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NORMAL FORM STRUCTURES IN EXTENSIVE FORM GAMES

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Abstract

Mailath, Samuelson, and Swinkels (1992) introduce the *normal form information set.* Normal form information sets capture situations in which players can make certain decisions *as if* they knew their opponents' had chosen from a particular subset of their strategies. In this paper, we say that an extensive form game *represents* a normal form game if, for each such situation, the corresponding choice in the extensive form is made with the player *knowing* that the opponents have chosen from the relevant subset. We show that normal form games exist that cannot be represented. We develop an algorithm that generates a representation whenever one exists and present a necessary and sufficient condition for a normal form game to be representable.

Keywords: representation, extensive form game, normal form game, information set, extensive form--normal form equivalence.

JEL **classification numbers:** C70, C72.

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I. Introduction

Introductions to game theory typically begin with the concept of a normal (or strategic) form game. It is then observed that a more detailed representation of the strategic interaction is available, the extensive form. For example, information sets in the extensive form provide information about when players move and what they know when they move. Those who have argued for confining analysis to the normal form, such as Kohlberg and Mertens (1986), take it for granted that the extensive form provides more information about interactions than the normal form, but argue that this extra information is strategically irrelevant.

In Mailath, Samuelson, and Swinkels (1992) (hereafter MSS), we showed that the normal form provides a detailed description of the information structure of a strategic interaction. We argued that the distinguishing feature of extensive form information sets is strategic independence: it is not that the choice of an action at an information set need not be made until that information set is reached that is important, but rather that the choice of action, *whenever taken*, "matters" (i.e., affects the outcome of the game) only if that information set is reached. We used this idea of strategic independence to define a pure strategy reduced normal form structure called the *normal form information set.*[1] Roughly, a normal form information set for player i is a subset of player i's strategies and a subset of the other players' strategies with the property that i's choice from his subset "matters" only if the other players choose from their subset.

For any single normal form information set of a reduced normal form, there is an extensive form game with that reduced normal form and a corresponding extensive form information set (MSS, Theorem 1). At this extensive form information set, the player has explicitly available to him the information that his opponents are playing from the relevant subset of their strategy profiles. However, it is not always possible to represent *all* normal form information sets, and hence the informational structure of the normal form, in a single extensive form game (MSS Example 7).

MSS raise the question of when such a representation is possible. That is the topic of this paper. If such a representation exists, then we can use the analytically convenient extensive form. If no such representation exists, then it may be necessary to confine analysis to the normal form, not because the extensive form contains irrelevant information, but because it conceals information.

Our basic notion of a representing extensive form can be described as follows: *If a particular choice between normal form strategies does not matter unless the opponents are playing within some subset of their strategy profiles, then in a representing extensive form the player can defer that choice*

[1]The (pure strategy) reduced normal form of a game is obtained by deleting, for each player, any pure strategy that is a duplicate of another pure strategy.

representation algorithm. Section VI defines the condition of generalized nested or disjoint (GND) and proves the **Representability Theorem**: *a normal form is representable if and only if it satisfies GND, which holds if and only if the representation algorithm is successful.* Section VII provides conditions under which the representing extensive form is "generic," in the sense of having distinct payoff vectors at different terminal nodes. Section VIII contains some remarks on the playability of representing extensive forms. Proofs are in the appendices.

II. Preliminaries

We denote the set of players by $N \equiv \{1,...,n\}$ and player i's pure strategy space by S_i, $i = 1,...,n$. The set of strategy profiles is given by $S \equiv S_1 \times ... \times S_n$. Player i's payoff function is written $\pi_i: S \rightarrow \Re$, with $\pi \equiv (\pi_1,...,\pi_n)$. A set of strategy profiles S and a payoff function π determine the normal form game (S,π). A subset of player i's strategy space will often be written X_i. Typical pure strategies for player i are denoted r_i, s_i, and t_i. As usual, a subscript $-i$ denotes $N\backslash\{i\}$ and a subscript $-I$ denotes $N\backslash I$. We reserve $\subset$ and $\supset$ for proper containment, using $\subseteq$ and $\supseteq$ when there is possible equality between sets.

Definition 1: *Two strategies* s_i, t_i *<u>agree on</u>* X_{-i} *if* $\pi(s_i,s_{-i}) = \pi(t_i,s_{-i})$ $\forall s_{-i} \in X_{-i}$. *The set* $X_i \subseteq S_i$ *agrees on* X_{-i} *if every* s_i, $t_i \in X_i$ *agree on* X_{-i}. *The normal form game* (S,π) *is a <u>pure strategy reduced normal form game (PRNF)</u> if for all* i, *no strategy* $s_i \in S_i$ *agrees with any element of* $S_i\backslash\{s_i\}$ *on* S_{-i}.

The PRNF of a normal form game (S',π') is that PRNF (S,π) in which each equivalence class of strategies in S'_i defined by agreement on S'_{-i} is represented by a single strategy $s_i \in S_i$. If $\sigma_i \in S'_i$ is contained in the equivalence class $s_i \in S_i$, we write $\sigma_i \in s_i$. We do not distinguish between PRNFs that differ only in the strategy labels.

The strategy space of the extensive form game Γ is denoted S^Γ. We will use Greek letters for pure strategies in S^Γ. If (S,π) is the PRNF of Γ, the normal form of Γ can be written as (S^Γ,π), where $\pi(\sigma_1,...,\sigma_n) = \pi(s_1,...,s_n)$ for $\sigma_i \in s_i$. Define H_i to be the set of information sets in Γ belonging to player i. For an information set h of Γ, denote the set of strategies in S^Γ consistent with reaching h by $S^\Gamma(h)$. Define $S(h) \equiv \{s \in S: \exists\sigma \in S^\Gamma(h) \text{ s.t. } \sigma_i \in s_i\ \forall i\}$. The sets $S^\Gamma(Y)$ and $S(Y)$ are defined analogously for arbitrary subsets Y of nodes of Γ. Denoting the set of player i's normal form strategies consistent with reaching h and which specify the action a at h by $S_i^\Gamma(h,a)$, then $S_i(h,a) \equiv \{s_i \in S_i: \exists\sigma_i \in S_i^\Gamma(h,a) \text{ s.t. } \sigma_i \in s_i\}$. All other extensive form notation is from Kreps and Wilson (1982). We restrict attention to finite of beliefs over $S_{-i}\backslash X_{-i}$. We can thus think of player i's choice in X_i as one of choosing behavior on X_{-i} and behavior off X_{-i} independently.

MSS prove the following:

Theorem 0: *The strategy subset* X *of the PRNF* (S,π) *is a normal form information set for player* i *if and only if there exists an extensive form game with PRNF* (S,π) *with an information set* h *for player* i *such that* $S(h) = X$.

Thus, for any single normal form information set, there is an extensive form which has a corresponding extensive form information set. In the next section, we analyze an example which shows that this need not be true for *collections* of normal form information sets.

III. A Game With No Representation

We begin with an example showing that even with a very unrestrictive notion of representation, and a very simple normal form game, representability can fail. It turns out that every normal form game that cannot be represented contains a structure similar to this game.

Abreu and Pearce (1986), in a discussion of axiomatic foundations of solution concepts, introduce the following game:

		2: L	R
1	T	a	a
	B	a	b

(G1)

We first look for choices where a player can say, "this choice will not matter unless my opponents are playing within some subset of their strategy profiles." The relevant such choices are captured by the normal form information sets $X = \{T,B\}\times\{R\}$ for player 1 and $Y = \{B\}\times\{L,R\}$ for player 2.

It is easy to find an extensive form game that represents X; i.e., a game in which, at the time player 1 chooses between T and B, he knows that player 2 has not chosen L (the first tree in Figure 1). It is similarly easy to find an extensive form game which represents Y (the second tree in Figure 1).

At a minimum, for a single extensive form to capture both players' strategic independences, and hence represent (G1), normal form information sets X and Y would both have to be represented. Neither of the trees in Figure 1 accomplishes this. The first extensive form in Figure 1 contains an information set at which player 2 chooses between L and R, but this information set does not give player 2 the

If new definitions restrict attention to information that is explicitly *known* to players, then we believe that the impossibility result will survive: Each player must have made his choice before the other can know that his choice matters. A redefinition may allow (G1) to be represented by introducing "as if" considerations, but we think the normal form information set and subgame already do this in the appropriate way. We explore representability precisely to determine when the "as if" knowledge of normal form strategic independences can be transformed into explicit knowledge in the extensive form.

