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A Note On Approximating Agreeing to Disagree Results With Common p-Beliefs*

by

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ABSTRACT

Monderer and Samet (1989) generalize Aumann's (1976) agreeing to disagree result for the case of beliefs. They show that if the posteriors of an event are "common p-belief" then they cannot differ by more than 2(1-p). We provide a different proof of this result with a lower bound of 1-p. An example which attains this bound is provided.

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Aumann's (1976) famous agreeing to disagree result states that the posteriors formed over an event X by rational players must coincide if they are commonly known. Monderer and Samet (1989) generalize this result for the case of beliefs. They define the notion "common p-belief" and show that if the posteriors of an event are common p-belief then they cannot differ by more than 2(1-p). In this short note we provide a different proof of this result which enables us to obtain a bound of 1-p over the difference between posteriors that can be sustained as common p-belief. We show that this is the best possible bound by an example which attains it. As opposed to Monderer and Samet's result, our result imposes some restrictions on posteriors which are common p-belief for $p \leq 1/2$.

Set-Up (following Monderer and Samet's formulation)

Let I be a finite set of players and let (Ω,Σ,μ) be a probability space, where Ω is a space of states, Σ is an σ-algebra of events, and μ is a probability measure on Σ (to be interpreted as a common prior). For each $i \in I$, Π_i is a partition of Ω into measurable sets with positive measure. Since Π_i is countable, we use the notation $\Pi_i = \{\Pi_i^1, \Pi_i^2, \ldots\}$. For $\omega \in \Omega$, denote by $\Pi_i(\omega)$ the element of Π_i containing ω. Π_i is interpreted as the information available to agent i; $\Pi_i(\omega)$ is the set of all states which are indistinguishable to i when ω occurs. We denote by $\mathcal{F}_i$ the σ-field generated by Π_i. That is, $\mathcal{F}_i$ consists of all unions of elements of Π_i. For $i \in I$, $E \in \Sigma$, $\omega \in \Omega$ and $p \in [0,1]$, we say that *"i believes E with probability at least p at ω"*, or simply *"i p-believes E at ω"* if $\mu(E|\Pi_i(\omega)) \geq p$. Denote by $B_i^p(E)$ the event "i p-believes E." That is,

$$B_i^p(E) = \{\omega : \mu(E|\Pi_i(\omega)) \geq p\}$$

Notice that this is an event (i.e. measurable with respect to Σ). Moreover, for any $E \in \Sigma$, it is also measurable with respect to $\mathcal{F}_i$.

Definition E is an *evident p-belief* if for each $i \in I$

$$E \subseteq B_i^p(E).$$

Definition An event *C is common p-belief at* ω if there exists an evident p-belief event E such that $\omega \in E$, and for all $i \in I$,

$$E \subseteq B_i^p(C).$$

The Result

We start by formulating the following proposition,

Proposition An event C is common p-belief at $\omega \in \Omega$ if and only if for all $i \in I$ there exists a set $\pi_i = \bigcup_{k \in K_i} \Pi_i^k$ for some $K_i \subseteq \mathbb{N}$ such that $\omega \in \pi_i$ and such that the following two conditions hold,

(i) For all $i \in I$, $\mu(\bigcap_{h \in I} \pi_h \mid \Pi_i^k) \geq p$ for all $\Pi_i^k \subseteq \pi_i$.

(ii) For all $i \in I$, $\mu(C \mid \Pi_i^k) \geq p$ for all $\Pi_i^k \subseteq \pi_i$.

