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# ON THE RANGES OF BAIRE AND BOREL MEASURES, WITH APPLICATIONS TO FINE AND TIGHT COMPARATIVE PROBABILITIES

by

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**Abstract.** Let X be a topological space, $\mu$ a probability measure defined on the Baire $\sigma$-field on X, and $\mu'$ a probability measure on the Borel $\sigma$-field which extends $\mu$. In the first part of the paper we deal with the relations existing between the ranges of $\mu$ and $\mu'$. In particular, we show cases in which these ranges coincide. In the second part we apply to comparative probability some of the tecniques previously used. In this part it is proved that if a comparative probability relation defined on a Boolean algebra satisfies well known conditions, namely fineness and tigthness (defined below), then the image of any probability measure that agrees with the relation is dense in the unit interval. This sharpens earlier results for comparative probability relations on power sets.

## Introduction

Let **X** be a topological space, $\mu$ a probability measure defined on the Baire $\sigma$-field on **X**, and $\mu'$ a probability measure on the Borel $\sigma$-field which extends $\mu$. In the first part of the paper we deal with the relations existing between the ranges of $\mu$ and $\mu'$. In particular, we show cases in which these ranges coincide. In the second part we apply to comparative probability some of the tecniques previously used. In this part it is proved that if a comparative probability relation defined on a Boolean algebra satisfies well known conditions, namely fineness and tigthness (defined below), then the image of any probability measure that agrees with the relation is dense in the unit interval. This sharpens earlier results for comparative probability relations on power sets.

The reader interested only in comparative probability can look just at proposition 2.1 and skip the rest of sections 2 and 3.

## 2. Preliminaries

We first introduce some preliminary notions. Let **X** be a topological space. The *Borel $\sigma$-field* (**Bor(X)** for short) is the smallest $\sigma$-field generated by the open sets in **X**. The *Baire $\sigma$-field* (**Bai(X)** for short) is the smallest $\sigma$-field w.r.t. which all continuous functions are measurable. A non-negative, finite, countably additive set function defined on **Bor(X)** (or on **Bai(X)** ) is a *Borel measure* (or a *Baire measure*).

In what follows an important role is played by $\tau$-additive Borel measures. To define them, we make use of nets, a generalization of the notion of sequence (cf. Kelley [5] ch.2). A binary relation $\succsim$ directs a set **D** if **D** is non-void and

(i) $\succsim$ is transitive;

(ii) $\succsim$ is reflexive;

(iii) if **m** and **n** are in **D**, then there exists a $\mathbf{p}\in\mathbf{D}$ such that $\mathbf{p}\succsim\mathbf{m}$ and $\mathbf{p}\succsim\mathbf{n}$.

A *directed* set is a pair $(\mathbf{D},\succsim)$ such that $\succsim$ directs **D**. A *net* is a pair $(\mathbf{S},\succsim)$ such that **S** is a function and $\succsim$ directs the domain of **S**. The set of non-negative integers are trivially directed by $\geq$. This explains why nets generalize sequences. A Borel measure $\mu$ is *weakly $\tau$-additive* if whenever $\{\mathbf{g}_\alpha\}$ is a net of open sets in **X** such that $\mathbf{g}_\alpha\subset\mathbf{g}_\beta$ for $\alpha\leq\beta$ and $\bigcup_\alpha\mathbf{g}_\alpha=\mathbf{X}$, then $\mu(\bigcup_\alpha\mathbf{g}_\alpha)=\sup\{\mu(\mathbf{g}_\alpha)\}$. If we do not require $\bigcup_\alpha\mathbf{g}_\alpha=\mathbf{X}$, we say that $\mu$ is *$\tau$-additive*.

A space $\mathbf{X}$ is called *(weakly) Borel measure-complete* if each Borel measure is (weakly) $\tau$-additive.

A Borel measure is *inner regular* if $\mu(g)=\sup\{\mu(f) : f \text{ closed}\}$ for all open sets $g\subset X$. A Borel measure $\mu$ is *compact inner regular* if $\mu(g)=\sup\{\mu(k) : k$ compact, $k\in Bor(X)\}$ for all open sets $g\subset X$.

A cardinal $\aleph$ is *real-valued measurable* if there is a nonzero finite diffused measure (i.e. giving measure zero to all singletons) defined for all subsets of a set $\mathbf{X}$ of cardinality $\aleph$.

A space $\mathbf{X}$ is called *weakly $\vartheta$-refinable* if for any open cover of $\mathbf{X}$ there is an open refinement $\mathcal{U}=\bigcup_{n=1}^{\infty}\mathcal{U}_n$ such that if $x\in X$ there is some $n$ for which $\{U\in\mathcal{U}_n : x\in\mathcal{U}_n\}$ is non-empty and finite. Weakly $\vartheta$-refinability is a rather weak property. For example, compact spaces, regular Lindelöf spaces, paracompact spaces, and metacompact spaces are all weakly $\vartheta$-refinable.