Finally, note that the game (G1) is not rectangular (i.e., $\pi(T,L) = \pi(B,L) = \pi(T,R)$, but $\pi(B,L) \neq \pi(B,R)$). Rectangularity is an important feature of PRNFs of extensive form games with perfect information (Gurvič (1982)).[3] An example showing that rectangularity is not sufficient for representability is provided by the following:

$$\begin{array}{ccc} & 2 & \\ & \begin{array}{c|c|c|} & \ell & r \\ \hline T & a & b \\ \hline B & c & c \\ \hline \end{array} & \\ 1 & & \\ & L & \end{array} \qquad \begin{array}{ccc} & 2 & \\ & \begin{array}{c|c|c|} & \ell & r \\ \hline T & d & b \\ \hline B & d & e \\ \hline \end{array} & \\ 1 & & \\ & R & \end{array} \qquad (G2)$$

$$3$$

The relevant information sets are $S_1 \times \{\ell L, Rl, Rr\}$ for player 1, $S_2 \times \{TL, TR, BR\}$ for player 2, and $S_3 \times \{T\ell, B\ell, Br\}$ for player 3. The game is not representable (Theorem 6). This game has the flavor of a three player Abreu-Pearce game, in that each player needs *both* of the others to have moved before he knows whether his information set has been reached.

IV. Representability

Not all extensive forms with a given pure strategy reduced normal form (PRNF) represent that PRNF. A representing extensive form must also capture the information structure (implied by the normal form information sets) of the PRNF.[4] In this section, we develop our notion of representability.

We begin with *weak representability*: a player *can* always defer a decision in the extensive form until he explicitly knows all the information that he could act as if he knew in the normal form (this will

[3] We thank Hervé Moulin for drawing our attention to Gurvič (1982).

[4] This interest in the informational structure of the normal form separates our work from that of Thompson (1952), Dalkey (1953), and Elmes and Reny (1991), who examine extensive form transformations that yield new extensive form games without altering the PRNF.

is not *forced* to choose between the elements of X_i. This is captured by requiring that there is an action at h for the player that preserves all his options in X_i, i.e., an action a with $X_i \subseteq S(h,a)$.[5]

IV.2 Strong Representability

It is entirely possible in the situation just described that there is another action $a' \in A(h)$ such that $S(h,a')$ contains some but not all of X_i. Thus, under weak representability, we only give the player the *option* of deferring a decision between the elements of X_i. A stronger notion of represent would *require* that the player not choose between the elements of X_i until the extra information is available.

Definition 4: *Let* Γ *be an extensive form game with PRNF* (S,π). *We say* Γ *strongly represents* (S,π) *if for each* $h \in H_i$, *each* $X \in N_i$ *with* $X \subseteq S(h)$ *and* $X_{-i} \subset S_{-i}(h)$, *and for all* $a \in A(h)$, *either* $X_i \subseteq S_i(h,a)$ *or* $X_i \cap S_i(h,a) = \varnothing$.

Consider again an information set X at which a player can decide "as if" he has more information available than is given to him by h. We now require that when choosing at h, the player *cannot* choose between the elements of X_i. This is captured by requiring that every action a at h satisfy either $X_i \subseteq S_i(h_i,a)$ or $X_i \cap S_i(h_i,a) = \varnothing$. Every strong representation is a weak representation.[6]

IV.3 Informational Dominance

Strong representability is an attractive notion. Unfortunately, strong representations do not always exist. Moreover, in many cases where there is no strong representation, there is still a satisfactory weak representation:

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[5] We have motivated weak representability as not forcing players to make decisions with less information than they could assume to be true in the normal form. But suppose that the elements of X_i agree on $S_{-i}(h)$. Then, a decision over the elements of X_i cannot affect payoffs, and so it seems less compelling to require that $X_i \subseteq S(h,a)$ for some action at h. However, in this case the logic of the extensive form, independent of any representability considerations, implies that if $X_i \cap S_i(h,a) \neq \varnothing$, then $X_i \subseteq S_i(h,a)$ (see Lemma A4 in Appendix 2). Since $X_i \cap S_i(h,a) \neq \varnothing$ must hold for some a (because $X_i \subseteq S_i(h) = \cup_{a \in A(h)} S_i(h,a)$), the requirement of Definition 3 is automatically satisfied in this case.

[6] Let Γ be a strong representation. Let $h \in H_i$, $X \in N_i$ with $X \subseteq S(h)$ and $X_{-i} \subset S_{-i}(h)$. Then, since $\cup_{a \in A(h)} S_i(h,a) = S_i(h)$, there is $a \in A(h)$ with $X_i \cap S_i(h,a) \neq \varnothing$. But, then by strong representability, $X_i \subseteq S_i(h,a)$, and so Γ is a weak representation.

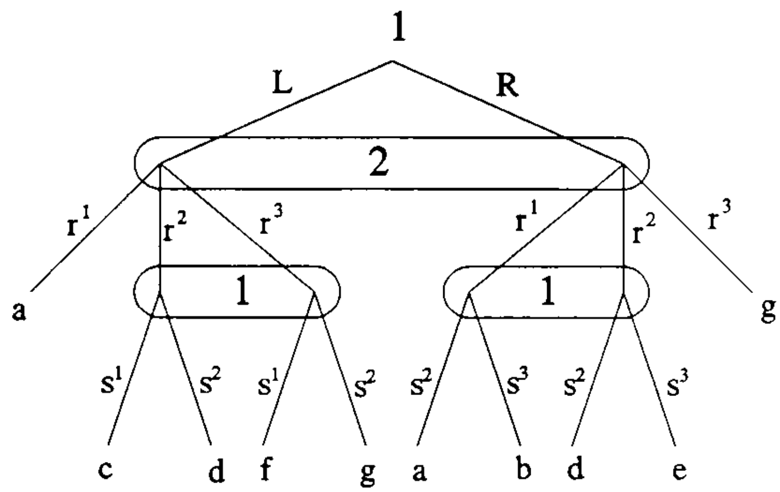

Figure 2—Note that L corresponds to $\{s^1,s^2\}$ and R to $\{s^2,s^3\}$.

strategies agree on a subset of $S_{-i}(h)$, for some $h \in H_i$, and there is an action at h that is consistent with at most one of the strategies. Game (G4) and the extensive form in Figure 3 suggest that this is not always desirable.

		2: r^1	2: r^2
1	s^1	a	d
	s^2	b	d
	s^3	c	d

(G4)

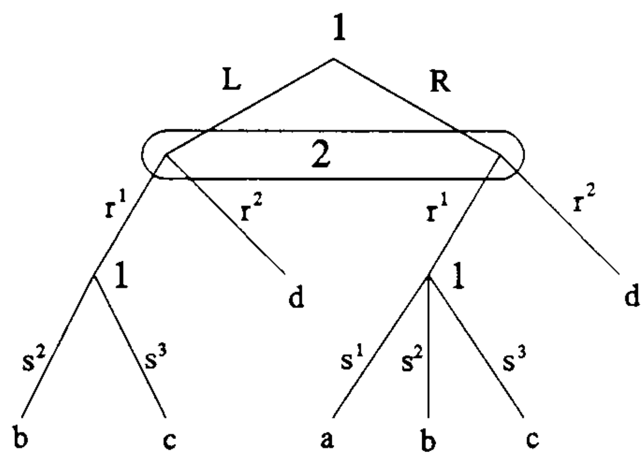

Figure 3—L corresponds to $\{s^2,s^3\}$ and R to $\{s^1,s^2,s^3\}$.

