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RESIDUAL MEASURES AND THE EXISTENCE AND RANGE OF PROBABILITY MEASURES ON BOOLEAN ALGEBRAS

by

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Abstract: a Borel probability measure is residual if it gives measure zero to all meager subsets. We first give some existence results about this class of measures. Then they are applied in order to get some non-existence results for probability measures defined on Boolean algebras. This is done on the basis of some duality methods. Finally we prove that the range of a nonatomic probability measure defined on a Boolean algebra which satisfies the c.c.c. is dense in the unit interval.

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ABSTRACT

A Borel probability measure is residual if it gives measure zero to all meager subsets. We first give some existence results about this class of measures. Then they are applied in order to get some non-existence results for probability measures defined on Boolean algebras. This is done on the basis of some duality methods. Finally we prove that the range of a nonatomic probability measure defined on a Boolean algebra which satisfies the c.c.c. is dense in the unit interval.

1. Introduction

Let **X** be a topological space and let **cl(a)** and **int(a)** denote, respectively, the closure and the interior of a subset **a⊆X**. A subset **a** is said to be nowhere dense if **intcl(a)=∅**, i.e. if the closure of **a** has empty interior. Nowhere dense subsets have no interior points. A subset **a** is said to be meager (or of the first category) if it can be represented as a countable union of nowhere dense subsets. They are the subsets of **X** which can be approximated by nowhere dense subsets. A probability measure is called residual if it gives measure zero to meager subsets. The interest for these measures is twofold.

(1) Nowhere dense subsets and meager subsets have been regarded as small in a topological sense. To help intuition, observe that a nowhere dense subset of the real line is a subset full of holes. On the other hand, from a measure-theoretic viewpoint the notion of small is represented by nullsets. So, quoting [11, p.4], "it is natural to ask whether these notions of smallness are related". The study of residual measures is of interest in a similar perspective. Indeed, with respect to a residual measure, meager Borel subsets are small not only in a topological sense, but also in a metric one.

(2) Let **A** be a Boolean σ-algebra and **P** a probability measure (not necessarly strictly positive) defined on it. By the Loomis-Sikorski representation theorem, to the pair **(A,P)** can be associated a measure space characterized by a residual measure. Using techniques of this type it is possible to derive some results about probability measures defined on Boolean algebras from propositions established for residual measures.

The paper is organized as follows. Section 2 contains some preliminary notions. Section 3 gives existence results concerning residual measures. In section 4 the results of section 2 are used in order to obtain some non-existence results for probability measures defined on Boolean algebras. Finally, in section 5 the constructions of the previous sections are used to get a result about the ranges of probability measures on Boolean algebras.

We follow the notation of [9]. This implies, for instance, that lower and upper cases will usually denote, respectively, sets and Boolean algebras.

2. Preliminary notions

We first define some classes of Borel probability measures.

Definition 2.1. *A Borel probability measure* **P** *defined on a topological* *space* **X** *is said to be regular if for all Borel subsets* **a** *we have*

$$P(a)=\sup\{P(c) : c\subseteq a \text{ and } c \text{ closed}\}=\inf\{P(v) : a\subseteq v \text{ and } v \text{ open}\}.$$

Definition 2.2. *A Borel probability measure* **P** *defined on a topological space* **X** *is said to be* τ*-additive if whenever* $\{g_a\}$ *is a net of open subsets such that* $g_\alpha \subset g_\beta$ *for* $\alpha\le\beta$*, then* $P(\bigcup_\alpha g_\alpha)=\sup\{P(g_\alpha)\}$.

Let **M** denote the σ-ideal of all meager Borel subsets. Next we define residual measures.

Definition 2.3. *A Borel probability measure* **P** *defined on a topological space* **X** *is said to be residual if* $P(a)=0$ *for all* $a\in M$.

Given a probability measure **P** on a field **A**, a subset $a\in A$ is called a **P**-atom if: (i) $P(a)>0$; (ii) if $b\subset a$ and $b\in A$, then either $P(b)=0$ or $P(a\setminus b)=0$.

