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ADDITIVE PROCESSES FROM THE POINT OF VIEW OF REGENERATION

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1. INTRODUCTION

A stochastic process is said to be regenerative if there exist stopping times from which onwards the future is a probabilistic replica of the original process. Such stopping times are called regeneration times, and the collection of all such times, considered as a random subset of $\mathbb{R}_+ = [0,\infty)$, is called a regeneration set.

Classical renewal theory is concerned with regeneration sets which are discrete subsets of $\mathbb{R}_+$, that is, subsets whose points are all isolated. Then, those points can be ordered, and the resulting sequence is called a renewal process.

The present note will be devoted to those regeneration sets which are not discrete. But our aim will be to obtain results which are directly related to the classical ones obtained in renewal theory.

In the classical case, a regeneration set G has the form

$$G = \{t \in \mathbb{R}_+ : t = Y_n \quad \text{for some } n \in \mathbb{N}\}$$

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where $(Y_n)_{n \in \mathbb{N}}$ is a renewal process. Of course, a renewal process is a discrete parameter increasing stochastic process with stationary and independent increments. A similar result holds in general: every regeneration set G has the form

$$G = \{t \in \mathbb{R}_+ : t = Y_s \text{ for some } s \in \mathbb{R}_+\}$$

where $(Y_s)_{s \in \mathbb{R}_+}$ is an increasing process with stationary and independent increments (such processes will be called additive).

This is the main characterization theorem on which everything else depends. It is due to MAISONNEUVE [9].

Section 2 introduces increasing additive processes in an elementary setting. In Section 3, the nature of a regeneration set is examined by using the known results about additive processes.

The role of the renewal counting process in renewal theory is taken over by local times. These are inverse times obtained from additive processes. They are introduced and discussed in Section 4.

Potential measures of increasing additive processes play the same role in this theory as the renewal measure does in the classical renewal theory. Section 5 is devoted to an account of the basic results concerning them. In particular, it turns out that a potential measure is a constant multiple of a renewal measure. And this in turn enables one to use classical renewal theory to obtain limit theorems of Section 6.

Finally in Section 7 we examine the equivalents of backward and forward recurrence times of classical renewal theory, and in Section 8 the delayed version of the forward recurrence time. By making use of sample paths, we are able to derive their distributions by simple arguments using renewal theory. This direct derivation is made possible by MAISONNEUVE's characterization. HOROWITZ [6] had obtained them by Markov process techniques by first computing infinitesimal generators, then the corresponding resolvent, and finally the transition probabilities. KINGMAN [8] obtained the same in a special case by employing triple Laplace transforms; his computational method, in addition to being lengthy, does not generalize.

This paper is an elementary introduction to regenerative processes. For a more careful treatment which reviews MAISONNEUVE's work in some detail, we refer to [3].

In general, if E is a topological space, $\underline{\underline{E}}$ denotes the σ-algebra of its Borel subsets. Some special notation: $\mathbb{N} = \{0,1,2,\ldots\}$, $\mathbb{R}_+ = [0,\infty)$, $\bar{\mathbb{R}}_+ = [0,\infty]$, $\mathbb{R}_s = (s,\infty)$, $\bar{\mathbb{R}}_s = (s,\infty]$.

2. INCREASING ADDITIVE PROCESSES

Let $(\Omega,\underline{\underline{M}},P)$ be a complete probability space, and let $Y = (Y_t)_{t \in \mathbb{R}_+}$ be a process with stationary and independent increments taking values in $\bar{\mathbb{R}}_+ = [0,\infty]$. Such a process is almost surely increasing, and we further assume that $Y_0 = 0$ and $t \to Y_t$ is right continuous almost surely. We will call such a process an increasing additive process.

It is well known that

$$(2.1) \qquad Q_t^\lambda = E[\exp(-\lambda Y_t)] = \exp(-t\lambda N^\lambda), \quad \lambda, t \in \mathbb{R}_+,$$

where

$$(2.2) \qquad \lambda N^\lambda = \lambda a + \int_{(0,\infty)} (1 - e^{-\lambda y}) L(dy)$$

for some constant $a \geq 0$ and some (positive) measure L on the Borel subsets of $\bar{\mathbb{R}}_0 = (0,\infty]$ satisfying

$$(2.3) \qquad \int_{\mathbb{R}_0} (y \wedge 1) L(dy) < \infty.$$

The constant a is called the <u>drift rate</u> of Y, and the measure L is called the Lévy measure of Y. (Conversely, if $a \geq 0$ is a constant and L is a measure on $\bar{\mathbb{R}}_0$ satisfying (2.3), then there is an increasing additive process Y satisfying (2.2).)

If L is finite, i.e., if $L(\bar{\mathbb{R}}_0) < \infty$, then $Y_t = at + Y_t^c$ where (Y_t^c) is a <u>compound Poisson process</u>: the times of its jumps form a Poisson process with intensity $L(\bar{\mathbb{R}}_0)$; and the magnitudes of the jumps are independent of each other and of the jump times, and have the common distribution $L(B)/L(\bar{\mathbb{R}}_0)$, $B \in \bar{\underline{\underline{R}}}_0$. The general case is a limit of the compound Poisson case in the following sense.

For $c > 0$ let Y_t^c be the sum of the magnitudes of all the jumps of Y occurring during $[0,t]$ whose magnitudes are greater than c, that is, let

$$(2.4) \qquad Y_t^c = \sum_{s \le t} (Y_s - Y_{s-}) I_{\{Y_s - Y_{s-} > c\}}, \qquad t \in \mathbb{R}_+;$$

and let ζ be the lifetime of Y^c, that is, let

$$(2.5) \qquad \zeta = \inf\{t: Y_t^c = +\infty\}.$$

Define

$$(2.6) \qquad Y_t^b = \begin{cases} Y_t - Y_t^c & \text{if } t < \zeta, \\ Y_{\zeta-} - Y_{\zeta-}^c & \text{if } t \ge \zeta. \end{cases}$$

Then,

$$Y_t = Y_t^b + Y_t^c$$

for all $t \in \mathbb{R}_+$.

As $c \downarrow 0$, (Y_t^c) increases to an increasing pure jump process (Y_t^d), and (Y_t^b) decreases to an increasing continuous process (Y_t^a). Since (Y_t^b) has stationary and independent increments for each c, (Y_t^a) also has stationary and independent increments. Now the continuity of (Y_t^a) implies that it must be of the form $Y_t^a = at$ for some constant $a \ge 0$.

The process $(Y_t^d)_{t \in \mathbb{R}_+}$ is called the <u>purely discontinuous component</u> of Y, and it is clear that $(Y_t) = (at) + (Y_t^d)$. The following well known proposition states that Y^c is a compound Poisson process "independent" of Y^b; therefore, Y is the limit of an increasing sequence of compound Poisson processes up to a deterministic drift term. Recall that $\overline{\mathbb{R}}_c = (c,\infty]$.

