Discussion Paper No. 1037

EXPECTED VALUE, EXPECTED UTILITY... OR SHOULD WE EXPECT SOMETHING ELSE*

by

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December 1992

* I wish to thank Morton Kamien and Roger Myerson for helpful comments. I am especially grateful to Itzhak Gilboa for stimulating discussions and numerous suggestions that significantly influenced the original version of the paper.

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ABSTRACT

A procedure for decision-making under risk is developed and axiomatized. It provides another explanation for the Allais paradox as well as justification for some other preference patterns that can not be represented by the expected utility model, but it includes expected utility representation of preferences as a particular case. The idea of the procedure is that evaluation of the lotteries takes two steps. First, a decision maker classifies a lottery as a "bad", "good" or "medium" one. Then comparing the lotteries the decision maker uses lexicographic ordering between the classes and expected utility value (with possibly different utility scales for different classes) within each of the three categories. The paper contains comparison of the suggested procedure with several other non-expected utility models. Many preference patterns that motivated the other models can be explained within the suggested procedure.

Key words: threshold, expected utility, Allais paradox.

The approach which is overwhelmingly used for decision-making under risk is expected utility maximization. There is a great bulk of literature that challenges almost all aspects of this theory. The first prominent strike is due to Allais (1953). A definitely incomplete list of more recent sources includes Handa (1977), Kahneman and Tversky (1979), Machina (1982), Quiggin (1982), Slovic and Lichtenstein (1983), Yaari (1987), Karni and Safra (1987), (1989), Camerer (1989), Segal (1989). Nevertheless the majority of applications still use that methodology.

In section 2 we develop a procedure that can explain preferences, incompatible with expected utility maximization, and which is yet very close to the expected utility model and includes it as a special case. This is because the expected utility model seems to work and to make sense as long as prizes and probabilities are not extreme.

The idea of the procedure is the following. A decision maker has lower and upper thresholds in both outcomes and probabilities. If the probability of payoff realization below the lower threshold in prizes is not negligible (i.e. above the lower threshold in probabilities) then the decision maker considers such a lottery a "bad" one. Similarly, if there are few chances for bad luck but sufficiently high probability to win something "really good" then the lottery is a "good" one. Finally, if the probability of any dramatic change in the current situation is small enough the decision maker thinks of such a lottery as of an "average" one. Then the decision maker uses possibly distinct utility scales to order the lotteries according to expected utility within each category.

This simple procedure may indeed be derived from the three intuitive axioms, which do not seem to be more restrictive than the other axioms commonly used and are attractive from the descriptive point of view. One of the axioms stipulates that a decision maker treats in a special way "unusually bad" and "unusually good" outcomes. The second axiom says that the decision maker takes expected utility into account. Finally, the third axiom demands compliance of the choice rule with first order stochastic dominance, which is a very mild rationality requirement.

Hints to the basic idea of our procedure could be found in classical writings. For example, Arrow (1971) cites Buffon and Cournot as having suggested the "general principle that events whose probability is sufficiently small are to be regarded as morally impossible." Arrow also refers to the Menger's argument that the amount which an individual is willing to pay for an uncertain outcome depends upon "the diminishing marginal utility of money, the ratio of the initial size of the fortune to the minimal subsistence level in comparison with chances of gain and the systematic undervaluation of both very large and very small probabilities." Besides, similar considerations underlied theories of cardinal utility by Handa (1977), Kahneman and Tversky (1979), Quiggin (1982), Yaari (1987).

The rest of the paper is organized as follows: section 1 contains notation and formal definitions; the axioms are introduced in section 2; the main results are derived in section 3; in section 4 we discuss properties of the proposed procedure and its relation to some other non-expected utility models; proofs of some technical results are put in the appendix.

1. NOTATION AND DEFINITIONS

Let $\mathbf{X}$ be a closed interval $[a,b] \subset \mathfrak{R}$. $\geq_X$ is a weak order on $\mathbf{X}$, coinciding with the natural order of the real numbers.[1]

Denote by **Q** the set of all probability distributions (lotteries) on **X** which is endowed with the σ-algebra induced by the Borel σ-algebra on [a,b]. $\geq_Q$ is a weak order on **Q**.

Given an arbitrary set S with a weak order $\geq_S$ we define, for $f,g \in S$ satisfying $f \geq_S g$, $(g,f) = \{a \in S \mid f >_S a >_S g\}$. (g,f) is then called an open $>_S$-interval. A closed (half-closed) $>_S$-interval is defined similarly.

For $x \in \mathbf{X}$ let Lx denote a lottery which assigns probability one to x.

For any $s,r \in \mathbf{Q}$ let us define $S^-(x) = \int_{\{y \leq_X x\}} s(y)dy$, $S^+(x) = 1 - S^-(x)$.

For $s,r \in \mathbf{Q}$, s dominates r ($s \geq^1 r$) if for all $x \in \mathbf{X}$, $R^-(x) \geq S^-(x)$. s strictly dominates r ($s >^1 r$) if at least for one x there is strict inequality.

For an open $>_Q$-interval (g,f), g^* will denote some element of $\inf_{>^1}(g,f)$; f^* will denote an element of $\sup_{>^1}(g,f)$.

A closed $>_Q$-interval [g,f] will be called a vNM-interval if it is a maximal $>_Q$-interval I, with which the von Neuman-Morgenstern axioms are satisfied, i.e.

i) $\forall q,r,s \in I$, $q \geq_Q s$, $\forall \alpha \in (0,1)$ $\alpha q + (1-\alpha)r \geq_Q \alpha s + (1-\alpha)r$;

ii) $\forall q,r,s \in I$, $q >_Q r >_Q s$ $\exists \alpha,\beta \in (0,1)$ such that $\alpha q + (1-\alpha)s >_Q r >_Q \beta q + (1-\beta)s$.

We call a non-empty open $>_Q$-interval (g,f) an open vNM-interval if $[g^*,f^*]$ is a vNM-interval and $[g,f] \neq [g^*,f^*]$. We will clarify this definition after lemma 2.3.

For $f,g \in \mathbf{Q}$ let us define $f \cap g$ and $f \cup g$ by their decumulative distributions as follows: $S^+ \cap R^+(x) = \min(S^+(x), R^+(x))$ and $S^+ \cup R^+(x) = \max(S^+(x), R^+(x))$.

2. AXIOMS

Throughout the paper we will assume (unless otherwise is stated) that $\geq_Q$ satisfies the following axioms:

Axiom A1. If $f,g \in Q$, $f \geq_Q g$ are not in a vNM-interval then either:

i) $\forall h \in Q$ such that $g \geq_Q h$, $[h,g]$ is in a vNM-interval; or

ii) $\forall h \in Q$ such that $h \geq_Q f$, $[f,h]$ is in a vNM-interval.

Axiom A2. If $f,g \in Q$ are in a vNM-interval then $f \cup g$ and $f \cap g$ are also in the same vNM-interval.

Axiom A3. For $f,g \in Q$ if $f \geq^1 g$ then $f \geq_Q g$ and if $f >^1 g$ then $f >_Q g$.

Axioms A1 - A3 suffice for our main result, so let us discuss their meaning in more details.

Axiom A1 implies that a decision maker splits the whole set of lotteries into not more than three subsets and within each of them he or she behaves like an expected utility maximizer. Why could one have preferences like that? Because in many instances it seems natural to apply some coarse, preliminary structure on the set of alternatives, i.e. if the lotteries are "sufficiently different" it is natural to distinguish several classes within them. Then, how many classes? Probably, not too many, since this division is preliminary and categorical. If so, why not just two of them: good and bad? In fact, in many cases this is a plausible assumption. However, in a more general case it is not enough, since possibility of both "very good" and "very bad" outcomes can drastically change one's perception about lotteries.

