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SOME THEOREMS ON THE EXISTENCE OF
COMPETITIVE EQUILIBRIUM

by

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ABSTRACT

The purpose of this paper is to point out a relationship between theorems on the existence of competitive equilibrium in economies with externalities, and recent results (pioneered by A. Mas-Colell) on the existence of equilibrium for economies in which consumer preferences are neither complete nor transitive. This observation leads both to a substantial strengthening of the theorem on the existence of equilibrium with externalities, and at the same time to a revealing perspective on the Mas-Colell theorem.

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I. INTRODUCTION

The purpose of this paper is to point out a relationship between theorems on the existence of competitive equilibrium in economies with externalities, and recent results (pioneered by A. Mas-Colell) on the existence of equilibrium for economies in which consumer preferences are neither complete nor transitive, [10], [7], and [14]. The analysis is restricted to pure exchange economies since the relationship we establish does not depend on production. This observation leads both to a substantial strengthening of the theorem on the existence of equilibrium with externalities, and at the same time to a revealing perspective on the Mas-Colell theorem.

The theorem on the existence of equilibrium with externalities goes back to McKenzie [11] and Arrow and Debreu [1]. McKenzie was the first to explicitly allow for externalities. Although the statement of the Arrow-Debreu Theorem does not include the possibility of externalities, a suitable reinterpretation of their proof yields equilibrium with dependent preferences. Arrow and Hahn [2], provide an alternative proof of existence with externalities. Our Theorem 1 gives a short (new, and very different) proof of that theorem for the case of pure exchange. In Theorem 2, we use a smoothing operation to prove existence with externalities under much weaker continuity assumptions than have usually been placed on utility functions, and this theorem is used

to prove our Theorem 3, which is of the type pioneered by A. Mas-Colell [10].

Mas-Colell proves the existence of equilibrium under very weak assumptions on preferences. The striking feature of his theorem is that the conditions he assumes do not guarantee that individual excess demand correspondences have a convex valued selection with closed graph. Because of this, most of the standard techniques for proving the existence of competitive equilibrium (e.g., [6], [12], and [4]) do not apply. However, a suitable reinterpretation of economies with non-complete and non-transitive preferences allows us to view them as standard economies with externalities; this is the technique used to deduce Theorem 3 from Theorem 2, and we believe that it helps to take some of the surprise (but none of the beauty!) out of Mas-Colell's result.

II. PRELIMINARIES

We consider pure exchange economies with n consumers and l commodities. The i^{th} consumer is specified by his consumption set X_i (a subset of \mathbb{R}_+^l), his initial holdings ω_i (a point in \mathbb{R}_+^l), and a preference indicator. Preference indicators take several different forms. In economies without externalities they are either a utility function $U_i: X_i \rightarrow \mathbb{R}$ or an irreflexive relation $P_i \subset X_i \times X_i$. The latter approach was used by A. Mas-Colell and is more general than the utility function formulation since P_i is not required to be asymmetric or transitive. If $U_i(x) > U_i(y)$, or alternatively if $(x,y) \in P_i$, then we say that the i^{th} consumer prefers x to y .

In economies with externalities we allow the preferences of each individual to depend on not only his own consumption, but the consumption of each consumer and prices. A price vector p is a point in $Q = R_+^l$. An allocation $x = (x_1, x_2, \dots, x_n) \in X = \prod X_i$ specifies a consumption for each consumer.

In economies with externalities a preference indicator is either a utility function $U_i: X \times Q \times X_i \rightarrow R$ or a preference correspondence $P_i: X \times Q \rightarrow X_i$.

The utility representation is interpreted as follows; for each allocation price pair $(\bar{x}, \bar{p}) \in X \times Q$, the function $U_i(\bar{x}, \bar{p}, \cdot)$ represents preference between pairs of points in X_i . In other words, for each state (\bar{x}, \bar{p}) , the consumer is considered to possess a ranking over points in his own consumption set. The ranking is conditioned by his perspective (which includes the i^{th} co-ordinate of \bar{x} .) But his own position in the state (\bar{x}, \bar{p}) will normally

be different than the consumption he evaluates. The latter form the domain of $U_i(\bar{x}, \bar{p}, \cdot)$. The representation P_i is new. It indicates for each pair $(x, p) \in X \times Q$ the points in X_i which the i^{th} consumer prefers to x_i .