(2.7) PROPOSITION. a) Y^c is a compound Poisson process: the times of its jumps form a Poisson process with parameter $L(\bar{\mathbb{R}}_c)$; the magnitudes of the jumps are independent of each other and of the jump times, and have the common distribution $L(B)/L(\bar{\mathbb{R}}_c)$, $B \in \bar{\underline{\underline{\mathbb{R}}}}_c$.

b) The lifetime ζ defined by (2.5) has the exponential distribution with parameter $L(\{\infty\})$ if this is positive; otherwise, if $L(\{\infty\}) = 0$, then $\zeta = +\infty$ almost surely.

c) Given ζ, Y^b is conditionally independent of Y^c. Y^b has the same probability law as $(\hat{Y}_{t \wedge \zeta})_{t \in \mathbb{R}_+}$ where $(\hat{Y}_t)$ is an increasing additive process independent of Y^c (and therefore of ζ) whose drift rate is a and whose Lévy measure is the restriction of L to $[0,c]$. □

In particular, if $L(\{\infty\}) = 0$, then $\zeta = +\infty$ almost surely, Y^c is finite valued, Y^b is independent of Y^c, and Y^b is an increasing additive process with infinite lifetime whose drift rate is a and Lévy measure is $\{L(B);\ B \subset [0,c] \text{ Borel}\}$. Complications arising from the possible finiteness of ζ can be lessened by thinking of Y constructively as follows. Suppose $L(\{\infty\}) > 0$, let L' be the restriction of L to $\mathbb{R}_0 = (0,\infty)$. Let Y' be an increasing additive process with drift rate a and Lévy measure L'; then Y' has infinite lifetime almost surely. Let ζ be a random variable <u>independent</u> of the process Y' and having the exponential distribution with parameter $L(\{\infty\})$. Define

$$(2.8) \qquad Y_t(\omega) = \begin{cases} Y'_t(\omega) & \text{if } t < \zeta(\omega), \\ +\infty & \text{if } t \geq \zeta(\omega). \end{cases}$$

Then, Y is an additive process with drift rate a and Lévy measure L.

3. REGENERATION SETS

As we had mentioned in the introduction, a random set G is a regeneration set if and only if G is the range of an increasing additive process (Y_s), that is,

$$G(\omega) = \{t\colon Y_s(\omega) = t \quad \text{for some } s \geq 0\}. \tag{3.1}$$

In this section we will discuss the structure of G in terms of that of (Y_s). We are assuming that Y has the drift rate a and Lévy measure L. The following summarizes the results.

(3.2) THEOREM. a) If $L(\{+\infty\}) > 0$ then G is almost surely bounded. Otherwise, if $L(\{+\infty\}) = 0$, G is almost surely unbounded.

b) If $a = 0$ and $L(\overline{\mathbb{R}}_0) < \infty$, then G is almost surely a discrete set, that is, every point is isolated. Otherwise, if either $a > 0$ or $L(\overline{\mathbb{R}}_0) = +\infty$ or both, G is almost surely a perfect set (that is, G has no isolated points).

c) If $a = 0$ then $\text{leb}(G) = 0$ almost surely. Otherwise, if $a > 0$, $\text{leb}(G) > 0$ almost surely.

d) If $a > 0$ and $L(\overline{\mathbb{R}}_0) < \infty$, then G is almost surely a countable union of intervals of form [ ), and thus the interior of G is non-empty. Otherwise, if either $a = 0$ or $L(\overline{\mathbb{R}}_0) = +\infty$ or both, G almost surely does not contain any open intervals, and its interior is empty.

PROOF. a) The image G of Y is contained in the interval $[0, Y_{\zeta-}]$ where $\zeta = \inf\{s\colon Y_s = +\infty\}$. The statement (a) now follows from the fact that ζ has the exponential distribution with parameter $L(\{\infty\})$.

b) If $a = 0$ and $L(\overline{\mathbb{R}}_0) < \infty$, then Y is a compound Poisson process, and its image G is a renewal process $\{T_n\}$, and hence almost surely is a discrete set. Otherwise, if either $a > 0$ or $L(\overline{\mathbb{R}}_0) = \infty$, $s \to Y_s$ is almost surely strictly increasing. For a "good" ω, let $t \in G(\omega)$. Then $t = Y_s(\omega)$ for some s. Let $s_n > s$, $s_n \downarrow s$. Since $s \to Y_s(\omega)$ is strictly increasing, $t_n = Y_{s_n}(\omega) > Y_s(\omega) = t$; and by the right continuity of $s \to Y_s(\omega)$, (t_n) decreases to t. Clearly each t_n belongs to $G(\omega)$ and hence $t \in G(\omega)$ is not isolated.

c) Let $t \in G(\omega)$ and choose s so that $Y_s(\omega) = t$. Pick $c > 0$ and define Y^c and Y^b respectively by (2.4) and (2.5). Since Y^c is a compound Poisson process, it has only finitely many jumps during $[0,s]$ and therefore

$$\text{leb}(G(\omega) \cap [0,t]) = \text{leb}\{u: Y_v^b = u \quad \text{for some } v \leq s\}.$$

Taking limits, as $c \downarrow 0$, we see that

$$\text{leb}(G(\omega) \cap [0,t]) = as.$$

This proves (c).

d) If $a > 0$ and $L(\overline{\mathbb{R}}_0) < \infty$, then $Y_t = at + Y_t^d$ where Y^d is a compound Poisson process. Then the assertion is obvious. If $a = 0$ and $L(\overline{\mathbb{R}}_0) < \infty$, then G is discrete and the result is again immediate. Next let $L(\overline{\mathbb{R}}_0) = \infty$. Then, Y has infinitely many jumps (almost surely) in any open interval $(s, s + \varepsilon)$; hence, its image G cannot contain any open intervals and its interior is empty. □

In terms of the parameters a and L of the process Y the situation may be reworded as follows (also see Figures 1, 2, 3 below).

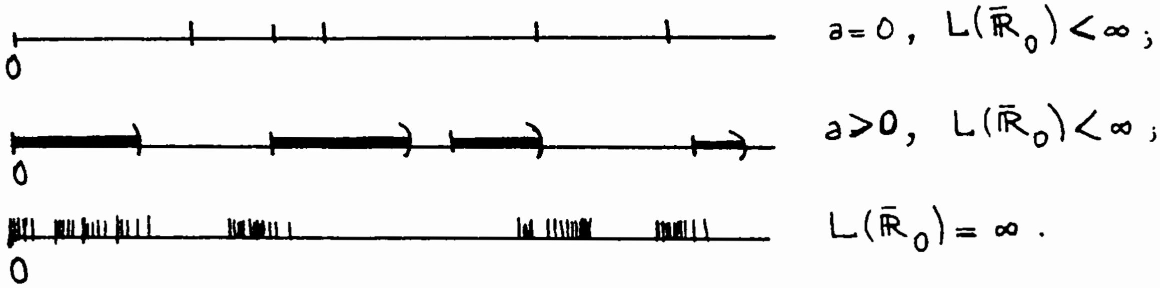

Figure 1. Regeneration set G

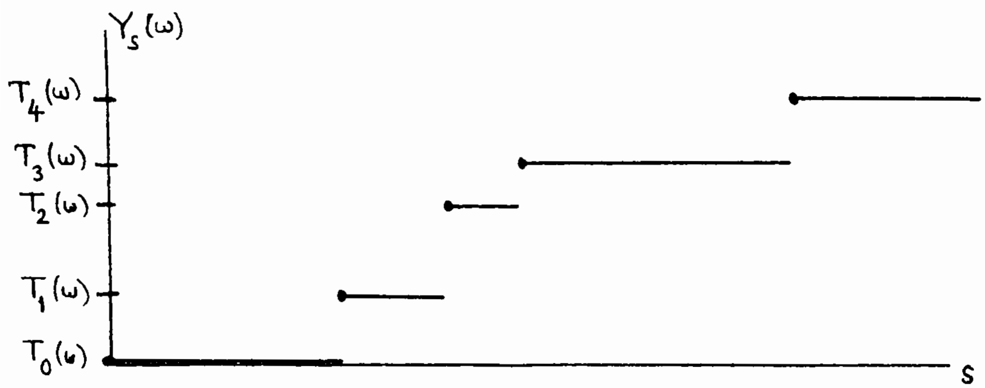

Figure 2. When $a=0$ and $L(\bar{\mathbb{R}}_0)<\infty$, Y is a compound Poisson process and G is the union of the points T_n of the renewal process (T_n).