For notation and basic results on Boolean algebras we refer to Koppelberg (1989). In what follows we do not require of the additive set functions defined on a Boolean algebra to be strictly positive (i.e. we do not require $P(a)=0$ iff $a=0$).

A *filter* in a Boolean algebra $\mathbf{A}$ is a subset $\mathbf{p}$ of $\mathbf{A}$ such that:

(i) $\mathbb{1}\in p$;

(ii) if $x\in p, y\in A$ and $x\leq y$, then $y\in p$;

(iii) if $x\in p$ and $y\in p$, then $x\wedge y\in p$.

A filter $\mathbf{p}$ of $\mathbf{A}$ is an *ultrafilter* if, for each $x\in A$, we have $x\in p$ or $-x\in p$, but not both. An algebra $\mathbf{A}$ of subsets of a space $\mathbf{X}$ is said to be *perfect* if every ultrafilter of $\mathbf{A}$ is determined by a point of $\mathbf{X}$, i.e. every ultrafilter has the form $\{a\in A: x_0\in X \text{ and } x_0\in a\}$. An algebra $\mathbf{A}$ of subsets of a space $\mathbf{X}$ is said to be *reduced* if for every pair of distinct points $x,y\in X$ there exists a set $a\in A$ such that $x\in a$ and $y\notin a$. Suppose that $\mathbf{A}$ is a perfect and reduced set algebra. Equipped with the topology of base $\mathbf{A}$, $\mathbf{X}$ becomes a Boolean space, i.e. a compact Hausdorff and totally disconnected space (i.e. each connected subspace contains at most one point).

## 2. Results

**Proposition 2.1.** *Let* $\mathbf{X}$ *be a topological space on which the open Baire subsets form a base, and let* $\mu'$ *be a Baire measure which has an inner regular and weakly* $\tau$*-additive Borel extension* $\mu''$. *Then* $R(\mu')=R(\mu'')$.

**Proof:** let $\mathbf{G(X)}=\{\mathbf{g}\subseteq\mathbf{X} : \mathbf{g}\in\mathbf{Bai(X)}$ and $\mathbf{g}$ open$\}$ and $\mathbf{F(X)}=\{\mathbf{c}\subseteq\mathbf{X} : \mathbf{c}\in\mathbf{Bai(X)}$ and $\mathbf{c}$ closed$\}$. Set $\Gamma(\mathbf{X})=\mathbf{G(X)}\cup\mathbf{F(X)}$. Clearly, a subset is in $\mathbf{G(X)}$ iff its complement is in $\mathbf{F(X)}$. By hypothesis, $\mathbf{G(X)}$ is a base in $\mathbf{X}$. Let $\mu$ be the restiction of $\mu'$ on $\Gamma(\mathbf{X})$. We prove that $\mathbf{R}(\mu)$ is dense in $\mathbf{R}(\mu'')$.

