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STRUCTURAL INDIFFERENCE IN NORMAL FORM GAMES

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Abstract

Refinements of the Nash equilibrium concept differ in which indifferences between strategies they select for evaluation. In this paper, we suggest that "structural" indifferences, or indifferences that arise out of the structure of the payoffs of the game independently of opponents' strategies, are worthy of special attention. We define an order over a player's strategies, called the *structural* order, by ranking strategies according to expected payoff under a belief about opponents' play and requiring that (only) structural indifferences be evaluated by appealing to higher-order beliefs about opponents' play. This order is robust to trembles in payoffs and beliefs and ranks strategy r_i ahead of s_i if and only if r_i receives a higher payoff along every sequence of trembles that converges (in a certain sense) to the beliefs. We use the structural order to define an equilibrium concept called the *structural indifference respecting equilibrium (SIRE)*. A proper equilibrium is SIRE but not conversely. We show that the lexicographic probability system used to describe beliefs about opponents' play when defining SIRE can always be taken to have disjoint supports. Finally, we argue that SIRE can be viewed as a normal form extension of the sequential equilibrium concept.

Keywords: refinements, proper equilibrium, sequential equilibrium, trembles, lexicographic probability systems, indifferences.

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1. Introduction

A Nash equilibrium is not strict if some player has multiple best responses to his opponents' equilibrium strategies. Such an equilibrium can require a player to fortuitously choose one of infinitely many best responses. Why does the player choose precisely the best response needed to support the equilibrium?

This question is at the heart of the equilibrium refinements literature. If a player has multiple best responses to his opponents' equilibrium strategies, then equilibrium refinements test choices among these best replies by appealing to beliefs about how the opponents would play if they did *not* follow their equilibrium strategies. These beliefs explicitly appear in most extensive form refinements. Blume, Brandenburger and Dekel (1991b) (hereafter BBDb) have shown that the normal form refinements of trembling hand perfection and properness can also be viewed as implicitly working with beliefs about out-of-equilibrium behavior.

We will refer to the process of justifying a choice from multiple best replies as *evaluating the indifference*. The point of departure for this paper is the question of *which* indifferences should be evaluated. Consider the following game:

1 \ 2	L	R
T	3,3	2,0
B	0,0	2,2

(G1)

In the Nash equilibrium given by (B,R), player 1 has multiple best responses, being indifferent between T and B (and hence any mixture of the two). A common argument is that player 1 should then choose between T and B by appealing to beliefs about player 2's behavior assuming that player 2 does *not* play R with probability one. Any such beliefs attach positive probability to L, at which point T becomes a strict best reply. In light of this, we would conclude that player 1 chooses T, upsetting the equilibrium (B,R).

Now consider the following game, which is obtained by "replacing" (B,R) in (G1) by a 2×2 game with unique Nash equilibrium payoffs of (2,2).

In the equilibrium given by ($\frac{1}{2}$M+$\frac{1}{2}$B,$\frac{1}{2}$C+$\frac{1}{2}$R), player 1 again has multiple best replies, being indifferent between all of his (pure and mixed) strategies. One could again argue that player 1 should appeal to beliefs about player 2's strategy that place some probability on L, upsetting the equilibrium (since such beliefs necessarily imply that the simultaneous use of M and B is an inferior response).

		2 L	C	R
	T	2,2	2,0	2,0
1	M	0,0	4,1	0,3
	B	0,0	1,3	3,1

(G2)

We find this argument less convincing here. One standard interpretation of equilibrium mixed strategies derives them as the limit of pure strategies in a sequence of games with incomplete information about payoffs (Harsanyi (1973)). In these incomplete information games, each type of player 1 has a strict best response. The equilibrium ($\frac{1}{2}M+\frac{1}{2}B,\frac{1}{2}C+\frac{1}{2}R$) then appears as the limit of strict Nash equilibria in a sequence of games of incomplete information. As a result, it is not obvious that the equilibrium ($\frac{1}{2}M+\frac{1}{2}B,\frac{1}{2}C+\frac{1}{2}R$) should be rejected.

If we allow indifferences to be broken by payoff perturbations, as the previous paragraph suggests, then it appears as if any pure strategy Nash equilibrium can be "justified" (Fudenberg, Kreps, and Levine (1988, Proposition 1)), including (B,R) in Game (G1). However, we are unwilling to allow such latitude in choosing perturbations. Instead, we require payoff perturbations to preserve any payoff ties that appear in the original specification of the game. We think it is important to restrict analysis to generic games. By generic, however, we mean generic in the assignment of utilities to economic outcomes for some player. This genericity allows the possibility that two or more strategy profiles may lead to identical economic outcomes for some player. In such a case, payoff ties will appear in the normal form that are "generic" ties and should be respected when constructing trembles.[1]

A similar argument can be constructed with belief rather than payoff uncertainty. In particular, suppose there are some player 1 types who believe player 2's strategy is close to $\frac{1}{2}C+\frac{1}{2}R$ but places slightly more probability on C, making M a best reply; and other player 1 types who believe that player 2's strategy places slightly more probability on R, making B a best response. We again have strict (with probability one) equilibria approaching ($\frac{1}{2}L+\frac{1}{2}B,\frac{1}{2}C+\frac{1}{2}R$). We might interpret this as reflecting a

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[1]Payoff trembles are thus intended to reflect uncertainty concerning how players assign utilities to economic outcomes. The identification of these outcomes themselves is less likely to be subject to uncertainty. The ties generated by identical outcomes then should not be broken in perturbed games. Fudenberg, Kreps and Levine (1988) devote the bulk of their analysis to perturbations of extensive form payoffs that necessarily do not affect ties in the normal form that are created by identical outcomes in the extensive form.

situation in which players are reasonably certain of the strategies their opponents' might play (i.e., reasonably certain of supports of distributions), but are not certain of the precise mixture played by opponents.

More generally, we believe that indifferences caused by opponents' mixed strategies are more fragile than those caused by payoff equalities. We will say that two strategies are *structurally indifferent* for player i, given a strategy profile for the other players, if and only if the two strategies give player i precisely the same payoff for every pure strategy profile in the support of the opponents' strategy. This paper then asks: What are the implications for equilibrium play in normal form games if we require players to use beliefs about out-of-equilibrium play that evaluate only structural indifferences?

We examine structural indifference in normal form games, and hence require a way of describing out-of-equilibrium beliefs in such games. We use the lexicographic probability systems of Blume, Brandenburger and Dekel (1991a,b). We are especially interested in the special case of lexicographic probability systems known as lexicographic *conditional* probability systems, in which the supports of the probability distributions describing a player's alternative theories of opponents' behavior are disjoint. In many cases we think these are the most natural descriptions of alternative theories of behavior.

We use lexicographic beliefs to induce two orders on strategies. In each case, the basic ordering technique is to place better replies ahead of inferior replies. If there are indifferences between strategies, an appeal is made to a higher belief level to resolve the indifference. In the lexicographic ordering (used by Blume, Brandenburger and Dekel (1991a,b)), every indifference is so evaluated. In the structural ordering, only structural indifferences are so evaluated. The structural ordering has two desirable features: First, unlike the lexicographic ordering, it is robust to small tie-preserving perturbations in payoffs or small "level-preserving" perturbations in beliefs. Second, the structural ordering ranks strategy r_i ahead of s_i if and only if r_i receives a higher payoff along *every* sequence of strategy trembles that converges (in a certain sense) to the lexicographic beliefs.

We use the structural ordering to define an equilibrium concept, the *Structural Indifference Respecting Equilibrium (SIRE)*. There are three interesting properties of SIRE. First, if *all* indifferences are structural, then SIRE coincides with properness. In the presence of indifferences that are not structural, SIRE is weaker than (but implied by) properness, because properness insists that all (rather than only structural) indifferences be evaluated. Second, unlike properness, any SIRE induced by a lexicographic probability system remains a SIRE when the lexicographic probability system is converted to a lexicographic *conditional* probability system in a natural way. Finally, SIRE induces a sequential equilibrium in every corresponding extensive form. In fact, SIRE is in some sense the normal form analog of sequential equilibrium, having an attractive interpretation using strategic independences (Mailath, Samuelson, Swinkels (1993), hereafter MSS).

