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"THE DIFFUSION OF CONSUMER DURABLES
IN A VERTICALLY DIFFERENTIATED OLIGOPOLY"

by

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Abstract

This article studies a vertically differentiated durable goods duopoly. We show that unlike in the monopoly case (Stokey, 1979), the open-loop equilibrium involves intertemporal price discrimination: firms sell in every period in which the high and low quality good are in direct competition, and the market is eventually saturated. With the exception of the initial price of the high quality good, raising the quality of the superior product may lower all prices in every period. Furthermore, when firms can change prices arbitrarily frequently, the game of endogenous quality choice leads firms to select qualities which are separated arbitrarily little. Both of these results stand in stark contrast with those of the static vertical product differentiation models of Gabszewicz and Thisse (1979) and Shaked and Sutton (1982). Finally, despite the fact that the equilibrium concept is open-loop, all but the introductory price of the high quality good converge to marginal cost in the limit as firms can change prices arbitrarily frequently.

1. Introduction

Consider a market for new durable goods, in which different quality levels within the same product category are simultaneously being offered for sale. An example would be the market for microcomputers, where consumers can purchase XT's, AT's, 386- and 486-based microcomputers at successively higher price points. All else equal, consumers consider microcomputers with a more powerful and faster central processing unit to be of superior quality: if offered a once-in-a-lifetime purchase opportunity, they would all be willing to pay at least as much for a more powerful unit as for a less capable one. However, because of differences in incomes and tastes, consumers will differ in their valuations for the respective quality levels. As emphasized in the literature on vertical product differentiation (Gabszewicz and Thisse (1979), Shaked and Sutton (1982)) such consumer heterogeneity creates opportunities for market segmentation: lower quality products can be sold to the poor at relatively low markups, whereas higher quality products are targeted towards the rich, thereby commanding comparatively higher margins.

Unlike in a market for nondurables, however, consumers shopping around for a durable are not just faced with the decision of *what* quality level to buy; they must also decide on *when* to purchase. The market for microcomputers, for instance, has traditionally seen price decreases which hover around 20 percent per year. A reasonable alternative to acquiring a 386 microcomputer at current prices is therefore to postpone the purchase, and buy a 486 microcomputer at next year's prices. Note that this is especially true for consumers for whom purchasing a 486 microcomputer in the current period remains out of reach, i.e., the poorer segment of the market. In addition, if all else remains equal, waiting to purchase the higher quality product in a later period becomes relatively more attractive the higher the rate of price decrease, the stronger the quality difference, and the less consumers discount the future. This willingness to substitute for the purchase of a lower quality product today with the purchase of a higher quality product in the future has important consequences: in the dynamic pricing environment that characterizes markets for durable goods, high and low quality products do not just compete in the current period (intratemporal competition) but also across periods (intertemporal competition). It is fair to say that the static notion of product quality gets somewhat blurred in the context of durable goods. For most people, delivery of a 486 (or even a more powerful microcomputer) five years from now is an inferior alternative to delivery of a 386 today. In effect, companies can be seen as offering a whole array of products, indexed by their delivery date: both the intrinsic quality level and the delivery date matter in terms of how consumers rank the different options. The relative ranking of these options in turn affects how products and firms compete in the market.

The intricate pattern of competition in a vertically differentiated durable goods oligopoly makes both optimal consumers purchase decisions and optimal firm pricing behavior a complex exercise. In order to decide whether or not to purchase in the current period, consumers must predict the prices of the array of qualities offered for sale in the next period. The optimal prices to charge in the next period will therefore not only depend on today's prices (as those affect the level of stocks carried over into the next period, and hence potential sales in that period), but also on expectations of prices two periods whence (as those affect whether or not consumers will further delay their purchases). In effect, to determine equilibrium, one must solve for the price paths of the various quality levels all at once.