Let $\delta\in\mathbf{R}(\mu'')$. By definition, there is a subset $\mathbf{a}\in\mathbf{Bor(X)}$ such that $\mu''(\mathbf{a})=\delta$. Suppose that $\mathbf{a}$ is an open subset for which there is a sequence $\{\mathbf{g}_\mathrm{i}\}_{\mathrm{i}=1}^{\infty}\subset\mathbf{G(X)}$ such that $\mathbf{a}=\bigcup_{\mathrm{i}=1}^{\infty}\mathbf{g}_\mathrm{i}$. Then $\mathbf{a}\in\mathbf{Bai(X)}$, and so $\mathbf{a}\in\Gamma(\mathbf{X})$. A similar reasoning shows that if $\mathbf{a}$ was a closed subset such that $\mathbf{a}=\bigcap_{\mathrm{i}=1}^{\infty}\mathbf{f}_\mathrm{i}$ with $\mathbf{f}_\mathrm{i}\in\mathbf{F(X)}$, then $\mathbf{a}\in\Gamma(\mathbf{X})$. Now, suppose that $\delta=\mu''(\mathbf{a})$ for an open subset $\mathbf{a}$ for which there is a set $\mathbf{I}\subset\mathbf{G(X)}$ such that $|\mathbf{I}|=\aleph_1$ and $\mathbf{a}=\bigcup\{\mathbf{f}:\mathbf{f}\in\mathbf{I}\}$. The set $\mathbf{I}$ can be put into a one-to-one correspondence with the ordinal numbers less than $\aleph_1$ so that can be written $\mathbf{I}=\{\mathbf{f}_\alpha:\alpha<\aleph_1\}$ (see proposition 3.27 in Rosenstein (1982)). Let $\mathbf{g}_\alpha=\bigcup\{\mathbf{f}_\beta:\beta\leq\alpha<\aleph_1\}$. $\{\mathbf{g}_\alpha:\alpha<\aleph_1\}$ is a monotone increasing net (cf. Kelley (1955) p.77) with respect to set inclusion and each $\mathbf{g}_\alpha$ is an open subset. Clearly we have $\mathbf{a}=\bigcup\{\mathbf{g}_\alpha:\alpha<\aleph_1\}$. By hypothesis, $\mu''$ is $\tau$-additive.Thus $\delta=\mu''(\mathbf{a})=\sup\{\mu''(\mathbf{g}_\alpha):\alpha<\aleph_1\}$. Hence, for every $\varepsilon>0$ there exists an ordinal number $\alpha'(\varepsilon)$ such that for all $\alpha\geq\alpha'(\varepsilon)$ we have $\mu''(\mathbf{g}_a)\in\mathbf{V}(\delta,\varepsilon)$. On the other hand, we know that $\mathbf{g}_\alpha\in\Gamma(\mathbf{X})$ for $\alpha<\aleph_1$. Therefore, $\delta$ is an accumulation point of $\mathbf{R}(\mu)$. Similarly, it can be shown that if $\mathbf{a}$ was a closed subset such that $\mathbf{a}=\bigcap\{\mathbf{f}:\mathbf{f}\in\mathbf{I}\}$ for a set $\mathbf{I}\subset\mathbf{F(X)}$ such that $|\mathbf{I}|=\aleph_1$, then $\delta$ would be an accumulation point of $\mathbf{R}(\mu)$.

To sum up, we have proved that if $\delta=\mu''(\mathbf{a})$ with $\mathbf{a}=\bigcup\{\mathbf{f}:\mathbf{f}\in\mathbf{I}\}$, $\mathbf{I}\subseteq\mathbf{G(X)}$ and $|\mathbf{I}|=\aleph_1$ (or $\mathbf{a}=\bigcap\{\mathbf{f}:\mathbf{f}\in\mathbf{I}\}$, , $\mathbf{I}\subset\mathbf{F(X)}$ and $|\mathbf{I}|\leq\aleph_1$), then $\delta$ is an accumulation point of $\mathbf{R}(\mu)$. In order to extend this conclusion to all $\mathbf{I}$ with $|\mathbf{I}|\leq|\mathbf{G(X)}|$ (or $|\mathbf{I}|\leq|\mathbf{F(X)}|$), we must use transfinite induction. We prove only the case $|\mathbf{I}|\leq|\mathbf{G(X)}|$. The case $|\mathbf{I}|\leq|\mathbf{F(X)}|$ can be proved analogously. We have already established that this hold for $\aleph_0$. Let $\beta$ be an ordinal, which is not a limit one. In order to extend the conclusion to $|\mathbf{I}|=\aleph_{\beta+1}$ when it holds for all $|\mathbf{I}|\leq\aleph_\beta$, it suffices to repeat the reasoning used to pass from $\aleph_0$ to $\aleph_1$. Now let $\gamma$ be a limit ordinal number. Suppose that $\delta=\mu''(\mathbf{a})$ for an open subset $\mathbf{a}$ for which there is a set $\mathbf{I}\subset\mathbf{G(X)}$ such that $|\mathbf{I}|=\aleph_\gamma$ and $\mathbf{a}=\bigcup\{\mathbf{f}:\mathbf{f}\in\mathbf{I}\}$. By now we know that can be written $\mathbf{I}=\{\mathbf{f}_\beta:\beta<\aleph_\gamma\}$. Set $\mathbf{g}_\alpha=\bigcup\{\mathbf{f}_\beta:\beta\leq\alpha<\aleph_\gamma\}$. We have $\mathbf{a}=\bigcup\{\mathbf{g}_\alpha:\alpha<\aleph_\gamma\}$. Furthermore, it holds $\delta=\mu''(\mathbf{a})=\sup\{\mu''(\mathbf{g}_\alpha):\alpha<\aleph_\gamma\}$ since each $\mathbf{g}_\alpha$ is an open subset and $\mu''$ is $\tau$-additive. Let $\delta_\alpha=\mu''(\mathbf{g}_\alpha)$ for $\alpha<\aleph_\gamma$. If our previous conclusion holds for all $|\mathbf{I}|<\aleph_\gamma$, we can say that $\delta_\alpha$ is an accumulation point of $\mathbf{R}(\mu)$ for all $\alpha<\aleph_\gamma$. Now, we have already dealt with a similar situation when we observed that for all $\alpha<\aleph_1$ the real number $\mu'(g_\alpha)$ was accumulation point of $\mathbf{R}(\mu)$. If we repeat the reasoning we have used at that time, it is a simple matter to check that $\delta=\mu''(\mathbf{a})$ is an accumulation point of $\mathbf{R}(\mu)$.