For example, suppose that a person is offered a lottery, the outcomes of which are to lose one's house, to get a TV set or to win $10,000,000. Someone who is scared by the very idea of the first outcome might say:

"Forget it! If a house is at stake, I don't want to participate in such lottery, regardless of what the other prizes are" - unless, of course the chances of the first outcome are negligible. On the other hand, a person who is very dissatisfied with his current well-being and always dreamed about luxurious life would, probably, exclaim: "Sure! If I can only get $10,000,000, I will certainly go for it!"

This suggests that introduction of three categories is sometimes more realistic. Now suppose that the first individual is forced to choose among the lotteries about all of which he would rather forget if he had such an option. Despite the fact that all of them are very undesirable, he would choose the least of the bads. And why not to use expected utility to determine the one? The same is equally applicable to the person who has only very favorable lotteries in the agenda.

Summing up, the first axiom assumes that a decision maker may distinguish between three groups of lotteries (which we would like to interpret as bad, average and good) and considers the lotteries within each group "close enough" to choose between them on the basis of their expected utility.

Axiom A2 is the most restrictive one. It says that if two lotteries are "close" (i.e. are in the same category, namely, a vNM-interval), then the lotteries obtained by their combination in two specific ways are also "close" to the original ones. How do we mutilate the original lotteries to get these combinations? We represent each lottery by its decumulative distribution function and take minimum or maximum of the decumulative distributions with which we start. This basically means the following. Suppose we start with two "bad" ("average" or "good") lotteries and try to change their status by substituting cumulative probability of some outcomes in one lottery by their cumulative probabilities in the other. The axiom implies that regardless of how we do this, the new lottery belongs to the same category. In other words, being "bad" or "good" (and, hence, "average") is a fundamental property of a distribution, which indeed makes lotteries similar in a sense that the two lotteries within the same category are robust with respect to substitution of probabilities from one of them by the probabilities from another. Alternative way to look at this axiom is to say that there is a specific part in a distribution which can make it bad, and (possibly another) part which can make it good. And these parts are the same in all distributions. Then A2 means that we cannot "fix" a bad lottery by transplanting to it parts from another bad lottery. In the same way one cannot spoil a good lottery transplanting to it parts from another good lottery. If both distributions are bad (good) then in both of them the parts responsible for that are bad (good). Assuming A2 we deny a possibility to combine something good out of two bads and vise a versa.

An implication of this axiom is that each category is defined by a threshold in utility and a threshold in probability, i.e. a decision maker compares a lottery with the threshold values in utility and probability and classifies it based on the results of such comparison. Is this a reasonable conclusion? We argue that it is. Firstly, it is a very simple way to classify alternatives. Secondly, people often think in terms of target values and aspiration levels, which bear the same idea of a threshold. Thirdly, it seems plausible to think of a "bad" ("good") lottery as of one which has "significant" chance to bring "unsatisfactory" ("very desirable") outcome.

The third axiom represents a mild and commonly used normative property. As a matter of fact, some theories, for example, prospect theory by Kahneman and Tversky (1979), are criticized for the lack of compliance with first order stochastic dominance.

The immediate consequences of axioms A1-A3 are summarized in the four lemmata below.

Lemma 2.1. For every $x,y \in \mathbf{X}$, $Lx >_Q Ly$ iff $x >_X y$.

Proof. Follows from axiom A3. //

Lemma 2.2. There are $f,g \in \mathbf{Q}$ such that $\mathbf{Q}$ is of one of the four possible forms:

$\mathbf{Q} = [Lx_0,g] \cup (g,f) \cup [f,Lx^0]$ or $\mathbf{Q} = [Lx_0,g) \cup [g,f] \cup (f,Lx^0]$ or

$\mathbf{Q} = [Lx_0,g) \cup [g,f) \cup [f,Lx^0]$ or $\mathbf{Q} = [Lx_0,g] \cup (g,f] \cup (f,Lx^0]$.

Proof is in the appendix.

Lemma 2.3. If $[r,s]$ is a vNM-interval then $q \in [r,s]$ iff $s \geq^1 q \geq^1 r$.

Proof is in the appendix.

Lemma 2.3 can help in understanding why we need g^* and f^* for open vNM-intervals. Let us consider $\mathbf{Q} = [Lx_0,g] \cup (g,f) \cup [f,Lx^0]$, where each $>_Q$-interval is a vNM-interval. Then, since $>_Q$ is a weak order, (g,f), by lemma 2.3, consists of all such lotteries q that neither $g \geq^1 q$ nor $q \geq^1 f$. But $\geq^1$ is incomplete and we have to use lotteries other than g and f in the modifications of Archimedean and Independence axioms, which characterize vNM-interval. Thus, $g^{*} \in \inf_{\geq^1}\{q\colon \text{not } g \geq^1 q\}$ and $f^{*} \in \sup_{\geq^1}\{q\colon \text{not } q \geq^1 f\}$. For example, let $\mathbf{X} = [-10;10]$, $g(0) = .6$, $g(10) = .4$, $f(-10) = .3$, $f(5) = .7$. A lottery s belongs to $[Lx_0, g]$ if $g \geq^1 s$, i.e. $S^-(0) \geq .6$. Similarly, $s \in [f, Lx^0]$ if $S^-(5) \leq .3$. Any lottery $r \in Q$ which is neither dominated by g nor dominates f belongs to (g,f). If we consider the closure of (g,f) in the topology induced by $\geq^1$ and denote it by $[g^*, f^*]$, then $g^*(-10) = .6$, $g^*(0) = .4$, $f^*(5) = .3$, $f^*(10) = .7$, i.e. $r \in (g,f)$ iff $R^-(0) < .6$ and $R^-(5) > .3$. Hence, $[g^*, f^*] \neq [g,f]$.

Lemma 2.4. Let $f, g \in Q$ then any of the "boundary" lotteries g, g^*, f, f^* may have in their support no more than two points and no more than one point $x \in \mathbf{X} \backslash \{x_0, x^0\}$.

Proof is in the appendix.

To "fix" the ends of vNM-intervals let us introduce a technical axiom A4.

Axiom A4. Each vNM-interval is either closed or open.[2]

Given axioms A1-A4 one can come up with a specific interpretation to the "boundary" lotteries g and f. We may think of lower and upper thresholds both in outcomes (x_l and x_h) and probabilities (p_l and p_h) with $g(x_l) = p_l$, $g(x^0) = 1-p_l$, $f(x_0) = 1-p_h$, $f(x_h) = p_h$, and imagine that if, for a given lottery, the probability to get something worse than x_l is greater than p_l then this lottery is a "bad" one (falls into $[Lx_0, g]$). If a lottery is not a bad one and has probability higher than p_h to bring an outcome better than x_h then it is a good one (falls into $[f, Lx^0]$). Otherwise a lottery is neither bad nor good (i.e. from (g,f)). In the non-trivial case when $supp(g) = \{y, x^0\}$ and $supp(f) = \{x_0, z\}$ for some $y, z \in \mathbf{X}$, one can set $x_l = y$, $x_h = z$, $p_l = G^-(y)$, $p_h = F^+(z)$. Coming back to the example after lemma 2.3, $x_l = 0$, $x_h = 5$, $p_l = .6$, $p_h = .7$.

If $supp(g) = \{x^0\}$ then $x_l = x_0$ and $p_l = 0$; if $supp(f) = \{x_0\}$ then $x_h = x^0$ and $p_h = 1$. In some proofs of section 3 we will use the representation of g,f by the lower and upper thresholds.