Given a utility function $U_i: X \times Q \times X_i \rightarrow R$, there is a naturally associated preference correspondence P_i which is defined by $P_i(x, p) = \{x_i' \in X_i: U_i(x, p, x_i') > U_i(x, p, x_i)\}$, where x_i in this case is the i^{th} coordinate of x .

A competitive equilibrium for an economy \mathcal{E} is an allocation price pair $(\bar{x}, \bar{p}) \in X \times Q$ such that for each i ,

$$e1) \quad \bar{p} \cdot \bar{x}_i = \bar{p} \cdot \omega_i,$$

$$e2) \quad \sum \bar{x}_i \leq \sum \omega_i, \text{ and}$$

$$e3) \quad \text{if } x_i \text{ is preferred (given } (\bar{x}, \bar{p})) \text{ to } \bar{x}_i, \text{ then } \bar{p} \cdot x_i > \bar{p} \cdot \omega_i.$$

Observe that " x_i is preferred (given (\bar{x}, \bar{p})) to \bar{x}_i " has a formal statement which depends on how preferences are specified. For example, with externalities and when preferences are represented by utility functions, the formal equivalent reads $U_i(\bar{x}, \bar{p}, x_i) > U_i(\bar{x}, \bar{p}, \bar{x}_i)$.

III. THEOREMS

Theorem 1

Let $\mathcal{E} = (X_i, U_i, \omega_i)$ be an economy which satisfies, for each i , $X_i = R_+^l$,

$U_i: X \times Q \times X_i \rightarrow R$ which satisfies

(strong monotonicity) for all $(y, p) \in X \times Q$, if $x_i \geq x_i'$ and $x_i \neq x_i'$, then $U_i(y, p, x_i) > U_i(y, p, x_i')$,

(convexity) for all $(y, p) \in X \times Q$, the set $C_i(y, p) = \{x_i \in X_i \mid U_i(y, p, x_i) > U_i(y, p, y_i)\}$ is convex,

(continuity) U_i is continuous, and

$\omega_i \in \text{int } X_i$.

Then \mathcal{E} has an equilibrium.

We employ a consequence of a fundamental theorem on the convexity and non-emptiness of the set of competitive equilibrium in an economy with no externalities, homogeneous utility functions, and proportional initial endowments of commodities. The theorem is due to E. Eisenberg [5], and a new proof was offered by J. Chipman [3]. A short and very direct proof of the result is provided in the appendix.

Theorem (Eisenberg) Let $\mathcal{E} = (X_i, U_i, \omega_i)$ be an economy which satisfies, for each i ,

$X_i \subset R_+^{\ell}$ is a closed convex cone ($X_i \neq \{0\}$),

$U_i: X_i \rightarrow R_+$ is continuous, concave, homogeneous of degree one, and nonconstant, and

$\omega_i = \beta_i \omega$, where $\sum \beta_i = 1, \beta_i > 0$, and $\omega \in \text{int } R_+^{\ell}$.

Let $\Delta = \{p \in Q \mid p \cdot \omega = 1\}$ and $W(\mathcal{E})$ denote the set of competitive equilibrium pairs for \mathcal{E} . Then $W(\mathcal{E})$ is compact, convex, and nonempty in $X \times \Delta$.

As an immediate consequence of this theorem we have

Lemma 1 Let $\mathcal{E} = (X_i, U_i, \omega_i)$ be an economy which satisfies, for each i ,

$X_i \subset R_+^{\ell}$ is a closed convex cone ($\{0\} \neq X_i$),

$U_i: X \times Q \times X_i \rightarrow R_+$ is continuous and nonconstant for

all $(x, p) \in X \times Q$, $U_i(x, p, \cdot)$ is homogeneous of degree one and concave, and

$\omega_i \in \text{int } R_+^{\ell}$.

Then there exists an equilibrium for \mathcal{E} .

Proof of Lemma Let $\hat{X} = \{(x_1, x_2, \dots, x_n) \in X : \sum x_i \leq \sum \omega_i\}$. We will construct a correspondence $\Psi: \hat{X} \times \Delta \rightarrow \hat{X} \times \Delta$ with closed graph whose values are nonempty convex sets in $\hat{X} \times \Delta$, and prove that its fixed points are competitive equilibria.