To keep the analysis manageable, we make a number of simplifying (but not entirely unreasonable) assumptions. There are two producers each offering a single quality level for sale in any of an infinite number of periods. We assume that the good is infinitely durable, so that we can concentrate on the pattern of first sales. This allows us to uncover some structural determinants of the diffusion curve. Our model has a continuum of consumers ranked according to their reservation prices for the low quality good. We assume that the same ranking applies to the reservation prices of the high quality good; to simplify matters further, we take the ratio of the respective reservation prices to be constant across all consumers. Finally, we look for an equilibrium in open-loop strategies. This forms a useful benchmark for further research, and facilitates comparison with the existing literature.

Perhaps the paper most closely related to ours is Stokey (1979), which considers a durable goods *monopolist* offering a single infinitely durable good for sale at each of an infinite number of time periods. Since high valuation consumers stand to lose more surplus from postponing a purchase in order to obtain a lower price, the monopolist has an opportunity to intertemporally price discriminate: by gradually lowering his price over time, he can induce high valuation customers to buy earlier, and at a higher price, than low valuation customers. Stokey proves the following remarkable result: the optimal amount of intertemporal price discrimination is *no* price discrimination. In essence, the monopolist makes a single take-it-or-leave-it offer in the initial period. Stokey's result can be generalized to the case where a monopolist sells a product line of vertically differentiated durables; again, the optimal open-loop strategy involves no intertemporal price discrimination.[1] In Section 3, we show that this result is particular to the monopoly context: in an oligopolistic market intertemporal price discrimination will necessarily be present.

Our model also has very different implications from those in the literature on static vertical product differentiation (Gabszewicz and Thisse (1979), Shaked and Sutton (1982)). In Section 4 we show that in a durable goods environment it is no longer true that the high quality producer specializes in serving the needs of the rich and that the low quality producer exclusively caters to the poor: over time, both firms will reach relatively rich as well as relatively poor consumer segments. In the Shaked and Sutton model, which assumes that goods are either nondurable or rented out, an increase in the quality of the superior product results in higher prices and higher profits for both firms.[2] We show in Section 5 that in our model quite the opposite conclusion may hold: if the low quality producer has a lead time of even a single period in bringing the product to the market, all prices and profits may decrease as a consequence of the quality increase! Finally, and perhaps most importantly, the celebrated Shaked and Sutton (1982) result (see also Gabszewicz and Thisse (1979)), that endogenous quality choice will lead firms to select quality levels that remain bounded away from each other, does not generalize to a durable

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[1]The same intuitive argument given in Section 3 to explain Stokey's result can be used to establish this proposition.

[2]While this need not be true in general, it will be the case whenever the reservation price function for the high quality good (and hence its demand curve) becomes more inelastic, without in addition shifting out too much. Both firms then compete less aggressively as a consequence of the quality increase. As in Shaked and Sutton, we will identify a quality increase with an equal proportional increase in consumers' valuations for the high quality product.

goods environment. In Sections 5 and 6, we show that as the discount factor between periods approaches one, the low quality producer will end up choosing a quality level arbitrarily close to that of its competitor. In this sense, the principle of differentiation does not hold. Even more surprisingly, despite the fact that in our model firms have the commitment power to stick to open-loop strategies, when the time interval between successive pricing rounds becomes arbitrarily small, the equilibrium of the price game with exogenously specified quality levels will also result in the perfectly competitive outcome.

2. The Model

We consider a market where two infinitely durable goods, H and L, are offered for sale at discrete moments in time, $t = 0, 1, \ldots$. The goods are indivisible; consumers face the choice of buying one unit of good H, buying one unit of good L, or not buying at all. There is a continuum of consumers, indexed by $q \in [0, 1]$. The reservation prices of consumer q are indicated by $f(q)$ for the low quality good (L), and $\mu f(q)$ for the high quality good (H). We assume that $f(q) = 1 - q$ and that $\mu \geq 1$. An increase in the parameter μ can thus be interpreted as an increase in the relative quality of the two products.[3] Except for the durable goods aspect, the preference structure just described is identical to the one studied in the vertical product differentiation models of Gabszewicz and Thisse (1979), and Shaked and Sutton (1982).