Let h be player 1's first information set in Figure 3. Then, $S(h) = S$. Thus, the choice of L gives up the alternative of s^1 before player 1 has all the information player 1 could have had (namely,

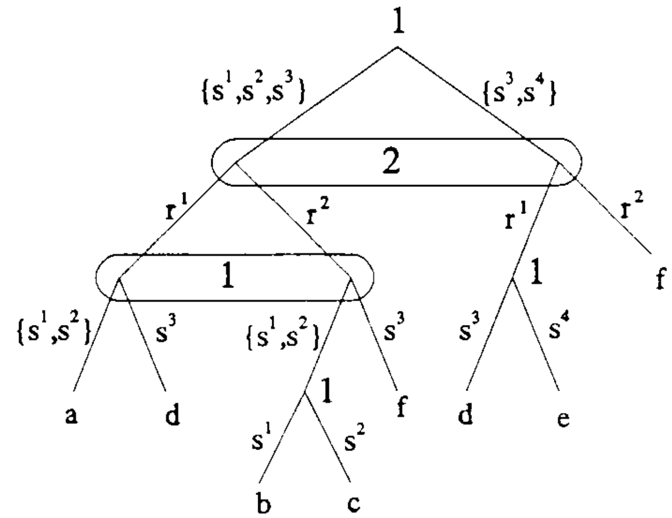

Figure 5

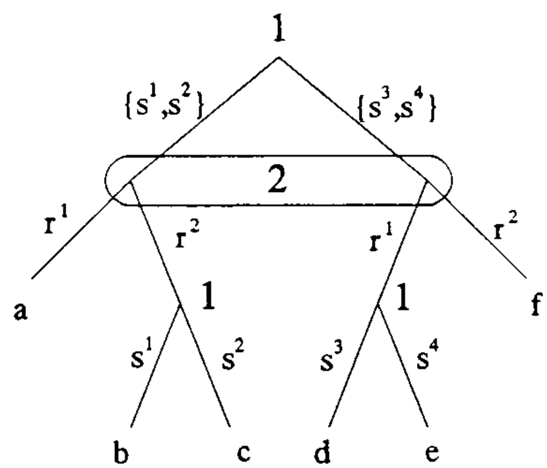

Figure 6

We formalize this with the following definition:

Definition 5: *Let* Γ *be an extensive form. Suppose* $h,h' \in H_i$, $h' \geq h$, $S^{\Gamma}_{-i}(h)=S^{\Gamma}_{-i}(h')$, *and* $a' \in A(h')$. *The sequence of actions ending in* a' *is* *<u>informationally dominated</u>* *if there is an action* a" *at* h *and* $K_i \subseteq S_i(h,a")$ *such that*

(D5.1) $S_i(h',a') \subset K_i$, *and*

(D5.2) *if* $\hat{h} \in H_i$ *is an information set following* a" *such that* $S_{-i}(h)=S_{-i}(\hat{h})$, *then there is an action* $\hat{a}$ *at* $\hat{h}$ *such that* $K_i \subseteq S_i(\hat{h},\hat{a})$.

not be a partition of Z_i. As we will see later, a PRNF is strongly representable if and only if it is weakly representable and $\Psi_i(Z)$ partitions Z_i for all i (Theorem 7).

Our discussion of (G4) and (G5) then suggests a representability notion in which one can never split apart elements of Ψ_i:

Definition 7: *Let* Γ *be a weak representation of* (S,π). *We say* Γ *parsimoniously represents* (S,π) *if for each* $h\in H_i$ *and each* $a\in A(h)$, $S_i(h,a)$ *is a union of elements of* $\Psi_i(S(h))$.

Figure 2, Figure 4, and Figure 6 are parsimonious representations. Figure 3 and Figure 5, in which informational dominances appear, are not. This is no coincidence:

Theorem 2: (T2.1) *A weak representation is a parsimonious representation if and only if it has no informational dominances.*

(T2.2) *If a game has a strong representation then any parsimonious representation of the game is also a strong representation.*

If both a weak but not parsimonious representation of a game and a parsimonious representation of a game exist, then the parsimonious representation is clearly preferable. In addition, we later prove that any time a game has a weak representation, then it also has a parsimonious representation (Theorems 3, 5, and 6). These observations suggest to us that parsimonious representability is the "right" notion.

The reader may wonder why our definition of parsimonious representation did not have each action at h equal to some element of $\Psi_i(S(h))$, and instead allowed for unions over elements of $\Psi_i(S(h))$. Consider the following game:

		2: L	2: R
1	T	3,3	3,3
	M	0,0	1,1
	B	4,2	2,0

(G6)

For this game, $\Psi_1(S)=\{\{T\},\{M\},\{B\}\}$. Thus, if we required 1's choices at S to be elements of $\Psi_1(S)$, the only allowable parsimonious representation (with no redundant strategies) would be Figure 7. There are, however, many orders in which player 1 could decide among {T,M,B}. For example, player 1 may first decide whether to use T, and then, if T is not chosen, decide between M and B, giving the

		2			
		r^1	r^2	r^3	r^4
	s^1	a	b	d	f
	s^2	a	b	e	f
1	s^3	c	c	d	f
	s^4	c	c	e	f

(G7)

Among player 1's normal form information sets are $S_1 \times S_2$, $S_1 \times \{r^1,r^2,r^3\}$, $S_1 \times \{r^1,r^2\}$, and $S_1 \times \{r^3\}$. The information set $S_1 \times S_2$ ignores the fact that when choosing from S_1, player 1 can proceed as if he knew that r^4 had not been chosen. The information set $S_1 \times \{r^1,r^2,r^3\}$ obscures the fact that, because $S_1 \times \{r^1,r^2\}$ and $S_1 \times \{r^3\}$ are information sets, player 1's decision can be described as an ex post decision contingent on observing $\{r^1,r^2\}$ or r^3. The description of player 1's decision problem that captures *all* of the information that player 1 can take to be available when deciding is provided by the information sets $S_1 \times \{r^1,r^2\}$ and $S_1 \times \{r^3\}$. The distinguishing feature of $S_1 \times \{r^1,r^2\}$ and $S_1 \times \{r^3\}$ among player 1's information sets is formalized as follows:

Definition 8: *The normal form information set,* $X_i \times X_{-i}$, *for player* i *is* <u>*strict*</u> *if (1) there does not exist* Y_{-i} *with* $X_i \times Y_{-i}$ *being a normal form information set for player* i *and* $Y_{-i} \subset X_{-i}$ *and (2) for some* $s_i, t_i \in X_i$ and $s_{-i} \in X_{-i}$, $\pi(s_i,s_{-i}) \neq \pi(t_i,s_{-i})$.[11] *The extensive form information set* h *is* <u>*strict*</u> *if* S(h) *is strict.*

For any non-strict information set $X \in N_i$, it follows immediately from Definition 6 that $\Psi_i(X) = X_i$. Thus, in a weak representation, at any non-strict information set, one choice of a player must be to defer *all* decisions. In a parsimonious representation, *all* choices at any non-strict information set must correspond to retaining all options. These observations suggest that non-strict information sets play no significant role in a representation. As we shall see, every parsimoniously (and hence weakly) representable game has a parsimonious representation that involves only strict information sets. In particular, the representation algorithm generates such a game.

How might one construct a representation for the game (G7)? The first information set in any (two player) extensive form corresponds to $S_1 \times S_2$. Since player 1 does not have $S_1 \times S_2$ as a strict

[11] This is to rule out, for example, $S_1 \times \{r^4\}$ as a strict information set in (G7).

reached at which neither of these can be done, then the algorithm is a failure, while otherwise the algorithm successfully generates an extensive form game.[12]

We now present the algorithm. If X_i is set of strategies for player i, then we define $I(X_i) \equiv \{s_{-i} \in S_{-i}: \pi(.,s_{-i})$ is constant on $X_i\}$. If X_i is strict then $X_{-i} \cap I(X_i) = \varnothing$.

Definition 9: *The Representation Algorithm is given by the following procedure*:

Initial Step: Create an initial node ω, and define $T(\omega) \equiv S_1 \times \ldots \times S_n$.

Recursion: If this is the first application of the recursion step, let $\Phi \equiv \{\omega\}$. Otherwise, let Φ be the collection of new nodes generated by the most recent application of the recursion step. Associated with each node $\zeta \in \Phi$ is $T(\zeta) \equiv T_1(\zeta) \times \ldots \times T_n(\zeta)$, where $T_i(\zeta) \subseteq S_i$ for each i.

For each $\zeta \in \Phi$, define $M(\zeta) \equiv \{i : T(\zeta) \subseteq T_i(\zeta) \times Y_{-i}$, for some Y_{-i}, where $T_i(\zeta) \times Y_{-i} \in N_i$ is strict$\}$. The set $M(\zeta)$ is the set of players who can make a decision at ζ.