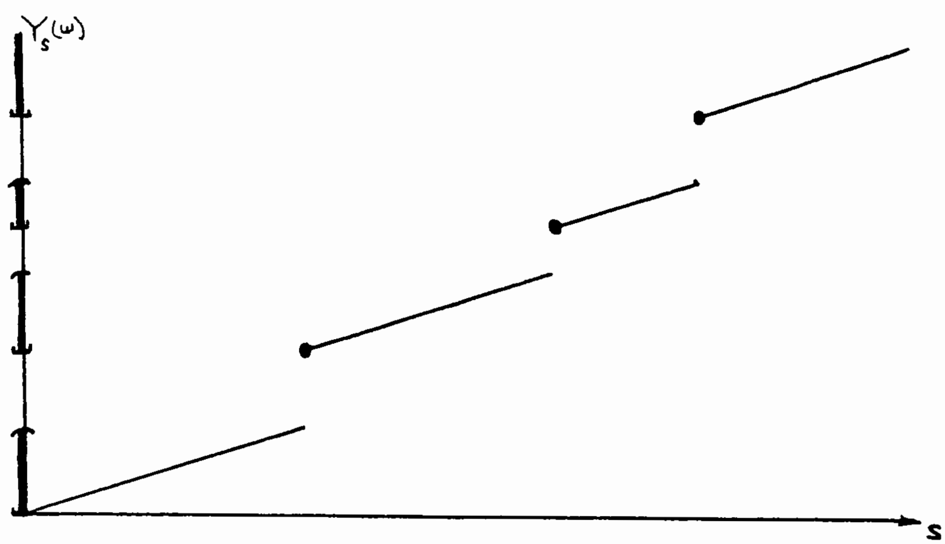

Figure 3. When $a>0$ and $L(\bar{\mathbb{R}}_0)<\infty$, G is a countable union of intervals each of which has positive length.

When $a = 0$ and $L(\bar{\mathbb{R}}_0) < \infty$, Y is a compound Poisson process and its image G is a renewal process. The distribution of the interrenewal times is given by $F(B) = L(B)/L(\bar{\mathbb{R}}_0)$, $B \in \underline{\underline{\bar{\mathbb{R}}}}_+$. This is the simplest case and has been studied in some detail in the past.

When $a > 0$ and $L(\bar{\mathbb{R}}_0) < \infty$, G is a union of intervals $[T_0,S_1),[T_1,S_2)$, $[T_2,S_3),\ldots$ where $0 = T_0 < S_1 < T_1 < S_2 < T_2 < S_3 < \ldots$ almost surely. The lengths of these intervals are independent and identically distributed with

$$P\{S_{n+1} - T_n > t\} = \exp[-L(\bar{\mathbb{R}}_0)/a]. \tag{3.3}$$

The lengths of the intervals contiguous to G, namely the intervals $[S_1,T_1)$, $[S_2,T_2),\ldots$, are independent of the lengths of the intervals of G and are independent and identically distributed with

$$P\{T_n - S_n \in B\} = L(B)/L(\bar{\mathbb{R}}_0). \tag{3.4}$$

Hence this case also is very simple.

The interesting problems (of a nature which classical renewal theory cannot handle directly) come up when $L(\bar{\mathbb{R}}_0) = +\infty$. Then, $s \to Y_s$ is strictly increasing and has infinitely many jumps in any open interval $(t,t+\varepsilon)$, $\varepsilon > 0$. Thus, its range G does not contain any open intervals. The complement of G is a countable union of intervals of form [ ), called the intervals contiguous to G. No two contiguous intervals have any end points in common. G has no isolated points, that is, every point of G is a point of accummulation of G. Moreover, G is right-closed: if $(t_n) \subset G(\omega)$ and $t_n \downarrow t$, then $t \in G(\omega)$ also. Stochastic structure of G, therefore, is of much more interest in this case where $L(\overline{\mathbb{R}}_0) = +\infty$. Further, if $a = 0$, $\mathrm{leb}(G) = 0$; if $a > 0$, $\mathrm{leb}(G) > 0$. Significance of this final point is with respect to computations: If $a > 0$, then $p(t) = P\{t \in G\} > 0$ for all t, and most computations become easier (this is KINGMAN's case of regenerative events [8]; p is then a p-function). If $a = 0$, then $p(t) = P\{t \in G\} = 0$ for all $t > 0$.

4. LOCAL TIME PROCESS

Let $(Y_t)_{t \in \mathbb{R}_+}$ be an increasing additive process with drift rate a and Lévy measure L. We define

$$(4.1) \qquad L_t(\omega) = \inf\{s \geq 0: Y_s > t\}, \qquad t \in \mathbb{R}_+,\ \omega \in \Omega.$$

The process $(L_t)_{t \in \mathbb{R}_+}$ is called the <u>local time</u> associated with (Y_s).

Our point of view is as follows: the process (Y_s) sets up a correspondence between "real time" and "local time" so that $Y_s(\omega)$ is the real time corresponding to the local time s (for the realization ω). Then $L_t(\omega)$ is the local time corresponding to the real time t. If Y were a discrete parameter process, that is, if the local time were measured in discrete units, then Y would be a renewal process and L_t would become the number of renewals during the real time [0,t].

The following theorem due to Lebesgue shows the relationship between (L_t) and (Y_s): The essential idea of the proof is the observation that, for every $t \in \mathbb{R}_+$,

$$(4.2) \qquad L_t = \text{leb}\{s: Y_s \leq t\} = \int_0^\infty 1_{[0,t]}(Y_s)ds.$$

For a complete proof we refer to DELLACHERIE [4, p. 91].

(4.3) THEOREM. The local time process (L_t) is almost surely increasing, right continuous, and has $L_0 = 0$. Further, $t \to L_t(\omega)$ is continuous if and only if $s \to Y_s(\omega)$ is strictly increasing. (Y_s) is related to (L_t) by

$$(4.4) \qquad Y_s(\omega) = \inf\{t \geq 0: L_t(\omega) > s\}, \qquad s \in \mathbb{R}_+,\ \omega \in \Omega.$$

Moreover, for any Borel measurable positive function f on $\mathbb{R}_+$,

$$(4.5) \qquad \int_0^\infty f(t)\,dL_t(\omega) = \int_0^{L_\infty(\omega)} f(Y_s(\omega))\,ds$$

(here $L_\infty(\omega) = \lim_{t\to\infty} L_t(\omega)$; $L_\infty(\omega) < +\infty$ if and only if $\zeta(\omega) = \inf\{s: Y_s(\omega) = +\infty\} < \infty$). □

The importance of local time is based on its facilitating our comprehension of the structure of a regeneration set. This is because of the following

(4.6) THEOREM. Let G be the range of (Y_s) and let J be the set of points of right increase for (L_t):

$$(4.7) \qquad J(\omega) = \{t: L_t(\omega) < L_{t+\varepsilon}(\omega) \quad \text{for all } \varepsilon > 0\}.$$

Then, for almost all $\omega \in \Omega$, $J(\omega) = G(\omega)$.