It is common, particularly in normal form games, to define equilibrium concepts in terms of trembles. Our work with lexicographic probability systems does not represent a departure from this tradition. Both the lexicographic and structural orderings over strategies can be obtained as the limits of sequences of trembles, and we characterize the limiting operations involved.

The next section introduces notation and basic concepts. Section 3 reviews the basic features of lexicographic probability orderings. Section 4 examines the lexicographic and structural orders. Section 5 introduces the Structural Indifference Respecting Equilibrium. Section 6 investigates the properties of SIRE. Section 7 examines trembles and Section 8 concludes.

2. Preliminaries

We examine finite normal form games. We denote the set of players by N and player i's (pure) strategy set by S_i, $i \in N$. The set of strategy profiles is given by $S \equiv \Pi_{i \in N} S_i$. A set of strategy profiles S and a payoff function $\pi: S \rightarrow \Re^N$ constitute the normal form game (S,π). A subset of player i's pure strategy space will be denoted X_i. Denote the set of probability distributions over a set X_i by $\Delta(X_i)$. Typical strategies for player i are r_i, s_i, and t_i. A subscript $-i$ denotes $N\backslash\{i\}$ and a subscript $-I$ denotes $N\backslash I$.

An essential ingredient in our approach to equilibrium analysis is the examination of ties in the normal form payoffs. A tie in payoffs creates a *structural indifference*:

Definition 1: Strategies r_i and s_i are *structurally indifferent* for player i on $Y_{-i} \subseteq S_{-i}$ if, $\forall\, s_{-i} \in Y_{-i}$,

$$\pi_i(r_i,s_{-i}) = \pi_i(s_i,s_{-i}).$$

We use the term *structurally indifferent* to emphasize that the indifference is due to the structure of payoff ties in the game, and is not simply the result of a fortuitous randomization by the other player(s). In generic games, these payoffs ties represent identical economic outcomes. Because players are more likely to be certain of economic outcomes than opponents' randomizations, structural indifferences are likely to be particularly important.

3. Lexicographic Probability Systems

In order to describe beliefs about out-of-equilibrium play, we use the notion of a *Lexicographic Probability System* (*LPS*) introduced by Blume, Brandenburger and Dekel (1991a) (hereafter BBDa).[2] Consider a finite state space Ω . In a game theoretic context, the appropriate choice for Ω is S_i, S_{-i}, or S, depending on the context. For example, the state space when describing player i's beliefs about opponents' play is the space of strategy choices for the other players, S_{-i}. Where convenient, we (like BBDb) will define the concepts for an arbitrary state space Ω .

Definition 2: A *Lexicographic Probability System* (*LPS*) on Ω is a K-tuple $\rho = (\rho^0,\ldots,\rho^{K-1})$, for some integer K, of probability distributions on Ω . A *Lexicographic Conditional Probability System* (*LCPS*) is an LPS with pairwise disjoint supports.

BBDb interpret an LPS ρ as follows (page 82) "The first component of the LPS can be thought of as representing the player's primary theory of how the game will be played, the second component the player's secondary theory, and so on." If the LPS is in fact an LCPS, then we can interpret ρ^1 as the player's belief over Ω given the information that a state (strategy combination if $\Omega = S_{-i}$) not in the support of ρ^0 occurs. Similarly, if the player is told that this theory is also incorrect, then ρ^2 is his belief, and so on. We believe LCPSs are a particularly attractive notion of alternative theories, because an LCPS makes an appeal to a higher level belief only when forced to by an event outside the support of the current level belief.

Each player i has an LPS ρ_{-i} describing his or her beliefs about S_{-i}. A collection of beliefs for all players, $(\rho_{-1},\ldots,\rho_{-N})$, is a *belief system*. We (like BBDb) will consider several restrictions on the belief system held by players, motivated by standard game theoretic considerations. To do this, we need an additional piece of notation. Given an LPS ρ and a vector $r \equiv (r^1,\ldots,r^{K-1}) \in (0,1)^{K-1}$, write $r\Box\rho$ for the probability distribution given by $(1-r^1)\rho^0 + r^1[(1-r^2)\rho^1 + r^2[(1-r^3)\rho^2 + r^3[\cdots + r^{K-2}[(1-r^{K-1})\rho^{K-2} + r^{K-1}\rho^{K-1}]\cdots]]]$. If $\{r(n)\}$ is a sequence satisfying $r(n) \in (0,1)^{K-1}$ and $r(n) \to 0$ as $n \to \infty$, then $r(n)\Box\rho$ is a sequence of distributions that "captures" the hierarchy of beliefs described by the LPS, in the sense that strategies are ranked the same way by the LPS ρ and the sequence of probability distributions

[2]BBDa provide an axiomatic characterization of LPS based on subjected expected utility theory. BBDb characterize trembling hand perfect and proper equilibrium in terms of LPSs. This section recaps the material from BBDa and BBDb that we need. An alternative approach to describing beliefs about out-of-equilibrium play is provided by Myerson's (1986) *conditional probability system* (which are isomorphic to the LCPSs defined in Definition 2). A discussion of these and other approaches to describing out-of-equilibrium behavior is contained in Hammond (1992).

$\{r(n)\Box\rho\}_{n=1}^{\infty}$. The following example illustrates this (as well as the structure of LPSs), while a formal statement of this relationship between LPSs and trembles is given by Lemma 2 in Section 7.2.

Example 1: Consider a three player game, with $S_1 = \{T,B\}$ and $S_2 = \{L,R\}$. Suppose that player 3's beliefs about player 1 and 2's behavior are given by the LPS ρ_{-3} on S_{-3}, where

$$\rho^0_{-3}(T,L) = 1,$$
$$\rho^1_{-3}(T,R) = 2/3, \quad \rho^1_{-3}(B,L) = 1/3,$$
$$\rho^2_{-3}(T,R) = 1/2, \quad \rho^2_{-3}(B,R) = 1/2,$$

with all other terms zero. Notice that ρ^1_{-3} is not a product measure on $S_1 \times S_2$.

Now consider the sequence $(r^1(n),r^2(n)) = (1/n,1/n)$. Then we obtain a sequence of probability distributions given by:

$$(r(n)\Box\rho_{-3})(T,L) = (1-n^{-1})\rho^0_{-3}(T,L) + n^{-1}(1-n^{-1})\rho^1_{-3}(T,L) + n^{-2}\rho^2_{-3}(T,R) = 1-n^{-1},$$
$$(r(n)\Box\rho_{-3})(T,R) = n^{-1}(1-n^{-1})(2/3) + n^{-2}(1/2),$$
$$(r(n)\Box\rho_{-3})(B,L) = n^{-1}(1-n^{-1})(1/3), \text{ and}$$
$$(r(n)\Box\rho_{-3})(B,R) = n^{-1}(1-n^{-1})(1/2).$$

Then the sequence of probability distributions $(r(n)\Box\rho_{-3})$ captures the hierarchy of beliefs described by ρ_{-3} in the sense player 3 "lexicographically prefers" (Definition 4 in Section 4 below) strategy $r_3 \in S_3$ to $s_3 \in S_3$ if and only if the expected value of r_3 given beliefs $(r(n)\Box\rho_{-3})$ is higher than the expected payoff of s_3 for all sufficiently large n. (This is Lemma 2 in Section 7.2).

Alternatively, consider the sequence given by $r^1(n) = (3n-2)/n^2$ and $r^2(n) = 4/(3n-2)$.[3] These again give a sequence of probability distributions $(r(n)\Box\rho_{-3})$ that capture the beliefs described by ρ_{-3}. In this case, we have $(r(n)\Box\rho_{-3}) = \sigma_1 \times \sigma_2$, where $\sigma^n_1(T) = 1-n^{-1}$, $\sigma^n_1(B) = n^{-1}$, $\sigma^n_2(L) = 1-2n^{-1}$ and $\sigma^n_2(R) = 2n^{-1}$. Because this sequence r(n) allows $(r(n)\Box\rho_{-3})$ to be written as a product measure, we say that the beliefs given by ρ_{-3} satisfy strong independence (see below). ■

We consider the following assumptions:

(I) Common Prior Assumption: There exists an LPS ρ on S such that for all i, ρ_{-i} is the marginal on S_{-i} of ρ.[4]

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[3]See the proof of Proposition 2 in the Appendix of BBDb for details of how these sequences are derived.