Remark. It also follows from the form of g and f and the facts that $f \geq_Q g$ and $x_h \geq_X x_l$ that $p_l + p_h > 1$. This in turn implies that for $q \in \mathbf{Q}$, if $q \geq^1 f$ then not $g \geq^1 q$.

3. RESULTS

First we wish to show that for each vNM-interval there is a von Neumann-Morgenstern utility function $u: Y \to \mathfrak{R}$, where $Y \subseteq \mathbf{X}$ is the only relevant subset of outcomes, such that $u(\cdot)$ represents $>_X$ on Y and $>_Q$, restricted to this vNM-interval, admits expected utility representation with $u(\cdot)$. We need several auxiliary lemmata.

Lemma 3.5. Assume $\mathbf{Q} = [Lx_0, g] \cup (g,f) \cup [f, Lx^0]$, where each $>_Q$-interval is a vNM-interval. Then

i) $x^0 \in supp(g)$ iff $\forall x \in \mathbf{X}\ \exists q \in [Lx_0, g]$ such that $x \in supp(q)$;

ii) $x_0 \in supp(f)$ iff $\forall x \in \mathbf{X}\ \exists r \in [f, Lx^0]$ such that $x \in supp(r)$;

iii) $\forall x \in \mathbf{X}\ \exists s \in (g,f)$ such that $x \in supp(s)$.

Proof in the appendix.

It can be similarly shown that if $y \in \sup_{>_X} \text{supp}(g)$ and $z \in \inf_{>_X} \text{supp}(f)$ then for all $q \in [Lx_0, g]$ $\text{supp}(q) \subset [x_0, y]$ and for all $r \in [f, Lx^0]$ $\text{supp}(r) \subset [z, x^0]$.

Lemma 3.6. Assume $Q = [Lx_0, g] \cup (g,f) \cup [f, Lx^0]$, where each $>_Q$-interval is a vNM-interval. Then $Lx \in [Lx_0, g]$ iff for every $y \in \text{supp}(g)$, $y \geq_X x$ and $Lx \in [f, Lx^0]$ iff for every $y \in \text{supp}(f)$, $x \geq_X y$.

Proof. By lemma 2.3, $Lx \in [Lx_0, g]$ iff $g \geq^1 Lx$, which means that $\forall y \in \text{supp}(g)$, $y \geq_X x$. Similarly, $Lx \in [f, Lx^0]$ iff $Lx \geq^1 f$, i.e. $\forall y \in \text{supp}(f)$, $x \geq_X y$. //

It follows from lemma 3.6 that $Lx \in (g,f)$ iff $x \in (x_l, x_h)$, where $x_l \in \inf_{>_X} \text{supp}(g)$ and $x_h \in \sup_{>_X} \text{supp}(f)$. $(x_l, x_h) \neq \emptyset$ since if $x_l \geq_X x_h$ then $g \geq^1 f$, i.e. $g \geq_Q f$ which contradicts the definition of an open vNM-interval.

Theorem A. If (g,f) is an open vNM-interval then there exists a measurable utility function $u\colon \mathbf{X} \to \mathbb{R}$, representing $>_X$ such that for all $s,r \in (g,f)$ $s >_Q r$ iff $\int_{\mathbf{X}} u(x)dS^-(x) > \int_{\mathbf{X}} u(x)dR^-(x)$.

Proof. As it was mentioned, $Lx \in (g,f)$ iff $x \in (x_l, x_h)$. Let $u(x_l) = 0$ and $u(x_h) = 1$. For any $x \in (x_l, x_h)$ assign $u(x) = \alpha$, where $\alpha \in (0,1)$ is uniquely determined by $\alpha Lx_h + (1-\alpha)Lx_l \sim_Q Lx$.

For any $u\colon \mathbf{X} \to \mathbb{R}$, measurable with respect to the σ-algebra underlying the definition of Q, define U: Q → ℜ by letting $U(s) = \int_{\mathbf{X}} u(x)ds(x)$, for every s ∈ **Q**. It follows that U(•) is linear. Define β and γ by the following relationships: $Lx_h \sim_Q \beta f^* + (1-\beta)g^*$ and $Lx_1 \sim_Q \gamma f^* + (1-\gamma)g^*$. Since $f^* = p_h Lx^0 + (1-p_h)Lx_h$ and $g^* = p_1 Lx_0 + (1-p_1)Lx_1$ and by definition of U(•), $u(x_0) = \gamma/[(\gamma-\beta)p_1] < 0$ and $u(x^0) = 1 + (1-\beta)/[(\beta-\gamma)p_h] > 1$.

For any $x \in (x_h, x^0)$, $p_h Lx + (1-p_h)Lx_h \in (g,f)$. Hence there is a unique $\alpha_x \in (0,1)$ such that $p_h Lx + (1-p_h)Lx_h \sim_Q \alpha_x f^* + (1-\alpha_x)g^*$. Assign $u(x) = 1 + (\alpha_x-\beta)/[(\beta-\gamma)p_h]$. $u(x) > 1$, since $\alpha_x > \beta$, by dominance. Similarly, for any $y \in (x_0, x_1)$, there is a unique $\alpha_y \in (0,1)$ such that $p_1 Ly + (1-p_1)Lx_1 \sim_Q \alpha_y f^* + (1-\alpha_y)g^*$. Assign $u(x) = (\alpha_y-\gamma)/[(\beta-\gamma)p_1] < 0$. It is easy to see that u(•) is continuous and represents $>_X$. That, in particular, implies that u(•) is bounded.

Let us now prove that U(•) represents $>_Q$ on (g,f). We start with:

Claim 1. For all $r \in (g,f)$, $r \sim_Q \alpha f^* + (1-\alpha)g^*$ implies $U(r) = U(\alpha f^* + (1-\alpha)g^*)$.

Proof. First, consider lotteries with finite support and use induction on the number of elements in the support. If r has only one point in its support the result follows from the definitions of u(•) and U(•).

Now let r contain two points x,y ∈ **X** in its support, i.e. $r = r(x)Lx + (1-r(x))Ly$ and $r \sim_Q \alpha_r f^* + (1-\alpha_r)g^*$. If both $x,y \in [x_1, x_h]$ we are done by the standard expected utility argument. Let only $x \in [x_1, x_h]$. Without loss of generality, let $y >_X x_h$. (The case when $x_1 >_Q y$ is proven similarly.) Then $r(x) \geq 1-p_h$ (otherwise not $f^* >^1 r$, i.e. $r \notin (g,f)$.) Notice that for some $\alpha_x, \alpha_y, \beta_y \in (0,1)$, $Lx \sim_Q \alpha_x Lx_h + (1-\alpha_x)Lx_1$; $p_h Ly + (1-p_h)Lx_h \sim_Q \alpha_y f^* + (1-\alpha_y)g^*$ and $p_h Ly + (1-p_h)Lx_1 \sim_Q \beta_y f^* + (1-\beta_y)g^*$. By definitions of u(•), U(•) and the properties of vNM-intervals, all substitutions above preserve utility level as well as equivalence. Hence, we can get rid of Ly in the representation of r, if $r(x)\alpha_x p_h/[1-p_h] + r(x)(1-\alpha_x)p_h/[1-p_h] \geq 1-r(x)$, which can be reduced to $r(x)p_h \geq (1-p_h)(1-r(x))$, which is always true. So, one gets a representation of r only through f^* and g^* which has the same utility as r. Then the coefficient by f^* has to be α_r, by the dominance argument, and we are done with this case.