So far we have considered only $\delta\in\mathbf{R}(\mu'')$ such that $\delta=\mu''(\mathbf{a})$ with $\mathbf{a}$ open or closed subset in $\mathbf{Bor(X)}$. Now suppose $\mathbf{a}\in\mathbf{Bor(X)}$, with $\mathbf{a}$ neither open nor closed. By hypothesis, $\mu$ is inner regular. So $\delta=\mu''(\mathbf{a})=\sup\{\mu''(\mathbf{c}):\mathbf{c}$ is closed and $\mathbf{c}\subset\mathbf{a}\}$. For every $\varepsilon>0$ there exists a closed subset $\mathbf{c}$ such that $\delta-\varepsilon<\mu''(\mathbf{c})\leq\mu''(\mathbf{a})$, i.e. for every $\varepsilon$-neighbourhood $\mathbf{V}(\delta,\varepsilon)$ there is $\mathbf{c}\in\mathbf{C}$ such that $\mu''(\mathbf{c})\in\mathbf{V}(\delta,\varepsilon)$. Otherwise, $\delta-\varepsilon$ should be an upper bound of the subset $\{\mu''(\mathbf{c}) : \mathbf{c}\in\mathbf{C}$ and $\mathbf{c}\subset\mathbf{a}\}$. On the other hand, we have proved that for every $\varepsilon'$-neighbourhood $\mathbf{V}(\mu''(\mathbf{c}),\varepsilon')$ there is $\delta_{\varepsilon'}=\mu(\mathbf{f}_{\varepsilon'})$, with $\mathbf{c}\subset\mathbf{f}_{\varepsilon'}$ and $\mathbf{f}_{\varepsilon'}\in\Gamma(\mathbf{X})$, such that $\mu(\mathbf{f}_{\varepsilon'})\in\mathbf{V}(\mu''(\mathbf{c}),\varepsilon')$. Taking $\varepsilon'<2\varepsilon$, it is easy to verify that $\delta_{\varepsilon'}\in\mathbf{V}(\delta,\varepsilon)$. Therefore $\delta=\mu''(\mathbf{a})$ is an accumulation point of $\mathbf{R}(\mu)$. To sum up, we have proved that $\mathbf{R}(\mu)$ is dense in $\mathbf{R}(\mu'')$. Since $\mathbf{R}(\mu')$ is a closed subset of the real line and $\mathbf{R}(\mu)\subseteq\mathbf{R}(\mu')$, it follows $\mathbf{R}(\mu')=\mathbf{R}(\mu'')$ and this completes the proof. ■

**Proposition 2.2.** *The following are both sufficient conditions on a space* **X** *for the open Baire subsets to be a base:*

*(i)* **X** *is normal;*

*(ii)* **X** *is locally compact.*

**Proof:** the proposition can be proved with an argument similar to that used on pp. 174-177 of Berberian (1965). ■

The next three results are simple consequences of the previous propositions.

**Corollary 2.3.** *Let* **X** *be a normal or a locally compact space. Let $\mu$ be a $\tau$-additive Baire measure on* **X**. *Then $\mu$ admits a Borel extension $\mu'$ such that* $\mathbf{R}(\mu)=\mathbf{R}(\mu')$.

**Proof:** by what proved on p.78 of Csizar (1970), $\mu$ has a $\tau$-additive Borel extension $\mu'$. Hence $\mu'$ is regular. A simple application of propositions 2.1 and 2.2 completes the proof. ■

**Corollary 2.4.** *Let $\mu'$ be a $\tau$-additive Borel measure defined on a topological space* **X**. *Let $\mu$ be the restriction of $\mu'$ to the Baire $\sigma$-algebra. Then* $\mathbf{R}(\mu)=\mathbf{R}(\mu')$ *if* **X** *is normal or locally compact.*

**Proof:** since the hypotheses imply that $\mu'$ is regular, a simple application of propositions 2.1 and 2.2 completes the proof. ■

**Corollary 2.5.** *Let $\mu'$ be a compact inner regular Borel measure defined on a topological space* X. *Let $\mu$ be the restriction of $\mu'$ to the Baire $\sigma$-algebra. Then* $\mathbf{R}(\mu)=\mathbf{R}(\mu')$ *if* X *is normal or locally compact.*

**Proof:** in view of the previous corollary, it suffices to note that a compact inner regular Borel measure is $\tau$-additive. ■

### 3. Applications of §2: topological conditions

**Proposition 3.1.** *Let* X *be a Borel measure-complete space which is normal or locally compact. Let $\mu$ be a Baire measure. Then every Borel extension $\mu'$ of $\mu$ is such that* $\mathbf{R}(\mu)=\mathbf{R}(\mu')$.