Choose $i \in M(\zeta)$ if $M(\zeta)$ is non-empty, and let X^ζ denote the unique strict normal form information set such that $T_i(\zeta) = X_i^\zeta$ and $T(\zeta) \subseteq X^\zeta$.[13] Player i is the player who moves at ζ. Give player i one edge from ζ for each element $T_i' \in \Psi_i(X^\zeta)$, with each edge leading to a distinct new node, and label the edges by the corresponding T_i'.

For the node η reached from ζ by the action $T_i' \in \Psi_i(X^\zeta)$, define $T(\eta) \equiv T_i' \times T_{-i}(\zeta)$. To interpret T, note that for any node ζ, if player j has moved prior to ζ, then $T_j(\zeta)$ is the name of the most recent action taken by j. If j has not moved prior to ζ, then $T_j(\zeta) = S_j$.

If $M(\zeta) = \varnothing$, then ζ is a *terminal node* and no player makes a decision there.

Halting: Let Φ^* be the collection of nodes generated by the most recent application of the recursion step. The algorithm halts if and only if for every node $\zeta \in \Phi^*$, $M(\zeta) = \varnothing$. Since (S,π) is finite, the above process must stop in a finite number of steps.

Evaluation: The algorithm is a *success* if, for each terminal node ζ, $T_{-i}(\zeta) \subseteq I(T_i(\zeta))$ for all i.

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[12]One aspect of the algorithm that is not captured by this example is the assignment of nodes to information sets. WE provide an example involving nontrivial information sets immediately following the formal statement of the algorithm.

[13]The uniqueness of X^ζ follows from Lemma A7 in Appendix 3.

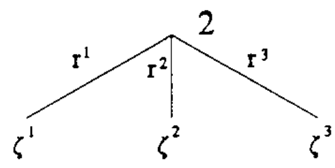

Figure 10

which contains $S_1 \times \{r^\tau\}$ is S, and so $X^{\zeta^\tau} = S$. Now, $\Psi_1(S) = \{X_1^1, X_1^2\}$ (because X^1 is an information set for 1 with $X_2^1 \subset S_2$ and similarly for X^2). Thus, at the node ζ^τ, player 1 chooses between X_1^1 and X_1^2, leading to two nodes $\zeta^{1\tau}$ and $\zeta^{2\tau}$ with $T(\zeta^{1\tau}) \equiv X_1^1 \times \{r^\tau\}$, and $T(\zeta^{2\tau}) \equiv X^2 \times \{r^\tau\}$, $\tau = 1,2,3$. See Figure 11.

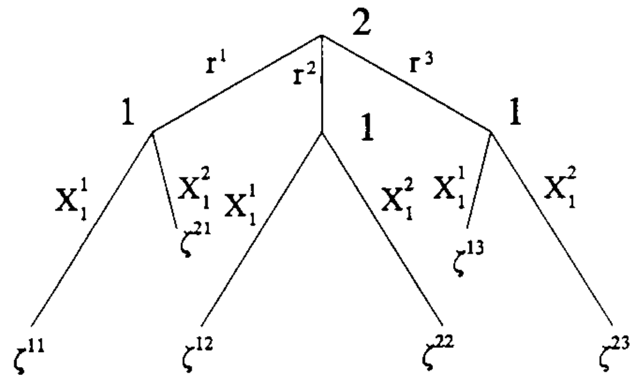

Figure 11

Now, $M(\zeta^{21}) = \varnothing$ and ζ^{21} receives a payoff of d, $M(\zeta^{13}) = \varnothing$ and ζ^{13} receives a payoff of c. However, $M(\zeta^{11}) = M(\zeta^{12}) = M(\zeta^{22}) = M(\zeta^{23}) = \{1\}$. Now, $X^{\zeta^{11}} = X^{\zeta^{12}} = X^1$, and $\Psi_1(X^1) = \{\{s^1\}, X_1^3\}$. So, 1 chooses between s^1 and X_1^3 at these two nodes. Similarly, 1 chooses between X_1^3 and s^4 at ζ^{22} and ζ^{23}. Denote the two nodes following $\zeta^{\kappa\tau}$ by $\zeta^{1\kappa\tau}$ and $\zeta^{2\kappa\tau}$, for $\kappa,\tau = 1,2$. See Figure 12. Then,

$$
\begin{aligned}
T(\zeta^{111}) &= \{s^1\}\times\{r^1\}, & T(\zeta^{211}) &= X_1^3\times\{r^1\},\\
T(\zeta^{112}) &= \{s^1\}\times\{r^2\}, & T(\zeta^{212}) &= X_1^3\times\{r^2\} = X^3,\\
T(\zeta^{122}) &= X_1^3\times\{r^2\} = X^3, & T(\zeta^{222}) &= \{s^4\}\times\{r^2\},\\
T(\zeta^{123}) &= X_1^3\times\{r^3\}, \text{ and} & T(\zeta^{223}) &= \{s^4\}\times\{r^3\}.
\end{aligned}
$$

In four of the new nodes (ζ^{111}, ζ^{112}, ζ^{222}, and ζ^{223}), the last choices of player 1 have led to a single strategy, so that no further choices can be made and payoffs can be assigned. Since $M(X_1^3\times\{r^1\})) = \varnothing$, player 1 cannot refine X_1^3, but a payoff of d can be assigned to ζ^{211}. Similarly, $M(X_1^3\times\{r^3\}) = \varnothing$ and ζ^{123} is assigned the payoff c. This leaves the two nodes, ζ^{212} and ζ^{122}, with $T(\zeta^{212}) = T(\zeta^{122}) = X^3$. The unique strict information set containing X^3 is X^3 itself, and so at each of these nodes player

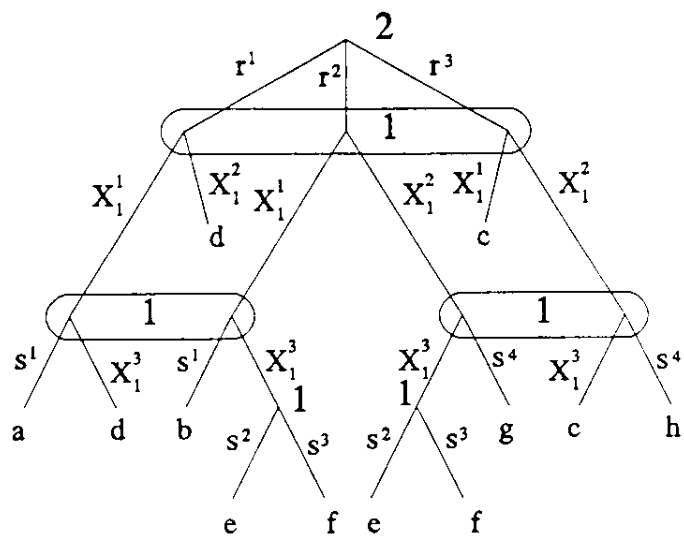

Figure 13

It is easily verified that the collection $\mathfrak{I}_i$ is precisely the set of information sets represented in the extensive form generated by a successful application of the algorithm. In addition, *any* weak representation contains an information set corresponding to each element of $\mathfrak{I}_i$ (see Lemma A12 in Appendix 6). Moreover, every information set in a parsimonious representation is related to a normal form information set in $\mathfrak{I}_i$ as follows:

Theorem 4. *Let* Γ *be a parsimonious representation of* (S,π). *Let* $h \in H_i$ *for some* i. *Then either*:

(T4.1) $S(h) \in \mathfrak{I}_i$,

(T4.2) h *is not strict and hence no decision is made at* h (i.e., $S_i(h,a) = S_i(h)\ \forall a \in A(h)$), or

(T4.3) h *is strict and follows an information set* $h' \in \mathfrak{I}_i$ *for which* $S_{-i}(h) = S_{-i}(h')$ *and* $S_i(h,a)$ *is a union over elements of* $\Psi_i(h')$ *for all* $a \in A(h)$.