For the proof we refer to ÇINLAR [3], or MAISONNEUVE [9]. We end this section with the following computational result.

(4.8) THEOREM. For any t, s $\in \mathbb{R}_+$

$$(4.9) \qquad P\{L_t \le s\} = P\{Y_s > t\}$$

except when $a > 0$, $L(\bar{\mathbb{R}}_0) < \infty$, and L has atoms. If $a > 0$ and $L(\bar{\mathbb{R}}_0) < \infty$ and L has atoms, (4.10) $$P\{L_t \leq s\} = P\{Y_s \geq t\}.$$

PROOF. Since $Y_s > t$ implies $L_t \leq s$ we can write $\{L_t \leq s\} = \{Y_s > t\} \cup \{L_t \leq s, Y_s \leq t\}$. Moreover, $Y(L_t) \geq t$ for all t, and $s \geq L_t$ implies $Y_s \geq t$, and by symmetry provided by (4.1) and (4.4), $Y_s \leq t$ implies $L_t \geq s$. Hence,

(4.11) $$\{L_t \leq s\} = \{Y_s > t\} \cup \{Y_s = t, L_t = s\}.$$

If Y is not strictly increasing, $L_t = s$ implies $Y_s > t$, and hence (4.9) follows from (4.11). If Y is strictly increasing, either $a > 0$ or $L(\overline{\mathbb{R}}_0) = +\infty$ or both. If $L(\overline{\mathbb{R}}_0) = \infty$, then $P\{Y_s = t\} = 0$ (see ESSEEN [4 , Remark 2] for a streamlined proof), and hence (4.11) yields (4.9) again. Finally, consider the case $a > 0$, $L(\overline{\mathbb{R}}_0) < \infty$; then, (Y_s) is the sum of (as) and a compound Poisson process. In this case also, $P\{Y_s = t\} = 0$ unless the distribution $F(\cdot) = L(\cdot)/L(\overline{\mathbb{R}}_0)$ of the magnitude of a jump has some atoms. This proves the first statement. When the conditions of the second statement hold, $s \to Y_s$ is strictly increasing because $a > 0$. Thus, $Y_s = t$ implies $L_t = s$, and (4.10) follows from (4.11).

(4.12) REMARK. We will see in the next section (see Theorem (5.8)) that when $a > 0$,

$$R(\{t\}) = \int_0^\infty P\{Y_s = t\} ds = 0.$$

Hence, for any $t \in \mathbb{R}_+$ and almost every $s \in \mathbb{R}_+$,

(4.13) $$P\{L_t \leq s\} = P\{Y_s > t\}.$$

5. POTENTIAL MEASURES

Throughout this section $(Y_t)_{t \in \mathbb{R}_+}$ is an increasing additive process, defined over a complete probability space $(\Omega, \underline{\underline{M}}, P)$, and having the drift rate a and Lévy measure L.

Perhaps the most important computational role in the theory of regeneration is played by the <u>potential measure</u> R defined by

$$(5.1) \qquad R(B) = E\left[\int_0^\infty 1_B(Y_s)ds\right] = \int_0^\infty P\{Y_s \in B\}ds, \qquad B \in \underline{\underline{\overline{\mathbb{R}}}}_+.$$

It is easy to see from (2.1) that the Laplace transform

$$(5.2) \qquad R^\lambda = \int_0^\infty e^{-\lambda t} R(dt), \qquad \lambda > 0,$$

satisfies

$$(5.3) \qquad R^\lambda N^\lambda = 1/\lambda, \qquad \lambda > 0,$$

where N^λ is as defined by (2.2).

Note that N^λ is the Laplace transform of a measure N on $\underline{\underline{\overline{\mathbb{R}}}}_+$ given by

$$(5.4) \qquad N(B) = a\varepsilon_0(B) + \int_B n(s)ds, \qquad B \in \underline{\underline{\overline{\mathbb{R}}}}_+,$$

where ε_0 is the Dirac measure concentrating its unit mass at 0, and where the density n of N on $\overline{\mathbb{R}}_0 = (0,\infty]$ is defined by (recall that $\overline{\mathbb{R}}_s = (s,\infty]$)

$$(5.5) \qquad n(s) = L(\overline{\mathbb{R}}_s), \qquad s \in \overline{\mathbb{R}}_0.$$

The function n is right continuous decreasing and finite valued on $(0,\infty]$.

In general, $n(0+) = \lim_{s \to 0} n(s)$ is infinite, but n is always integrable near the origin, and in fact

(5.6) $$\int_{(0,t]} n(s)ds = \int_{(0,t]} x\,L(dx) + tn(t).$$

Hence, the measure N is finite on compact subsets of $\bar{\mathbb{R}}_0$.

It is clear that R also is finite on compact subsets of $\bar{\mathbb{R}}_+$, and hence the convolution $R * N$ of R and N is well defined. It follows from (5.3) that we have

(5.7) $$R * N(B) = \text{leb}(B), \qquad B \in \underline{\bar{\mathbb{R}}}_+.$$

The following theorem strengthens this result.

(5.8) THEOREM. If $a = 0$, then for every $t \in \mathbb{R}_0$

(5.9) $$\int_{[0,t]} R(ds)n(t - s) = 1.$$

If $a > 0$, then R is absolutely continuous (with respect to the Lebesgue measure), and possesses a continuous function r as its derivative. Then, for every $t \in \mathbb{R}_+$,

(5.10) $$ar(t) + \int_0^t r(s)n(t - s)ds = 1.$$

PROOF. First, suppose $a > 0$. Then (5.4) and (5.7) imply that $aR(B) \leq \text{leb}(B)$, which shows that R is absolutely continuous. Thus, R admits a bounded function r as its derivative, and this r can be chosen to satisfy, in view of (5.7), the equation (5.10) for every $t \in \mathbb{R}_+$. The function n is integrable near the origin (see (5.6) for this), and r is bounded. Hence, their convolution defines a continuous function, that is, the second term of the left side in (5.10) is a continuous function in t. Hence, r must be continuous.

Next, consider the case $a = 0$. Then, (5.7) implies that (5.9) holds for (Lebesgue) almost every $t > 0$. That this implies the same for all t is put next as a separate proposition. □

(5.11) PROPOSITION. If (5.9) holds for almost all $t > 0$, then it holds for all $t > 0$.

PROOF of this innocent looking assertion is very treacherous. The first complete proof was given by KESTEN [7]. Somewhat shorter proofs may be found in ASSOUAD [1] and BRETAGNOLLE [2]. Theorem (5.8) and the proof of the case $a > 0$ are due to NEVEU [10].