[4]The marginal of an LPS $(\rho^0,...,\rho^{K-1})$ is the LPS whose κ^{th} probability distribution is the marginal of ρ^κ, $\kappa=0,...,K-1$.

(2) Strong Independence: There exists $\{r(n)\}$, $r(n) \to 0$, such that $r(n)\square\rho$ is a product measure for all n.[5]

(3) Full Support: For all i and $s_{-i} \in S_{-i}$, there exists κ such that $\rho^{\kappa}_{-i}(s_{-i}) > 0$.

The first condition is the usual requirement that different players have the same beliefs about the behavior of other players. We denote the lexicographic belief system that players other than i hold on S_i by ρ_i. The second condition ensures that player i believes that the other players are independently choosing strategies. Note that ρ^{κ}_{-i} need not be a product measure on S_{-i} for $\kappa \geq 1$ (consider ρ^{1}_{-3} in Example 1 above). The third condition ensures that a player can evaluate the relative likelihood of any two strategy profiles chosen by the other players.

4. Orders

We now examine orders over strategy spaces induced by belief systems. Two orders are of interest, the *lexicographic* order and the *structural* order.[6] In the *lexicographic* order (studied by BBDa and BBDb), two strategies are first evaluated using a player's first level belief about his opponents. If one strategy receives a higher payoff under this belief, then it is ranked ahead of the other in the lexicographic order. If they are indifferent under this belief, than they are evaluated under the second level belief. This process of appealing to higher level beliefs continues until a payoff difference appears, with this difference sufficing to rank the strategies (and the strategies being indifferent if no such level exists).

Definition 3: Given a lexicographic conditional probability system ρ, define

$$k(r_i,s_i) \equiv \max_{1\leq k\leq K}\{k\colon \sum_{s_{-i}\in S_{-i}} \pi_i(r_i,s_{-i})\rho^{\kappa}_{-i}(s_{-i}) = \sum_{s_{-i}\in S_{-i}} \pi_i(s_i,s_{-i})\rho^{\kappa}_{-i}(s_{-i}), \quad \kappa = 0,\ldots,k-1\}.$$

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[5]This is equivalent to requiring that the non-Archimedean probability measure equivalent to ρ is a product measure. Different notions of independence for lexicographic probability systems are discussed in BBDa and BBDb (see also Battigalli (1992) and Hammond (1992)), with strong independence being the most stringent. This is the appropriate notion of independence for analysis of refinements like trembling hand perfection and properness.

[6]Okada (1988) also defines an order over strategies, saying that strategy s_i lexicographically dominates t_i with respect to q^{*}_{-i} if $s_i \succeq_L t_i$ (see Definition 4 below) for every lexicographic probability system ρ_{-i} that has the following properties: $\rho^0_j = q^*_j$ for all $j \neq i$, ρ_j has exactly two belief levels for each $j \neq i$, and ρ_j has full support for each $j \neq i$. Okada shows that perfect equilibrium strategies are lexicographically undominated with respect to the equilibrium but the converse can fail.

Note that $k(r_i,s_i) = K$ if

$$\sum_{s_{-i}\in S_{-i}} \pi_i(r_i,s_{-i})\rho^{\kappa}_{-i}(s_{-i}) = \sum_{s_{-i}\in S_{-i}} \pi_i(s_i,s_{-i})\rho^{\kappa}_{-i}(s_{-i}), \quad \text{for all } \kappa.$$

Definition 4: Given a lexicographic conditional probability system ρ, *the lexicographic ordering* $\geq_L$ on S_i is given by

(4.1) $r_i \succ_L s_i$ if $k(r_i,s_i) < K$ and, for $\kappa = k(r_i,s_i)$,

$$\sum_{s_{-i}\in S_{-i}} \pi_i(r_i,s_{-i})\rho^{\kappa}_{-i}(s_{-i}) > \sum_{s_{-i}\in S_{-i}} \pi_i(s_i,s_{-i})\rho^{\kappa}_{-i}(s_{-i}),$$

(4.2) $r_i \sim_L s_i$ if $k(r_i,s_i) = K$, and

(4.3) $r_i \geq_L s_i$ for $r_i,\ s_i \in S_i$ if $r_i \sim_L s_i$ or $r_i \succ_L s_i$.

We have argued that not all indifferences were created equal. This motivates our definition of the following order, which we call the *structural order*. The structural order resembles the lexicographic order in that cases of indifference can prompt an appeal to a higher level belief in order to evaluate two strategies, but under the structural order this occurs only if the two strategies are structurally indifferent.

Definition 5: Given a lexicographic conditional probability system ρ, *the structural (partial) ordering* $\geq_S$ on S_i is given by $r_i \geq_S s_i$ if

(5.1) $r_i \geq_L s_i$, and

(5.2) for all $\kappa < k(r_i,s_i)$, r_i and s_i are structurally indifferent on the support of ρ^{κ}_{-i}.

Note that while $\geq_L$ is a complete (or total) order, $\geq_S$ in general will only be a partial order. In particular, if two strategies r_i and s_i have equal expected payoffs according to ρ^{0}_{-i} but are not structurally indifferent on the support of ρ^{0}_{-i}, then they are not comparable under $\geq_S$. If two strategies r_i and s_i are not comparable under $\geq_S$, then for some k, the expected payoffs to r_i and s_i are equal under all ρ^{κ}_{-i} for $\kappa \leq$ k, while r_i and s_i are indifferent but not structurally indifferent on the support of ρ^{k}_{-i}.

The next result, which follows immediately from the definitions, provides one reason why we find $\geq_S$ an attractive theory of how players might use levels of beliefs about play to evaluate indifferences over strategies: $\succeq_S$ is robust to both trembles in beliefs about opponents' play and trembles in payoffs that preserve ties.[7] It is easy to construct examples showing that neither result holds for $\succeq_L$.

Proposition 1: (1.1) *Fix an LPS* ρ. *There exists* $\epsilon > 0$ *such that* $r_i \succeq_S s_i$ *(given* ρ*) if and only if* $r_i \succeq_S s_i$ *(given* σ*) for all LPSs* σ *satisfying* $\text{supp}(\sigma^\kappa) = \text{supp}(\rho^\kappa)$ *and* $\sup_{s_{-i} \in S_{-i}} |\sigma^\kappa(s_{-i}) - \rho^\kappa(s_{-i})| < \epsilon$ *for all* κ.

(1.2) *Fix payoffs* π. *There exists* $\epsilon > 0$ *such that* $r_i \succeq_S t_i$ *if and only if* $r_i \succeq_S t_i$ *on any game with payoffs* π^* *satisfying* $|\pi - \pi^*| < \epsilon$ *and* $\pi_i(s) = \pi_i(s') \Rightarrow \pi_i^*(s) = \pi_i^*(s')$.

The failure of $\succeq_S$ to be complete requires discussion. In decision theory, preferences are taken as fixed. In contrast, a player's preferences over his or her strategy space in a game are to a large extent endogenous. They are determined by the player's analysis of the game and beliefs about the opponents' play. The incompleteness of $\succeq_S$ does *not* mean that the player cannot compare two strategies that are incomparable under $\succeq_S$. Rather, it reflects the idea that the player is not able to evaluate an indifference that appears in one level of beliefs by simply going to the next level. Some other means of evaluating the two strategies may then come into play. We do not view $\succeq_S$ as being a complete description of a player's evaluation of a game, just as any theory that fails to identify a unique strategy for each player in every game is incomplete. We do view $\succeq_S$ as a good candidate for how the player will use beliefs about opponent's play to evaluate strategies.