Definition 2.4. *A Borel probability measure* **P** *defined on a topological space* **X** *is said to be nonatomic if there are no* **P**-*atoms.*

In the sequel we will make use also of the notion of nonmeasurable cardinal. Indeed, suppose that on the power set of a space **X** there exists a diffuse probability measure **P**, i.e. a measure giving measure zero to all singletons. In such a case the cardinal number of **X** is called measurable. Instead, a cardinal $\aleph$ is called nonmeasurable if this does not hold for all spaces of cardinality $\aleph$.

Now we turn to Boolean algebras. For basic notions and results we refer to [9]. We begin with a definition ($\mathbb{I}$ is the unit element of **A**).

Definition 2.5. *A probability measure* **P** *on a Boolean algebra* **A** *is a real-valued function such that:*

$P(a)\ge 0$ *for all* $a\in A$;

$P(\sum_{n=1}^{\infty} a_n)=\sum_{n=1}^{\infty} P(a_n)$ *whenever* $\{a_n\}_{n=1}^{\infty}\subset A$ *is a set of pairwise disjoint elements for which* $\sum_{n=1}^{\infty} a_n$ *exists;*

$P(\mathbb{I})=1$.

It is worth noting that **P** has not been defined as a strictly positive measure (i.e. it is not required $P(a)=0$ iff **a** is the zero element).

It is important to distinguish the previous definition from the usual definition of probability measures on fields of subsets. Let **A** be a field of subsets. A real-valued set function **P** on **A** is countably additive if: (iii')

$\mathbf{P}(\bigcup_{n=1}^{\infty}\mathbf{a}_n)=\sum_{n=1}^{\infty}P(a_n)$ whenever $\{a_n\}_{n=1}^{\infty}\subset A$ is a set of pairwise disjoint elements such that $\bigcup_{n=1}^{\infty}a_n\in\mathbf{A}$. Condition (iii') is quite different from condition (iii) in definition 2.5, as was stressed in [7]. In fact, $\sum_{n=1}^{\infty}a_n$, the supremum of $\{\mathbf{a}_n\}_{n=1}^{\infty}$ in **A**, can exists even when $\bigcup_{n=1}^{\infty}\mathbf{a}_n$ is not in **A**. Instead, if $\bigcup_{n=1}^{\infty}\mathbf{a}_n\in\mathbf{A}$, then $\bigcup_{n=1}^{\infty}\mathbf{a}_n=\sum_{n=1}^{\infty}\mathbf{a}_n$. Therefore, if a set function satisfies (iii), then it satisfies (iii'), while a set function which satisfies (iii') can fail to satisfy (iii). We illustrate this failure with what would be otherwise a counterexample to the results of next section. Let $\mathbf{A}^*$ be the smallest field of subsets of $[0,1)$ generated by the intervals $[a,b)$, with $0\leq a<b\leq 1$. The field $\mathbf{A}^*$ is both separable and atomless (these notions are introduced below). It is known that each element of $\mathbf{A}^*$ can be expressed as a finite and disjoint union of half-open intervals of the form $[a,b)$. Consequently, each element of $\mathbf{A}^*$ contains an open subset. Now, let **m** be the restriction of the Lebesgue measure on $\mathbf{A}^*$. We want to show that **m** does not satisfy condition (iii) of definition 2.5. In fact, let us construct a subset of $[0,1)$ in a way similar to that of the Cantor set. However, this time at stage **n** we delete 2^{n-1} intervals of the form $[a,b)$ and of length $\alpha(3^{-n})$, with $0<\alpha<1$. We denote by $c^*_{\alpha,n}$ what remains after stage **n** and we set $c^*_a=\bigcap_{n=1}^{\infty}c^*_{\alpha,n}$. The subsets $c^*_{\alpha,n}$ belong to $\mathbf{A}^*$ for all $n\geq 1$. Moreover, we have $c^*_\alpha\leq c_\alpha$, where c_α is defined below in the second remark on p. 7. Thus $cl(c^*_\alpha)\leq c_\alpha$, and so c^*_α is nowhere dense. This implies that c^*_α does not contain any open subset. Therefore, c^*_α does not belong to $\mathbf{A}^*$ because we have already seen that each element of $\mathbf{A}^*$ has a non-empty interior. Furthermore, for the same reason all subsets of c^*_α do not belong to $\mathbf{A}^*$. To sum up, $\bigcap_{n=1}^{\infty}c^*_{\alpha,n}=c^*_\alpha\notin\mathbf{A}^*$ and $\wedge_{n=1}^{\infty}c^*_{\alpha,n}=\varnothing$. In section 2 we said that $m(c_\alpha)=1-\alpha$. Looking at the construction of c^*_α, it is easy to see that the continuity of **m** implies $m(c_\alpha)=m(c^*_\alpha)$. Therefore, using again the continuity of **m**, we have $\lim_{n\to\infty}m(c^*_{\alpha,n})=m(c^*_\alpha)=1-\alpha$. But $m(\wedge_{n=1}^{\infty}c^*_{\alpha,n})=m(\varnothing)=0$. Hence, $\lim_{n\to\infty}m(c^*_{\alpha,n})>m(\wedge_{n=1}^{\infty}c^*_{\alpha,n})$, and it easy to verify that this violates condition (iii) of definition 2.5. This is the result we wanted to prove.