The only case left is when $x_1 >_X x$ and $y >_X x_h$. Then $p_1 \geq r(x) \geq (1-p_h)$, and hence $f >_Q r(x)Lx_1 + (1-r(x))Ly \geq_Q r \geq_Q r(x)Lx + (1-r(x))Lx_h >_Q g$. Therefore, there exist $\delta \in (0,1)$ such that $r \sim_Q \delta[r(x)Lx_1 + (1-r(x))Ly] + (1-\delta)[r(x)Lx + (1-r(x))Lx_h]$. In this representation we have two lotteries which fall under the previous case. So each of them can be represented by equivalent mixture of f^* and g^*, which can then be cobmined retaining equivalence due to the properties of vNM-intervals.

Suppose the claim is true for all lotteries with $n \geq 2$ points in the support. Consider $r \sim_Q \delta f^* + (1-\delta)g^*$, for some $\delta \in (0,1)$, which support contains n+1 points. There are three cases to consider.

a) If there are at least three points in the support of r which are from the interval $[x_1,x_h]$, one can substitute them by the equivalent mixture of Lx_1 and Lx_h. This would reduce r to r', such that r' has not more than n elements in its support. So we are done.

b) The support of r contains at most one $x \in \mathbf{X}$ such that $x >_X x_h$ and at most one $y \in \mathbf{X}$ such that $x_1 >_X y$, while a) is not the case. Then there exists $z \in [x_1,x_h] \cap \text{supp}(r)$. Define $q,s \in \mathbf{Q}$ as follows: $q(y) = r(y)$, $s(y) = 0$, $q(x) = 0$, $s(x) = r(x)$, $q(z) = s(z) = r(x) + r(z)$, $q(t) = s(t) = r(t)$, for all $t \in \mathbf{X}\backslash\{x,y,z\}$. Then $f >_Q s >_Q r >_Q q >_Q g$. Hence there is $\varepsilon \in (0,1)$ such that $r \sim_Q \varepsilon s + (1-\varepsilon)q$. Both s and q have only n points in their supports, therefore each of them can be substituted by a mixture of f^* and g^*. After collecting all the terms one gets a coefficient of δ for f^*, due to dominance argument. Since all substitutions preserved utility level, we get that $U(r) = U(\delta f^* + (1-\delta)g^*)$.

c) There are at least two elements $x,y \in \mathbf{X}$ in the support of r such that either $x,y >_X x_h$ or $x_1 >_X x,y$ and a) is not the case. Without loss of generality let $x >_X y >_X x_h$. Define $q,s \in Q$ as follows: $s(y) = 0$, $q(x) = 0$, $q(y) = s(x) = r(x) + r(z)$, $q(t) = s(t) = r(t)$, for all $t \in \mathbf{X}\setminus\{x,y\}$. Then $f >_Q s >_Q r >_Q q >_Q g$. Hence there is $\varepsilon \in (0,1)$ such that $r \sim_Q \varepsilon s + (1-\varepsilon)q$ and we can continue as in case b).

To extend the proof to the lotteries with infinite support we have to first prove the following.

Claim 2. Let $r,s \in (g,f)$, have finite supports and $r \sim_Q \alpha f^* + (1-\alpha)g^*$, $s \sim_Q \beta f^* + (1-\beta)g^*$, then for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $\sup_{x\in\mathbf{X}}|R^-(x)-S^-(x)| < \delta$ then $|\alpha-\beta| < \varepsilon$.

Proof. We already know that $U(\cdot)$ represents $>_Q$ for lotteries with finite support. Denote $\max_{x\in\mathbf{X}}|u(x)|$ by M. If $\sup_{x\in\mathbf{X}}|R^-(x)-S^-(x)| < \delta$ then $|U(r)-U(s)| = |\int_{\mathbf{X}}u(x)dR^-(x)-\int_{\mathbf{X}}u(x)dS^-(x)| < 2M\delta$. On the other hand, $U(r) = \alpha U(f^*) + (1-\alpha)U(g^*)$, and $U(s) = \beta U(f^*) + (1-\beta)U(g^*)$, i.e. $|U(r)-U(s)| = |\alpha-\beta| * (U(f^*)-U(g^*))$. Set $\varepsilon = 2M\delta/(U(f^*)-U(g^*))$ to complete the proof of claim 2.

Now let us consider a lottery $h \in (g,f)$ with infinitely many points in its support. We can approximate h by sequences (s_k) and (r_k) of lotteries such that s_k and r_k have only k elements in their supports and for all k, $h \geq^1 s_{k+1} \geq^1 s_k$ and $r_k \geq^1 r_{k+1} \geq^1 h$. Since $f^* \geq^1 h \geq^1 g^*$ and both f^* and g^* have no more that two points in their supports, one can easily ensure that (s_k) and (r_k) are both contained in (g,f) and also that for any $\varepsilon > 0$ there is $K \in \mathbf{N}$

such that if $k > K$ then $\sup_{x \in \mathbf{X}} |R_k^-(x) - S_k^-(x)| < \varepsilon^3$. The latter implies that (S_k^-) and (R_k^-) converge pointwise to H^-. Let $h \sim_Q \alpha_h f^* + (1-\alpha_h)g^*$ and for all $k \in \mathbf{N}$, $s_k \sim_Q \alpha_k f^* + (1-\alpha_k)g^*$, $r_k \sim_Q \beta_k f^* + (1-\beta_k)g^*$. If $\lim(\alpha_k) = \alpha$ and $\lim(\beta_k) = \beta$, we want to show that $\alpha = \alpha_h = \beta$. Notice that, by stochastic dominance, $\beta \geq \alpha_h \geq \alpha$. By claim 2, for any $\varepsilon > 0$ there is $K \in \mathbf{N}$ such that if $k > K$ then $|\alpha_k - \beta_k| < \varepsilon$, i.e. $\beta = \alpha = \alpha_h$. On the other hand, the by dominated convergence theorem, $U(h) = \lim U(s_k)$, which completes the proof of claim 1.

To complete the proof of the theorem, let $q, r \in (g,f)$, $q >_Q r$, $q \sim_Q \alpha f^* + (1-\alpha)g^*$ and $r \sim_Q \beta f^* + (1-\beta)g^*$. By linearity of $U(\cdot)$, $U(q) - U(r) = (\alpha - \beta)(U(f^*) - U(g^*))$. And since $\alpha > \beta$, $U(q) > U(r)$.

Now let $r \in (g,f)$. Then $r \sim_Q \int_{\mathbf{X}} Lx dR^-(x)$, i.e. $U(r) = U(\int_{\mathbf{X}} Lx dR^-(x))$. Using linearity of $U(\cdot)$ again, one gets that $U(r) = \int_{\mathbf{X}} u(x) dR^-(x)$. //

Theorem B. If $[Lx_0, g]$ is a vNM-interval then there exists a utility function $u: \mathbf{X} \to \mathfrak{R}$ representing $>_X$ on $Y \subseteq \mathbf{X}$, where $Y = [x_0, \sup_{>_X} \mathrm{supp}(g)]$, and such that for all $s, r \in [Lx_0, g]$, $s >_Q r$ iff $\int_{\mathbf{X}} u(x) dS^-(x) > \int_{\mathbf{X}} u(x) dR^-(x)$.

Proof. Assign $u(x_0) = 0$. Take any $x \in \mathbf{X} \setminus \{x_0\}$. There is a unique maximal β such that $g \geq_Q \beta Lx + (1-\beta)Lx_0$. We can define β by setting $\beta = G^+(x) > 0$. Then indeed, $g \geq^1 \beta Lx + (1-\beta)Lx_0$ and hence $g \geq_Q \beta Lx + (1-\beta)Lx_0$. Also for every $\alpha > \beta$, not $g \geq^1 \alpha Lx + (1-\alpha)Lx_0$, i.e. not $g \geq_Q \alpha Lx + (1-\alpha)Lx_0$.