(5.12) COROLLARY. R has atoms if and only if $a = 0$, $n(0+) < \infty$, and L has atoms.

PROOF. If $a > 0$, R is absolutely continuous. If $a = 0$ and $n(0+) = \infty$, then

$$R(\{t\})n(0) \leq \int_{[0,t]} R(ds)n(t-s) = 1$$

implies that $R(\{t\}) = 0$. If $a = 0$ and $n(0+) < \infty$, then Y is a compound Poisson process and R is the renewal measure corresponding to the distribution $F(\cdot) = L(\cdot)/n(0+)$. Now the result is evident. □

Next we consider the connection between R and the local time process (L_t) associated with (Y_s) (see (4.1) for the definition). It follows from (4.2) and (5.1) that

$$E[L_t] = R([0,t]), \qquad t \in \mathbb{R}_+. \tag{5.13}$$

Moreover, by the generalized version of the Fubini's theorem applied to (4.5) we have

$$\begin{aligned} E\Big[\int_0^\infty f(Y_s)ds\Big] &= E\Big[\int_0^\infty f(t)dL_t\Big] \\ &= \int_0^\infty f(t)R(dt) \end{aligned} \tag{5.14}$$

for any positive Borel measurable function f defined on $\overline{\mathbb{R}}_+$ with $f(+\infty) = 0$.

We close this section with a computational result of some importance. The potential measure R plays the same role in the present theory as the renewal measure does in renewal theory. Indeed, R can be expressed as a renewal measure in a number of ways. The following is one which is very useful (see the next section for another such representation).

Let $c > 0$ be such that $n(c) > 0$, and define the processes (Y_s^c) and (Y_s^b) as in (2.4) and (2.5). Let

$$S = \inf\{s \geq 0: Y_s^c > 0\} = \inf\{s: Y_s - Y_{s-} > c\}; \tag{5.15}$$

and

(5.16) $$G_c(B) = P\{Y_S^b \in B\}, \quad F_c(B) = P\{Y_S^c \in B\}, \qquad B \in \overline{\mathbb{R}}_+.$$

It follows from Proposition (2.6) that S has the exponential distribution with parameter $n(c)$; Y_S^c is independent of Y_S^b and S, and has the distribution $H_c = L/n(c)$ on $\overline{\mathbb{R}}_c$. These and other facts listed in (2.6) yield the following: For $\lambda \geq 0$,

(5.17) $$H_c^\lambda = E[\exp(-\lambda Y_S^c)] = \int_{(c,\infty]} e^{-\lambda y} L(dy)/n(c);$$

(5.18) $$F_c^\lambda = E[\exp(-\lambda Y_S)] = H_c^\lambda G_c^\lambda$$

(5.19) $$\begin{aligned} G_c^\lambda &= E[\exp(-\lambda Y_S^b)] \\ &= \int_0^\infty n(c)e^{-n(c)u} du \exp\{-u[\lambda a + \int_{(0,c]} \left(1 - e^{-\lambda y}\right) L(dy)]\} \\ &= n(c)[n(c) + \lambda a + \int_{(0,c]} \left(1 - e^{-\lambda y}\right) L(dy)]^{-1} \\ &= n(c)[\lambda N^\lambda + n(c)H_c^\lambda]^{-1}. \end{aligned}$$

Now let R_c be the renewal measure corresponding to R_c, that is,

(5.20) $$R_c = \sum_{n=0}^\infty F_c^{n*}.$$

Then, the Laplace transform of R_c is $R_c^\lambda = (1 - F_c^\lambda)^{-1}$. This fact along with (5.3),(5.18), and (5.19) proves the following important observation.

(5.21) PROPOSITION. Let $c > 0$ be such that $n(c) > 0$. Then,

$$R = \frac{1}{n(c)} R_c * G_c . \qquad \square$$

The preceding proppsition shows that R is (up to a constant multiplicative term) the renewal measure of a delayed renewal process. This delayed renewal process $(T_n)_{n=1,2,...}$ is given by

(5.22) $$T_n = Y_{S_n^-}, \qquad n = 1,2,...,$$

where S_n is the instant of the n^{th} jump of Y whose magnitude exceeds c.

In fact, noting that $\frac{1}{n(c)} = E[S]$, we may reword (5.21) as follows:

(5.23) $$E[L_t] = E[S]E[N_t]$$

where L_t is the local time at t, N_t is the number of jumps with magnitude greater than c that Y makes before reaching the real time level t, and the $S_{n+1} - S_n$ are the local times between these jumps.

6. RENEWAL LIMIT THEOREMS

Let (Y_s) be as in the previous section and let R be its potential measure. The potential measure plays the same role in the theory of continuous regeneration as the renewal measure does in the theory of discrete regeneration (which is usually called renewal theory). In fact R itself is the renewal measure of a delayed renewal process:

Let $S_1, S_2, \ldots$ be the points of a Poisson process with parameter 1 independent of (Y_s), and put $T_1 = Y(S_1)$, $T_2 = Y(S_2), \ldots$. Then, (T_n) is a (delayed) renewal process and a simple computation shows that its renewal measure is exactly R, that is, for every $B \in \overline{\underline{\mathbb{R}}}_+$,

$$(6.1) \qquad R(B) = E\Big[\int_0^\infty 1_B(Y_s)ds\Big] = E\Big[\sum_{n=1}^\infty 1_B(T_n)\Big].$$

Note that the expected value of the "interrenewal times" $T_{n+1} - T_n$ is

$$(6.2) \qquad m = a + \int_{\mathbb{R}_0} xL(dx) = a + \int_{\mathbb{R}_0} n(s)ds.$$

The following renewal limit theorems are, then, simply restatements of the classical limit theorems (see, for instance, FELLER [5] for proofs and also for the definition of direct Riemann integrability). We exclude

$$(6.3) \qquad \text{CASE: } a = 0 \text{ and } n(0+) = L(\overline{\mathbb{R}}_0) < \infty$$

for reasons of simplicity. This is the case where the range of Y is that of a renewal process and falls in the classical renewal theory case. We will omit the proofs of the following two main theorems (only note that R has no atoms when case (6.3) is excluded; see Corollary (5.12) for this).

(6.4) THEOREM. Suppose Case (6.3) does not hold. Then R has no atoms and

(6.5) $$\lim_{t \to \infty} R(t + B) = \frac{1}{m} \operatorname{leb}(B)$$

for any finite interval $B \subset \mathbb{R}_+$. In particular, if $a > 0$, for the density r of R we have

(6.6) $$\lim_{t \to \infty} r(t) = \frac{1}{m}.$$

(6.7) THEOREM. Suppose (6.3) does not hold. If $n(\infty) > 0$, then

(6.8) $$\lim_{t \to \infty} R * g(t) = g(\infty)/n(\infty)$$

provided that $g(\infty) = \lim_{t \to \infty} g(t)$ exists. If $n(\infty) = 0$, then

(6.9) $$\lim_{t \to \infty} R * g(t) = \frac{1}{m} \int_0^\infty g(s)ds$$

provided that g be directly Riemann integrable.

Next is a refinement of Theorem (6.4).