Following [9], **Ult(A)** denotes the Stone space of **A** and **Clop(Ult(A))** the dual algebra of **Ult(A)**. In next section we will make use of a generalization of the Loomis-Sikorski representation theorem, proved in [14]. To report this generalization we need some further notions. A subset of **Ult(A)** is said to be σ-closed provided it is the intersection of countably many subsets in **Clop(Ult(A))**. A subset of **Ult(A)** is said to be σ-nowhere dense provided it is a subset of a nowhere dense σ-closed set. Clearly, a σ-nowhere dense subset is nowhere dense. Finally, a subset of **Ult(A)** is said to be of the σ-category if it is the union of countably many subsets σ-nowhere dense in **Ult(A)**. Observe that a σ-category subset is meager in **Ult(A)**. Let $\mathbf{M}_0$ be the σ-ideal of all subsets of the σ-category in σ**(Clop(Ult(A)))**, the σ-field generated by **Clop(Ult(A))**. Set $\mathcal{F}=\{(u\cup v_1)-v_2$ with $u\in$**Clop(Ult(A))** and $v_1,v_2\in \mathbf{M}_0\}$. $\mathcal{F}$ is a field contained in σ**(Clop(Ult(A)))**. Let **s:A→Clop(Ult(A))** be the Stone isomorphism. Let **h** be the σ-homomorphism from $\mathcal{F}$ onto the quotient algebra $\mathcal{F}/\mathbf{M}_0$ defined by $h(a)=\{b\in\mathcal{F} : b\Delta a\in \mathbf{M}_0\}$, where $a\in\mathcal{F}$. Let π be a homomorphism from **A** onto $\mathcal{F}/\mathbf{M}_0$ defined by $\pi(a)=h(s(a))$, where $a\in\mathbf{A}$. π is called the canonical homomorphism. The version of Sikorski's result we are interested in is the following one:

Proposition 2.6 (Sikorski). *The canonical homomorphism is a σ-isomorphism from* **A** *onto* $\mathcal{F}/\mathbf{M}_0$.

A Boolean algebra satisfies the countable chain condition, c.c.c. for short, if each pairwise disjoint family in **A** is at most countable. For this class of Boolean algebras in [14] it is proved the next result:

Proposition 2.7 (Sikorski). *A Boolean algebra* **A** *satisfies the c.c.c. if and only if every nowhere dense subset in* **Ult(A)** *is σ-nowhere dense.*

Therefore, if a Boolean algebra **A** satisfies the c.c.c., then a subset of **Ult(A)** is meager iff it is σ-nowhere dense. Let **M'** be the σ-ideal of all meager subsets in σ**(Clop(Ult(A)))**. By proposition 2.7, $\mathcal{F}/\mathbf{M}_0=\mathcal{F}/\mathbf{M}'$ in a Boolean algebra which satisfies the c.c.c..

Two other notions in which we are interesed are those of separable Boolean algebras and of atomless Boolean algebras. A Boolean algebra **A** is said to be separable if there exists a countable set **D** of nonzero elements of **A** which is dense in **A**, i.e. such that for every $a\in\mathbf{A}$, with $a\neq\mathbb{O}$, there is an element $a'\in\mathbf{D}$ with $a'\leq a$. A Boolean algebra is said to be atomless if it has no atoms. An atom is a nonzero element $a\in\mathbf{A}$ such that for every element $a'\in\mathbf{A}$ with $a'\leq a$ we have either $a'=a$ or $a'=\mathbb{O}$, where $\mathbb{O}$ is the zero element of **A**.