Furthermore, there is a unique $\gamma \in (0,1)$ such that $\beta Lx + (1-\beta)Lx_0 \equiv L_\beta x \sim_Q \gamma g + (1-\gamma)Lx_0$. Now assign $u(x) = \gamma/\beta$.

We will denote the unique β and γ associated with a specific $x \in \mathbf{X}$ by β_x and γ_x.

To see that $u(\cdot)$ represents $>_X$ on Y notice that if $x >_X y$ then $\beta_x \leq \beta_y$, by dominance argument. If in addition $L_\beta x \geq_Q L_\beta y$, $u(x) > u(y)$, we are done. However, it could be that $L_\beta y >_Q L_\beta x$.

Claim 1. Let $x,y \in Y$ be such that $x >_X y$ and $L_\beta y >_Q L_\beta x$. Then $\gamma_x/\beta_x > \gamma_y/\beta_y$.

Proof. Since $x >_X y$, $\beta_x < \beta_y$. Also $\beta_y Ly + (1-\beta_y)Lx_0 >_Q \beta_x Lx + (1-\beta_x)Lx_0$ and $\beta_x Lx + (1-\beta_x)Lx_0 \sim_Q \gamma_x g + (1-\gamma_x)Lx_0$, $\beta_y Ly + (1-\beta_y)Lx_0 \sim_Q \gamma_y g + (1-\gamma_y)Lx_0$, with $\gamma_y > \gamma_x$.

There exists the unique $\delta \in (0,1)$ such that $\beta_x Lx + (1-\beta_x)Lx_0 \sim_Q \delta[\beta_y Ly + (1-\beta_y)Lx_0] + (1-\delta)Lx_0 = \delta\beta_y Ly + (1-\delta\beta_y)Lx_0$. Since $Lx >_Q Ly$, $\beta_x < \delta\beta_y$.

Moreover, $\delta\beta_y Ly + (1-\delta\beta_y)Lx_0 \sim_Q \delta\gamma_y g + (1-\delta\gamma_y)Lx_0$, i.e. $\beta_x Lx + (1-\beta_x)Lx_0 \sim_Q \delta\gamma_y g + (1-\delta\gamma_y)Lx_0$. Since γ_x is unique, $\gamma_x = \delta\gamma_y$ and, thus, $\gamma_x/\beta_x > \delta\gamma_y/\delta\beta_y = \gamma_y/\beta_y$. The claim is proven.

Define $U: Q \to \mathbb{R}$ by letting $U(s) = \int_{\mathbf{X}} u(x)dS^{-}(x)$, for every $s \in Q$. It follows that $U(\cdot)$ is linear. Now without loss of generality we can assign $u(x) = 0$ for all $x \in \mathbf{X}\backslash Y$ and set $u(x^0) = (1-p_1u(x_1))/(1-p_1)$, which implies $U(g) = 1$. Let us show that for all $s,r \in [Lx_0,g]$, $s >_Q r$ iff $\int_{\mathbf{X}} u(x)dS^{-}(x) > \int_{\mathbf{X}} u(x)dR^{-}(x)$.

We start with the proof of

Claim 2. For all $r \in [Lx_0,g]$, $r \sim_Q \alpha g + (1-\alpha)Lx_0$ implies $U(r) = U(\alpha g + (1-\alpha)Lx_0)$.

Proof. First, consider lotteries with finite support and use induction on the number of elements in support. If r has only one point in its support the result follows from the definitions of $u(\cdot)$ and $U(\cdot)$. Suppose the claim is true for all lotteries with n points in the support. Consider r which support contains n+1 points.

Notice that since $r \in [Lx_0,g]$, there is at least one point y in the support of r such that $x_1 \geq_X y$. If all $y \in supp(r)$ are such that $x_1 \geq_X y$ then we are done since for every such y there exists $\gamma_y \in (0,1)$ such that $Ly \sim_Q \gamma_y g + (1-\gamma_y)Lx_0$. Let there is $x \in supp(r)$ such that $x \geq_X x_1$. r can be represented as $r = (\gamma_1 Ly_1 + \ldots + \gamma_n Ly_n) + \alpha Lx$, where $x \geq_X y$ for all $y \in supp(r)$. Consider $r' = (\gamma_1/(1-\alpha))Ly_1 + \ldots + (\gamma_n/(1-\alpha))Ly_n$. $r' \in [Lx_0, g]$ since $r >^1 r'$. Hence, there exists $\alpha' \in (0,1)$ such that $r' \sim_Q \alpha' g + (1-\alpha')Lx_0$ and by assumption $U(r') = U(\alpha' g + (1-\alpha')Lx_0)$. By the properties of a vNM-interval, there exists $\delta \in (0,1)$ such that $r \sim_Q \delta g + (1-\delta)Lx_0$ then since $r = (1-\alpha)r' + \alpha Lx$, $(1-\alpha)\alpha' g + (1-\alpha)(1-\alpha')Lx_0 + \alpha Lx \sim_Q \delta g + (1-\delta)Lx_0$.

Since $\beta_x Lx + (1-\beta_x)Lx_0 \sim_Q \gamma_x g + (1-\gamma_x)Lx_0$ one can substitute in every lottery within a vNM-interval $(\alpha\gamma_x/\beta_x)g + [\alpha(1-\gamma_x)/\beta_x]Lx_0$ for $\alpha Lx + [\alpha(1-\beta_x)/\beta_x]Lx_0$.

If we have $\alpha(1-\beta_x)/\beta_x \leq (1-\alpha)(1-\alpha')$ then $[(1-\alpha)\alpha' + (\alpha\gamma_x/\beta_x)]g + [(1-\alpha)(1-\alpha') + \alpha(1-\gamma_x)/\beta_x]Lx_0 \sim_Q \delta g + (1-\delta)Lx_0$. By the stochastic dominance argument, the corresponding coefficients are equal and so are the utilities, which proves the claim.

Next we will show that $\alpha(1-\beta_x)/\beta_x > (1-\alpha)(1-\alpha')$ might not be the case. First, recall that for any lottery from our vNM-interval the total weight of the outcomes which are better than x_1 has not to exceed $\beta_x = 1-p_1$. Assume that $\alpha(1-\beta_x)/\beta_x > (1-\alpha)(1-\alpha')$ and use the equivalence $\beta_x Lx + (1-\beta_x)Lx_0 \sim_Q \gamma_x g + (1-\gamma_x)Lx_0$ to increase the coefficient by Lx as much as possible.

There are two cases. If $(1-\alpha)\alpha'/\gamma_x < (1-\alpha)(1-\alpha')/(1-\gamma_x)$, i.e. $\gamma_x > \alpha'$, we need to have $(1-\alpha)\alpha'\beta_x/\gamma_x + \alpha < \beta_x$, which can be reduced to $\gamma_x/\beta_x > (1-\alpha)\alpha'/(\beta_x-\alpha)$. From $\alpha(1-\beta_x)/\beta_x > (1-\alpha)(1-\alpha')$ we obtain that $(1-\alpha)\alpha' > (\beta_x-\alpha)/\beta_x$, which leads to $\gamma_x > 1$. A contradiction.

Now consider the case when $\alpha' > \gamma_x$. Then $r \sim_Q [(1-\alpha)\alpha' - \gamma_x(1-\alpha)(1-\alpha')/(1-$

$\gamma_x)]g + [\alpha + \beta_x(1-\alpha)(1-\alpha')/(1-\gamma_x)]Lx + (1-\beta_x)(1-\alpha)(1-\alpha')/(1-\gamma_x)Lx_0$ and we need to have $\beta_x[(1-\alpha)\alpha' - \gamma_x(1-\alpha)(1-\alpha')/(1-\gamma_x)] + \alpha + \beta_x(1-\alpha)(1-\alpha')/(1-\gamma_x) < \beta_x$. After simplifications one gets that either $\alpha = 0$ (a trivial case) or $\beta_x = 1$. The latter means that $p_1 = 0$ and one cannot have $p_1 + p_h > 1$ as required. A contradiction.