**Proof:** similar to the one of corollary 2.3. ■

Hereditarily Lindelöf spaces are an example of Borel measure-complete spaces.

**Proposition 3.2.** *Let* X *be a weakly Borel measure-complete space which is normal or locally compact. Let $\mu$ be a Baire measure. Then every inner regular Borel extension $\mu'$ of $\mu$ is such that* $\mathbf{R}(\mu)=\mathbf{R}(\mu')$.

**Proof:** it suffices to note that by theorem 4.3 of Gardner (1975) a regular weakly $\tau$-additive measure is $\tau$-additive. ■

The following spaces are weakly Borel measure-complete:

(i) Lindelöf spaces (and so compact spaces);

(ii) Weakly $\vartheta$-refinable spaces which contain no discrete subspace of real-valued measurable power (see Gardner (1975), theorem 3.9).

Lastly, as an immediate consequence of (ii) we have:

**Corollary 3.3.** *Let* X *be an hereditarily weakly $\vartheta$-refinable space which contains no discrete subspace of real-valued measurable power. Let $\mu$ be a Baire measure. If* X *is normal or locally compact, then every Borel extension $\mu'$ of $\mu$ is such that* $\mathbf{R}(\mu)=\mathbf{R}(\mu')$.

### 4. Comparative probability

This section is devoted to some preliminaries about comparative probability. So, let **A** be a Boolean algebra and $\precsim^{*}$ a comparative probability relation (CP-relation, for short) defined on **A**. Let **P** be a finitely additive (f.a. for short) probability measure defined on **A**. If we have

$$\mathbf{a} \precsim^* \mathbf{b} \text{ only if } \mathbf{P(a)\le P(b)} \qquad \mathbf{(a,b\in A)}$$

we say that **P** *almost agrees* with $\precsim^*$. If we have

$$\mathbf{a} \precsim^* \mathbf{b} \text{ if and only if } \mathbf{P(a)\le P(b)} \qquad \mathbf{(a,b\in A)}$$

we say that **P** *agrees* with $\precsim^*$.

Perhaps the main theme of the literature on comparative probability has been to find conditions to impose on $\precsim^*$ in order to get an agreeing f.a probability measure. A first set of necessary conditions is the following ($\mathbb{0}$ and $\mathbb{1}$ are, respectively, the zero and the unit element of **A**):

$\mathbf{A}_1$: $\precsim^*$ is a linear preorder.
$\mathbf{A}_2$: $\mathbf{b} \precsim^* \mathbf{c}$ if and only if $\mathbf{b \vee c} \precsim^* \mathrm{c \vee d}$, where $\mathbf{b \wedge d = c \wedge d = \mathbb{0}}$ and $\mathbf{b,c,d \in A}$.
$\mathbf{A}_3$: $\mathbb{0} \precsim^* \mathbf{b}$ for all $\mathbf{b \in A}$.
$\mathbf{A}_4$: $\mathbb{0} <^* \mathbb{1}$.

Following Niiniluoto (1972) we refer to a CP-relation that satisfies $\mathbf{A}_1$-$\mathbf{A}_4$ as a *CP-structure*. However, $\mathbf{A}_1$-$\mathbf{A}_4$ does not form a set of sufficient conditions for the agreeing problem. Several conditions, sufficient or both necessary and sufficient, have been proposed. A recent survey can be found in Fishburn (1986).

Here we are interested in the following three conditions:

**(i)** A CP-relation $\precsim^*$ is *fine* if for every nonzero $\mathbf{a \in A}$ there is a finite partition $\{\mathbf{a}_i\}_{i=1}^m$ such that $\mathbf{a}_i \precsim^* \mathbf{a}$ for $\mathbf{1 \le i \le m}$.

**(ii)** A CP-relation $\precsim^*$ is *superfine* if for every nonzero $\mathbf{a \in A}$ there is a finite partition $\{\mathbf{a}_i\}_{i=1}^m$ such that $\mathbf{a}_i <^* \mathbf{a}$ for $\mathbf{1 \le i \le m}$.

**(iii)** Two elements $\mathbf{b,c \in A}$ are almost equivalent if the following two conditions are satisfied:

**(a)** $\mathbf{c} \precsim^* \mathbf{b \vee e}$ for all nonzero $\mathbf{e}$ such that $\mathbf{b \wedge e = \mathbb{0}}$.
**(b)** $\mathbf{b} \precsim^* \mathbf{c \vee f}$ for all nonzero $\mathbf{f}$ such that $\mathbf{c \wedge f = \mathbb{0}}$.