In view of the results of section 2 we are particularly interested in the Boolean algebras with separable Stone spaces. To deal with them we introduce a cardinal function. So, let **s** be a countable subset of the Stone space **Ult(A)**. Let **d(·)** be the cardinal function on **A** defined by $d(A)=\min\{|s| : s$ is a dense subset of **Ult(A)**$\}$. Clearly, **Ult(A)}** is separable iff $d(A)=\aleph_0$.

Remark. Unlike [9], we will denote by **Bai(X)** the Baire σ-field of **X** and not the σ-field of subsets with the Baire property. Moreover, we will call Baire subsets the elements of **Bai(X)**.

3. Existence of residual measures

We begin by proving a quite useful existence result.

Proposition 3.1 *Let* **X** *be a separable* $\mathbf{T_1}$ *space. Then there exists a residual regular measure iff* **X** *has an isolated point.*

Proof: if **x** is an isolated point, the Dirac probability measure δ_x is a residual regular measure. For the converse, suppose that **X** is perfect (i.e. **X** has no isolated points). Since **X** is a $\mathbf{T_1}$ space, all singletons $\{x\}$, for $x \in X$, are closed subsets. Since **X** is perfect, no singleton can be at the same time both open and closed. So $\mathrm{int}(\{x\})=\emptyset$ and the singletons are nowhere dense subsets. Let **s** be a countable dense subset of **X** and let **P** be a regular residual measure. We have $P(s)=0$ since $P(\{x\})=0$, the singletons being nowhere dense subsets. Then $P(s)=\sum_{x\in s}P(\{x\})=0$. Since **s** is a dense subset, $X\setminus s$ does not contain any nonempty open subset. This implies that all closed subsets **c** contained in $X\setminus s$ are such that $\mathrm{int}(c)=\emptyset$, i.e. they are nowhere dense subsets. Then $P(c)=0$. Since **P** is regular, we have:

$$P(X\setminus s)=\sup\{P(c) : c\subset X\setminus s \text{ and } c \text{ closed}\}.$$

It follows $P(X\setminus s)=0$ and so $P(X)=P(X\setminus s)+P(s)=0$ for any regular residual measure **P** ■

Remark. After having proved proposition 3.1, I have found a similar result in [5, p.113]. However my proof is different from that given in [5].

Remark. The Borel measure on **[0,1]** is not residual. In fact, there exists a nowhere dense subset with positive Borel measure. This set is constructed like the Cantor set, except that now at each stage are deleted intervals of the form **(a,b)** of length $\alpha(3^{-n})$, with $0<\alpha<1$. The set has Borel measure $1-\alpha$, and it is nowhere dense. We denote this set by c_α.

In [4, proposition 5] it is stated that in a perfect metric space cannot exist a nonzero residual measure which is compact regular. In [2, proposition 4a] it is proved that there exists a nonzero τ-additive residual measure on a metric space **X** iff **X** has an isolated point. We can give a more general result.

Proposition 3.2. *Let* X *be a metric space. Then there exists a nonzero residual measure iff* X *has an isolated point.*

Proof: in [4, p.249] it is proved that a perfect metric space contains a dense subset s which is meager. Let P be a residual measure. Then $P(s)=0$. Since s is dense, we have $int(X\setminus s)=\emptyset$. Moreover, since X is a metric space, P is regular. Therefore, in order to complete the proof it suffices to follow the same reasoning employed in the final part of the proof of proposition 3.1. ■

In proposition 3.2 we have considered metric spaces. Similar results hold for other topological spaces, as shows the next proposition. Here T_3 means regular and T_1.

Proposition 3.3. *Let* X *be a* T_3 *separable space. Then there exists a residual measure iff* X *has an isolated point.*

Proof: a separable space has the countable chain property. Therefore it follows from [2, proposition 2a] and from [1, proposition 6] that every residual measure is τ-additive. On the other hand, by [6, theorem 5.4] every τ-additive probability measure on a regular space is a regular probability measure. A simple application of proposition 3.1 completes the proof. ■

Now we turn to nonatomic residual measures. The next proposition contains some non-existence results for this class of measures. In particular, points (i) and (iv) will be employed in section 4. We denote by $Is(X)$ the set of all isolated points of the space X.