To extend the proof of the claim for the lotteries with infinite support one may repeat the reasoning used in the proof of the theorem A.

Now let $q, r \in [Lx_0, g]$, $q >_Q r$, $q \sim_Q \alpha g + (1-\alpha)Lx_0$ and $r \sim_Q \beta g + (1-\beta)Lx_0$. By linearity of $U(\cdot)$, $U(q) = \alpha$ and $U(r) = \beta$, $\alpha > \beta$, i.e. $U(q) > U(r)$.

Take $s \in (g,f)$. Then $s \sim_Q \int_{\mathbf{X}} Lx dS^-(x)$, i.e. $U(s) = U(\int_{\mathbf{X}} Lx dS^-(x))$. Using linearity of $U(\cdot)$ again, one gets that $U(s) = \int_{\mathbf{X}} u(x) dS^-(x)$. //

Similarly, we can establish existence of expected utility representation for the lotteries from the interval $[f, Lx^0]$.

To summarize the results above we can formulate

Theorem C. $>_Q$ is a weak order on **Q** satisfying axioms A1-A4 iff

i) **Q** $= [Lx_0, g] \cup (g,f) \cup [f, Lx^0]$, where each $>_Q$-interval is a vNM-interval;

ii) for all $q, r \in$ **Q**, $q \in [Lx_0, g]$ iff $g >^1 q$ and $r \in [f, Lx^0]$ iff $r >^1 f$;

iii) there exist utility functions u_1, u_2, u_3 from **X** into $\Re$ such that u_1 provides expected utility representation for $>_Q$ on $[Lx_0, g]$, u_2 on (g,f) and u_3 on $[f, Lx^0]$.

iv) There are lower and upper thresholds in outcomes ($x_1, x_h \in$ **X**, $x_h \geq_X x_1$) and probabilities ($p_1, p_h \in [0,1]$, $p_1 + p_h > 1$) such that if

each $s \in$ **Q** is evaluated by a pair $v(s) = (v_1(s), v_2(s))$, where $v_1(s) =$

1 if $S^{-}(x_1) \geq p_1$; $v_1(s) = 3$ if $S^{-}(x_h) \leq 1\text{-}p_h$; $v_1(s) = 2$ otherwise. $v_2(s) = \int_{\mathbf{X}} u_{v_1}(x)dS^{-}(x)$, where u_1, u_2 and u_3 are as in iii),

then for all $q, r \in Q$, $q >_Q r$ iff $v(q)$ lexicographically exceeds $v(r)$.

Proof. The "only if" part follows from lemmata 2.2, 2.3, the remark closing section 2, and theorems A and B. The "if" part can easily be checked directly. //

Notice that $S^{-}(x_h) \leq 1\text{-}p_h$ implies $S^{-}(x_1) < p_1$.

We will refer to the procedure for lotteries' evaluation described in part iv) of the theorem C as procedure P.

4. PROPERTIES OF PROCEDURE P AND ITS COMPARISON WITH OTHER MODELS

Let us first briefly discuss some superficial features of P.

The fact that a person first cares about the lower threshold and only after that about the upper one bears some flavor of risk aversion. The defence of this point can be split into two arguments. First of all, Q might be a vNM-interval itself. However, when it is not the case a decision maker first concentrates on unfavorable outcomes in his evaluation of a lottery. In that case a natural justification for this procedure is an evolutionary argument along the following lines: if someone does not succeed in getting 10 million dollars he can try it once again later, but if he cannot survive today he will not have another chance either to survive or to become a millionaire.

Despite close resemblance to the expected utility model our procedure may violate both the Archimedean and the Independence axioms.

Observation 4.1. The procedure P may violate the Archimedean axiom.

Proof. Suppose that $x_l = 0$; $p_l = 0.4$; $x_h = 5$; $p_h = 0.7$ and s, q, r are as below:

s:

x	-10	0	100
p	0.2	0.2	0.6

q:

x	2
p	1

r:

x	1
p	1

$v(s) = <1, 58>$; $v(q) = <2, 2>$, $v(r) = <2, 1>$. Hence, $q >_Q r >_Q s$.

$\alpha q + (1-\alpha)s$:

x	-10	0	2	100
p	$0.2(1-\alpha)$	$0.2(1-\alpha)$	α	$0.6(1-\alpha)$

$[\alpha Q + (1-\alpha)S]^{-}(0) = 0.4(1-\alpha) < 0.4 = p_l$. Thus, $v_1(\alpha q + (1-\alpha)s)$ is either 3 (hence $\alpha q + (1-\alpha)s >_Q r$) or 2. In the latter case $v_2(\alpha q + (1-\alpha)s) = 58 - 56\alpha > 2 > 1$ and again $\alpha q + (1-\alpha)s >_Q r$. Therefore the Archimedean axiom is violated. //

In fact P violates not only the Independence axiom but also a weaker Betweenness axiom.

Definition 4.2. Betweenness. For all $q,r \in Q$, if $q >_Q r$ then for all $\alpha \in (0,1)$ $q >_Q \alpha q + (1-\alpha)r >_Q r$.

Observation 4.3. P may violate betweenness.

Proof. Let $x_l = 0$; $p_l = 0.6$; $x_h = 10$; $p_h = 0.5$.

r:

x	9
p	1

q:

x	-10	5	10
p	0.3	0.2	0.5

$v(r) = <2, 9>$; $v(q) = <3, 3>$. Hence, $q >_Q r$.

For any $\alpha \in (0,1)$, $t = \alpha q + (1-\alpha)r$:

x	-10	5	9	10

p	0.3α	0.2α	$1-\alpha$	0.5α

$v(t) = \langle 2, 9-6\alpha \rangle$. Hence, $r >_Q t$. //

Lack of so many desirable properties makes P less attractive from the normative viewpoint. Yet we argue that it is descriptively plausible.

Let us try to use P to explain the Allais paradox as presented in Kahneman and Tversky (1979). (In the sequel we will always assume that **X** is a compact subset of $\mathbb{R}$ containing 0 and $u(x) = x$, $\forall x \in$ **X** unless **X** and $u(\cdot)$ are specified explicitly. Hence we will sometimes not distinguish between outcomes and their utilities to simplify notation. Even in such a restricted form our model can explain many paradoxes.)

Consider the four lotteries:

A:

x	4000	0
p	0.8	0.2

B:

x	3000
p	1

C:

x	4000	0
p	0.2	0.8

D:

x	3000	0
p	0.25	0.75

where x denotes a prize (in $) and p - the probability of getting that prize.

Empirical evidence shows that for most individuals $A < B$ and $D < C$ (with the symbol "<" standing for the "less preferred than" relation. Such a preference pattern violates the independence axiom (and hence the expected utility representation), since in the presence of the lottery

E:

x	0
p	1

we have $0.25A + 0.75E = C$ and $0.25B + 0.75E = D$.

We will apply P to a general form of the parodox written as follows:

A:

x	Z	0
p	α	$1-\alpha$

B:

x	Y
p	1

C:

x	Z	0
p	$\alpha\beta$	$1-\alpha\beta$

D:

x	Y	0
p	β	$1-\beta$

where $Z > Y \gg 0$, $0 \ll \alpha < 1$, $0 < \beta \ll 1$, $\alpha Z > Y$, where notation $a > b \gg c$ conveys the idea that b is "much closer" to a than to c.