There is an obvious completion of $\succeq_S$ that is in the spirit of $\succeq_S$: Define a new indifference relation as $r_i \sim_S' s_i$ if either $r_i \sim_S s_i$ or r_i and s_i are not comparable under $\succeq_S$. If $r_i \sim_S' s_i$, then r_i and s_i have equal expected payoffs under ρ_{-i}^κ for $\kappa \leq k(r_i,s_i)$. The relation $\sim_S'$ in conjunction with $\succeq_S$ yields a complete relation denoted $\succeq_S'$ in the obvious manner: $r_i \succeq_S' s_i$ if $r_i \sim_S' s_i$ or $r_i \succeq_S s_i$. The next example shows that this relation need not equal $\succeq_L$ for *any* LPS, and the subsequent discussion shows that $\succeq_S'$ need not be transitive.

Example 2: Consider the game (G3). Suppose a lexicographic conditional probability system {L,C,R} is given by $\rho_2^0(L) = \rho_2^0(C) = ½$ and $\rho_2^1(R) = 1$. Note first that A,B $\succ_S$ C,D $\succ_S$ E $\succ_S$ F, and that A and

[7] Notice that in Proposition (1.1), the supports of the distributions that characterize the successive levels of belief in the tremble σ must agree with those of the LPS, though there is room for (small) differences in probabilities within those supports. This is a generalization of our suggestion in the introduction that players may be reasonably certain of which strategies their opponents will possibly play, i.e., certain of supports, but uncertain concerning distributions within supports. Notice also that the payoff trembles in Proposition (1.2) preserve any ties in normal form payoffs.

		L	C	R	
	A	1	2	1	
	B	2	1	0	
	C	0	1	0	
1	D	1	0	1	(G3)
	E	−1	−1	−1	
	F	−1	−1	−2	

B, and C and D are not comparable under $\succeq_S$. Completing this order as suggested above yields A $\sim_S^*$ B $\succ_S^*$ C $\sim_S^*$ D $\succ_S^*$ E $\succ_S^*$ F. It is then straightforward to verify that any LPS that yields A $\sim_L$ B and C $\sim_L$ D must never (i.e., at no level) attach positive probability to R. However, if R is never allocated positive probability, it must be that E $\sim_L$ F, preventing any LPS from generating the order $\succ_S^*$. ■

BBDa provide an axiomatic foundation for decision making that *implies* that agents have preferences that can be represented by an LPS and $\succeq_L$. It is important to understand the relationship between $\succeq_S$ and the axiomatic foundation discussed in BBDa. The preferences represented by $\succeq_S$ violate Axiom 1 in BBDa (which requires that the decision maker's preferences be a complete order). It may appear as if the potential failure of $\succeq_S$ to be a complete order is irrelevant, since $\succeq_S$ can be extended to the complete order $\succeq_S^*$. However, this extension may force transitivity to fail. Consider again (G3). BBDa's axiom 1 requires a complete preference order on the space of *all* mappings from the state space Ω into the space of outcomes.[8] In the game theoretic context, Ω is S_{-i} and the space of outcomes is payoffs. Hence, in order to be consistent with Axiom 1, $\succeq_S^*$ must continue to be a complete and transitive order when (G3) is augmented to include the strategies G and H, where:

	L	C	R
G	1	2	0
H	2	1	1

[8]More specifically, they take as their space of outcomes objective probability distributions with finite support over some set of pure consequences.

We then have A $\succ_S^{*}$ G $\sim_S^{*}$ H $\succ_S^{*}$ B. Since A $\sim_S^{*}$ B, $\succeq_S^{*}$ is not transitive.

5. Equilibrium

Our examination of equilibrium refinements builds on the idea of a lexicographic Nash equilibrium. BBDb define a *lexicographic Nash equilibrium* as:

Definition 6: A belief system $(\rho_{-1},\ldots,\rho_{-N})$ is a *lexicographic Nash equilibrium* if for all i,

(6.1) $\rho^0_{-i}(s_{-i}) = \Pi_{j\neq i}\rho^0_j(s_j)$

(6.2) $\rho^0_i(s_i) > 0 \Rightarrow s_i \succeq_L t_i\ \forall t_i \in S_i$.

A lexicographic Nash equilibrium is a hierarchy of beliefs with the property that a single product measure gives the first level beliefs of all players and the strategies contained in these first level beliefs are best replies.[9] It is straightforward that a lexicographic Nash equilibrium is a Nash equilibrium and that for any Nash equilibrium, there is a corresponding belief system (in fact, there are generally many) that is a lexicographic Nash equilibrium. For example, ρ can be chosen so that first level beliefs match the Nash equilibrium behavior and there are no higher level beliefs. We could use our order $\succeq_S$ to define an identical concept by replacing (6.2) with $\rho^0_i(s_i) > 0 \Rightarrow \forall t_i \in S_i$, either $s_i \succeq_S t_i$ or s_i and t_i are incomparable under $\succeq_S$.

Equilibrium refinements require that an equilibrium survive when restrictions are placed on the beliefs that support it as a lexicographic Nash equilibrium. As a first step, we might strengthen (6.1) to the requirement that $(\rho_{-1},\ldots,\rho_{-N})$ satisfy the common prior, strong independence, and full support assumptions. BBDb (Proposition 7) show that the result is an equilibrium concept equivalent to perfect equilibrium.

As a next step, we might require that there be some consistency between beliefs and the way that players who hold these beliefs compares strategies. These restrictions can be viewed as restricting out-of-equilibrium behavior as well as equilibrium behavior. In the normal form, this means that the equilibrium must not only identify a strategy for each player, but must also say something about how players rank, or would choose from, their remaining strategies contingent upon not following equilibrium behavior. To describe this ranking of remaining strategies, we need an order on Ω that reflects the relative likelihood of states in Ω .

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[9]The common prior and full support conditions are not imposed.

Definition 7: Given an LPS ρ on Ω and $\omega,\omega' \in \Omega$, write $\omega \geq_\rho \omega'$ if

$$\min\{\kappa : \rho^\kappa(\omega) > 0\} \leq \min\{\kappa : \rho^\kappa(\omega') > 0\}.$$

The order $\geq_\rho$ captures the ranking on states induced by the order in which these states appear in the levels of the belief systems. As usual, the strict version of this order is obtained by $\omega >_\rho \omega'$ if $\omega \geq_\rho \omega'$ holds but $\omega' \geq_\rho \omega$ does not. Similarly, $\omega =_\rho \omega'$ if $\omega \geq_\rho \omega'$ and $\omega' \geq_\rho \omega$ both hold. The order $\geq_\rho$ is a complete and transitive order on Ω.

We then capture restrictions on belief systems with:

Definition 8: The belief system $(\rho_{-1},\ldots,\rho_{-N})$ *respects an order* $\succeq$ if, for all i, for all r_i, $s_i \in S_i$,

$$r_i \succ s_i \;\Rightarrow\; r_i >_{\rho_i} s_i.$$

If ρ respects $\succeq_L$, then ρ respects preferences in the sense of Definition 4 of BBDb. If ρ respects an order $\succeq$, then whenever $r_i \succ s_i$, the beliefs held by the other players are that player i is infinitely more likely to play r_i than s_i.

We now define an equilibrium concept based on the requirement that players resolve indifference by appealing to higher order beliefs, and that beliefs be consistent with the resulting ranking of strategies. However, we require that players only evaluate indifferences that are structural, which is to say that we require beliefs to respect the order $\succeq_S$. The following section, in the course of exploring this equilibrium concept, identifies how it would differ if we required beliefs to respect $\succeq_L$.

Definition 9: The belief system $(\rho_{-1},\ldots,\rho_{-N})$ is a *Structural Indifference Respecting Equilibrium* (*SIRE*) if it is a lexicographic Nash equilibrium that satisfies the common prior assumption, strong independence, and full support, and respects $\succeq_S$.