All players in the game share a common discount factor between periods, denoted by $\delta \in (0, 1)$. When a consumer of type q purchases the low quality good in period t at a price p_t^L, he derives a net surplus of $\delta^t(f(q) - p_t^L)$. Likewise, the net surplus obtained from purchasing the high quality good in period t at the price p_t^H is $\delta^t(\mu f(q) - p_t^H)$. Note that in order to maximize his discounted net surplus each consumer faces the dual decision of when to purchase and what to purchase.

A high and a low quality producer serve this market at a common constant marginal cost, which

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[3]A model in which the reservation prices for the low and high quality goods are given by $\mu^H f(q)$ and $\mu^L f(q)$ (with $\mu^H \geq \mu^L$) can be transformed into our model by setting $\mu = \mu^H/\mu^L$ and by measuring prices (and hence profits) in multiples of μ^L.

is assumed to equal zero. The timing of moves within a period is as follows. First, firms simultaneously announce their prices. Consumers who did not buy in previous periods then decide whether or not to make a purchase in the current period. Let s_t^H and s_t^L denote the sales of good H and L in period t; profits in period t are then $\pi_t^H = p_t^H s_t^H$ for firm H and $\pi_t^L = p_t^L s_t^L$ for firm L. Firms maximize their net present value of profits, $\Pi^H = \Sigma_{t=0}^{\infty} \delta^t \pi_t^H$ and $\Pi^L = \Sigma_{t=0}^{\infty} \delta^t \pi_t^L$. We assume throughout the paper that firms compete in prices and that their strategies can be functions of calendar time only. Formally, we therefore examine the open-loop equilibria of our game.

3. The Nonequivalence of Renting and Selling

Stokey (1979) considers a model of a monopolist selling a single infinitely durable good over an infinite period of time to a demand curve f(q), and shows the following remarkable result: the optimal (open-loop) sales policy involves no intertemporal price discrimination. The durable good is introduced at p*, the monopoly price relative to the static demand curve f(q), and prices are never cut thereafter.

Stokey's result can be understood intuitively by instead considering the rental version of her model, where a monopolist lessor leases the good in every period to a rental demand curve $(1 - \delta)f(q)$.[4] Clearly, the optimal rental policy consists of charging - in each period - the monopoly price on the rental demand curve, $r^* = (1 - \delta)p^*$. Since any sales policy can always be mimicked by an appropriately chosen rental policy,[5] profits in the sales model can be no higher than the net present value of the optimal profit stream in the rental model. That level of profits can be achieved, however, by charging p* in the initial period and never cutting the price thereafter.

We will now show that the equivalence between the rental and the sales outcome breaks down in

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[4]Observe that the rental demand curve is equivalent to the one in the sales model. Indeed, if a consumer's reservation price for renting the good for one period equals $(1 - \delta)f(q)$, then the maximum net present value he would be willing to pay for the privilege of owning the good in perpetuity equals $\Sigma_{t=0}^{\infty} \delta^t(1 - \delta)f(q) = f(q)$.

[5]This can be accomplished by setting $r_t = p_t - \delta p_{t+1}$ (with the proviso that in any period k in which sales are zero, p_k is chosen to be the minimum price which does not induce sales in that period).

the case of a duopoly, i.e., that the sales equilibrium necessarily exhibits intertemporal price discrimination whereas the rental equilibrium does not. First consider the duopoly rental model. We will argue that the open-loop equilibrium of the infinitely repeated rental game is unique and consists of merely repeating the unique one-period equilibrium. To characterize the one-period rental equilibrium, observe that in any equilibrium H necessarily sells to a nonempty interval of customers $[0, \underline{q}]$ whereas L sells to a nonempty interval $[\underline{q}, \bar{q}]$, with $\bar{q} < 1$. Let r^H and r^L be the rental prices of good H and good L, respectively. Consumer $\underline{q}$ must be indifferent between renting H and L, so that $(1 - \delta)\mu f(\underline{q}) - r^H = (1 - \delta)f(\underline{q}) - r^L$, or $\underline{q} = 1 - (r^H - r^L)/(1 - \delta)(\mu - 1)$. Consumer $\bar{q}$, on the other hand, is indifferent between renting good L and not renting at all; hence, $\bar{q} = 1 - r^L/(1 - \delta)$. Profits are given by: $\pi^H = r^H\underline{q}$ and $\pi^L = r^L(\bar{q} - \underline{q})$. Since π^H and π^L are strictly concave functions of r^H and r^L, respectively, the unique rental equilibrium can be found by solving the system of equations $\partial\pi^H/\partial r^H = 0$ and $\partial\pi^L/\partial r^L = 0$:

$$\bar{r}^H = 2(1 - \delta)\mu(\mu - 1)/(4\mu - 1) \quad \text{and} \quad \bar{r}^L = (1 - \delta)(\mu - 1)/(4\mu - 1).$$

Observe that the outcome consisting of charging the rentals $\bar{r}^H$ and $\bar{r}^L$ at all times is achievable in the sales model. Firms just price their goods at $\bar{p}^H = \bar{r}^H/(1 - \delta)$ and $\bar{p}^L = \bar{r}^L/(1 - \delta)$ in the initial period, and never cut their prices thereafter. One might therefore naively believe that this outcome is also an equilibrium of the sales game. Indeed, it can be shown that the sequence $p_t^L = \bar{p}^L$ $(t \geq 0)$ is a best response to the sequence $p_t^H = \bar{p}^H$ $(t \geq 0)$.[6] However, in response to firm L charging $\bar{p}^L$, firm H

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[6]To see this, one first shows that there cannot be three or more successive periods in which L sells positive quantities. Suppose that m, m + r and m + r + s are three periods with positive sales $(m \geq 0, r \geq 1$ and $s \geq 1)$ such that q_1 is indifferent between purchasing L in periods m and m + r, and such that q_2 is indifferent between purchasing L in periods m + r and m + r + s. Then $q_1 = 1 - (p_m^L - \delta^r p_{m+r}^L)/(1 - \delta^r)$ and $q_2 = 1 - (p_{m+r}^L - \delta^s p_{m+r+s}^L)/(1 - \delta^s)$. Setting the derivative of π^L with respect to p_{m+r}^L equal to zero yields $p_m^L(1 - \delta^s) - p_{m+r}^L[\delta^r(1 - \delta^s) + (1 - \delta^r)] + p_{m+r+s}^L\delta^s(1 - \delta^r) = 0$. Observe that this last expression is equal to $(1 - \delta^r)(1 - \delta^s)s_{m+r}^L$, where $s_{m+r}^L = q_2 - q_1$ is the sales level of firm L in period m + r. Consequently, optimization implies that $s_{m+r}^L = 0$.

Similarly, it can be shown there cannot be two successive periods with positive sales, for if m + r were the last period of positive sales, then $q_2 = 1 - p_{m+r}^L$, and hence $\partial\pi^L/\partial p_{m+r}^L = 2\delta^r(p_m^L - p_{m+r}^L)/(1 - \delta^r) = 0$ implying $s_{m+r}^L = 0$.

With only one period of positive sales, m is necessarily equal to zero (since charging the same can do better than charging $\bar{p}^H$ forever. Specifically, suppose firm H deviates in period n by charging $p_n^H = \mu(1 - \bar{q}) - \epsilon$ for some small $\epsilon > 0$, where n is the unique solution to $\delta^n \mu < 1 \le \delta^{n-1}\mu$.[7] Then firm H generates positive sales in period n (it sells to the interval $[\bar{q} - \epsilon\delta^n/(1 - \delta^n\mu), \bar{q} + \epsilon/\mu]$) but its sales in period 0 remain unaffected.[8] Hence we obtain:

Proposition 1: Any (open-loop) equilibrium of the duopoly sales game with $\mu > 1$ involves intertemporal price discrimination. Hence, as long as $\mu > 1$, the rental and the sales model are not equivalent.