Proposition 3.4. (i) *Let* X *be a* T_3 *separable space. Then there is no nonzero nonatomic residual measure.*

(ii) *Let* X *be a separable metric space. Then there is no nonzero nonatomic residual measure.*

(iii) *Let* X *be a* T_3 *space such that* $Is(X)$ *forms a dense subset. If* $|Is(X)|$ *is nonmeasurable, then there is no nonatomic residual measure.*

(iv) *Let* X *be a compact Hausdorff space such that* $Is(X)$ *forms a dense subset. Then there is no nonatomic and regular residual measure.*

Proof: (i) let P be a nonatomic residual measure. Since P is nonatomic, $P(\{x\})=0$ for all $x\in X$. In particular, if s is the countable dense subset of X which exists by hypothesis, we have $P(s)=\sum_{x\in s}P(\{x\})=0$. Observe that if there are isolated points in $\mathbf{X}$, they are all contained in $\mathbf{s}$. In fact $\mathbf{int(X\setminus s)=\varnothing}$. We have seen in the proof of proposition 3.3 that all residual measures on a separable $\mathbf{T_3}$ space are regular. Therefore a reasoning similar to the one employed in the proof of proposition 3.1 shows that $\mathbf{P(int(X\setminus s))=0}$.

(ii) The proof is similar to the one of point (i).

(iii) Let $\mathbf{P}$ be a nonatomic residual measure. Clearly the cardinal number of each pairwise disjoint family of open subsets is at most $|\mathbf{Is(X)}|$. By [2, proposition 2a] every residual measure on $\mathbf{X}$ is τ-additive. In particular, $\mathbf{P}$ is τ-additive. Let $\mathbf{I\subset Is(X)}$ with $|\mathbf{I}|=\aleph_0$. From point (i) we know that $\mathbf{P(I)=0}$. Now suppose that $\mathbf{I\subset Is(X)}$ with $|\mathbf{I}|=\aleph_1$. Let us denote the ordinal numbers by greek letters. It is known that the set $\mathbf{I}$ can be put into a one-to-one correspondence with the ordinal numbers less than $\aleph_1$ so that can be written $\mathbf{I=\{x_a:\alpha<\aleph_1\}}$ (see [12, proposition 3.27]). Set $\mathbf{h_\beta=\bigcup\{x_\alpha:\alpha\leq\beta<\aleph_1\}}$. $\mathbf{\{h_\beta:\beta<\aleph_1\}}$ is a monotone increasing net with respect to set inclusion. Every isolated point is a clopen subset. So $\mathbf{h_\beta}$ is an open subset for every ordinal number β. By definition, we have $\mathbf{I=\bigcup\{h_\beta:\beta<\aleph_1\}}$. Therefore it holds $\mathbf{P(I)=sup\{P(h_\beta):\beta<\aleph_1\}}$ because $\mathbf{P}$ is τ-additive. From what we have already proved for $|\mathbf{I}|=\aleph_0$, we can say that $\mathbf{P(h_\beta)=0}$ for each $\beta<\aleph_1$. Therefore $\mathbf{P(I)=0}$. This implies that every subset of $\mathbf{Is(X)}$ of cardinality $\aleph_1$ has measure zero. Using transfinite induction, it is easy to extend this conclusion to all cardinals $\aleph\leq|\mathbf{Is(X)}|$. In particular $\mathbf{P(Is(X))=0}$. Since $\mathbf{Is(X)}$ is a dense subset, we have $\mathbf{int(X/Is(X))=\varnothing}$. Since all τ-additive measures on a T_3 space are regular, a reasoning similar to the one employed in the proof of proposition 3.1 shows that $\mathbf{P(int(X\setminus s))=0}$.

(iv) A compact regular Borel probability measure is τ-additive. Therefore the proof is similar to the one of point (iii). ∎

4. The existence of probability measures on Boolean algebras

The existence results proved in the previous section have interesting applications in the study of probability measures defined on Boolean algebras. We begin with a consequence of proposition 3.3.