6. SIRE

(6.1) Strategic Independence

Our original interest in SIRE was motivated by its being a normal form extension of the extensive form sequential equilibrium concept. MSS introduce the idea of a normal form information set:

Definition 10: Two strategies t_i and r_i are structurally indifferent for *every* player on X_{-i} if $\forall j$, $\pi_j(t_i,s_{-i}) = \pi_j(r_i,s_{-i})$ $\forall s_{-i} \in X_{-i}$. The set $X \subseteq S$ is *normal form information set for player i* if $X = X_i \times X_{-i}$ and

$\forall$ r_i, $s_i \in X_i$, $\exists$ $t_i \in X_i$ such that t_i and r_i are structurally indifferent for every player on X_{-i} and t_i and s_i are structurally indifferent for every player on $S_{-i} \backslash X_{-i}$.

If X is a normal form information set for player i, then when player i is evaluating strategies in X_i, we can think of the player as separately choosing an X_{-i}-equivalence class of strategies (i.e., those strategies in X_i that are structurally indifferent on X_{-i}) and an $S_{-i} \backslash X_{-i}$-equivalence class (i.e., those strategies in X_i that are structurally indifferent on $S_{-i} \backslash X_{-i}$). The optimality of a particular X_{-i}-equivalence class depends only on player i's beliefs over X_{-i} (since for any two X_{-i}-equivalence classes, one can choose a strategy from each that are structurally indifferent on $S_{-i} \backslash X_{-i}$). Similarly, the optimality of a $S_{-i} \backslash X_{-i}$-equivalence class depends only on the beliefs over $S_{-i} \backslash X_{-i}$. We can thus think of player i's choice from X_i as one of choosing behavior conditional on X_{-i} and behavior conditional off X_{-i} independently.

Extensive form information sets exhibit a similar independence property, in that decisions at different extensive form information sets can be made independently of one another. This is no coincidence. MSS show that for every normal form game and normal form information set, there is a corresponding extensive form game containing a corresponding extensive form information set. Furthermore, a normal form game contains a normal form information set for every extensive form information set appearing in any corresponding extensive form.

MSS introduce an equilibrium concept called *normal form sequential equilibrium* that essentially requires optimal play at every *normal form information set*. Let $\rho_i|_X$ denote the conditional distribution $\rho_i^\kappa(.|X)$ for the smallest κ for which $\mathrm{supp}(\rho_i^\kappa) \cap X_i \neq \varnothing$, and similarly for $\rho_{-i}|_X$. Then MSS's definition of normal form sequentiality is equivalent to:[10]

Definition 11: A belief system ρ is a *normal form sequential equilibrium* if ρ satisfies the common prior assumption, strong independence, and full support and, for any normal form information set X for player i, $\rho_i|_X$ is a best reply to $\rho_{-i}|_X$.

Normal form sequential equilibrium is the normal form counterpart of sequential equilibrium, in the sense that a strategy profile and belief system is a normal form sequential equilibrium if and only if it induces a sequential equilibrium in every corresponding extensive form game (MSS, Theorems 7 and 8).

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[10]This immediately follows from Definition 11 of MSS and the equivalence between trembles and LPSs that is described in Section 7.

The equivalence between extensive form and normal form sequentiality exploits the fact that if strategies s_i and t_i give equal payoffs (to *all* players) whenever the opponents choose from some set $S_{-i}\backslash X_{-i}$, then player i can make the choice between s_i and t_i "as if" opponents have chosen from X_{-i}. From player i's perspective, it is only necessary to require ties in player i's payoffs. This will lead us to SIRE, which is a natural extension of normal form sequentiality.

Definition 12: The set $X \subseteq S$ is *strategically independent for player i* if $X = X_i \times X_{-i}$ and $\forall\, r_i, s_i \in X_i$, $\exists\, t_i \in X_i$ such that t_i and r_i are structurally indifferent for player i on X_{-i} and t_i and s_i are structurally indifferent for player i on $S_{-i}\backslash X_{-i}$.

The following is then immediate from the definitions:

Proposition 2. *A belief system ρ is a SIRE if and only if ρ satisfies the common prior assumption, strong independence, and full support, and, for any* X *strategically independent for player* i, $\rho_i|_X$ *is a best reply to* $\rho_{-i}|_X$.

Hence, SIRE strengthens normal form sequentiality by replacing normal form information sets with strategically independent sets.[11] It is immediate from Definition 11 and Proposition 2 that any SIRE is a normal form sequential equilibrium, and hence induces a sequential equilibrium in every extensive form.

Corollary: *If a belief system ρ is a SIRE of the normal form* (S,π), *then the outcome of ρ is a sequential equilibrium outcome of every extensive form with normal form* (S,π).

(6.2) Proper Equilibrium

BBDb show that if we require that players resolve *all* indifferences by appealing to higher order beliefs, then we obtain proper equilibrium. It is then no surprise that SIRE is closely related to proper equilibrium. Rather than using trembles, it is possible from Proposition 8 (BBDb) to define properness as follows:

[11] Note that the acronym SIRE serves equally well for *Strategic Independence Respecting Equilibrium*.

Definition 13: The belief system $(\rho_{-1},\ldots,\rho_{-N})$ is *proper* if it is a lexicographic Nash equilibrium that satisfies the common prior assumption, strong independence, and full support, and respects $\succeq_L$.

The requirement that the belief system respect $\succeq_L$ forces each player to evaluate all indifferences, including those generated by opponents' randomizations.

Our next theorem shows that SIRE and proper coincide if all indifferences are structural, and that otherwise proper is a stricter requirement than SIRE because proper evaluates more indifferences:

Definition 14: A belief system ρ is *pure* if for every player i $\in$ N and every κ, supp(ρ_i^κ) is a singleton.

Proposition 3: (3.1) *The belief system ρ is a SIRE if it is proper, but the reverse implication can fail.*

(3.2) *Suppose the belief system ρ is pure. Then ρ is proper if and only if it is SIRE.*

(3.3) *If ρ is SIRE, then ρ_i^0 cannot attach positive weight to a strategy weakly dominated by another pure strategy.*

Proof: (3.1) That a proper equilibrium is SIRE follows immediately from $r_i \succeq_S s_i \Rightarrow r_i \succeq_L s_i$. The equilibrium (½M+½B,½C+½R) in (G1) is SIRE but not proper.

(3.2) Let ρ be pure. Then two strategies have equal expected values under ρ_{-i}^κ if and only if they are structurally indifferent on the support of ρ_{-i}^κ, so that $r_i \succeq_L s_i \Leftrightarrow r_i \succeq_S s_i$.

(3.3) Suppose s_i is weakly dominated by r_i. If the domination is strict, then s_i must yield strictly lower payoffs than r_i, so s_i cannot receive positive weight in ρ_i^0. So, suppose s_i and r_i yield the same payoffs on some $X_{-i} \subseteq S_{-i}$. If the support of ρ_{-i}^0 is not a subset of X_{-i}, then s_i has a strictly lower payoff than r_i, so s_i cannot receive positive weight in ρ_i^0. Finally, if the support of ρ_{-i}^0 is a subset of X_{-i}, then r_i and s_i are structurally indifferent on the support of ρ^0_{-i} and the two strategies are compared under ρ_{-i}^1. Either s_i has a strictly lower payoff than r_i under ρ^1_{-i} or they have the same payoff, in which case r_i and s_i must be structurally indifferent on the support of ρ_{-i}^1. But now, r_i and s_i are compared under ρ^2_{-i}, and so on. ■

(6.3) Lexicographic Conditional Probability Systems

LCPSs are a particularly attractive notion of alternative theories. One of the advantages of SIRE is that is that any SIRE generated by an LPS can also be generated by an LCPS. In contrast, BBDb use the game (G4) to show that the restriction to LCPSs precludes both a characterization of properness and existence of lexicographic Nash equilibrium (when the LCPSs must respect $\succeq_L$ and have full support).