Let $x_l = 0$; $p_l > 1-\alpha\beta$; $x_h < Y$; $p_h > \alpha$, then $v(A) = \langle 2;\ \alpha Z\rangle$, $v(B) = \langle 3;\ Y\rangle$; $v(C) = \langle 2;\ \alpha\beta Z\rangle$ and $v(D) = \langle 2;\ \beta Y\rangle$ agreeing with the Allais paradox preference pattern. Needless to say, the choice of utility function as well as any of the four parameters for this example is not unique.

This builds a link to prospect theory by Kahneman and Tversky (1979) and its generalization cumulative prospect theory (Tversky and Kahneman (1990)). One of the principles observed by Tversky and Kahneman (1990) is diminishing sensitivity which they illustrate by the following example:

f:	x	25,000	25,000	25,000
g:	x	25,000	0	75,000
f':	x	0	25,000	25,000
g':	x	0	0	75,000
	p	0.01	0.89	0.10

Experimental data confirm that prevailing preferences express that f $>_Q$ g but g' $>_Q$ f' which is inconsistent with expected utility theory.

Once again we would like to apply P to a general form of the preferences above. Assume that

f:	x	Z	Z	Z
g:	x	Z	0	Y
f':	x	0	Z	Z
g':	x	0	0	Y
	p	α	β	$1-\alpha-\beta$

where $Y \gg Z \gg 0$, $0 < \alpha \ll 1-\alpha-\beta \ll \beta$ (i.e. $1-\alpha-\beta$ is "sufficiently" different from both α and β.) Let $(1-\alpha-\beta)u(Y) > (1-\alpha)u(Z)$; $x_l = 0$; $p_l > \alpha+\beta$; $x_h \leq Z$; $p_h > 1-\alpha$.

Then according to P, $v_1(f) = 3$ while $v_1(g) = 2$ (hence f $>_Q$ g) but g' $>_Q$ f' since $v_1(f') = v_1(g') = 2$ and $v_2(g') = (1-\alpha-\beta)u(Y) > (1-\alpha)u(Z) = v_2(f')$.

This example is noteworthy because it is inconsistent with Loomes's and Sugden's (1983) regret theory or Fishburn's SSB (1984) model.

Another model for resolution of preference "paradoxes" was proposed by Schmeidler (1989) who introduced non-additive probabilities. Since his model deals with choice under uncertainty rather than risk it is not always easy to interpret his results in the terms we employ. However, there is a feature in Schmeidler's model, namely, that sometimes an individual simultaneously prefers to buy insurance for some risky outcome and gamble on another, which can be explained by P.

Suppose a decision maker is not sure what he is going to do next Thursday. With probability .5 he may receive a new piece of equipment and work on it. If the new equipment does not arrive he will participate in a small lottery. The new equipment when arrived may fail with probability .01, which will result in the loss of $1,000. If it operates normally then the decision maker can make a profit of $50. However, he has an option to buy a warranty for only $10.50, which will guarantee his profit if the equipment arrives. An entry to the lottery costs $10, and if lucky the decision maker can win $100, which has probability .2. The decision maker may also sell his right to enter the lottery to his friend for $12. He has to decide whether to buy the warranty for equipment and/or to sell the right to enter the lottery.

Available options can be represented by the following lotteries:

A:

x	-1,000	50	-10	100
p	0.005	0.495	0.4	0.1

B:

x	39.5	12
p	0.5	0.5

,

C:

x	39.5	-10	100
p	0.5	0.4	0.1

D:

x	-1,000	50	12
p	0.005	0.495	0.5

,

where A corresponds to not buying any insurance at all, B corresponds to a purchase of insurance for both risky outcomes while C and D only for one of them. Suppose that $x_l = 0$; $p_l = 0.5$; $x_h = 20$; $p_h = 0.6$. Then $v(A) = (2;\ 25.75)$, v(B) = (2; 25.75), v(C) = (3; 25.75); v(D) = (2; 25.75). Hence C is preferred over all other lotteries. (Note that v(B) = v(D) = v(A) only because of linearity of utility function assumed for simplicity.) This example may give an insight at why we observe that the same people who insure their cars are buying lottery tickets.

One of the main advantages of Yaari's (1987) dual choice theory is that it can separate two conceptually different properties of risk aversion and diminishing marginal utility of wealth (which are mixed together in the classical expected utility theory). P also can separate these two effects to a certain extent. As shown by the previous example, even when a utility function is linear in prizes a person might not be risk neutral. The attitude towards risk may partially be reflected through the thresholds that characterize an individual.

Chew (1983) axiomatized the generalized quasilinear mean. All properties of this functional described as desirable in its application in decision theory except betweenness are met by P. Chew also criticizes the other theories including prospect theory as well as approaches by Handa (1977), Karmarkar (1978) and Machina (1982) for inability to explain St.Petersburg's paradox described, for example, in Luce and Raiffa (1957). St. Petersburg's paradox is perfectly explained by P.

Comparing procedure P with the model of combined expected utility and maximin criteria suggested by Gilboa (1988) notice that they both explain the Allais paradox and both allow violations of continuity and independence in particular cases, but there is still one difference we would like to pay attention to. Consider an example from Gilboa (1988):

$$A_\alpha: \begin{array}{c|c|c|} x & 4,000,000 & 0 \\ \hline p & 1-\alpha & \alpha \end{array} \qquad B: \begin{array}{c|c|} x & 3,000,000 \\ \hline p & 1 \end{array} \qquad A: \begin{array}{c|c|} x & 4,000,000 \\ \hline p & 1 \end{array},$$

where a prize of 0 basically plays the role of the consequence of death. According to Gilboa's model for any $\alpha > 0$, $A_\alpha <_Q B$ and $B <_Q A$. In our model discontinuity arises not at $\alpha = 0$ but could occur twice: when $1 - \alpha = p_h$ and when $\alpha = p_l$. Typically neither of these switches happen at $\alpha = 0$. This seems to be more plausible since people definitely participate in events with strictly positive probability of death even with relatively low complimentary prizes, provided that the probability of death is small enough (for example, we cross streets and take planes). Hence we would expect that one will prefer to cross the street and then get 4,000,000 (i.e. A_α) rather than just receive 3,000,000 (i.e. B).

What we argue here is that the thresholds in utility and probability are interrelated and that for at least some of their particular choices a certainty effect is indeed an "almost certainty" one. We believe that this fact is more apparent and easier to capture when the desirable prize in A_α is close to $\sup_{<}\{x \in \mathbf{X}\}$, the upper threshold in prizes is high and the difference in utilities between the favorable outcomes in A_α and B is substantial.

We would also like to mention that prospect theory (Kahneman and Tversky (1979)) as well as the theories by Handa (1977), Karmarkar (1978) and Yaari (1987) might violate first order stochastic dominance. Specific examples and more detailed discussion can be found in Quiggin (1982).

Cumulative prospect theory by Tversky and Kahneman (1990) discusses a weighing function used in lotteries' evaluation. They argue that it is not well-behaved for very low or very high probabilities which could be either greatly overweighed or neglected altogether. P does approximately the same. Namely, it overweighs probability close to 1 when the outcome associated with that probability is below the lower threshold or better than the upper threshold while chances to occur below the lower threshold are negligible. P underevaluates small probabilities associated with unfavorable outcomes if the chance to get a very desirable prize is high enough.

In this section we tried to show that P is not refuted by experimental evidence. In this connection we would like to cite Camerer (1989) who processed a large amount of experimental data to compare different theories and came to a conclusion that each theory can account for some of expected utility violations but not all. In that respect procedure P is not an exception.