First, for each i , define the continuous function $\rho_i : \Delta \rightarrow R_+^l$ by $\rho_i(p) = (p \cdot \omega_i / p \cdot \omega) \omega$. Now, for each $(x, p) \in \hat{X} \times \Delta$, consider the economy $\mathcal{G}(x, p) = (X_i, U_i(x, p, \cdot), \rho_i(p))$. By Eisenberg's Theorem $W(\mathcal{G}(x, p))$ is convex and nonempty for each $(x, p) \in \hat{X} \times \Delta$; furthermore, since U_i is continuous for each i , the correspondence $\psi : \hat{X} \times \Delta \rightarrow \hat{X} \times \Delta$ defined by $\psi(x, p) = W(\mathcal{G}(x, p))$ has a closed graph. Thus, by Kakutani's fixed point theorem there exists $(\bar{x}, \bar{p}) \in \psi(\bar{x}, \bar{p})$. Finally, since for each i , $\bar{p} \cdot \rho_i(\bar{p}) = \bar{p} \cdot (\bar{p} \cdot \omega_i / \bar{p} \cdot \omega) \omega = \bar{p} \cdot \omega_i$, the fact that (\bar{x}, \bar{p}) is an equilibrium for $\mathcal{G}(\bar{x}, \bar{p})$ means that (\bar{x}, \bar{p}) is an equilibrium for \mathcal{G} . This completes the proof of the Lemma.

Proof of Theorem 1 Define the economy $\tilde{\mathcal{G}} = (X_i, V_i, \omega_i)$ as follows. For each consumer i and each pair $(x, p) \in X \times Q$, the $V_i(x, p, \cdot)$ indifference surface (in X_i) through x_i is the boundary of $C_i(x, p)$. Furthermore, we require that for each i , $V_i(x, p, \cdot)$ is homogeneous of degree one, which is possible by strict monotonicity, and is normalized by the condition $V_i(x, p, e) = 1$, where e denotes the vector of ones in R_+^l . $\tilde{\mathcal{G}}$ satisfies the conditions of Lemma 1, and so there exists competitive equilibrium for $\tilde{\mathcal{G}}$, which we denote by (\bar{x}, \bar{p}) . But since $U_i(\bar{x}, \bar{p}, x_i) > U_i(\bar{x}, \bar{p}, \bar{x}_i)$ if and only if $V_i(\bar{x}, \bar{p}, x_i) > V_i(\bar{x}, \bar{p}, \bar{x}_i)$, (\bar{x}, \bar{p}) is also an equilibrium for \mathcal{G} . This completes the proof.

It is commonly assumed in economies with externalities that the utility functions U_i are continuous. We show in the next theorem that this can be weakened. The weaker assumption allows us to establish a version of the Mas-Colell theorem.

Theorem 2 Let $\mathcal{E} = (X_i, U_i, \omega_i)$ satisfy the same conditions as in Theorem 1 with continuity replaced by

$$(*) \quad \{(y, p, x_i) \mid U_i(y, p, x_i) > U_i(y, p, y_i)\}$$

is open in $X \times Q \times X_i$.

Then \mathcal{E} has an equilibrium.

We first establish a lemma which allows us to replace each U_i with a sequence of continuous utility functions.

Lemma 2 Let U_i satisfy the conditions of Theorem 2. Then there exists a sequence $\{V^m\}$ of continuous functions $V^m: \text{int } X \times Q \times (\text{int } X_i) \rightarrow \text{int } R_+$ satisfying:

- 1) $V^m(y, p, \cdot)$ is concave and homogeneous of degree one for each y, p , and
- 2) For all $z_i > 0$, $y > 0$, and $\{(y^m, p^m)\}$, if $U_i(y, p, z_i) > U_i(y, p, y_i)$ and $\lim (y^m, p^m) = (y, p)$, then $V^m(y^m, p^m, z_i) > V^m(y^m, p^m, y_i^m)$ for all m sufficiently large.