		2		
		L	C	R
1	T	1, −1	−1,1	1, −3
	B	−1,1	1, −1	0,2

(G4)

This game has a unique Nash (and hence trembling hand perfect and proper) equilibrium: $((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2},0))$. Player 2's beliefs about player 1 are completely described by 1's equilibrium distribution. A player 1 LPS ρ_2 that is consistent with the equilibrium play and respects $\geq_L$ is $\rho_2^0 = (\frac{1}{2},\frac{1}{2},0)$ and $\rho_2^1 = (0,\frac{1}{3},\frac{2}{3})$. Note that ρ^0 and ρ^1 have overlapping supports. An LCPS must specify (0,0,1) at the second level, forcing player 1 to strictly prefer (under $\geq_L$) T to B, which is inconsistent with equilibrium. SIRE, on the other hand, would not force player 1 to evaluate the indifference between T and B since T and B are not structurally indifferent on {L,C}.

To see that the restriction to LCPSs is without loss of generality as far as SIRE is concerned, define the following:

Definition 15: Denote by $\hat{\rho}$ the LCPS obtained from the LPS ρ by defining $\hat{\rho}^{\kappa}_{-i}(.) = \rho^{\kappa}_{-i}(.\,|S_{-i}\backslash \cup_{\ell<\kappa}\text{supp}(\rho^{\ell}_{-i}))$ for all κ.

Note that $\hat{\rho}^0 = \rho^0$, so that the LPS and LCPS must agree at the first level.

Proposition 4: *The LPS ρ is a SIRE if and only if the LCPS $\hat{\rho}$ implied by ρ is a SIRE.*

Proof: Suppose $r_i \geq_S s_i$ and suppose r_i and s_i are structural indifferent on supp(ρ^{κ}_{-i}) for $\kappa < k$, and are not structurally indifferent on supp(ρ^{k}_{-i}). Then the sign of $\sum \rho^{\kappa}_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})]$ equals the sign of $\sum \hat{\rho}^{\kappa}_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})]$ for $\kappa \leq k$. Thus, the ordering $\geq_S$ implied by ρ_{-i} agrees with that implied by $\hat{\rho}_{-i}$. ■

7. Trembles

It is common to explain out-of-equilibrium behavior, both in normal and extensive form games, in terms of trembles. BBDb show that the lexicographic order can be formulated in terms of sequences of probability distributions or trembles, in the sense that the lexicographic order induced by an LPS is identical to the ordering by expected payoffs induced by a sequence of trembles that "converges" to the

LPS in a certain sense (Lemma 2 below). This section introduces an alternative notion of convergence of a sequence of trembles to an LPS. We then use this convergence notion to show that the structural order can also be formulated in terms of sequences of trembles. We restrict attention throughout to lexicographic probability systems satisfying the common prior, strong independence, and full support assumptions.

(7.1) Probability Sequences

A *probability sequence*, P, is a collection of *independent* probability distributions $\{\{P_i^n\}_{n=1}^{\infty}, i\in N\}$, such that each P_i^n is a completely mixed probability distribution on S_i. Given a probability sequence P, we define P^n and P_{-i}^n in the obvious manner, i.e., $P^n(s) = \Pi_{i\in N}P_i^n(s_i)$ and $P_{-i}^n(s_{-i}) = \Pi_{j\neq i}P_j^n(s_j)$. Let $P_{-i} \equiv \{P_{-i}^n\}$ and $P_i \equiv \{P_i^n\}$.

We now introduce two senses in which a probability sequence can be equivalent to an LPS.

Definition 16: The LPS ρ_{-i} on S_{-i} and probability sequence P_{-i} are *limit equivalent* if for all $i\in N$, $s_{-i}, t_{-i}\in S_{-i}$,

$$s_{-i} >_{\rho_{-i}} t_{-i} \Leftrightarrow \lim_{n\to\infty} \frac{P_{-i}^n(t_{-i})}{P_{-i}^n(s_{-i})} = 0$$

and

$$P_{-i}^n|_{A^\kappa} \to \rho_{-i}^\kappa|_{A^\kappa}, \quad \forall\kappa,$$

where $A^0 = \mathrm{supp}(\rho_{-i}^0)$ and $A^\kappa = \mathrm{supp}(\rho_{-i}^\kappa)\backslash\cup_{\ell<\kappa}A^\ell$. If P_{-i} is limit equivalent to ρ_{-i} for all i, then P is *limit equivalent* to ρ, written $P \approx_\lambda \rho$.

Note that a probability sequence can be limit equivalent to many different LPSs, but to only one LCPS. In fact, a probability sequence is limit equivalent to an LCPS if and only if it is limit equivalent to *every* LPS implying that LCPS (see Lemma 1 below). This is important because, as we saw in Proposition 4, SIRE can be defined using LCPSs.

The second notion of equivalence is the one used by BBDb:

Definition 17: The LPS ρ on S and probability sequence P are *tail equivalent*, written $P \approx_\tau \rho$, if there exists n' and a sequence $\{r(n)\}$ with $r(n)\in(0,1)^{K-1}$ and $r(n)\to 0$ such that $P^n = r(n)\square\rho$ for $n \geq n'$.

The proof of the following is obvious from the definitions, and describes the relationship between limit and tail equivalence for LPSs and LCPSs:[12]

Lemma 1: (1.1) *A probability sequence* P *is limit equivalent to the LPS* ρ *if and only if* P *is limit equivalent to the LCPS* $\hat{\rho}$ *implied by* ρ.

(1.2) *Tail equivalence implies limit equivalence.*

(1.3) *A probability sequence* P *is tail equivalent to a LCPS* $\hat{\rho}$ *if and only if* P *is limit equivalent to* $\hat{\rho}$ *and there exists* n' *such that for all* n $\geq$ n',

$$P^{n}_{-i}|_{A^{\kappa}} = \hat{\rho}^{\kappa}_{-i}|_{A^{\kappa}}, \quad \forall \kappa,$$

where $A^{\kappa} = \mathrm{supp}(\hat{\rho}^{\kappa}_{-i})$.

Example 3: To illustrate the differences between limit and tail equivalence, consider again Game (G4). Suppose P^n_2 = ((n−4)/(2n), (n+2)/(2n), 1/n). Now, $\{P^n_2\}$ is limit equivalent to any LPS (ρ^0_2,ρ^1_2) = ((½,½,0), (p,q,r)) for which r $\neq$ 0. In particular, P^n_2 is limit equivalent to both ((½,½,0), (0,0,1)) and ((½,½,0), (0,¾,¼)). On the other hand, P^n_2 is tail equivalent to ((½,½,0),((0,¾,¼)) but not to ((½,½,0), (0,0,1)). The expected payoff to T under P^n_2 is less than the expected payoff to B for all n $\geq$ 4, and reflecting this, B $\succ_L$ T under the LPS ((½,½,0), (0,¾,¼)) (see Lemma 2 below). However, T and B are not comparable according to $\succeq_S$. Consider now the sequence of probability distributions Q^n_2 = ((n+2)/(2n), (n−4)/(2n), 1/n). While the sequence $\{Q^n_2\}$ is also limit equivalent to any LPS (ρ^0_2,ρ^1_2) = ((½,½,0), (p,q,r)) for which r $\neq$ 0, it is tail equivalent to ((½,½,0),((¾,0,¼)) but not to ((½,½,0), (0,0,1)). The expected payoff to T under Q^n_2 is greater than the expected payoff to B for all n $\geq$ 4, and reflecting this, T $\succ_L$ B under the LPS ((½,½,0), (¾,0,¼)). As Proposition 5 will make clear, the existence of two probability sequences both limit equivalent to the LPS ((½,½,0), (0,0,1)) implying different rankings over strategies is central to the incomparability of the two strategies under $\succeq_S$. ■

(7.2) Lexicographic Order

The following proposition (a minor extension of BBDb (Proposition 1)) indicates that if P is tail equivalent to ρ, then P precisely captures the lexicographic order on the strategy spaces.