Proposition 4.1. *Let* $\mathbf{A}$ *be an atomless Boolean algebra with* $\mathbf{d(A)=\aleph_0}$. *Then there is no nonzero probability measure on* $\mathbf{A}$.

Proof: let $\mathbf{Bai(Ult(A))}$ be the Baire σ-field of $\mathbf{Ult(A)}$. It is known that $\mathbf{Bai(Ult(A))}$ coincides with the σ-field generated by $\mathbf{Clop(Ult(A))}$, i.e. $\mathbf{\sigma(Clop(Ult(A)))}$. Then $\mathbf{M'}$ is the collection of all meager Baire subsets. Let $\mathbf{P}$

be a probability measure on $\mathbf{A}$. Set $\mathbf{P}_\nu=\mathbf{P}\circ\pi^{-1}\circ\mathbf{h}:\mathcal{F}\to[0,1]$ and $\mathbf{P}_s=\mathbf{P}\circ s^{-1}:\mathbf{Clop(Ult(A))}\to[0,1]$. Using propositions 2.6 and 2.7, it is easy to verify that $\mathbf{P}_s$ and $\mathbf{P}_\nu$ are probability measure respectively on $\mathbf{Clop(Ult(A))}$ and on $\mathcal{F}$. Let $\mathbf{P}^\#$ be the restriction of $\mathbf{P}_\nu$ to $\mathbf{Clop(Ult(A))}$. If $\mathbf{a}\in\mathbf{A}$, we have:

$$\mathbf{P}^\#(s(a))=\mathbf{P}_\nu(s(a))=\mathbf{P}(\pi^{-1}(h(s(a)))=\mathbf{P}(\pi^{-1}(\pi(a)))=\mathbf{P}(a)=\mathbf{P}_s(s(a)).$$

Therefore $\mathbf{P}^\#\equiv\mathbf{P}_s$, and $\mathbf{P}_\nu$ is the probability measure which extends $\mathbf{P}_s$ on $\mathcal{F}$.

Clearly $\mathbf{M'}\subset\mathcal{F}$ and it is easy to check that we have $\mathbf{P}_\nu(\mathbf{v})=0$ for all $\mathbf{v}\in\mathbf{M'}$. From the inclusions $\mathbf{Clop(Ult(A))}\subset\mathcal{F}\subset\mathbf{Bai(Ult(A))}$ it follows $\sigma(\mathcal{F})=\mathbf{Bai(Ult(A))}$. By Caratheodory extension theorem there is a unique probability measure $\mathbf{P'}$ which extends $\mathbf{P}_s$ from $\mathbf{Clop(Ult(A))}$ to $\mathbf{Bai(Ult(A))}$. Clearly $\mathbf{P'}$ coincides with $\mathbf{P}_\nu$ on $\mathcal{F}$. The Boolean algebra $\mathbf{A}$ satisfies the countable chain condition (c.c.c.). Suppose not. Let $\{a_i\}_{i\in I}\subset\mathbf{A}$ be a set of pairwise disjoint and nonzero elements with $|I|>\aleph_0$. Then $\{s(a_i)\}_{i\in I}$ is a set of pairwise disjoint elements in $\mathbf{Ult(A)}$. Moreover $s(a_i)\neq\varnothing$ for all $i\in I$ since a_i is nonzero. But $\mathbf{Ult(A)}$ is separable and so it satisfies the c.c.c. for topological spaces. Since the $s(a_i)$ are clopen subsets, this implies that at most a countable number of the $s(a_i)$ are non-empty. This contradiction shows that $\mathbf{A}$ satisfies the c.c.c.. Therefore, by proposition 2.7 every nowhere dense subset in $\mathbf{Ult(A)}$ is σ-nowhere dense, i.e. it is contained in a closed nowhere dense subset $\mathbf{c}$ such that $\mathbf{c}=\bigcap_{i=1}^\infty u_i$ for some $u_i\in\mathbf{Clop(A)}$. Clearly $\mathbf{c}\in\mathbf{Bai(Ult(A))}$. Let $\mathbf{a}\subseteq\mathbf{Ult(A)}$ be a meager subset. Then $a=\bigcup_{i=1}^\infty a_i$, with a_i nowhere dense subsets. By proposition 2.7 we have $a=\bigcup_{i=1}^\infty a_i\subseteq\bigcup_{i=1}^\infty c_i$, where the c_i are closed and Baire nowhere dense subsets. Since $\bigcup_{i=1}^\infty c_i\in\mathbf{Bai(Ult(A))}$, it follows that every meager subset is contained in a Baire meager subset. Let $\mathbf{Bor(Ult(A))}$ be the Borel σ-field of $\mathbf{Ult(A)}$. Since $\mathbf{Ult(A)}$ is compact and Hausdorff, there exists a unique regular extension $\mathbf{P''}$ from $\mathbf{Bai(Ult(A))}$ to $\mathbf{Bor(Ult(A))}$ (see [3, p.183]). Let $\mathbf{M}$ be the σ-ideal of all meager Borel subsets of $\mathbf{Ult(A)}$. If $\mathbf{a}\in\mathbf{M}$, we know that there is a subset $\mathbf{a'}\in\mathbf{M'}$ such that $\mathbf{a}\subseteq\mathbf{a'}$. Since $\mathbf{P'(a')}=0$, it follows that $\mathbf{P''(a)}=0$, i.e. $\mathbf{P''}$ is a residual measure. Furthermore, $\mathbf{P''}$ is just the unique extension of $\mathbf{P'}$. In fact, by now it is clear that every extension of $\mathbf{P'}$ is residual. And we have seen in the proof of proposition 3.3 that each residual measure on a σ-field like $\mathbf{Bor(Ult(A))}$ is regular. But, there is a unique regular extension.