Hence, for each player i and $h \in H_i$, either (T4.1) S(h) is an element of $\mathfrak{I}_i$, and so h also appears in the representation generated by the algorithm; (T4.2) no options are given up by any action at h; or (T4.3) there is h' (with $S(h') \in \mathfrak{I}_i$) preceding h, such that player i has learned nothing between h' and h, and such that the choices i makes at h correspond to choices he would make at h' in representations generated by the algorithm.[14] Figure 8 is an illustration of the last possibility. Thus,

[14]More precisely, the choices i makes at h correspond to the choices he would make at the extensive form information set that corresponds to S(h') in representations generated by the algorithm.

The necessity of GND is less obvious. For any particular X in Ξ, there is no reason to believe that a node corresponding to X will appear in *any* weak representation. The proof of necessity involves finding, for any $X \in \Xi$, and any representation, a node y with the properties that the player i moving at y (if any) has $S_i(y)=X_i$, while the remaining players have $S_j(y) \supseteq X_j$. Since y is a node in a representation, either someone moves at y, in which case (D11.1) will be the satisfied for the player who moves, or y is a terminal node, in which case (D11.2) must hold for S(y) and thus for $X \subseteq S(y)$. The formalization of this argument in Appendix 6 allows us to prove:

Theorem 6: *If* (S,π) *is weakly representable, then* $\mathfrak{I}$ *is GND.*

Thus, if the algorithm fails, we have reached a node that is essentially an n-person Abreu-Pearce game: a node where some player must move and yet no player can. Theorems 3, 5, and 6 together constitute:

The Representability Theorem: *The following are equivalent:*

(i) (S,π) *is weakly representable;*

(ii) (S,π) *satisfies GND;*

(iii) *the representability algorithm is successful on* (S,π); *and*

(iv) (S,π) *is parsimoniously representable.*

Remark: It is easily verified that if the definition of Ξ is modified such that for each player except some particular player i, $\Psi_j=S_j$, then GND is always satisfied. Correspondingly, if the representation algorithm is modified such that the first step is that all players apart from i choose simultaneously over all their strategies, then the algorithm will always terminate successfully, generating a game which represents i's strategic independences. Thus, we can always represent the decision problem of a single player. There is then a strong sense in which the force of both GND and the representation algorithm involves the weaving together of separate players' decision problems.

VIII. Strong Representability

We return now to the notion of strong representability. We first show how the previous analysis can be adapted to this representability notion. We then turn to the connection between strong representations and extensive form games which are "generic" in the sense that for any two distinct terminal nodes z and z', $\pi(z) \neq \pi(z')$.

IX. Playable Representations

We conclude with some remarks on another stronger but more speculative notion of representability, namely that the extensive forms be playable.[16] Intuitively, a game is playable if allowing players to condition their decisions on the *time* that they are asked to make a choice does not convey any additional information.

To illustrate the difference between representability and playability, consider the following game:

Player 1 (rows), player 2 (columns), player 3 chooses L or R: (G9)

L

	ℓ'	r'
ℓ	a	a
r	b	c

R

	ℓ'	r'
ℓ	d	f
r	e	f

3

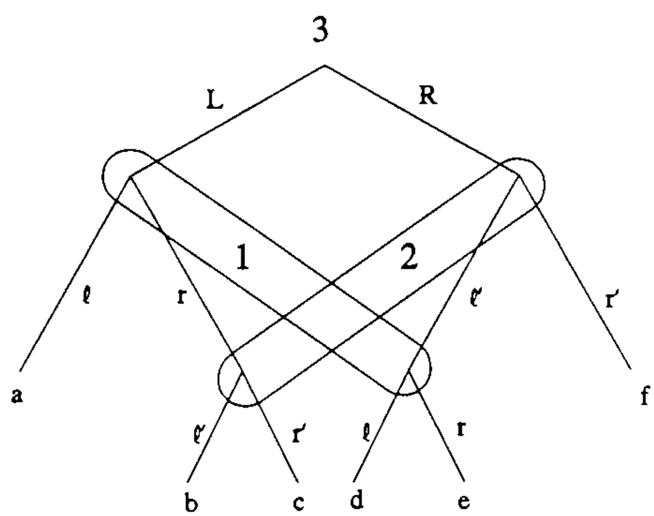

Figure 14

The only information set in player 1's complete collection is $S_1 \times \{\ell'L, r'L, \ell'R\}$. Similarly, for player 2, we have $S_2 \times \{Rl, \ell R, Rr\}$; and for player 3 we have S. This collection of information sets satisfies GND. Applying the algorithm to this game yields the extensive form representation in Figure 14.

If this extensive form game is played in real time by soliciting decisions from players when their information set has been reached (and players know this process), then at least one of players 1 or 2, when asked to move, can infer from the time at which he is asked whether he is the first of players 1 or

---

[16] We thank Elon Kohlberg for drawing our attention to the issue of playability.

While playability is an intriguing and plausible requirement, we have not pursued it here because it is not common to restrict attention to playable extensive forms,[17] and such an investigation is beyond the scope of this paper.

[17]Non-playable extensive form games have appeared in, for example, Kreps and Wilson (1982, figure 8) and Kreps and Ramey (1987, figure 2).

Now suppose an extensive form game with nature, Γ, represents game (G1). The argument proceeds in three steps. First, suppose that the game is such that for each player i,information set h for i and nodes x and y, if $x \in h$ and there exists a strategy profile s for the players and two realizations ω_1 and ω_2 for nature such that given s, node x is reached if nature chooses ω_1 and node y is reached if nature chooses ω_2, then y is also in h (i.e., no player can condition choices on the realization of nature). Then we can construct an extensive form without nature by collapsing nature's moves at each information set into a single move and attaching expected payoffs to terminal nodes. This new game will have precisely the same information sets for players 1 and 2 as the original and also the same PRNF. We then have an extensive form game without nature that represents game (G1), contradicting the first part of the proof. Say that player i can *condition on nature* at information set h if, at h, player i can exclude some realizations of nature. Thus, there must exist a player i in Γ and an information set at which i conditions on nature.

The extensive form Γ must have an information set corresponding to either h^X of h^Y that follows a move by nature on which player 1 or 2 (respectively) can condition. If not, we can construct a new extensive form game by arbitrarily fixing the behavior of players at all information sets at which they can condition on nature. This new game will still have information sets corresponding to h^X and h^Y, will have the same PRNF as Γ, and has only information sets at which players do not condition on realizations of nature. The previous paragraph again yields a contradiction. Thus, one player, say 1, must have an information set corresponding to h^X that follows an information set h at which some player conditions on nature.

Since player i conditions on nature at h and h^X follows h, there are four strategy profiles s^1, s^2, s^3, and s^4 such that for some of the realizations of nature's move in N^h (the set of realizations which make h reachable), all four strategy profiles reach h^X and s^1 and s^2 give outcome A and s^3 and s^4 give outcome B, and such that, averaging over all moves by nature, $\pi(s^1) = \pi(s^2) = a$ and $\pi(s^3) = \pi(s^4) = b$.[18] Denote by N^1 the set of realizations for which s^1 and s^2 reach the same terminal node, and s^3 and s^4 reach the same terminal node. Furthermore, for at least one of the realizations of nature's move, $\omega' \notin N^h$, there is an information set $h' \neq h$ reached by all four strategy profiles s^τ, $\tau = 1,\ldots,4$, with s^1 and s^3 yielding a different outcome from s^2 and s^4. [If not, then all the payoffs following nature's realizations not in N^h yield the identical payoffs. But then these realizations could be eliminated and payoffs rescaled, yielding a game in which no one conditions on nature, a contradiction.] Call these outcomes C and D. Denote by N^2 the set of realizations for which s^1 and s^3 reach the same terminal node, and s^2 and s^4

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[18]If it was known that the extensive form only had as outcomes a or b, then the argument in this paragraph could be simplified, but other outcomes (including the zero vector) may also appear in the extensive form.