APPENDIX

Proof of the lemma 2.2. If Q is a vNM-interval itself choose $g = Lx_0$, $f = Lx^0$ and Q may be represented as $Q = [Lx_0, Lx_0] \cup (g,f) \cup [Lx^0, Lx^0]$. Suppose that there are $q,r \in Q$, $q >_Q r$ such that q and r are not in a vNM-interval. Without loss of generality, let, by axiom A1, $[Lx_0, r]$ be in a vNM-interval. Denote it $[Lx_0, g|$, meaning that it might be $[Lx_0, g)$ or $[Lx_0, g]$. Note that $q \notin [Lx_0, g|$.

If $Q \backslash [Lx_0, g| = |g, Lx^0]$ is a vNM-interval then $Q = [Lx_0, g] \cup (g, Lx^0) \cup [Lx^0, Lx^0]$ or $Q = [Lx_0, g) \cup [g, Lx^0) \cup [Lx^0, Lx^0]$. Suppose that $|g, Lx^0]$ is not a vNM-interval, i.e. there exist $s,t \in |g, Lx^0]$, $s >_Q t$ such that s and t are not in a vNM-interval. Since $[Lx_0, s]$ is not in a vNM-interval, $[t, Lx^0]$ is in a vNM-interval. Let us call it $|f, Lx^0]$.

Finally, consider $|g,f| \equiv Q \backslash ([Lx_0, g| \cup |f, Lx^0])$. The only way axiom A1 can be met is if $|g,f|$ is a vNM-interval. Indeed, let there be $h,w \in |g,f|$,

$h >_Q w$ such that h and w are not in a vNM-interval. Then neither $[Lx_0, w]$, nor $[h, Lx^0]$ is in a vNM-interval.

Thus, $Q = [Lx_0, g| \cup |g, f| \cup |f, Lx^0]$, where each $>_Q$-interval is a vNM-interval. //

Proof of the lemma 2.3. Suppose there exist $q \in [r,s]$ such that not $s \geq^1 q$. Then $s \cup q >^1 s$ and, by axiom A3, $s \cup q >_Q s$, while, by axiom A2, $s \cup q$ is in the same vNM-interval, which contradicts maximality of vNM-intervals.

Similarly, if for $s \in [r,s]$ not $q \geq^1 r$ then $r >^1 q \cap r$, i.e. $r >_Q q \cap r$. But $q \cap r \in [r,s]$. A contradiction.

Conversely, if $s \geq^1 q \geq^1 r$ then, by axiom A3, $q \in [r,s]$. //

Proof of the lemma 2.4. We will prove the lemma for g, since modifications required for the other lotteries are rather straightforward. Assume without loss of generality that (g,f) is an open vNM-interval. Suppose that $G^-(\cdot)$ increases at exactly two points $x, y \in \mathbf{X} \setminus \{x^0\}$. Suppose $F^-(x) < G^-(x)$. Take $r, s \in \mathbf{Q}$ such that $F^-(x) < R^-(x) < G^-(x)$, $S^-(x) > G^-(x)$, $S^-(y) < G^-(y)$ and $R^-(z) = 1$ for some $z \in (x,y)$. Then $r, s \in (g,f)$. Also $g >^1 r \cap s$, i.e. $r \cap s \in [Lx_0, g]$ in contradiction to axiom A2.

Let $F^-(x) \geq G^-(x)$. Notice that if $F^-(x) \geq G^-(x)$ then $F^-(y) < G^-(y)$ (otherwise $g >^1 f$.) Take $r, s \in \mathbf{Q}$ such that $R^-(x) < G^-(x)$, $S^-(x) > G^-(x)$, $F^-(y) < S^-(y) < G^-(y)$ and for some $z \in (x,y)$, $R^-(z) = 1$. Then $r, s \in (g,f)$. Also $g >^1 r \cap s$, i.e. $r \cap s \in [Lx_0, g]$ in contradiction to axiom A2.

If $G^-(\cdot)$ changes its value at more than two points one can easily find a contradiction, by a similar construction.

Now let $G^-(\cdot)$ increase at only one point $x \in \mathbf{X} \setminus \{x^0\}$. Then for any $s, r \in$

(g,f) neither $g \geq^1 r$ nor $g \geq^1 s$, i.e. $R^-(x) < G^-(x)$, $S^-(x) < G^-(x)$ and hence $r \cap s \in (g,f)$ in compliance with axiom A2. //

Proof of the lemma 3.5. Assume that for every $x \in \mathbf{X}$ there is $q_x \in [Lx_0, g]$ such that $x \in supp(q_x)$. Take $x = x^0$. But $g \geq^1 q_x$. Therefore, $x^0 \in supp(g)$. Conversely, assume $x^0 \in supp(g)$. Fix $x' \in \mathbf{X}$. Take $q \in \mathbf{Q}$ such that $q(x) = g(x)$, if $x \notin \{x', x^0\}$, $q(x^0) = 0$; $q(x') = g(x') + g(x^0) > 0$. Then $g \geq^1 q$, i.e. $q \in [Lx_0, g]$ and $x' \in supp(q)$.

ii) is proven similarly.

iii) $f >_Q g$, thus, not $g \geq^1 f$ and there exists $y \in \mathbf{X}$ such that $G^-(y) > F^-(y)$. Since G^- and F^- are continuous from the right there is an interval $[y,z) \subset \mathbf{X}$ such that for every $x \in [y,z)$ $G^-(x) > F^-(x)$. Take $s \in \mathbf{Q}$ such that $s(x)$ is strictly increasing and for every $x \in [y,z)$, $G^-(x) > S^-(x) > F^-(x)$. Then neither $s \geq^1 f$ nor $g \geq^1 s$, i.e. $s \in (g,f)$ and $supp(s) = \mathbf{X}$. //

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ENDNOTES

1. This allows for the following trivial generalization. Let **X** be a set of outcomes, or prizes. $\geq_X$ is a weak order on **X** represented by a utility function w: $\mathbf{X} \to \mathfrak{R}$ whose range is a closed interval. Corresponding strong preference $>_X$ and indifference $\sim_X$ are defined as usual. Let $x_0 \in \inf_{>_X} \mathbf{X}$ and $x^0 \in \sup_{>_X} \mathbf{X}$. We will assume without loss of generality that $\mathbf{X} = \mathbf{X}|\sim_X$, i.e. all indifference classes in **X** contain only one element. So we may identify **X** with an interval [a,b], determined by the range of w(•).

2. Axiom A4 reduces **Q** to the form $\mathbf{Q} = [Lx_0, g] \cup (g,f) \cup [f, Lx^0]$. We will accept it for the rest of the paper by default, mentioning here that three other forms of **Q** may be obtained through modification of this technical axiom (of course the results and their proofs must be adjusted accordingly with the necessary changes being straightforward).

If we use

Axiom A4'. A vNM-interval has a semi-open vNM-interval adjacent to itself iff it is closed

instead of axiom A4 then $\mathbf{Q} = [Lx_0, g) \cup [g,f] \cup (f, Lx^0]$.

Similarly,

Axiom A4". Each vNM-interval either contains Lx^0 or is semi-open corresponds to $\mathbf{Q} = [Lx_0, g) \cup [g,f) \cup [f, Lx^0]$ (where $[f, Lx^0]$ is possibly a singleton).

Finally, $\mathbf{Q} = [Lx_0, g] \cup (g,f] \cup (f, Lx^0]$ (where $[Lx_0, g]$ might be a singleton) if we employ

Axiom A4'''. Each vNM-interval either contains Lx_0 or is semi-open.

3. To ensure that both (s_k) and (r_k) are in (g,f) it suffices to select s_1 and r_1 such that their supports contain x_l and x_h respectively. The standard procedure of constructing sequences (s_k) and (r_k) in a way that guarantees convergence in the sup norm can be found, for example, in Halmos (1974), p.115, exercise 6.