Proof of Lemma 2 Define $\varphi : \text{int}(\text{int } X) \times Q \times (\text{int } X_i) \rightarrow \text{int } R_+$ by

$$\varphi(y, p, x_i) = \min \lambda$$

$$\text{subject to } U_i(y, p, (1/\lambda)x_i) \leq U_i(y, p, y_i).$$

This is a formal description of a method to obtain a homogeneous representation for U_i (see proof of Theorem 1). One can easily verify that φ is well defined, and that for each (y, p) , $\varphi(y, p, \cdot)$ is concave and homogeneous of degree one. Also, φ is lower semicontinuous (in particular, $\varphi(\cdot, \cdot, x_i)$ is measurable with respect to $\ell(n+1)$ - dimensional Lebesgue measure μ).

Let $B_{1/m}(y, p)$ denote the open ball of radius $1/m$ with center (y, p) intersected with $(\text{int } X) \times X_i$. Define

$$V^m(y, p, x_i) = \int_{B_{1/m}(y, p)} \frac{\varphi(\cdot, \cdot, x_i)}{\varphi(\cdot, \cdot, e)} \mu.$$

The integrand is a bounded measurable function for each x_i , hence V^m is well defined. $V^m(y, p, \cdot)$ is concave and homogeneous of degree one, and a routine argument shows $V^m(\cdot, \cdot, x_i)$ is continuous. Continuity of V^m then follows from a theorem of Rockafellar [13], p.89, which states: for $f: X \times Y \rightarrow R$, if $f(\cdot, y)$ is concave on an open convex set X , and $f(x, \cdot)$ is continuous on a locally compact space Y , then f is continuous on $X \times Y$.

Given $z_i, y, p, \{(y^m, p^m)\}$ (as in 2)), to show $\{V^m\}$ satisfies 2) it is sufficient to show that

$$3) \quad \varphi(y', p', z_i) > \varphi(y', p', y_i^m) \quad \text{holds for each}$$

$$(y', p') \in B_{1/m}(y^m, p^m) \quad \text{for large } m.$$

From (*), there exists $\eta > 0$, $\delta > 0$ such that $(y', p', z'_i) \in B_\delta(y, p) \times B_\eta(z_i)$ implies $U_i(y', p', z'_i) > U_i(y', p', y'_i)$. Choose $\eta < \|z_i\|$. Since $U_i(y', p', \varphi(y', p', z_i)^{-1} z_i) \cong U_i(y', p', y'_i)$ by definition, it follows that $\|z_i - \varphi(y', p', z_i)^{-1} z_i\| > \eta$ for each $(y', p') \in B_\delta(y, p)$. Solving for φ , we get

$$4) \quad \varphi(y', p', z_i) > \frac{\|z_i\|}{\|z_i\| - \eta} \quad \text{for each } (y', p') \in B_\delta(y, p) \quad .$$

Define λ_m to be the smallest number λ such that $y_i^m \cong \lambda y'_i$ holds for each $(y', p') \in B_{1/m}(y^m, p^m)$. Since y_i^m converges to a positive vector, it is straightforward to verify that λ_m is well defined for large m and that $\lambda_m \rightarrow 1$ as $m \rightarrow \infty$. From monotonicity, $y_i^m \cong \lambda_m y'_i$ implies $\varphi(y', p', y_i^m) \cong \varphi(y', p', \lambda_m y'_i)$, and $\varphi(y', p', \lambda_m y'_i) = \lambda_m \varphi(y', p', y'_i) \cong \lambda_m$, so

$$5) \quad \varphi(y', p', y_i^m) \cong \lambda_m \quad \text{for each } (y', p') \in B_{1/m}(y^m, p^m) \quad \text{and } \lambda_m \rightarrow 1$$

as $m \rightarrow \infty$.

Choose m large enough so that $\lambda_m \leq \frac{\|z_i\|}{\|z_i\| - \eta}$ and $B_{1/m}(y^m, p^m) \subset B_\delta(y, p)$. Then 4) and 5) combine to yield 3).