(6.10) THEOREM. Suppose (6.3) does not hold and

(6.11) $$v^2 = \int_{(0,\infty]} x^2 L(dx) < \infty.$$

Then $m < \infty$ and

(6.12) $$\lim_{t \to \infty} \left\{R([0,t]) - \frac{t}{m}\right\} = v^2/2m^2.$$

PROOF. By (2.10) we have $R * N$ equal to the Lebesgue measure. If $v^2 < \infty$, then $m < \infty$ obviously, and the total mass of N is $N(\mathbb{R}_+) = m$ (see (5.4) and (6.2) for this). Hence,

(6.13) $$R([0,t]) - \frac{t}{m} = R * g(t)$$

with

(6.14) $$g(t) = \frac{N((t,\infty))}{N([0,\infty))} = \frac{1}{m}\int_t^\infty n(s)ds.$$

Now g is monotone decreasing and is integrable with

(6.15) $$\int_0^\infty g(t)dt = \frac{1}{m}\int_0^\infty sn(s)ds = \frac{1}{m}v^2.$$

Thus g is directly Riemann integrable, and (6.9) applies to yield the desired result. □

7. BACKWARD AND FORWARD RECURRENCE TIMES

Let (Y_s) be an increasing additive process with drift rate a and Lévy measure L, and let G be the regeneration set associated with it, that is, $G = \{t: Y_s = t \text{ for some } s\}$. The points of G are to be thought of as points of regeneration. Then, for $\omega \in \Omega$,

$$W_t(\omega) = \inf\{u > t: u \in G(\omega)\} - t \tag{7.1}$$

is the waiting time from t until "the time of the next regeneration," and

$$V_t(\omega) = t - \sup\{u \leq t: u \in G(\omega)\} \tag{7.2}$$

is the elapsed time since "the last regeneration." In terms of the process (Y_s) and its local time (L_t), we have

$$V_t = t - Y_{L_t -}, \quad W_t = Y_{L_t} - t. \tag{7.3}$$

In the case where G is a discrete set (that is, if Y is a compound Poisson process, or in terms of the parameters of Y, if $a = 0$ and $L(\overline{\mathbb{R}}_0) < \infty$), then $G = \bigcup_n [T_n]$ where (T_n) is a renewal process. In this case, the backward and forward recurrence times are easy to examine (and are already studied in some detail). We will omit this case from further consideration by making the following

(7.4) ASSUMPTION. Either $a > 0$ or $L(\overline{\mathbb{R}}_0) = +\infty$ or both.

The process $(V_t)_{t \in \mathbb{R}_+}$ is right continuous; $V_0 = 0$; $t \to V_t$ increases with slope one over all intervals contiguous to G; $V_t = 0$ for $t \in \bar{G}$ where $\bar{G}$ is the closure of G. In the converse direction, if the process (V_t) is known, the regeneration set G is obtained as follows: The set $\{t: V_t = 0\}$ is the closure $\bar{G}$ of G, and G is obtained from $\bar{G}$ by removing from $\bar{G}$ those points $t \in \bar{G}$ which are the left end points of intervals contiguous to $\bar{G}$.

The process $(W_t)_{t \in \mathbb{R}_+}$ is right continuous; $W_0 = 0$; $t \to W_t$ decreases at rate one over any interval contiguous to G; if $[S,T)$ is a contiguous interval, $W_S = T - S$ and $W_{T-} = 0$; if t belongs to G then $W_t = 0$; or conversely, if $W_t = 0$, then $t \in G$.

This section is devoted to computational issues regarding (V_t) and (W_t).

The method of the computations below rests on the following observation: If the waiting time $W_t(\omega)$ is greater than c, then t belongs to a contiguous interval whose length is greater than c.

Let $c > 0$ be such that $n(c) > 0$; and let U and T be the left end point and the right end point of the first contiguous interval of G with length strictly greater than c. In other words, if $S = \inf\{s: Y_s - Y_{s-} > c\}$, we have $U = Y_{S-}$, $T = Y_S$. It follows from Proposition (2.6) that U and $T - U$ are independent and have the respective distributions G_c and H_c defined by (5.17) and (5.19). We utilize these facts to obtain the following.

(7.5) THEOREM. For any $t > 0$, $b \geq 0$, and $c \geq 0$,

$$P\{V_t \geq b,\ W_t > c\} = \int_0^{t-b} R(du)n(t + c - u).$$

(7.6) REMARK. Integral over an empty set is zero. Regarding the limits of integration, we note that whether the end points are included or not makes no difference, since R has no atoms under Assumption (7.4) by Corollary (5.12).

PROOF. We will prove the theorem for $c > 0$. Having done that, the proof for $c = 0$ follows from the monotone convergence theorem applied to the right side written for $c_k \downarrow 0$ (note that $n(t + c_k - u)$ increases with k since n is a decreasing function).

For $c > 0$, if $n(c) = 0$ then both sides vanish. Therefore we assume that $c > 0$ and $n(c) > 0$. Let the probability on the left be denoted by $f(t)$. Conditioning on T, by the regeneration property at T,

$$(7.7) \qquad P\{V_t \geq b,\ W_t > c \mid T\} = f(t - T) \quad \text{on } \{T \leq t\},$$

and therefore,

$$(7.8) \qquad P\{V_t \geq b,\ W_t > c,\ T \leq t\} = \int_0^t F_c(du) f(t - u),$$

where $F_c = G_c * H_c$ (see (5.18) also). On the other hand,

$$(7.9) \qquad P\{V_t \geq b,\ W_t > c,\ T > t\}$$

$$= P\{U \leq t - b,\ T > t + c\}$$

$$= \int_0^{t-b} G_c(dy) H_c\big((t + c - y, \infty]\big) = G_c * h(t)$$

where

$$(7.10) \qquad h(u) = 1_{[b,\infty]}(u) H_c\big((u,\infty]\big) = 1_{[b,\infty]}(u) \cdot n(u + c)/n(c).$$

Putting (7.8) and (7.9) together we obtain

(7.11) $$f = G_c * h + F_c * f$$

which is a renewal equation. It is well known that its solution is

(7.12) $$f = R_c * (G_c * h), \quad \text{where } R_c = \sum_{n=0}^{\infty} F_c^{n*}.$$

The desired result that $f(t) = R * h(t)n(c)$ now follows from (7.10), (7.12), and Proposition (5.21). □

(7.13) COROLLARY. For any $t > 0$ and $c \geq 0$,

$$P\{W_t > c\} = \int_0^t R(du)n(t + c - u).$$

(7.14) PROPOSITION. Let

(7.15) $$p(t) = P\{W_t = 0\}.$$

If the drift rate a is zero then $p(t) = 0$ for all $t > 0$. Otherwise, if $a > 0$, $p(t) > 0$ for all $t \geq 0$ and

(7.16) $$p(t) = ar(t)$$

where r is the density of the potential measure R.

PROOF. If $a = 0$ then $R * n(t) = 1 - p(t) = 1$ for all t by Theorem (5.8). If $a > 0$, by (5.8) again, R has a continuous derivative and (5.10) shows that $p = ar$ as claimed. There remains to show that $p(t) > 0$ for all t.