Proof (⇒): C is common p-belief at $\omega \in \Omega$ implies the existence of an evident p-belief event E such that $\omega \in E$ and $E \subseteq B_i^p(C)$. For all $i \in I$, define $\pi_i = \bigcup_{\substack{k \in \mathbb{N} \\ \mu(\Pi_i^k \cap E) > 0}} \Pi_i^k$. Note that $E \subseteq \pi_i$ up to measure zero for all $i \in I$ (i.e., there exists $E' \in \Sigma$ such that $\mu(E \triangle E') = 0$ and $E' \subseteq \pi_i$ for all $i \in I$). Therefore, $E \subseteq \bigcap_{h \in I} \pi_h$ up to measure zero. For $\omega \in E$, $\mu(E \mid \Pi_i(\omega)) \geq p$ for all $i \in I$ since E is evident p-belief. But for all $i \in I$, by definition of π_i, $\Pi_i(\omega) = \Pi_i^k$ for some $\Pi_i^k \subseteq \pi_i$, and moreover, for all $\Pi_i^k \subseteq \pi_i$ there exist an $\omega \in E$ such that $\Pi_i(\omega) = \Pi_i^k$. Therefore, $\mu(\bigcap_{h \in I} \pi_h \mid \Pi_i^k) \geq \mu(E \mid \Pi_i^k) \geq p$ for all $\Pi_i^k \subseteq \pi_i$ where the first inequality is due to $E \subseteq \bigcap_{h \in I} \pi_h$. Similarly, we establish (ii): $\omega \in E$ implies $\mu(C \mid \Pi_i(\omega)) \geq p$, and by the same argument $\mu(C \mid \Pi_i^k) \geq p$ for all $\Pi_i^k \subseteq \pi_i$.

(⇐): Let $E = \bigcap_{i \in I} \pi_i$, $\omega \in E$, so it is enough to show that E is an evident p-belief and that

$E \subseteq B_i^p(C)$ for all $i \in I$. For $\omega \in E$ and $i \in I$, $\mu(E \mid \Pi_i(\omega)) = \mu(E \mid \Pi_i^k)$ for some $\Pi_i^k \subseteq \pi_i$, and $\mu(E \mid \Pi_i^k) = \mu(\bigcap_{h \in I} \pi_h \mid \Pi_i^k) \geq p$ by (i). Similarly, For $\omega \in E$ and $i \in I$, $\mu(C \mid \Pi_i(\omega)) = \mu(C \mid \Pi_i^k)$ for some $\Pi_i^k \subseteq \pi_i$, and $\mu(C \mid \Pi_i^k) \geq p$ by (ii). Hence, $E \subseteq B_i^p(C)$. ■

We now turn to the agreeing to disagree result.

Fix an event $X \in \Sigma$ and define functions f_i for all agents i by

$$f_i(\omega) = \mu(X \mid \Pi_i(\omega))$$

$f_i(\omega)$ is i's posterior probability of X. Let r_i, $i \in I$, be numbers in the interval [0,1], and consider the event,

$$C = C(\{r_i\}_{i \in I}) = \bigcap_{i \in I} \{\omega \in \Omega : f_i(\omega) = r_i\}.$$

Theorem If C is common p-belief at $\omega \in \Omega$, then $|r_i - r_j| \leq 1-p$ for all $i,j \in I$. That is, if the posteriors of the event X are common p-belief at some $\omega \in \Omega$, then they cannot differ by more than 1-p.

Proof Suppose that $C = C(\{r_i\}_{i \in I})$ is common p-belief at $\omega \in \Omega$. Assume also that $p > 0$ (the conclusion is trivial for $p=0$). By the proposition, for all $i \in I$ there exists a set $\pi_i = \bigcup_{k \in K_i} \Pi_i^k$ where $K_i \subseteq \mathbb{N}$ and $\omega \in \pi_i$ such that the following two conditions hold,

(i) For all $i \in I$, $\mu(\bigcap_{h \in I} \pi_h \mid \Pi_i^k) \geq p$ for all $\Pi_i^k \subseteq \pi_i$.

(ii) For all $i \in I$, $\mu(C \mid \Pi_i^k) \geq p$ for all $\Pi_i^k \subseteq \pi_i$.

Aggregating $\mu(\bigcap_{h \in I} \pi_h \mid \Pi_i^k) \geq p$ over $\Pi_i^k \subseteq \pi_i$ yields,

$$\mu(\pi_j \mid \pi_i) \geq \mu(\bigcap_{h \in I} \pi_h \mid \pi_i) \geq p \qquad \text{for all } i,j \in I.$$

which implies,

$$\mu(\pi_i \cap \pi_j) \geq p \cdot \text{Max}\{\mu(\pi_i), \mu(\pi_j)\} \quad \text{for all } i,j \in I.$$

and,

$$\frac{1-p}{p} \mu(\pi_i \cap \pi_j) \geq \mu(\pi_i \setminus \pi_j) \qquad \text{for all } i,j \in I. \qquad (*)$$

For $\omega \in \pi_i$, $f_i(\omega) = r_i$. Otherwise, i would not p-believe C at ω, which contradicts (ii). Therefore, for all $i \in I$, $\mu(X \mid \pi_i) = r_i$.