Proof of Theorem 2 For each U_i choose a sequence $\{V_i^m\}$ of functions satisfying the conclusion of Lemma 2. Let $\{Y^m\}$ be a sequence of closed convex cones whose union is $(\text{int } R_+^l) \cup \{0\}$. Then by Lemma 1, each economy $\mathcal{E}^m = (Y^m, V_i^m, \omega_i)$ has an equilibrium (\bar{x}^m, \bar{p}^m) , with \bar{p}^m in Δ . (V_i^m is not defined at origin, but this is irrelevant since each $\omega_i > 0$.) Let (\bar{x}, \bar{p}) be a limit point of the sequence $\{(\bar{x}^m, \bar{p}^m)\}$. We show that (\bar{x}, \bar{p}) is an equilibrium of \mathcal{E} . Clearly e1) and e2)

are satisfied, so we need only show e3). Choose x_i such that $U_i(\bar{x}, \bar{p}, x_i) > U_i(\bar{x}, \bar{p}, \bar{x}_i)$. Choose $t \in (0,1)$ such that $U_i(\bar{x}, \bar{p}, tx_i + (1-t)(1/2)\omega_i) > U_i(\bar{x}, \bar{p}, \bar{x}_i)$. Then $tx_i + (1-t)(1/2)\omega_i \in Y^m$ for large m , so by Lemma 2 $V_i^m(\bar{x}^m, \bar{p}^m, tx_i + (1-t)(1/2)\omega_i) > V_i^m(\bar{x}^m, \bar{p}^m, \bar{x}_i^m)$ for m large, which implies by e3) that $\bar{p}^m(tx_i + (1-t)(1/2)\omega_i) > \bar{p}^m \cdot \omega_i$.

Thus $t\bar{p} \cdot x_i + (1-t)(1/2)\bar{p} \cdot \omega_i \geq \bar{p} \cdot \omega_i$, which implies $\bar{p} \cdot x_i > \bar{p} \cdot \omega_i$.

This completes the proof.

Using standard arguments, Theorem 2 can be strengthened to allow for a less restrictive monotonicity assumption and more general consumption sets. This is not pursued, since our purpose at this point of the paper is to concentrate on the continuity properties of utility functions with externalities. In any case a more general theorem is presented later (Theorem 5).

With a strong monotonicity assumption and R_+^ℓ for consumption sets, we will now prove the theorem (due to A. Mas-Colell [10]) on the existence of equilibrium in economies with nontransitive and noncomplete preferences. Our simple proof is based on the idea of viewing such economies as externalities economies with complete and transitive preferences.

Theorem 3 Let $\mathcal{E} = (X_i, P_i, \omega_i)$ be an economy which satisfies, for each i ,

$P_i \subset X_i \times X_i$ which satisfies

$$X_i = R_+^\ell$$

(strong monotonicity) $x_i \geq y_i$ and $x_i \neq y_i$ implies $(x_i, y_i) \in P_i$,

(convexity) for all $y_i \in X_i$, the set $P_i(y_i) = \{x_i \in X_i \mid (x_i, y_i) \in P_i\}$

is convex, and

(continuity) P_i is open in $X_i \times X_i$, and

$$\omega_i \in \text{int } X_i.$$

Then, \mathcal{E} has a competitive equilibrium.

Proof: For each consumer i , define a map $V_i: X_i \times X_i \rightarrow \mathbb{R}$ as follows. For each $x_i \in X_i$ let $V_i(x_i, \cdot)$ be the unique homogeneous of degree one function whose indifference curve through x_i is the boundary of $P_i(x_i)$ and satisfies $V_i(x_i, e) = 1$. Note that, for any $(x_i, x'_i) \in X_i \times X_i$, $V_i(x'_i, x_i) > V_i(x'_i, x'_i)$ is equivalent to $x_i \in P_i(x'_i)$. Define, for each i , the map $U_i: X \times Q \times X_i \rightarrow \mathbb{R}$ as the trivial extension of V_i ; i.e., for each $(x, p, x'_i) \in X \times Q \times X_i$, let $U_i(x, p, x'_i) = V(x_i, x'_i)$. Then the economy $\tilde{\mathcal{E}} = (X_i, U_i, \omega_i)$ will satisfy the conditions of Theorem 2, and thus has an equilibrium (\bar{x}, \bar{p}) . But since $U_i(\bar{x}, \bar{p}, x_i) > U_i(\bar{x}, \bar{p}, \bar{x}_i)$ is equivalent to $V_i(\bar{x}_i, x_i) > V_i(\bar{x}_i, \bar{x}_i)$, which is equivalent to $x_i \in P_i(\bar{x}_i)$, (\bar{x}, \bar{p}) is also an equilibrium for \mathcal{E} . This completes the proof.

The Mas-Colell theorem is more general than our Theorem 3 in two respects. First, strong monotonicity can be weakened to require only that preferences are nonsatiated, and second, consumption sets are required only to be closed, convex, and bounded below. The techniques used by Mas-Colell [10] and Gale and Mas-Colell [7] are sufficient to establish the following result, which is explicitly observed in [14].