To sum up, we have proved that the existence of a nonzero probability measure $\mathbf{P}$ on $\mathbf{A}$ implies the existence of a unique nonzero residual and regular measure $\mathbf{P''}$ on $\mathbf{Bor(Ult(A))}$. However, by proposition 3.3 we have $\mathbf{P''}\equiv 0$.

Therefore, by contraposition, **P≡0**. ■

In [8, theorem 3.2] it was proved that on a separable atomless Boolean algebra there is no nonzero probability measure. This result is now an easy consequence of our proposition 4.1, as shown by next corollary.

Corollary 4.2. *Let* **A** *be a separable atomless Boolean algebra. Then there is no nonzero probability measure on* **A**.

Proof: in view of the proof of proposition 4.1 it suffices to show that **Ult(A)** is separable. By hypothesis there exists a countable dense subset **D** of **A**. Let $\mathbf{s:A\to Ult(A)}$ be the Stone isomorphism. For every $\mathbf{f\in Clop(Ult(A))}$ there exists a subset $\mathbf{v\in s(D)}$ such that $\mathbf{\emptyset\neq v\subset f}$. Since **Clop(Ult(A))** is a base for **Ult(A)**, it follows that **s(D)** is a pseudobase for **Ult(A)**. Let $\mathbf{x_v\in v}$ for every $\mathbf{v\in s(D)}$. It is easy to check that $\mathbf{\{x_v\}_{v\in s(D)}}$ is a countable dense subset of **Ult(A)**. ■

While [8, theorem 3.2] is a consequence of proposition 4.1, the following result shows that the converse is not true.

Proposition 4.3. *There exists an atomless Boolean algebra* **A** *with* $\mathbf{d(A)=\aleph_0}$*, but not separable.*

Proof: let us consider the generalized Cantor space $\mathbf{2^I}$ with $\mathbf{I=2^{\aleph_c}}$. According to [13, theorem 14.3] its dual field $\mathbf{Clop(2^I)}$ is a free Boolean algebra with $\mathbf{2^{\aleph_c}}$ free generators. So $\mathbf{Clop(2^I)}$ is atomless (cf. [9, proposition 9.11]) and has a separable Stone space ([9, corollary 9.7a] and [13, §14E]). However $\mathbf{Clop(2^I)}$ is not a separable Boolean algebra (cf. [8, p.479]). ■

Remark. As shown by corollary 4.2 and proposition 4.3, proposition 4.1 extends the result of Horn and Tarski. Moreover, the proof they gave was algebraic. Instead, our proof of proposition 4.1 is mainly of topological nature.

The next proposition shows a rather interesting fact. It says that we can keep the result of proposition 4.1 by replacing the atomlessness assumption on **A** with a nonatomicity assumption on the probability measure.