Pick any $i,j \in I$, and assume without loss of generality that $r_i \leq r_j$. We bound r_i from below and r_j from above to obtain the desired bound. First, consider r_i:

$$r_i = \mu(X \mid \pi_i) = \frac{\mu(\pi_i \cap \pi_j) \cdot \mu(X \mid \pi_i \cap \pi_j) + \mu(\pi_i \setminus \pi_j) \cdot \mu(X \mid \pi_i \setminus \pi_j)}{\mu(\pi_i \cap \pi_j) + \mu(\pi_i \setminus \pi_j)}$$

$$\geq \frac{\mu(\pi_i \cap \pi_j) \cdot \mu(X \mid \pi_i \cap \pi_j)}{\mu(\pi_i \cap \pi_j) + \mu(\pi_i \setminus \pi_j)}$$

and by (*),

$$\geq \frac{\mu(\pi_i \cap \pi_j) \cdot \mu(X \mid \pi_i \cap \pi_j)}{\mu(\pi_i \cap \pi_j) + \frac{1-p}{p} \mu(\pi_i \cap \pi_j)}$$

$$= p\mu(X \mid \pi_i \cap \pi_j).$$

On the other hand,

$$r_j = \mu(X \mid \pi_j) = \frac{\mu(\pi_i \cap \pi_j) \cdot \mu(X \mid \pi_i \cap \pi_j) + \mu(\pi_j \setminus \pi_i) \cdot \mu(X \mid \pi_j \setminus \pi_i)}{\mu(\pi_i \cap \pi_j) + \mu(\pi_j \setminus \pi_i)}$$

$$\leq \frac{\mu(\pi_i \cap \pi_j) \cdot \mu(X \mid \pi_i \cap \pi_j) + \mu(\pi_j \setminus \pi_i)}{\mu(\pi_i \cap \pi_j) + \mu(\pi_j \setminus \pi_i)}.$$

Note, however, that for all $a \geq 0$, $b \leq 1$, and c, $\frac{ab+c}{a+c}$ increases with c. This, together with (*) implies,

$$\leq \frac{\mu(\pi_i \cap \pi_j)\cdot\mu(X \mid \pi_i \cap \pi_j) + \frac{1-p}{p}\mu(\pi_i \cap \pi_j)}{\mu(\pi_i \cap \pi_j) + \frac{1-p}{p}\mu(\pi_i \cap \pi_j)}$$

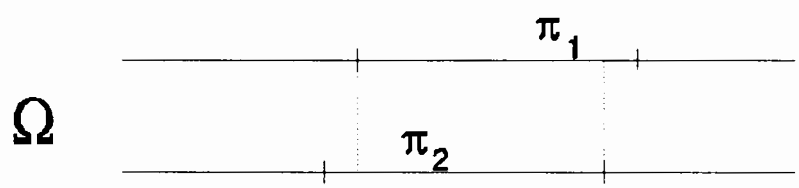
$$= p\mu(X \mid \pi_i \cap \pi_j) + 1 - p.$$

Therefore, for any $i,j \in I$,

$$|r_i - r_j| \leq 1-p. \qquad \blacksquare$$

An Example

Let (Ω,Σ,μ) be a probability space, let $I=\{1,2\}$ and pick any $0 \leq p \leq 1$. The players have information structures represented by the following figure which shows only π_1, π_2; the elements of each player's information structure that are relevant to the example.



Furthermore, assume $\mu(\pi_1 \cap \pi_2) = p\mu(\pi_1) = p\mu(\pi_2)$. Consider the following event, $X=\pi_1$. For player 1, $\mu(X \mid \pi_1)=1$, and for player 2 $\mu(X \mid \pi_2)=p$. Moreover, it can be verified that these posteriors are common p-belief for any $\omega \in \pi_1 \cap \pi_2$ (by setting $E=\pi_1 \cap \pi_2$). It follows that the bound obtained above cannot be improved in general.

References

R. Aumann (1976) Agreeing to Disagree. *Annals of Statistics 4*, 1236-1239.

D. Monderer and D. Samet (1989) Approximating Common Knowledge with Common Beliefs. *Games and Economic Behavior 1*, 170-190.