Theorem 4 Let $\mathcal{E} = (X_i, P_i, \omega_i)$ be an economy which satisfies for each i ,

X_i is closed, convex, and bounded below, ($X = \prod X_i$),

there exists $x_i \in X_i$ such that $x_i < \omega_i$, and

$P_i : X \times Q \rightarrow X_i$ satisfying

(continuity) P_i has an open graph in $X \times Q \times X_i$,

(convexity) for each $(y,p) \in X \times Q$, $P_i(y,p)$

is convex and nonempty, and

(irreflexivity) for each $(y,p) \in X \times Q$, $y_i \notin P_i(y,p)$.

Then \mathcal{E} has a competitive equilibrium.

This yields

Theorem 5 Let $\mathcal{E} = (X_i, U_i, \omega_i)$ be an economy which satisfies, for each i ,

X_i is closed, convex, and bounded below ($X = \prod X_i$),

$U_i : X \times Q \times X_i \rightarrow \mathbb{R}$ which satisfies

(continuity) $\{(y,p,x_i) \mid U_i(y,p,x_i) > U_i(y,p,y_i)\}$

is open in $X \times Q \times X_i$, and

(convexity) for all $(y,p) \in X \times Q$, the set

$\{x_i \in X_i : U_i(y,p,x_i) > U_i(y,p,y_i)\}$ is

convex and nonempty, and

there exists $x_i \in X_i$ such that $x_i < \omega_i$.

Then \mathcal{E} has a competitive equilibrium.

To deduce Theorem 5 from Theorem 4 define, for each i , the preference correspondence $P_i : X \times Q \rightarrow X_i$ by $P_i(y,p) = \{x_i \mid U_i(y,p,x_i) > U_i(y,p,y_i)\}$.

Then the economy $\tilde{\mathcal{E}} = (X_i, P_i, \omega_i)$ satisfies the conditions of Theorem 4, and an equilibrium for $\tilde{\mathcal{E}}$ is an equilibrium for \mathcal{E} .

APPENDIX

Before proving Eisenberg's Theorem we will review some duality theory of concave programming.

A concave programming problem is defined as

A1) maximize $f(z)$

subject to $g(z) \leq b$

$z \in Z,$

where Z is a convex set in $\mathbb{R}^n,$

$f : Z \rightarrow \mathbb{R}$ is concave,

$g : Z \rightarrow \mathbb{R}^m$ is convex, and

$b \in \mathbb{R}^m.$

The kernel $\varphi : Z \times \mathbb{R}_+^m \rightarrow \mathbb{R}$ associated with A1) is defined by $\varphi(z,y) = f(z) + y \cdot (b - g(z)).$ A point $(\bar{z}, \bar{y}) \in Z \times \mathbb{R}_+^m$ is a saddle point of φ if $\varphi(\bar{z}, y) \geq \varphi(\bar{z}, \bar{y}) \geq \varphi(z, \bar{y})$ holds for all $(z,y) \in Z \times \mathbb{R}_+^m.$

Duality Theorem Suppose there exists a $z^0 \in Z$ such that $g(z^0) < b.$ Then

A2) \bar{z} is a solution to A1) if and only if there exists a \bar{y} such that (\bar{z}, \bar{y}) is a saddle point of φ (Karlin [8]), and

A3) if $Z = \mathbb{R}_+^n$ and φ is continuously differentiable, then (\bar{z}, \bar{y}) is a saddle point of φ if and only if

$$\frac{\partial \varphi}{\partial z_i}(\bar{z}, \bar{y}) \leq 0 \quad \text{for each } i,$$

$$\sum \bar{z} \frac{\partial \varphi}{\partial z_i}(\bar{z}, \bar{y}) = 0,$$

$$g(\bar{z}) \leq b \text{ and } \bar{y} \cdot (b - g(\bar{z})) = 0, \text{ (Kuhn-Tucker [9]).}$$

Proof of Eisenberg's Theorem Associated with the economy \mathcal{G} we consider the concave programming problem

$$\begin{aligned} \text{A4) } & \text{maximize } \sum \beta_i \ln U_i(x_i) \\ & \text{subject to } \sum x_i \leq \omega \text{ and } x \in X. \end{aligned}$$

and its kernel

$$\text{A5) } \varphi_e(x, p) = \sum \beta_i \ln U_i(x_i) + p \cdot (\omega - \sum x_i).$$

We will show that (\bar{x}, \bar{p}) is a competitive equilibrium (with $\bar{p} \in \Delta$) if and only if (\bar{x}, \bar{p}) is a saddle point of φ_e . Then, the fact that $W(\mathcal{G})$ is nonempty, compact, and convex will follow from standard arguments. (The idea of using A4) is due to Eisenberg [5].)