Proposition 4.4. *Let* **A** *be a Boolean algebra with* $\mathbf{d(A)=\aleph_0}$*. Then there is no nonzero nonatomic probability measure on* **A**.

Proof: suppose that **P** is a nonzero probability measure on **A**. From the proof of proposition 4.1 we know that this implies the existence of a nonzero, regular and residual measure **P"** on **Bor(Ult(A))**. Now suppose that **P** is nonatomic. Set $P_s = P \circ s^{-1}$**:Clop(Ult(A))→[0,1]**. We already know that P_S coincides with the restiction of **P"** on **Clop(Ult(A))**. It is a simple matter to verify that P_S is nonatomic. Let **g** be a open subset of **Ult(A)**. We have $g = \bigcup_{i \in I} f_i$ with $f_i \in$ **Clop(Ult(A)** because **Clop(Ult(A)** is a base. Let $g_\alpha = \bigcup \{f_\beta : \beta \leq \alpha$ and $\alpha \leq |g|\}$. Since **P"** is τ-additive, we have $P''(g) = \sup\{P''(g_\alpha) : \alpha \leq |g|\}$. If **P"(g)>0**, then a reasoning similar to the one employed in the proof of proposition 3.4(iv) allows us to say that there is some $f_i \in$ **Clop(Ult(A)** such that $P_S(f_i) > 0$. Since P_S is nonatomic, f_i is not a P_S-atom. A fortiori, f_i is not a **P"**-atom. Therefore **g** is not a **P"**-atom. Let **c** be a closed subset of **Ult(A)**. We have **P"(c)=P"(int(c))** because the boundary **b(c)** is a nowhere dense Borel subset. Suppose **P"(c)>0**. Then **P"(int(c))>0** and **c** is not a **P"**-atom. Let **a∈Bor(Ult(A))**. Since **P"** is regular we have **P"(A)=sup{P"(c): c⊂a** and **c∈Ult(A)}**. If **P"(a)>0**, then **P"(c)>0** for some **c**. Therefore **a** is not a **P"**-atom and this implies that **P"** is nonatomic. Hence, by proposition 3.4(i) we have **P"≡0** and so, by contraposition, **P≡0**. ■

Proposition 4.4 shows a class of Boolean algebras which do not admit nonatomic probability measures. A special case of interest is now considered.

Corollary 4.5. *Let* **A** *be an atomic Boolean algebra which satisfies the c.c.c.. Then there is no nonzero nonatomic probability measure on* **A.**

Proof: it is known that if a Boolean algebra is atomic, then the subset of the isolated point is dense in the Stone space. Since **A** satisfies the c.c.c., then there are at most a countable number of isolated points. Therefore, **Is(Ult(A))** is a countable dense subset of **Ult(A)**, and so $d(A) = \aleph_0$. ■

5. The range of a nonatomic probability measure

We prove the following result:

Proposition 5.2. *Let* **P** *be a nonatomic probability measure defined on a Boolean algebra* **A** *which satisfies the c.c.c.. Then* **R(P)** *is a dense subset of* **[0,1]**.

Proof: from the proof of proposition 4.4 we know that **P** implies the existence of a regular, nonatomic and residual probability measure **P"** on **Bor(Ult(A))**. Since **X** is compact, **P"** is also compact regular and so τ-additive.

Therefore, since **A** is a base, the result can be proved with the same reasoning used in [10, proposition 2.1 and lemma 5.1]. ■

Remarks. (i) In looking at this result it is important to keep in mind the distinction between measures on Boolean algebras and measures on fields outlined in section 2. In fact, the example given there shows that this result is not true for probability measures defined on fields. **(ii)** Let **Bor(ℝ)** be the Borel σ-field of the real line. Let **N** be the σ-ideal of the subsets with Lebesgue measure zero. The quotient algebra **Bor(ℝ)/N** does not have a separable Stone space (cf. [13, p.95]). But **Bor(ℝ)/N** satisfies the c.c.c.. In fact, on **Bor(ℝ)/N** there exists a nonzero strictly positive σ-finite measure induced by the Lebesgue measure. This example shows that proposition 4.4 does not conflict with proposition 5.2.

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