If for every pair $\mathbf{b,c}$ of almost equivalent elements we have $\mathbf{b \approx c}$, we say that $\precsim^*$ is *tight*.

As it is well known, Savage (1954) used **(ii)** and **(iii)** to solve the agreeing problem on a power set. In particular, he proved the following theorem:

**Proposition 4.1 (Savage).** *Let* $\precsim^*$ *be a CP-structure defined on a power set* **B**. *We have:*

*(i) if* $\precsim^*$ *is superfine, then there exists a f.a. probability measure* **P** *on* **B** *which almost agrees with* $\precsim^*$*;*

*(ii) if* $\precsim^*$ *is both superfine and tight, then there exists a f.a. probability measure* **P** *on* **B** *which agrees with* $\precsim^*$*;*

*(iii) let* **P** *be as in either (i) or (ii). Then for all real numbers* $0 \leq \lambda \leq 1$ *and for all* $b \in B$*, there is an element* $c \in B$ *such that* $c \subseteq b$ *and* $P(c) = \lambda P(b)$.

**Remark.** If $\precsim^*$ is fine but not superfine, then the almost agreeing f.a. probability measure of point (i) does not have the property stated in point (iii). This was pointed out in Niiniluoto (1972).

Savage's result was extended in Wakker (1981) to arbitrary Boolean algebras, not necessarily $\sigma$-complete set algebras. Moreover, this was done using fine CP-structures instead of superfine ones.

**Proposition 4.2 (Wakker).** *Let* $\precsim^*$ *be a CP-structure defined on a Boolean algebra* **A**. *We have:*

*(i) if* $\precsim^*$ *is fine, then there exists a f.a. probability measure* **P** *on* **A** *which almost agrees with* $\precsim^*$.

*(ii) if* $\precsim^*$ *is both fine and tight, then there is a f.a. probability measure* **P** *on* **A** *which agrees with* $\precsim^*$.

Therefore, Wakker (1981) generalized part (i) and (ii) of Savage's theorem. The purpose of next sections is to show which form takes part (iii) of Savage's theorem when arbitrary Boolean algebras are considered. In particular, the main result we prove is the following theorem (**R(P)** denotes the range of **P**):

**Proposition 4.3.** *Let* $\precsim^*$ *be a fine and tight CP-structure on a Boolean algebra* **A**. *Let* **P** *be the f.a. probability measure that, by proposition 4.2, agrees with* $\precsim^*$ *on* **A**. *Then* **R(P)** *is a dense subset of* **[0,1]**.

**Remark.** The importance of results that extend Savage's framework to arbitrary Boolean algebras is not merely technical. For example, suppose that a decision maker is interested in an infinite family of events $\{a_i\}_{i \in I}$, where, for simplicity, we assume that the $a_i$ are subsets of a given state space $\Omega$. Suppose he wants to express in a quantitative way his beliefs about the events in $\{a_i\}_{i\in I}$. To do this using Savage's tools, he must consider the set $\sigma$-algebra generated by $\{a_i\}_{i\in I}$, and define a CP-structure on all elements of this $\sigma$-algebra. This can be quite difficult. So, it could be of interest to know what can be done by considering just the set algebra generated by $\{a_i\}_{i\in I}$. This can be a significant improvement because the algebra generated by $\{a_i\}_{i\in I}$ can have less cardinality than the $\sigma$-algebra. For instance, let us consider $\Omega=\{1,2,\ldots,n,\ldots\}$. Suppose one is interested in the singletons $\{i\}$, with $i\in\Omega$. Clearly the set $\sigma$-algebra generated by them is just the power set. So it has cardinality $\aleph_1$. Instead, the set algebra generated by the singletons has cardinality $\aleph_0$.

## 5. Some lemmas

This section contains some lemmas needed to prove proposition 4.3. Let **X** be a Boolean space with base **A** and let **P** be a f.a. probability measure defined on **A**. Since **A** is perfect and reduced, it is known that there is a unique $\sigma$-additive probability measure **P'** which extends **P** on $\sigma$(**A**), the set $\sigma$-algebra generated by **A** (see Sikorski (1969) pp. 202-204). Moreover, there exists a unique regular and $\sigma$-additive probability measure **P"** which extends **P** on **Bor(X)**, the Borel $\sigma$-algebra of **X** (see Ash (1972) p.183). The next lemma contains some properties of the ranges of **P**, **P'**and **P"**.

**Lemma 5.1.** *Let* **P** *be an infinitely many valued f.a. probability measure defined on a perfect and reduced set algebra* **A**. *Then:*

*(a)* **R(P)** *is a dense subset of* **R(P")**.

