contradicts the initial assumption. If party c chose policy 1 and we have $k_1^c - 1 > k_1^a$ then case (ii) applies and we have $\mu_c = 1 - x(a,c)$. For party c to win we must have $k_0^a = k_0^b$ and $1 - x(a,c) \geq \frac{1}{3}$ for $\left(k_0^c, k_1^c - 1\right)$. This condition $2k_0^a + k_1^a - k_0^c + 1 \geq k_0^c + k_1^c$ contradicts the initial assumption. If party c chose policy 1 and we have $k_1^c - 1 = k_1^a$ by previous lemmata this case is not possible.

(iii) For parties a and c the contradiction follows directly.

Suppose that parties' performance has been such that $k_0^b + k_1^b > k_0^a + k_1^a + k_0^c + k_1^c + 1$. It is necessary that, given the performances of parties a and c, party b wins an election with either $k_0^b - 1$ or $k_1^b - 1$.. If party b chose policy 1 then case (iii) applies and we have $\mu_b = x(b,c)$. For party b to win we need $x(b,c) \geq \frac{1}{3}$ for $(k_0^b, k_1^b - 1)$.. This condition $2k_1^c + k_0^c - k_1^b + 2 \geq k_0^b + k_1^b$ contradicts the initial assumption. If party b chose policy 0 and we have $k_0^b - 1 > k_0^a$ then case (iii) applies and we have $\mu_b = x(b,c)$. For party b to win we need $x(b,c) \geq \frac{1}{3}$ for $(k_0^b - 1, k_1^b)$. This condition $2k_1^c + k_0^c - k_1^b + 1 \geq k_0^b + k_1^b$ contradicts the initial assumption. If party b chose policy 0 and we have $k_0^b - 1 = k_0^a$ then case (v) applies and we have $\mu_b = x(b,c)$. For party b to win we need $x(b,c) \geq \frac{1}{2}$ for $(k_0^b - 1, k_1^b)$. This condition $k_1^c + k_0^c + 1 \geq k_0^b + k_1^b$ contradicts the initial assumption.

(iv) For parties a and b the contradiction follows directly.

Suppose that parties' performance has been such that $k_0^c + k_1^c > k_0^a + k_1^a + k_0^b + k_1^b + 1$. It is necessary that, given the performances of parties a and b, party c wins an election with either $k_0^c - 1$ or $k_1^c - 1$. If party c chose policy 0 then case (iv) applies and we have $\mu_c = 1 - x(a,c)$. For party c to win we need $1 - x(a,c) \geq \frac{1}{3}$ for $(k_0^c - 1, k_1^c)$. This condition $2k_0^a + k_1^a - k_0^c + 2 \geq k_0^c + k_1^c$ contradicts the initial assumption. If party c chose policy 1 and we have $k_1^c - 1 > k_1^a$ then case (iv) applies and we have $\mu_c = 1 - x(a,c)$. For party c to win we need $1 - x(a,c) \geq \frac{1}{3}$ for $(k_0^c, k_1^c - 1)$. This condition $2k_0^a + k_1^a - k_0^c + 1 \geq k_0^c + k_1^c$ contradicts the initial assumption. If party c chose policy 1 and we have $k_1^c - 1 = k_1^a$ then case $(viii)$ applies and we have $k_0^c = k_0^a - 1$. The contradiction follows directly.

For cases (v) to $(viii)$ the contradiction follows directly.♠

Proof of Theorem 3:

An implication of Theorem 1 is that the given strategies yield the following payoffs: $\Pi^a = \Pi^b = \frac{1}{4}$ and $\Pi^c = \frac{1}{2}$ up to any permutation of parties. If party a (or b) deviates we will have a situation as the one described in Theorem 2: one party mixes between policies 0 and 1 and the other two parties always choose the same policy. In this case we have already seen that the limit frequency of times in office of the party that mixes, in this case party a, is one fourth. Therefore, parties a and b have no incentive to deviate. Now consider a deviation of party c.

Lemma 2 in the appendix proves that at each $t > 0$ an for all $p \in \{a, b, c\}$, $\frac{k^p(t)}{t} \leq \frac{1}{2} + \frac{1}{t}$. Therefore, $\limsup_{t\to\infty} \frac{k^p(t)}{t}$ for any party is bounded by $\frac{1}{2}$, for every strategies of its opponents. Since in this case the limit frequency of party c is already one half, it does not have any incentive to deviate.♠

Proof of Remark:

Suppose that the winners of the first three elections are a,b and c respectively. Assume that parties a and b are stagewise vote-maximizers. Suppose that parties a and b have chosen policy 0 the first time they took office (it is stagewise maximizing). Further, suppose that they choose the following (stagewise maximizing) strategy for $t > 2$: if it is indifferent between the two policies in terms of stagewise maximization then it will choose the policy that party c chose at $t = 2$ (the third election). In this case we have that $x^a \in X^a$ and $x^b \in X^b$. Given the strategies of parties a and b, it pays to party c not to maximize next election votes the first time it takes office (by choosing policy 1) in order to make sure that its limit frequency of times in office will be one half.