Applying the regeneration property at t, we have

$$(7.17)\qquad p(t+s) = P\{W_{t+s} = 0\} \geq P\{W_t = 0,\ W_{t+s} = 0\} = p(t)p(s).$$

On the other hand, as $t \downarrow 0$,

$$(7.18)\qquad 1 - p(t) = \int_{[0,t]} R(du)n(t-u) = \int_{(0,t]} r(u)n(t-u)du \to 0;$$

since r is continuous and n is integrable near zero. So, $p(t) \to 1$ as $t \to 0$. This fact along with (7.17) shows that $p(t) > 0$ for all t.

(7.19) REMARK. We in fact showed that, when $a > 0$, $p(t)$ is a continuous strictly positive function on $\mathbb{R}_+$ with $p(0) = 1$.

The following puts together (7.13) and (7.14):

(7.20) PROPOSITION. For any $t \geq 0$ and $B \in \overline{\mathbb{R}}_+$

$$P\{W_t \in B\} = p(t)\varepsilon_0(B) + \int_0^t R(du)L(t-u+B).$$ □

Proof is evident. Next is a similar result for (V_t):

(7.21) PROPOSITION. For any $t \geq 0$ and $B \in \overline{\mathbb{R}}_+$,

$$P\{V_t \in B\} = p(t)\varepsilon_0(B) + \int_0^t R(du)n(t-u)1_B(t-u).$$

PROOF. If $W_t(\omega) = 0$, then $t \in G(\omega)$, which implies that $t \in \bar{G}(\omega)$, which is the same as $V_t(\omega) = 0$. Hence, $\{W_t = 0\} \subset \{V_t = 0\}$. Therefore, for $b > 0$, Theorem (7.5) gives

$$(7.22)\qquad P\{V_t \geq b\} = P\{V_t \geq b,\ W_t > 0\} = \int_0^t R(du)n(t-u)1_{[b,\infty]}(t-u).$$

As $b \downarrow 0$, $1_{[b,\infty]}(t-u)$ increases to $1_{\mathbb{R}_+}(t-u)$. By the monotone convergence theorem, then, (7.22) implies also that

$$(7.23)\qquad P\{V_t > 0\} = \int_0^t R(du)n(t-u) = P\{W_t > 0\} = 1 - p(t).$$

The result desired now is immediate from (7.22) and (7.23). □

We sum up these computations in the following theorem; the proof is evident and will be omitted.

(7.24) THEOREM. For any $t \geq 0$, and any Borel measurable positive function f on $\bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$,

$$E[f(V_t,W_t)] = p(t)f(0,0) + \int_0^t R(du)\int_0^\infty L(t-u+dx)f(t-u,x).$$

Next is a formula for the probability that the regeneration set G has no points in the interval $(s,t]$. For any $u \in (s,t]$ this event is equal to $\{V_u \geq u-s,\ W_u > t-u\}$. Thus, the probability sought is, by Theorem (7.5),

$$(7.25)\qquad P\{G \cap (s,t] = \emptyset\} = \int_{[0,s]} R(dv)n(t-v).$$

We end this section with a discussion of the limiting behaviors of V_t and W_t as $t \to \infty$. This is largely an application of the renewal limit theorems of the previous section. In particular, recall that m (defined by (6.2)) is the mean rate of increase of Y.

(7.26) THEOREM. If $m < \infty$ then for any $b \geq 0$ and $c \geq 0$

$$\lim_{t \to \infty} P\{V_t \geq b,\ W_t > c\} = \frac{1}{m} \int_{b+c}^{\infty} n(s)ds.$$

PROOF. Since $m < \infty$, $n(\infty) = 0$ and (6.9) holds. We apply (6.9) to the formula provided by (7.5) after noting that $g = 1_{[b,\infty]}\, n(\cdot + c)$ is directly integrable. The latter fact follows from the monotonicity of n coupled with the integrability of g:

$$\int_0^{\infty} g(s)ds = \int_{b+c}^{\infty} n(s)ds \leq \int_0^{\infty} n(s)ds = m < \infty. \qquad \square$$

As a corollary, it is interesting to note that, when $m < \infty$,

$$(7.27) \qquad p(t) = P\{W_t = 0\} = P\{V_t = 0\} \to \frac{a}{m}, \qquad t \to \infty.$$

which also follows directly from the facts that $p(t) = ar(t)$ and $r(t) \to 1/m$ as $t \to \infty$ by (6.6).

Next we consider the case $m = +\infty$.

(7.28) THEOREM. If $m = \infty$ then for any $c \geq 0$

$$\lim_{t \to \infty} P\{W_t > c\} = 1.$$

PROOF. Consider $R([t,t+c+b])$ for $b > 0$. Conditioning on W_t we obtain, since $t + W_t$ is a regeneration time,

$$R(t + [0,c+b]) = E\Big[\int_0^\infty 1_{t+[0,c+b]}(Y_s)ds\Big]$$

$$\geq \int_0^c P\{W_t \in dx\}E\Big[\int_0^\infty 1_{[0,c+b-x]}(Y_s)ds\Big]$$

$$\geq \int_0^c P\{W_t \in dx\}R([0,b])$$

$$= P\{W_t \leq c\}R([0,b]).$$

When $m = +\infty$, $R(t + [0,c+b]) \to 0$ as $t \to \infty$ by Theorem (6.4); hence, $P\{W_t \leq c\} \to 0$ as $t \to \infty$. □

(7.29) REMARK. It is now easy to see that Theorem (7.26) holds for $m = \infty$ also provided that we interpret $\frac{\infty}{\infty} = 1$ there.

(7.30) REMARK. Note that, when $m < \infty$, the measure N given by (5.4) has total mass $N(\overline{\mathbb{R}}_+) = m$. It is then easy to see from Theorem (7.26) that, for any compact $B \subset \mathbb{R}_+$,

$$\lim_{t\to\infty} P\{W_t \in B\} = \frac{1}{m} N(B). \tag{7.31}$$

As should be expected, we will see in the next section that N is an invariant measure for the "Markov process (W_t)" even when $m = \infty$.

8. DELAYED PROCESSES

In this section we will discuss briefly those regeneration sets which do not contain the point 0 necessarily. Let G^0 be a perfect regeneration set defined over the probability space $(\Omega^0, \underline{\underline{H}}^0, P)$ and which includes 0 almost surely. Let

$$(8.1) \qquad \Omega = \overline{\mathbb{R}}_+ \times \Omega^0, \quad \underline{\underline{H}} = \underline{\underline{\overline{\mathbb{R}}}}_+ \otimes \underline{\underline{H}}^0, \quad P^\phi = \phi \times P;$$

and for $\omega = (x, \omega^0) \in \Omega$, put

$$(8.2) \qquad G(\omega) = \begin{cases} x + G^0(\omega^0) & \text{if } x < \infty \\ \emptyset & \text{if } x = +\infty. \end{cases}$$

If Y^0 is the increasing additive process defined over $(\Omega^0, \underline{\underline{H}}^0, P)$ whose range is G^0, then for each $\omega \in \Omega$, $G(\omega)$ is the range of the process $(Y_t)_{t \in \mathbb{R}_+}$ defined on $(\Omega, \underline{\underline{H}})$ by

$$(8.3) \qquad Y_t(\omega) = x + Y_t^0(\omega^0) \qquad \text{if } \omega = (x, \omega^0).$$

It is clear that, over the probability space $(\Omega, \underline{\underline{H}}, P^\phi)$, (Y_t) is an increasing right continuous process with stationary and independent increments whose initial distribution is ϕ.