Lemma 2: *Suppose* ρ *is an LPS satisfying the common prior, strong independence, and full support assumptions. Suppose* P *is tail equivalent to* ρ. *Then there exists* n* *such that, for all* i,

[12]Since $\hat{\rho}^{\kappa}$ is an LCPS, the A^{κ} in Lemma 1 are as defined in Definition 16.

$$r_i \succ_L s_i \textit{ if and only if } \sum P^n(s_{-i})\pi_i(r_i,s_{-i}) > \sum P^n(s_{-i})\pi_i(s_i,s_{-i}) \textit{ for all } n > n^*.$$

Proof: The proof of Proposition 1 in BBDb applies here with the modification that (using their notation) n^* is chosen so that $n > n^*$ implies $r^k(n) < r^*\ \forall n > n^*$, where r^* solves $(1-r^*)B + r^*W > 0$. ■

(7.3) Structural Order

A lexicographic order induced by a lexicographic probability system ρ ranks r_i ahead of s_i if and only if r_i receives a higher payoff than s_i along some probability sequence that is tail equivalent to ρ. This implies that if r_i receives a higher payoff than s_i along *some* probability sequence that is tail equivalent to ρ, then r_i receives a higher payoff than s_i along *every* probability sequence that is tail equivalent to ρ. Tail equivalence thus places significant structure on a probability sequence. Our next proposition shows that a result similar to Lemma 2 holds for the structural order and limit equivalence. Hence, we can formulate the structural order entirely in terms of sequences of trembles, with no distinctions between types of indifference appearing in this formulation. The distinction between structural and nonstructural indifferences arises naturally out of the notion of limit equivalence.

There is an important difference, however, between tail and limit equivalence, and correspondingly between the lexicographic and structural orders. In contrast to tail equivalence, two probability sequences that are limit equivalent to the same LPS ρ can disagree in their expected payoff rankings of strategies (as illustrated in example 3). The next proposition states that the structural order induced by an LPS ρ ranks r_i ahead of s_i if and only if r_i receives a higher payoff than s_i along *every* probability sequence that is limit equivalent to ρ. Note that, unlike for the lexicographic order and tail equivalence, *every* cannot be replaced by *some*. While tail equivalence implies limit equivalence, there are probability sequences that are limit but not tail equivalent to ρ, and the structural order requires more stringent conditions than the lexicographic order to rank strategies. The structural order can then decline to rank strategies that are ranked under the lexicographic order.

Proposition 5: *Suppose ρ is an LPS satisfying the common prior, strong independence, and full support assumptions. Then for all* i, $r_i \succ_S s_i$ *if and only if for all* $P \approx_\lambda \rho$ *there exists* n^* *such that*

$$\sum P^n(s_{-i})\pi_i(r_i,s_{-i}) > \sum P^n(s_{-i})\pi_i(s_i,s_{-i}) \textit{ for all } n > n^*.$$

Proof: The result is trivial for r_i and s_i structurally indifferent on S_{-i}. So suppose r_i and s_i are not structurally indifferent on S_{-i}. Let k be the largest index satisfying: r_i and s_i are structurally indifferent on $\mathrm{supp}(\rho^\kappa_{-i})$ for $\kappa < k$. Let $A^0 = \mathrm{supp}(\rho^0_{-i})$, $A^\kappa = \mathrm{supp}(\rho^\kappa_{-i}) \setminus \cup_{\ell<\kappa} A^\ell$, for all κ.

($\Rightarrow$) Suppose $P \approx_\lambda \rho$ and $r_i \succ_S s_i$. Then r_i has a strictly higher expected payoff than s_i under ρ^k_{-i}. Define $C = \sum \rho^k_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})]$ and $B = \max|\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})|$. Note that $A^k \neq \varnothing$, so that $\epsilon(n) = P^n_{-i}(A^k)/\rho^k_{-i}(A^k)$ is well defined. Since P^n is completely mixed, $\epsilon(n) \neq 0$. Choose n' so that for $n > n'$ and for all $s_{-i} \notin \cup_{\kappa\leq k}\mathrm{supp}(\rho^\kappa_{-i})$, $P^n_{-i}(s_{-i}) < C\epsilon(n)/(3B|S_{-i}|)$. Since $\rho \approx_\lambda P$, there is an n'' such that for $n > n''$,

$$\Big|\sum_{s_{-i}\in A^k} \rho^k_{-i}(s_{-i})[\pi_i(r_i,s_{-i})-\pi_i(s_i,s_{-i})] - (\epsilon(n))^{-1}\sum_{s_{-i}\in A^k} P^n_{-i}(s_{-i})[\pi_i(r_i,s_{-i})-\pi_i(s_i,s_{-i})]\Big| < C/3.$$

Set $n^* = \max\{n',n''\}$. Since r_i and s_i are structurally indifferent on $\mathrm{supp}(\rho^\kappa_{-i})$ for $\kappa < k$,

$$\sum P^n_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] = \sum_{s_{-i}\notin\bigcup_{\kappa<k}\mathrm{supp}(\rho^\kappa_{-i})} P^n_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})]$$

$$= \sum_{s_{-i}\in A^k} P^n_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] + \sum_{s_{-i}\notin\bigcup_{\kappa\leq k}\mathrm{supp}(\rho^\kappa_{-i})} P^n_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})]$$

$$> \sum_{s_{-i}\in A^k} P^n_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] - C\epsilon(n)/3$$

$$> \epsilon(n)(C - C/3 - C/3) = \epsilon(n)C/3 > 0.$$

($\Leftarrow$) Suppose for all $P \approx_\lambda \rho$ and for n sufficiently large,

$$\sum P^n_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] > 0. \tag{1}$$

We suppose $r_i \succ_S s_i$ does not hold and derive a contradiction. If $r_i \succ_S s_i$ does not hold, then:

$$\begin{aligned}&\sum \rho^\kappa_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] = 0 \quad \text{for all } \kappa < k, \text{ and}\\ &\sum \rho^k_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] \leq 0.\end{aligned} \tag{2}$$

The definition of k and (1) implies:

$$\sum_{s_{-i}\in A^k} P^n_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] + \sum_{s_{-i}\notin\bigcup_{\kappa\leq k}\mathrm{supp}(\rho^\kappa_{-i})} P^n_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] > 0.$$

Dividing by $P^n_{-i}(A^k)$ and taking limits yields

$$\sum_{s_{-i}\in A^k} \rho^k_{-i}(s_{-i})[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] \geq 0.$$

Combining with (2) yields

$$\sum_{s_{-i}\in A^k} \rho^k_{-i}(s_{-i})[\pi_i(r_i,s_i)-\pi_i(s_i,s_{-i})] = 0. \tag{3}$$

We now argue that there exists a probability sequence Q that is limit equivalent to ρ but reverses the inequality in (1), which is a contradiction.

Define $k_j(s_j) \equiv \min\{\kappa : \rho_j^\kappa(s_j) \neq 0\}$ and $k_{-i}(s_{-i}) \equiv \min\{\kappa : \rho_{-i}^\kappa(s_{-i}) \neq 0\}$. Note that $k = k_{-i}(s_{-i})$ $\forall s_{-i} \in A^k$. Since ρ is strongly independent, there exists $r(n) \in (0,1)^{K-1}$, $r(n) \to 0$ as $n \to \infty$, such that $r(n)\Box\rho = \Pi_i(r(n)\Box\rho)_i = \Pi_i(r(n)\Box\rho_i)$. Fixing $s_{-i} \in S_{-i}$, and letting $k_j = k_j(s_j)$, $k_{-i} = k_{-i}(s_{-i})$, we have

$$r^1(n)\cdots r^{k_{-i}}(n)\,\rho_{-i}^{k_{-i}}(s_{-i}) = \prod_{j\neq i} r^1(n)\cdots r^{k_j}(n)\Big((1-r^{k_j+1}(n))\,\rho_j^{k_j}(s_j) + r^{k_j+1}(n)[\ldots$$
$$+\, r^{K-2}(n)[(1-r^{K-1})\rho_j^{K-2}(s_j) + r^{K-1}\rho_j^{K-1}(s_j)]\ldots]\Big).$$