Suppose (\bar{x}, \bar{p}) is a saddle point of φ_e . Consider the kernel $\hat{\varphi}_e(\lambda_1, \lambda_2, \dots, \lambda_n, p) = \sum \beta_i \ln \lambda_i U_i(\bar{x}_i) + p \cdot (\omega - \sum \lambda_i \bar{x}_i)$. Since (\bar{x}, \bar{p}) is a saddle point of φ_e , and $\lambda_i U_i(\bar{x}_i) = U_i(\lambda_i \bar{x}_i)$, it follows that $\bar{\lambda}_i = 1$ for each i and $p = \bar{p}$ form a saddle point of $\hat{\varphi}_e$. Writing out the Kuhn-Tucker conditions A3), we obtain $\beta_i - \bar{p} \cdot \bar{x}_i \leq 0$ for each i , $\sum (\beta_i - \bar{p} \cdot \bar{x}_i) = 0$, $\sum \bar{x}_i \leq \omega$ and $\bar{p} \cdot (\omega - \sum \bar{x}_i) = 0$. Then e2) is satisfied, and a computation shows the e1) is also and that $\bar{p} \in \Delta$. We will now establish e3). Again, since (\bar{x}, \bar{p}) is a saddle point of φ_e , we have $\varphi_e(\bar{x}, \bar{p}) \geq \varphi_e(x, \bar{p})$ for each $x \in X$. For the special case of $x = (\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n)$, this yields $\beta_i \ln U_i(\bar{x}_i) \geq \beta_i \ln U_i(x_i) + \beta_i - \bar{p} \cdot x_i$ for each i , where $\beta_j = \bar{p} \cdot \bar{x}_j$ and $\bar{p} \cdot \omega = \sum \beta_i$, and establishes e3).

To prove the converse suppose $(\bar{x}, \bar{p}) \in X \times \Delta$ is a competitive equilibrium. From the Duality Theorem, for each i there exists a $\bar{\lambda}_i$ such that

$$\text{A6) } \beta_i \ln U_i(\bar{x}_i) \geq \beta_i \ln U_i(x_i) + \bar{\lambda}_i (\beta_i - \bar{p} \cdot x_i)$$

for each $x_i \in X_i$. From A6) and the fact that $\theta U_i(\bar{x}_i) = U_i(\theta \bar{x}_i)$ for each

$\theta > 0$, it follows that the function h defined by $h(\theta) = \beta_i \ln \theta U_i(\bar{x}_i) + \bar{\lambda}_i (\beta_i - \bar{p} \cdot \theta \bar{x}_i)$ has an interior maximum at $\theta = 1$. Thus $h'(1) = 0$, or $\beta_i - \bar{\lambda}_i \bar{p} \cdot \bar{x}_i = 0$, which by e1) yields $\bar{\lambda}_i = 1$. By summing both sides of A6) and substituting 1 for $\bar{\lambda}_i$ and $\bar{p} \cdot \omega$ for $\sum \beta_i$, we obtain

$$A7) \quad \sum \beta_i \ln U_i(\bar{x}_i) \geq \sum \beta_i \ln U_i(x_i) + \bar{p} \cdot (\omega - \sum x_i)$$

for each $x \in X$. Then, e1) and e2) imply $\bar{p} \cdot (\omega - \sum \bar{x}_i) \geq \bar{p} \cdot (\omega - \sum \bar{x}_i) = 0$ for each $\bar{p} \in R_+^l$, which with A7) yields $\varphi_e(\bar{x}, \bar{p}) \geq \varphi_e(\bar{x}, \bar{p}) \geq \varphi_e(x, \bar{p})$ for each $(x, \bar{p}) \in X \times R_+^l$.

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