Proof: Since $T_i \in \Psi_i(Y)$, there exists $X \in N_i$ such that $X \subseteq Y$, $X_{-i} \subset Y_{-i}$ and $X_i = T_i$. Suppose the lemma is false and define $Z_i \equiv \{s_i \in Y_i : \exists s_i' \in X_i,\ s_i \text{ agrees with } s_i' \text{ on } Y_{-i}\}$. Then $X_i \subset Z_i$. Let $r_i, s_i \in Z_i$. Then there exist $r_i', s_i' \in X_i$ with r_i' agreeing with r_i on Y_{-i} and s_i' agreeing with s_i on Y_{-i}. Since X is an information set, there exists $t_i' \in X_i$ agreeing with r_i' on X_{-i} and agreeing with s_i' on $S_{-i} \backslash X_{-i}$. Since Y is an information set, there exists $t_i \in Y_i$ agreeing with t_i' on Y_{-i} (and so $t_i \in Z_i$) and agreeing with s_i on $S_{-i} \backslash Y_{-i}$. In fact, t_i also agrees with s_i on $Y_{-i} \backslash X_{-i}$ (since t_i agrees with t_i' on Y_{-i}, t_i' agrees with s_i' on $S_{-i} \backslash X_{-i}$ and s_i' agrees with s_i on Y_{-i}), so that t_i agrees with s_i on $S_{-i} \backslash X_{-i}$. It is easy to check that t_i agrees with r_i on X_{-i}. Thus, $Z_i \times X_{-i}$ is a normal form information set with $Z_i \times X_{-i} \subseteq Y$. This contradicts the assumption that $X_i = T_i \in \Psi_i(Y)$. ■

Lemma A4: *Let* h *be an information set for* i *in an extensive form with normal form* (S,π), *with* $S(h) = X$. *Let* a *be an action at* h *and let* $s_i \in X_i$ *be such that* $s_i \in S_i(h,a)$. *Then,* $t_i \in S(h,a)$ *for all* $t_i \in X_i$ *which agree with* s_i *on* X_{-i}.

Proof: Suppose $t_i \in X_i$, t_i agrees with s_i on X_{-i}, but $t_i \notin S_i(h,a)$. There is some behavior strategy τ_i for i which makes h reachable and $\tau_i \in t_i$. Similarly, since $s_i \in S_i(h,a)$, there is some behavior strategy σ_i for i which makes h reachable, takes the action a at h, and $\sigma_i \in s_i$. Let ρ_i be the behavior strategy which specifies the same actions as σ_i on and after h and as τ_i elsewhere. Since t_i and s_i agree on X_{-i}, $\rho_i \in t_i$. But this contradicts the assumption that $t_i \notin S_i(h,a)$. ■

Lemma A5: *Let* $h, h' \in H_i$ *for some* i *with* $h' \succ h$. *Assume that* $S_{-i}(h) = S_{-i}(h')$ *and that* $S_i(h')$ *is a union over elements* $\{T^1,...,T^K\} \subseteq \Psi_i(S(h))$. *Then,* $\Psi_i(S(h')) = \{T^1,...,T^K\}$.

Proof: Consider first any T^k such that the elements of T^k agree on $S_{-i}(h)$. Then, the elements of T^k agree on $S_{-i}(h')$, and so since $T^k \subseteq S_i(h')$, $\Psi_i(S(h'))$ contains some element T'^k with $T^k \subseteq T'^k$. Consider now T^k such that there is $Y_{-i} \subset S_{-i}(h)$ with $T^k \times Y_{-i} \in N_i$. Again, since $T^k \subseteq S_i(h')$, $\Psi_i(S(h'))$ must contain some element T'^k with $T^k \subseteq T'^k$. Conversely, let $T' \in \Psi_i(S(h'))$. Trivially, $T' \subseteq S_i(h)$. Thus, if the elements of T' agree on $S_{-i}(h) = S_{-i}(h')$ or if $T' \times Y_{-i}$ is an information set for i with $Y_{-i} \subset S_{-i}(h')$, then some element of $\Psi_i(S(h'))$ contains all of T'. This establishes the result. ■

Lemma A6: *Let* Γ *be a parsimonious representation of* (S,π), *and let* $h, h' \in H_i$ *be such that* $h' \succ h$, *and* $S_{-i}(h) = S_{-i}(h')$. *Let* $a' \in A(h')$. *Then* $S_i(h')$ *and* $S_i(h',a')$ *are unions over elements of* $\Psi_i(S(h))$.

Proof: Let $h = h^0 \prec h^1 \prec ... \prec h^K = h'$, be the sequence of information sets for i between h and h', let a^k, $k = 0,...,K-1$ be the action at h^k which leads toward h^{k+1}, and let $a^K = a'$. Note that $S_{-i}(h^k) = S_{-i}(h)$ for each k.

(Only if) Assume Γ is a parsimonious representation. We will show that Γ has no informational dominances. Let h, h' and a' be as described in Definition 5. Then, by Lemma A6, $S_i(h',a')$ is a union of elements of $\Psi_i(S(h))$.

Let K_i satisfy $S_i(h',a') \subset K_i \subseteq S_i(h)$. Then since $S_i(h',a')$ is a union of elements of $\Psi_i(S(h))$, there is $T_i^* \in \Psi_i(S(h))$ such that $T_i^* \subset K_i$. We will show that any behavior strategy for i that makes h reachable, must involve i giving up some elements of K_i at an information set where he knows no more than he knows at h. This will establish that no K_i can satisfy Definition 5, and thus that Γ has no informational dominances.

Note first that not all elements of K_i can agree on $S_i(h)$ (if they did, then each element of K_i agrees with every element of $S_i(h',a')$, and so by Lemma A2, $K_i \subseteq S_i(h',a')$, which is a contradiction). Choose $s_{-i}^* \in S_{-i}(h)$ such that not all of K_i agrees on s_{-i}^*.

Consider any behavior strategy σ_i for i that makes h reachable. Let $h'' \succeq h$ satisfy

(1) $h'' \in H_i$ is reachable given σ_i,

(2) $s_{-i}^* \in S_{-i}(h'')$,

(3) $K_i \subseteq S_i(h'')$, and

(4) there does not exist $h''' \succ h''$ satisfying (1)-(3).

Conditions (1)-(3) are satisfied by h and Γ is finite, so such an h" exists. Because $T_i^* \subset K_i \subseteq S_i(h'')$, Lemma A2 implies $S_{-i}^{\Gamma}(h'') = S_{-i}^{\Gamma}(h)$. Now, assume that $a'' = \sigma_i(h'')$ has $K_i \subseteq S_i(h'',a'')$. The information set h" cannot be the last information set for i reached by $(\sigma_i, \sigma_{-i}^*)$, because $K_i \subseteq S_i(h'',a'')$ and not all elements of K_i agree on s_{-i}^*. Let $h''' \succ h''$ be the next information set for player i reachable by $(\sigma_i, \sigma_{-i}^*)$. Then h''' clearly satisfies (1)-(3), contradicting the choice of h".

So, for any behavior strategy σ_i which makes h reachable, there is some point where i gives up some element of K_i knowing no more than he did at h. Thus, Γ has no informational dominances. ■

Appendix 3: Proof of Theorem 3

Lemma A7: *If* X *and* Y *are distinct normal form information sets for player* i *with* $X_i = Y_i$ *and* $X_{-i} \cap Y_{-i} \neq \varnothing$, *then either* $X_i \times (X_{-i} \cap Y_{-i})$ *is a normal form information set for player* i *or* $X_{-i} \cap Y_{-i} \subseteq I(X_i)$. *Hence, neither* X *nor* Y *can be strict.*

Proof: Suppose it is not the case that $X_{-i} \cap Y_{-i} \subseteq I(X_i)$. If $r_i, s_i \in X_i$, then there exists $t_i' \in X_i$ such that t_i' agrees with r_i on X_{-i} and t_i' agrees with s_i on $S_{-i} \backslash X_{-i}$. Moreover, there exists $t_i \in X_i$ such that t_i agrees with t_i' on Y_{-i} and t_i agrees with s_i on $S_{-i} \backslash Y_{-i}$. This implies t_i agrees with r_i on $X_{-i} \cap Y_{-i}$ and t_i agrees with s_i on $S_{-i} \backslash (X_{-i} \cap Y_{-i})$, and so condition (D2.2) of Definition 2 is satisfied. ■

Observe that

(1) for information sets $h \in H_i$ that have no predecessors for i, $X_i(h) = S_i$,

(2) for all $h \in H_i$, $X_i(h) = \cup\{T_i \in \Psi_i(X(h))\}$, and

(3) each information set h' for i that is an immediate successor for i to h and the action $a \in \Psi_i(X_i(h))$ has $X_i(h') = a$.