Throughout the following we write P^x for P^ϕ when $\phi = \varepsilon_x$ where ε_x is the Dirac measure putting its unit mass at x. We define the local time process (L_s) as before, by formula (4.1), but for the present Y. Define

$$(8.4) \qquad W_t = Y_{L_t} - t, \qquad t \in \mathbb{R}_+.$$

Then (W_t) is a right continuous Markov process. In fact, it can be shown to be a Hunt process if we had the proper machinery set up. Let (P_t) be the transition semi-group of (W_t), that is,

$$(8.5) \qquad P_t(x,B) = P^x\{W_t \in B\}, \qquad x \in \mathbb{R}_+, \quad B \in \underline{\underline{\mathbb{R}}}_+.$$

(8.6) PROPOSITION. We have

$$P_t(x,B) = \begin{cases} \varepsilon_{x-t}(B) & \text{if } t \leq x, \\ F(t-x,B) & \text{if } t \geq x \end{cases}$$

for all $t \in \mathbb{R}_+$ and $B \in \underline{\underline{\mathbb{R}}}$, where

$$(8.7) \qquad F(t,B) = p(t)\varepsilon_0(B) + \int_0^t R(ds)L(t - s + B)$$

in the notations of Section 7 for the process Y^0.

PROOF is immediate from the observation that, if $W_0 = x$, then $W_t = x - t$ for $t \leq x$, and at x there starts an ordinary regeneration set G^0.

Consider next the α-potential operators U^α of (W_t) defined by

$$(8.8) \qquad U^\alpha f(x) = E^x\Big[\int_0^\infty e^{-\alpha t} f(W_t)dt\Big], \qquad x \in \mathbb{R}_+,$$

for any non-negative Borel measurable function f on $\mathbb{R}_+$. Then,

$$(8.9) \qquad U^\alpha f(x) = \int_0^\infty e^{-\alpha t} \int P_t(x,dy)f(y)dt,$$

and carrying out the integrations involved we obtain

$$(8.10) \qquad U^{\alpha}f(x) = f^{\alpha}(x) + e^{-\alpha x}[af(0) + \int_0^{\infty} L(dz)f^{\alpha}(z)]R^{\alpha}$$

where R^{α} is as defined by (5.2) (see also (5.3) and (2.2)), and where

$$f^{\alpha}(x) = \int_0^{x} e^{-\alpha t} f(x - t)dt, \qquad x \in \mathbb{R}_+.$$

For arbitrary initial distributions ϕ we have

$$(8.11) \qquad \phi P_t(B) = P^{\phi}\{W_t \in B\} = \int_{\mathbb{R}_+} \phi(dx)P_t(x,B)$$

and

$$(8.12) \qquad \phi U^{\alpha}f = E^{\phi}\Big[\int_0^{\infty} e^{-\alpha t} f(W_t)dt\Big] = \int_{\mathbb{R}_+} \phi(dx)U^{\alpha}f(x).$$

We close this section by showing that the measure N defined by (5.4) is the unique invariant measure for (W_t) (up to multiplication by a constant of course). A measure ν is said to be invariant for (W_t) or for (P_t) if

$$(8.13) \qquad \nu P_t(B) = \int \nu(dx)P_t(x,B) = \nu(B), \qquad B \in \underline{\underline{R}}_+.$$

(8.14) THEOREM. The measure N defined by (5.4) for the process $(Y_t^0) = (Y_t - Y_0)$ is the unique invariant measure for (W_t).

PROOF. Let f be a bounded Borel measurable function with a compact support. To show that N is invariant, it is sufficient to show that $NP_t(f) = N(f)$.

Now by Proposition (8.6),

$$(8.15)\qquad NP_t(f) = \int N(dx) \int P_t(x,dy)f(y)$$

$$= \int_{[0,t]} N(dx)F(t-x,f) + \int_{(t,\infty)} N(dx)f(x-t)$$

$$= N * F(t,f) + \int_{(t,\infty)} n(x)f(x-t)dx.$$

Note that

$$F(t,f) = f(0) + R * g(t)$$

where

$$g(t) = \int L(t+dx)[f(x) - f(0)].$$

This, together with the fact that $N * R$ is equal to the Lebesgue measure by (5.7), implies

$$(8.16)\qquad N * F(t,f) = f(0)N([0,t]) + \int_0^t g(t-s)ds$$

$$= f(0)[a + \int_0^t n(s)ds] + \int_0^t du \int_{(u,\infty)} L(dz)[f(z-u) - f(0)]$$

$$= af(0) + \int_0^t du \int_{(u,\infty)} L(dz)f(z-u)$$

$$= af(0) + \int_{(0,t]} L(dz) \int_0^z f(z-u)du + \int_{(t,\infty)} L(dz) \int_0^t f(z-u)du.$$

On the other hand,

$$(8.17)\qquad \int_{(t,\infty)} n(x)f(x-t)dx = \int_{(t,\infty)} L(dz) \int_t^z f(x-t)dx.$$

Putting (8.16) and (8.17) into (8.15), we obtain

$$NP_t(f) = af(0) + \int_{(0,t]} L(dz) \int_0^z f(s)ds + \int_{(t,\infty)} L(dz) \int_{z-t}^z f(s)ds$$

$$+ \int_{(t,\infty)} L(dz) \int_0^{z-t} f(s)ds$$

$$= af(0) + \int_{(0,\infty)} L(dz) \int_0^z f(s)ds$$

$$= af(0) + \int_{(0,\infty)} n(s)f(s)ds = \int N(dx)f(x) = N(f).$$

Hence N is an invariant measure.

To show uniqueness, let ν be an invariant measure. Then $\nu(f) = \nu P_t(f)$ for all Borel measurable non-negative functions f. It follows from (8.9) that

$$\alpha\nu U^\alpha(f) = \nu(f) \tag{8.18}$$

for all $\alpha > 0$. In particular, for

$$f(x) = e^{-\lambda x}, \quad x \geq 0, \tag{8.19}$$

we have, by using (8.10),

$$(\alpha - \lambda)U^\alpha f(x) = e^{-\lambda x} - e^{-\alpha x} + e^{-\alpha x}R^\alpha(\alpha N^\alpha - \lambda N^\lambda) \tag{8.20}$$

$$= e^{-\lambda x} - e^{-\alpha x}R^\alpha(\lambda N^\lambda).$$

Hence, for f defined by (8.19), letting ν^{λ} denote $\nu(f)$, we see from (8.18) and (8.20) that

$$(\alpha - \lambda)\nu^{\lambda} = \alpha\nu^{\alpha} - \alpha\nu^{\alpha}R^{\alpha}(\lambda N^{\lambda}),$$

or equivalently,

$$\lambda\nu^{\lambda}R^{\lambda} = \alpha\nu^{\alpha}R^{\alpha}. \tag{8.21}$$

Since this holds for all α and λ positive, denoting the constant $\alpha\nu^{\alpha}R^{\alpha}$ by c, we see that

$$\lambda\nu^{\lambda} = c\cdot(R^{\lambda})^{-1} = c\lambda N^{\lambda}. \tag{8.22}$$

Hence, ν is a constant multiple of N. This completes the proof. □