*(b)* **R(P')=R(P")**.

**Proof:** since **X** is compact Hausdorff, there exists a unique regular and $\sigma$-additive probability measure **P"** defined on **Bor(X)** which extends **P'**. Since **X** is compact, **P"** is also compact-regular and so $\tau$-additive. Therefore, being **A** a base, the proof-tecnique of proposition 2.1 can be applied. This time **A**, being an algebra of clopen subsets, plays the role of $\Gamma$(**X**). We have to take care of one minor difference here. For, let **a** be an open subset for which there is a sequence $\{f_i\}_{i=1}^{\infty}\subseteq A$ such that $a=\bigcup_{i=1}^{\infty} f_i$. Now **a** is not necessarly in **A** because **A** is not $\sigma$-complete. Hence set $g_N=\bigcup_{i=1}^{N} f_i$. We have $g_N\in A$ and $\lim_{N\to\infty} g_N=a$. Since **P'** is countably additive, we have $P''(a)=P'(a)=\lim_{N\to\infty}P'(g_N)=\lim_{N\to\infty}P(g_N)$. Therefore **P"(a)** is an accumulation point of **R(P)**, and from now on the proof proceeds analogously to the one of proposition 2.1. ∎

Let us consider a f.a. probability measure **P** defined on a Boolean algebra **B**. Let **Ult(B)** be the Stone space of **B**, **Clop(Ult(B))** the dual algebra of **Ult(B)** and $\mathbf{s:B \to Ult(B)}$ the Stone map. It is known that **Clop(Ult(B))** is a perfect and reduced set algebra. So **Ult(B)**, equipped with the topology of base **Clop(Ult(B))**, is a Boolean space. Let $\mathbf{P_S}$ be defined on **Clop(Ult(B))** by $\mathbf{P_S(s(a))=P(a)}$ for $\mathbf{a \in B}$. It is a simple matter to check that $\mathbf{P_S}$ is a f.a. probability measure. Since **Clop(Ult(B))** is perfect and reduced, we know that there is a unique countably additive extension $\mathbf{P'_S}$ of $\mathbf{P_S}$ on $\sigma$**(Clop(Ult(B)))**. Therefore, being $\mathbf{R(P_S)=R(P)}$ by definition, as a direct consequence of lemma 5.1 we have the following result.

**Lemma 5.2.** *Let* **P** *be an infinitely many valued f.a. probability measure defined on a Boolean algebra* **B**. *Then* **R(P)** *is a dense subset of* $\mathbf{R(P'_S)}$.

## 6. Main results on comparative probability

As we have said in the introduction, our main result in comparative probability is proposition 4.3. Indeed, it is a simple consequence of next proposition. In fact, a tight CP-structure is atomless (cf. theorem 3 in Wakker (1981)).

**Proposition 6.1.** *Let* $\precsim^*$ *be a fine and atomless CP-structure defined on a Boolean algebra* **A**. *Let* **P** *be the f.a. probability measure that by theorem 4.2 almost agrees with* $\precsim^*$. *Then* **R(P)** *is a dense subset of* **[0,1]**.

**Proof:** the first part of the proof mimics, mutatis mutandis, the proof of lemma 1 in Saks (1933). So, let $\mathbf{a \in A}$ with $\mathbf{a >^* \mathbb{0}}$. By lemma 1 in Wakker (1981) there is a sequence $\mathbf{\{a_n\}_{n=1}^{\infty}}$ such that $\mathbf{a_1=a}$, $\mathbf{a_{n+1} \le a_n}$, $\mathbf{a_n >^* \mathbb{0}}$ and $\mathbf{a_{n+1} \precsim^* a_n - a_{n+1}}$. Then $\mathbf{P(a_{n+1})+P(a_n-a_{n+1}) \le P(a_n)}$. Hence $\mathbf{P(a_{n+1}) \le (1/2)P(a_n)}$. Therefore, $\mathbf{P(a_{n+1}) \le 2^{-n-1}P(a_1)}$. Let $\varepsilon>0$. For **n** large enough, $\mathbf{2^{-n-1}P(a_1)}<\varepsilon$ and so $\mathbf{P(a_{n+1})}<\varepsilon$. Since $\precsim^*$ is fine, there is a finite partition $\mathbf{\{b_i\}_{i=1}^{m}}$ such that $\mathbf{b_i \precsim^* a_{n+1}}$ for $\mathbf{1 \le i \le m}$. Then $\mathbf{P(b_i) \le P(a_{n+1})}<\varepsilon$ for $\mathbf{1 \le i \le m}$. Since the Stone map $\mathbf{s:A \to Ult(A)}$ is an isomorphism, we have $\mathbf{s(A)=s(\bigvee_{i=1}^{n} b_i)=\bigcup_{i=1}^{n} s(b_i)=Ult(A)}$ and $\mathbf{\emptyset=s(b_i \wedge b_j)=s(b_i) \cap s(b_j)}$. Hence $\mathbf{\{s(b_i)\}_{i=1}^{n}}$ is a finite partition of **Ult(A)** contained in **Clop(Ult(A)))**. Moreover, given $\varepsilon>0$ from the definition of $\mathbf{P_S}$ it follows $\mathbf{P_S(s(b_i))=P(b_i)}<\varepsilon$ for **i=1,...,n**. Consequently, $\mathbf{P'_S(s(b_i))}<\varepsilon$ for **i=1,...,n**. So, given $\varepsilon>0$ there is a finite partition of **Ult(A))** contained in $\sigma$**(Clop(Ult(A)))** whose elements are less probable that $\varepsilon$. Now, define on