If at $t = 2$ and $t = 3$ party c uses a stagewise maximizing strategy it will choose policy 1, since parties a and b have chosen policy 0. At $t = 4$ there is a tie among the three parties. If parties a (or b) wins this election it will be indifferent between policies 0 and 1 in terms of vote maximization and it will choose policy 1 and from then on we will have the following result: parties a and c choose always policy 1 and win one fourth of the times each and party b chooses always policy 0 and wins one half of the times. If at $t = 4$ party c wins the election then at $t = 5$ party a (or b) will win and we will have the situation described above.

If at $t = 2$ party c chooses policy 0, instead of a vote maximizing policy, at $t = 3$ there is a tie among the three parties. If party a (or b) wins the election it will be indifferent and will choose policy 0, imitating party c at $t = 2$. If the next time that party c wins an election it chooses policy 1 then the result will be as follows: parties a and b choose always policy 0 and win one fourth of the times each and party c chooses always policy 1 and wins one half of the times.♠

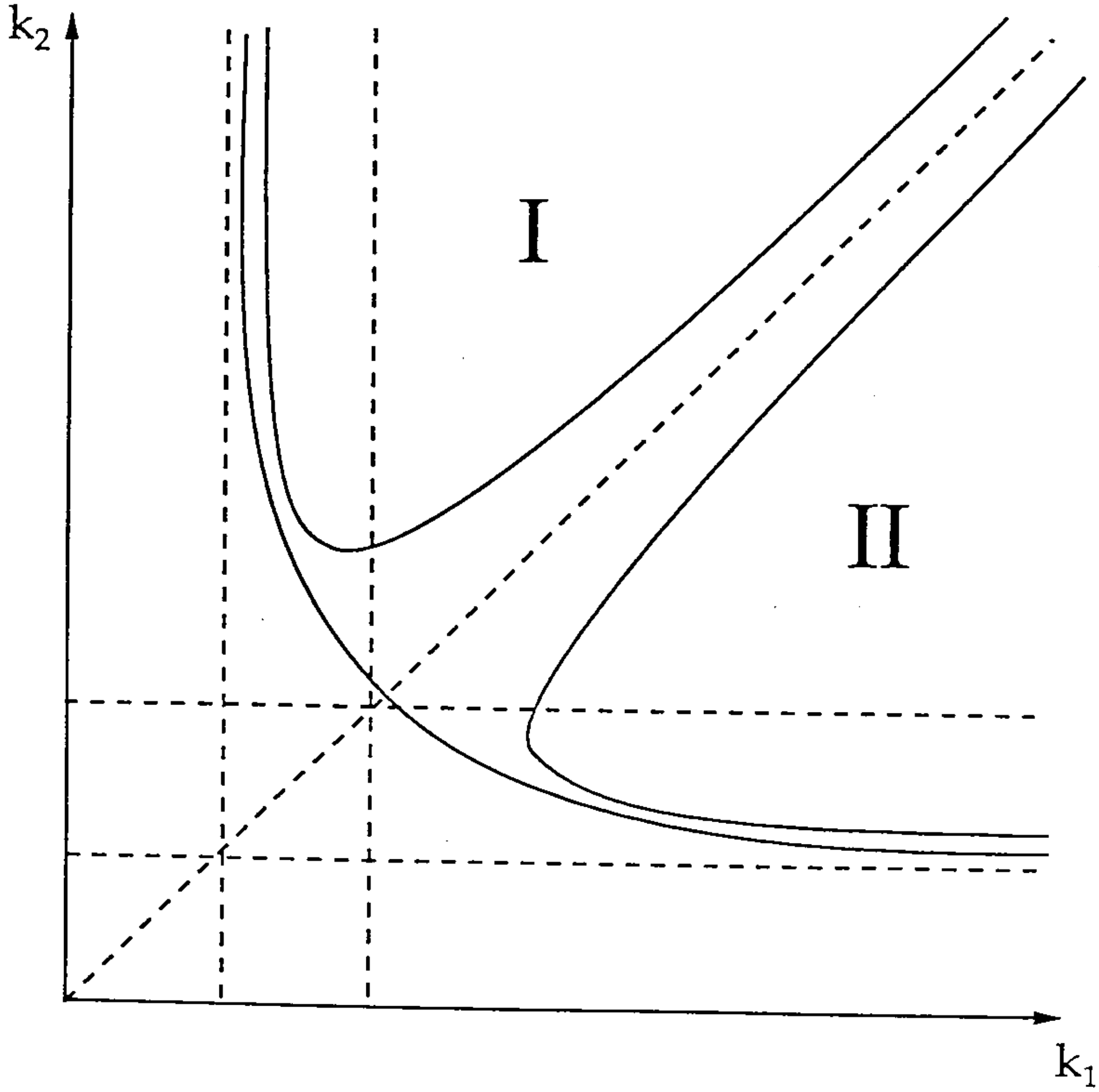

Figure 1