Induction then shows that $\Sigma_i(s_i)$ is non-empty for each $s_i \in S_i$: At each information set $h \in H_i$ for i which has no predecessors for i, observations (1) and (2) imply the existence of an action a with $s_i \in a$. Choose some such action. Now, consider any information set h' for i such that at i's most recent predecessor h, an action a was taken with $s_i \in a$. Then, by observation (3), $s_i \in X_i(h')$. But, then observation (2) implies the existence of an action a' at h' with $s_i \in a'$.

Let $\sigma \in \Sigma(s)$, and let ζ be the terminal node reached by σ. Then, $s \in T(\zeta)$ and so the payoff assigned to ζ is $\pi(s)$ (this payoff is unique since the algorithm is a success). Thus $\pi^\Gamma(\sigma) = \pi(s)$ for all $\sigma \in \Sigma(s)$. So, in particular, for each s_i, all elements of $\Sigma_i(s_i)$ agree on $\cup_{s_{-i} \in S_{-i}} \Sigma_{-i}(s_{-i})$, and $\Sigma_i(s_i) \cap \Sigma_i(s_i') = \varnothing\ \forall s_i \neq s_i' \in S_i$. Assume that we knew that for all i, $\cup_{s_{-i} \in S_{-i}} \Sigma_{-i}(s_{-i}) = S^\Gamma_{-i}$. Then, we can replace each $\Sigma_i(s_i)$ by a single strategy which we can call s_i. Clearly the game that results from this operation is (S,π).

To show that the PRNF of Γ is (S,π), it thus remains to show that there exists $s_i \in S_i$ with $\sigma_i \in \Sigma_i(s_i)$. For an arbitrary element σ_i of S_i^Γ, we will show that

$$\varnothing \neq \cap\{\sigma_i(h) : h \in H_i(\sigma_i)\}.$$

This is enough, since $s_i \in \cap\{\sigma_i(h) : h \in H_i(\sigma_i)\}$ if and only if $\sigma_i \in \Sigma_i(s_i)$.

We proceed by induction. Fix an information set $h' \in H_i$, and let $\sigma_i(h') \in \Psi_i(X(h'))$ be the action taken at h'. Let $h^1,\ldots,h^m \in H_i$ be the immediate successors for i to h' which are reachable under σ_i. Assume that for $j = 1,\ldots,m$, there is $s_i^j \in \cap\{\sigma_i(h) : h \in H_i(\sigma_i),\ h \geq h^j\}$.[20] We will find $t_i \in \cap\{\sigma_i(h) : h \in H_i(\sigma_i),\ h \geq h'\}$.

Let $Y^j \equiv X(h^j)$, $j = 1,\ldots,m$. By the specification of the algorithm, each Y^j is strict, and for each i, $Y_i^j = \sigma_i(h') \equiv Y_i$. Lemma A7 implies that for all $j,k \in 1,\ldots,m$, either $Y^j = Y^k$ or Y^j and Y^k are disjoint. The way in which the algorithm groups nodes together into information sets precludes $Y^j = Y^k$ if $j \neq k$. By Lemma A8, there is thus $t_i \in Y_i$ such that t_i agrees with s_i^j on Y^j_{-i}, $j = 1,\ldots,m$.

We wish to show that $t_i \in \cap\{\sigma_i(h) : h \in H_i(\sigma_i)\ h \geq h'\}$. Once again we proceed by induction. Note first that $t_i \in \sigma_i(h')$ (since $\sigma_i(h') = Y_i$). So, let $h \geq h'$ have $h \in H_i(\sigma_i)$, and assume that $t_i \in \sigma_i(h)$. We will show that for any immediate successor h" for i to h with $h'' \in H_i(\sigma_i)$, $t_i \in \sigma_i(h)$.

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[20]To get the induction started, note that, for each $j = 1,\ldots,m$, when h^j has no successors that are reachable under σ_i, then trivially, there is $s_i^j \in Y_i$ with $s_i^j \in \sigma_i(h^j)$.

Proof of Theorem 4: Let $h^0, h^1, \ldots, h^K$ be any sequence of information sets and let $a^0, a^1, \ldots, a^K$ be a sequence of actions for player i with h^0 having no predecessor for i and with h^k following h^{k-1} and a^{k-1}, $k = 1, \ldots, K$. Because all of player i's information sets appear in some such sequence, it suffices to show that Theorem 4 holds for all h^k in this sequence.

Let $h^{k'}$ be the last information set in this sequence such that $S_i(h^k) = S_i$. Then for all $h^k \prec h^{k'}$, $S_i(h^k, a^k) = S_i$ (by Lemma A9.2). If h^k is strict, it is in $\mathcal{G}_i$ (by (D10.1)) and thus (T4.1) holds. If h^k is not strict, $\{S_i\} = \Psi_i(S_i(h^k))$ (by Lemma A9.1), and because Γ is a parsimonious representation, every $a \in A(h^k)$ must satisfy $S(h^k, a) = S_i$. Thus, either (T4.1) or (T4.2) holds for all $h^k \prec h^{k'}$.

At $h^{k'}$, $S_i \notin \Psi_i(h^{k'})$, since otherwise $h^{k'+1}$ would also satisfy $S_i(h^{k'+1}) = S_i$, a contradiction to $h^{k'}$ being the last such information set. Hence, $h^{k'}$ is strict (Lemma A9.1) and since $S_i(h^{k'}) = S_i$, we have $S(h^{k'}) \in \mathcal{G}_i$ (by (D10.1)).

Now let k' be such that $S(h^{k'}) \in \mathcal{G}_i$. Let $h^{k''} \geq h^{k'}$ be the last information set after $h^{k'}$ such that $S_{-i}(h^{k''}) = S_{-i}(h^{k'})$. For $k' \leq k \leq k''$, if h^k is strict, then Lemma A6 implies that h^k satisfies (T4.3). If h^k is not strict, then (as before), h^k satisfies condition (T4.2).

At $h^{k''}$, it must be that $S_i(h^{k''}, a^{k''}) = T_i^*$ for some $T_i^* \in \Psi_i(S(h^{k''}))$, since otherwise, by Lemma A2, $h^{k''+1}$ also has $S_{-i}(h^{k''+1}) = S_{-i}(h^{k'})$. Let $k''' > k''$ be the last information set in the sequence such that $S_i(h^{k'''}) = T_i^*$. (The information set $h^{k''+1}$ has $S_i(h^{k''+1}) = T_i^*$, so k''' exists.) Then, for $k'' < k < k'''$, arguing as before, if h^k is not strict it satisfies condition (T4.2) and if it is strict, then it satisfies (T4.3). Finally, $h^{k'''}$ must be strict (since otherwise $h^{k'''+1}$ has $S_i(h^{k'''+1}) = T_i^*$, by Lemma A9.1, a contradiction to the choice of k'''), and hence $S_i(h^{k'''}) \in \Psi_i(S(h^{k''}))$. Since Γ is a parsimonious representation, $S_i(h^{k'''}) \in \Psi_i(S(h^{k'}))$, and so $S(h^{k'''}) \in \mathcal{G}_i$, satisfying (T4.1).

Repeating this argument shows that every information set in $h^0, \ldots, h^K$ satisfies the conditions of Theorem 4, establishing the result. ■

Appendix 5: Proof of Theorem 5

It suffices to show that for every node ζ of the tree generated at each step of the algorithm, $S(\zeta) \in \Xi$. Since $S(\omega) = S \in \Xi$, where ω is the initial node, it suffices to show that if $S(\zeta) \in \Xi$, i moves at ζ, and η immediately follows ζ, then $S(\eta) \in \Xi$. Now, because the algorithm causes η to follow ζ, it must be that $S(\zeta) = X_i \times X_{-i}$ and $S(\eta) = Y_i \times X_{-i}$, where $Y_i \in \Psi_i(Z)$ where $Z_i = X_i$, $X_{-i} \subseteq Z_{-i}$, $Z \in \mathcal{G}_i$, and $S(\eta) \subset Z$. Then from the definition of Ξ, $S(\eta) \in \Xi$. ■

$T_i \in \Psi_i(S(h'))$ has $X_i \subseteq T_i$. Since $X_i = S_i(h')$, $T_i \subseteq S_i(h')$ must hold, giving $X_i = T_i$ and again establishing the result. ■

Lemma A12: *If* Γ *is a weak representation for* (S,π), *then for each element* X *of* $\mathfrak{I}_i$, *there is* $h \in H_i$ *such that* $S(h) = X$.