$\sigma(\mathbf{Clop(Ult(A))})$ a CP-structure $<^{\#}$ as: $\mathbf{a}<^{\#}\mathbf{b}$ iff $\mathbf{P'_S(a)<P'_S(b)}$ for $\mathbf{a,b}\in\sigma(\mathbf{Clop(Ult(A))})$. From what just proved it follows easily that $<^{\#}$ is superfine. By proposition 4.1(ii), which can be easily adapted to set $\sigma$-algebras, it follows that the unique f.a. probability measure that agrees with $<^{\#}$ has range $[0,1]$. Hence, by the definition of $\mathbf{P'_S}$ it follows $\mathbf{R(P'_S)=[0,1]}$. Obviously $\mathbf{R(P)}$ is not finite. In fact, suppose on the contrary that $\mathbf{R(P)}$ was a finite set and let $\delta=\min\{r : r\in\mathbf{R(P)}\}$. It is easy to see that there exists a subset $\mathbf{a}\in\mathbf{A}$ such that $0<\mathbf{P(a)}<\delta$, which is a contradiction. Now a direct application of lemmma 5.2 completes the proof ∎

We conclude with a corollary.

**Corollary 6.2.** *Let* $\precsim^*$ *be a superfine CP-structure defined on a Boolean algebra* $\mathbf{A}$. *Let* $\mathbf{P}$ *be the f.a. probability measure that, by part (i) of proposition 4.2, almost agrees with* $\precsim^*$. *Then* $\mathbf{R(P)}$ *is a dense subset of* $[0,1]$.

**Proof:** in view of proposition 6.1 it suffices to prove that $\precsim^*$ is atomless. Now, suppose on the contrary that $\mathbf{a}>^*\mathbb{0}$ is an atom. Since $\precsim^*$ is superfine, there is a finite partition $\{\mathbf{b}_i\}_{i=1}^m$ such that $\mathbf{b}_i<^*\mathbf{a}$ for $1\le i\le m$. Set $\mathbf{d}_i=\mathbf{b}_i\cap\mathbf{a}$. Clearly $\mathbf{d}_i$ is nonzero for some $\mathbf{i}$, say $\mathbf{i'}$, because $\{\mathbf{d}_i\}_{i=1}^m$ is a partition of $\mathbf{a}$. So, suppose $\mathbf{d}_{i'}\neq\mathbb{0}$. We have $\mathbf{d}_{i'}\le(\mathbf{b}_{i'}\cap\mathbf{a})\le\mathbf{b}_{i'}$ and so $\mathbf{d}_{i'}\precsim^*(\mathbf{b}_{i'}\cap\mathbf{a})\precsim^*\mathbf{b}_{i'}$. Since $\mathbf{b}_{i'}<^*\mathbf{a}$, this implies $\mathbf{d}_{i'}<^*\mathbf{a}$. Then $\mathbf{d}_{i'}\approx^*\mathbb{0}$ because $\mathbf{a}$ is an atom and $\mathbf{d}_{i'}\subseteq\mathbf{a}$. Let $\mathbf{I}=\{\mathbf{i} : \mathbf{d}_i\neq\mathbb{0}\}$. We already know that $\mathbf{I}\neq\varnothing$. Suppose $\#(\mathbf{I})=\mathbf{k}$. Then $\mathbf{a}=\bigvee_{i=1}^k\mathbf{d}_i$. Since $\mathbf{d}_i\approx^*\mathbb{0}$ for $1\le i\le \mathbf{k}$, by $C_4$ p.195 of Fishburn (1970) we have $\bigvee_{i=1}^k\mathbf{d}_i\approx^*\mathbb{0}$, and so $\mathbf{a}\approx^*\mathbb{0}$. This contradiction proves the corollary. ∎

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