(We follow the convention that $r^1(n)\cdots r^0(n) = 1$.) Dividing both sides by $r^1(n)\cdots r^{k_{-i}}(n)$ and taking n to infinity shows that

$$\rho_{-i}^{k_{-i}}(s_{-i}) = \alpha(k^{-i}, k_{-i}) \times \prod_{j\neq i} \rho_j^{k_j}(s_j), \tag{4}$$

where k^{-i} is the vector $(k_j)_{j\neq i}$ and

$$\alpha(k^{-i}, k_{-i}) = \lim_{n\to\infty} \left[\frac{\prod_{j\neq i} r^1(n)\cdots r^{k_j}(n)}{r^1(n)\cdots r^{k_{-i}}(n)} \right] \neq 0.$$

Let $A_j^k \equiv \{s_j \in S_j : (s_j, s_{-\{i,j\}}) \in A^k$ for some $s_{-\{i,j\}}\}$. Define $u(s_{-i}) \equiv \alpha(k^{-i}(s_{-i}), k_{-i}(s_{-i})) \times [\pi_i(r_i, s_{-i}) - \pi_i(s_i, s_{-i})]$ and consider the function $\Phi : \Pi_{j\neq i}\Re_+^{A_j^k} \to \Re$, given by, for $p_{-i} = (p_j)_{j\neq i}$, $p_j \in \Re_+^{A_j^k}$:

$$\Phi(p_{-i}) = \sum_{s_{-i}\in A^k} \Big(\prod_{j\neq i} p_j(s_j)\Big) u(s_{-i}).$$

From (3) and (4), $\Phi(p_{-i}^*) = 0$, where $p_j^*(s_j) = \rho_j^{k_j(s_j)}(s_j)$. Since $u(s_{-i}) \neq 0$ for at least one $s_{-i} \in A^k$, Φ is not identically zero on any neighborhood of p^*. Fix $\delta > 0$ and p^0 such that $|p^* - p^0| < \delta$ and $\Phi(p^0) \neq 0$. Define $p^j \equiv (p_1^*, \ldots, p_j^*, p_{j+1}^0, \ldots, p_N^0)$, for $j = 1, \ldots, N$, so that $p^N \equiv p^*$. Let j' be first index such that $\Phi(p^{j'}) = 0$. Since $\Phi((1-\lambda)p^{j'-1} + \lambda p^{j'})$ is affine in λ and $\Phi(p^{j'-1}) \neq 0$, there is a $\lambda \in \Re$ such that $p^\delta = (1-\lambda)p^{j'-1} + \lambda p^{j'}$ satisfies $\Phi(p^\delta) < 0$ and $|p^* - p^\delta| < \delta$. Since $p_j^*(s_j) \neq 0$, $p_j^\delta(s_j) \neq 0$ for all $s_j \in A_j^k$ and j. Setting

$$\beta_j = \left[\sum_{\{s_j\in A_j^k : k_j(s_j)=0\}} p_j^\delta(s_j)\right]^{-1} \times \left[\sum_{\{s_j\in A_j^k : k_j(s_j)=0\}} \rho_j^0(s_j)\right],$$

we have $\Phi(\beta_1 p_1^\delta, \ldots, \beta_N p_N^\delta) \equiv -\eta_\delta < 0$, and $\beta_j \to 1$ as $\delta \to 0$.

We now define a LPS for player j, $\sigma_j(.;\delta)$, as follows. First set $\sigma_j^{k_j(s_j)}(s_j;\delta) = \beta_j p_j^{\delta}(s_j)$, and $\sigma_j^{\kappa}(s_j;\delta) = \rho_j^{\kappa}(s_j)$ for $\kappa \neq k_j(s_j)$. By construction, $\Sigma\sigma_j^0(s_j;\delta) = 1$. If $\Sigma\sigma_j^{\kappa}(s_j;\delta) \neq 1$ for some $\kappa \geq 1$, then adjust $\sigma_j^{\kappa}(s_j;\delta)$ for s_j with $k_j(s_j) = 0$ so that $\sigma_j^{\kappa}(.;\delta)$ is a probability distribution. The necessary adjustment is feasible for δ small.

Consider, for fixed δ, the probability sequence $\{P_j^n(.;\delta)\}$ given by $P_j^n(.;\delta) = r(n)\square\sigma(.;\delta)$. We now argue that for n sufficiently large,

$$\sum P_{-i}^n(s_{-i};\delta)[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] < 0. \tag{5}$$

This is clearly equivalent to:

$$\sum (r^1(n)\cdots r^k(n))^{-1} P_{-i}^n(s_{-i};\delta)[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] < 0.$$

Using the fact that $k_j(s_j) \equiv \min\{\kappa:\rho_j^{\kappa}(s_j) \neq 0\} = \min\{\kappa: \sigma_j^{\kappa}(s_j) \neq 0\}$ and the definition of $P_j^n(.;\delta)$, the left hand side converges to

$$\sum_{s_{-i}\in A^k} \left[\prod_{j\neq i} \sigma_j^{k_j(s_j)}(s_j)\right] \alpha(k^{-i}(s_{-i}),k_{-i}(s_{-i}))[\pi_i(r_i,s_{-i}) - \pi_i(s_i,s_{-i})] = -\eta_\delta < 0,$$

and so (5) holds for n sufficiently large.

Let $Q_j^m = P_j^{n(m)}(.;1/m)$, where $\{n(m)\}_{m\uparrow\infty}$ is an increasing sequence with the property that (5) holds when $\delta = 1/m$ and $n = n(m)$. It is immediate that $\{Q_j^m\}$ is limit equivalent to ρ and reverses the inequality in (1). This is the desired contradiction and so $r_i \succ_S s_i$. ■

8. Conclusion

The motivation for this paper was a suspicion that all indifferences are not created equal, in the sense that players might be "more indifferent" in cases of structural indifference. We have pursued this by examining the structural order and structural indifference respecting equilibrium, each of which requires that structural indifferences *only* be broken by appealing to higher order beliefs. We regard the properties of SIRE, such as its relationship to sequential equilibrium and the ability to work with lexicographic *conditional* probability systems, as an indication that structural indifferences embody interesting phenomena.

We close with a final example illustrating the distinction between $\succeq_S$ and the order implied by a particular SIRE. (Notice that any SIRE implicitly involves such an order.) Consider the following game:

2

$.5^0$ $.5^0$ 1^1

			L	C	R
	$.5^2$	A	1,1	0,0	0,0
	$.5^2$	B	0,0	1,1	1,0
1	1^1	C	1,1	0,1	3,2
	1^0	D	0,1	1,1	2,0

Let ρ be as indicated, so that

$$\rho_1^0(D) = 1, \qquad p_2^0(L) = \rho_2^0(C) = .5,$$
$$\rho_1^1(C) = 1, \qquad \rho_2^1(R) = 1, \text{ and}$$
$$\rho_1^2(B) = \rho_1^2(A) = .5.$$

Then (D, .5L+.5C) is a SIRE, with

$$C \succ_S A,$$
$$D \succ_S B,$$
$$L \sim_S C \succ_S R.$$

The choices embodied in the SIRE suggest completing the order as:

$$D \succeq_S^{\cdot} C \succ_S^{\cdot} B \sim_S^{\cdot} A.$$

However, it is easy to verify that no probability sequence limit equivalent to ρ can produce this completed order, nor can any sequence of tie-preserving payoff perturbations. Any such probability sequence, for example, must attach higher probability to C than to L, in order for the payoff to D to at least equal that of C, but then B must receive a higher payoff than A, preventing $A \sim_S^{\cdot} B$.

We take this example as an indication that one should not look to trembles to provide a complete description of play. Instead, we view trembles as establishing necessary conditions for a theory of play, but as leaving unresolved some issues for which one must then turn to auxiliary considerations, much as does any theory that fails to yield a unique prediction on all games. The structural order thus readily groups strategies into levels of likelihood, but is not always specific as to precise probabilities within levels. Given this lack of specificity, it is comforting that the structural order is robust to all (limit equivalent) strategy perturbations. This suggests that the structural order successfully captures the robust implications of any theory of opponents' out-of-equilibrium play.

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