Proof: First, consider any $X \in \mathfrak{I}_i$ for which $X_i = S_i$. We first argue that there is $h \in H_i$ with $S_i = S_i(h)$ and $X_{-i} \cap S_{-i}(h) \neq \varnothing$. For this it suffices to find an information set h for player i that has no predecessor for i (ensuring $S_i(h) = S_i$) with $s^*_{-i} \in S_{-i}(h)$ for some $s^*_{-i} \in X_{-i}$. Assume there is no such h. Then, $s^*_{-i} \notin S_{-i}(h)$ for any information set $h \in H_i$. But, then $s^*_{-i} \in I(S_i)$, contradicting that X was strict. Hence, we have $S_i(h) = S_i$ and $S_{-i}(h) \cap X_{-i} \neq \varnothing$. The strictness of X and Lemma A7 then imply that $X = S(h)$.

The proof is now extended to all elements of $\mathfrak{I}_i$ by repeatedly applying Lemma A11. ■

Proof of Theorem 6: Suppose Γ weakly represents (S,π) and $X^* \in \Xi$. Then, for each i, either $X^*_i = S_i$ or $\exists Z^i \in \mathfrak{I}_i$, $X^* \subseteq Z^i$, $X^*_i \in \Psi_i(Z^i)$. By Lemma A12, for each Z^i, there exists $\underline{h}^i$ such that $S(\underline{h}^i) = Z^i$. Fix such a Z^i for each i with $X^*_i \neq S_i$.

Claim 1: If $X^*_i \neq S_i$, there exists $h^i \succeq \underline{h}^i$ and an action a at h^i with $S_i(h^i,a) = X^*_i$, and such that $S_{-i}(h^i) = S_{-i}(\underline{h}^i) = Z^i_{-i}$. Further, h^i is strict.

Proof: A simple induction using Lemma A11. □

Fix such an h^i for each i for whom $X^*_i \neq S_i$. Let $P(X^*)$ be the set of nodes x in Γ satisfying the following two properties:

(P1) Let $x' \prec x$, let i be the player who moves at x', and let a' be the move at x' that keeps x reachable. Then:

(P1.i) If $X^*_i = S_i$, then $S_i(x',a') = S_i$;

(P1.ii) If $X_i \neq S_i$ and if h^i is not reached before x, then h^i is reachable given x' and a'.

(P2) If $X_i \neq S_i$, and the path leading to x first reaches h^i for some i, then at every node x' with $h^i \preceq x' \prec x$, the action a' at x' that keeps x reachable satisfies $S_i(x',a') = X^*_i$.

Clearly, the initial node is in $P(X^*)$. Because the set of nodes is finite, we can choose $y^* \in P(X^*)$ such that y^* has no successors in $P(X^*)$.

strategy profile $s \in X$, there is a behavior strategy profile $\sigma \in s$ which reaches y^*, and a behavior strategy profile $\rho \in s$ which reaches h^j. If h^j has been excluded prior to y^*, then because it was not excluded by player j, it must be that σ_{-i} but not ρ_{-i} is consistent with reaching h^j. Lemma A1 then applies, so that h^j is not strict, which is a contradiction to Claim 1. □

Claim 5: $X_j^* = S_j(h^{y*})$
Proof: Clear from (P2) and Claim 4. □

Claim 6: $S(h^{y*}) \in \vartheta_j$.
Proof: If $S(h^{y*})$ is not strict, then by Lemma A9.1, $\{X_j^*\} = \Psi_j(S(h^{y*}))$. Since Γ is a weak representation and $X_j^* = S_j(h^{y*})$ (Claim 5), some choice a at h^{y*} has $X_j^* \subseteq S_j(h^{y*},a) \subseteq S_j(h^{y*}) = X_j^*$. By (P2), this contradicts that y^* has no successor in $P(X^*)$, so $S(h^{y*})$ must be strict. In addition, by construction $X_j^* \in \Psi_j(Z^j)$ (with $Z^j \in \vartheta_j$) and $S(h^{y*}) \subseteq Z^j$ (because $h^j \prec y^*$ (Claim 4) and hence $S(h^{y*}) \subseteq S(h^j) \subseteq Z^j$). Then, by D10.2, $S(h^{y*}) \in \vartheta_j$. □

Since $X^* \subseteq S(h^{y*})$ (Claim 2), $X_j^* = S_j(h^{y*})$ (Claim 5), and $S(h^{y*}) \in \vartheta_j$ (Claim 6), X^* satisfies (11.i) and we are done.

■

Appendix 7: Proofs of Theorem 7 and Theorem T2.2

Proof of Theorem 7: (If) Since GND holds, the algorithm is a success. But, at each step of the algorithm, each edge corresponds to a single element of $\Psi_i(S(h))$. Since these elements do not overlap, the algorithm has generated a strong representation.

(Only if) Let Γ be a strong representation. Then, since every strong representation is a weak representation, GND is satisfied. Assume $\Psi_i(Z)$ is not a partition for some element of ϑ_i and some i. Then in particular, there is s_i a member of two distinct elements X_i and Y_i of $\Psi_i(Z)$. By strong representability, $X_i \cup Y_i \subseteq S(h,a)$ for any $h \in H_i$ with $S(h)=Z$ and a with $s_i \in S_i(h,a)$. But, then Lemma A10.3 applies to show that the next information set reached from this action also has both X_i and Y_i in Ψ_i. Continuing in this way, there is never any way to separate the X_i and Y_i. But, the elements of $X_i \cup Y_i$ cannot all agree on Z_{-i}, otherwise $X_i = Y_i$ (by Lemma A3). So, X_i and Y_i must be separated at some point, which is a contradiction. ■

$\in N_i$ with $W_i \subseteq Y_i$ and $Y \subset X$. Then there again exists T_i with is $W_i \subseteq T_i \in \Psi_i(X)$. By strong representability, $W \subseteq S(h,a)$ or $W \cap S(h,a) = \varnothing$, completing the proof.

■

Appendix 9: Proof of Theorem 9

Rectangularity follows from the observation that the set of strategy profiles that reaches any node must have a cross product structure. Consider any information set in h, and action a at h, and assume that a contains some but not all of some element $T_i \in \Psi_i(S(h))$. Let s_i be an element of T_i which is in a, and t_i an element of T_i which is not in a. Then, s_i and t_i must agree for some $s_{-i} \in S_{-i}(h)$. But, some terminal node following a must contain (s_i,s_{-i}), while some node not following a must contain (t_i,s_{-i}). These nodes must then have identical payoffs, contradicting the hypothesis. ■

References

Abreu, D. and D. Pearce (1984), "On the Inconsistency of Certain Axioms of Solution Concepts for Non-cooperative Games," *Journal of Economic Theory*, 34, 169-174.

Dalkey, N. (1953), "Equivalence of Information Patterns and Essentially Determinate Games," in *Contributions to the Theory of Games, Volume II. Annals of Mathematical Studies 28*. ed. by H.W. Kuhn and A.W. Tucker, Princeton NJ: Princeton University Press.

Elmes, S. and P. Reny (1990), "On the Strategic Equivalence of Extensive Form Games," mimeo, Columbia University and the University of Western Ontario.

Gurvič, V. A. (1982), "On The Normal Form of Positional Games," *Soviet Math. Dokl.*, 25, 572-575.

Kohlberg, E. and J.-F. Mertens, "On the Strategic Stability of Equilibria," *Econometrica*, 54, 1003-1038.

Kreps, D. and R. Wilson (1982), "Sequential Equilibria," *Econometrica*, 50, 863-894.

Kreps, D. and G. Ramey (1987), "Structural Consistency, Consistency, and Sequential Rationality," *Econometrica*, 55, 1331-1348.

Mailath, G.J., L. Samuelson, and J.M. Swinkels (1993), "Extensive Form Reasoning in Normal Form Games," *Econometrica*, forthcoming.

Thompson, F.B. (1952), "Equivalence of Games in Extensive Form," Research Memorandum 759, The Rand Corporation.