We now turn to the issues of existence and characterization of the equilibria of the sales game.

4. The Solution of the Sales Game

We first derive the candidate equilibrium by solving the equations describing the first order conditions for an interior solution, and then show that this candidate is indeed the unique equilibrium of the game. In describing the candidate, different cases arise according to the values of the parameters δ and μ. Let n be the unique solution to $\delta^n\mu < 1 < \delta^{n-1}\mu$.[9] The value of n has the following interpretation: when L lowers its price in time period t slightly, it attracts customers who would otherwise have bought the high quality good in period t + (n - 1) or in period t + n.

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single price earlier does not lose sales, and hence improves profits). We conclude that firm L will charge $\bar{p}^L$ in period zero.

[7]The reason for choosing period n is that charging the price $\mu(1 - \bar{q}) - \epsilon$ for any period k ($1 \le k \le n - 1$) would not only completely wipe out L's sales, but would also cut into H's sales in period 0.

[8]Interestingly, in the hybrid game where firm L leases and firm H sells, the unique equilibrium has $r_t^L = \bar{r}^L$ and $p_t^H = \bar{p}^H$. The reason that firm H cannot profitably deviate in the hybrid game, but can do so in the sales game, is that in the sales game firm L only fixes the net present value of the rentals, $\bar{p}^L$, whereas in the hybrid game it fixes all future rentals. By deviating in the manner described above, firm H can affect the composition of the stream of implicit rentals in the sales game. More specifically, firm H is able to raise the implicit rental charged by firm L in period 0, implying that firm L loses sales in that period.

[9]The solution for the case where $\delta^k\mu = 1$ for some $k \ge 0$ can be obtained by considering the limit of the solution for Case 1 or Case 2, given below.

4.1 Case 1: $\delta\mu < 1 < \mu$.

The condition $\delta\mu < 1 < \mu$ has specific implications for how the time pattern of sales of high and low quality goods is distributed across consumers' types. Consider the premium that a consumer would be willing to pay for the privilege of acquiring the high quality good in period t rather than purchasing the low quality good in the same period, $\mu f(q) - f(q) = (\mu - 1)f(q)$. Since $\mu > 1$, this premium is positive and decreasing in q. Hence in any interior equilibrium,[10] and for every $t \geq 0$, there exists a consumer $\underline{q}_t$ who is exactly willing to pay the premium implicitly charged by the market, $p_t^H - p_t^L$. Consumers $q < \underline{q}_t$ prefer purchasing H in period t to purchasing L in the same period; consumers $q > \underline{q}_t$ prefer the opposite. Similarly, consider the premium a consumer would pay for being able to purchase L in period t rather than having to wait to buy H in period t + 1, $f(q) - \delta\mu f(q) = (1 - \delta\mu)f(q)$. Using a similar argument as above, this shows that for every $t \geq 0$ there exists a consumer $\bar{q}_t$ who is exactly willing to pay the premium charged by the market, $p_t^L - \delta p_{t+1}^H$. Consumers $q < \bar{q}_t$ prefer to purchase L in period t rather than waiting for H in period t + 1; consumers $q > \bar{q}_t$ prefer to wait instead.

We conclude that the sales pattern in the case $\delta\mu < 1 < \mu$ must be as depicted in Figure 1. Interestingly, unlike in the static models of vertical price differentiation, the high quality firm does not specialize in selling to high valuation customers, and neither does the low quality firm cater to low valuation customers only. Instead, the market areas of the two firms form an *interlinked pattern* so that both firms sell to relatively rich as well as relatively poor consumers.

<Insert Figure 1 about here>

Sales in period t are given by $s_t^L = \bar{q}_t - \underline{q}_t$ (for $t \geq 0$), $s_t^H = \underline{q}_t - \bar{q}_{t-1}$ (for $t \geq 1$) and $s_0^H = \underline{q}_0$, where $\underline{q}_t$ and $\bar{q}_t$ were defined above by:

[10] By an interior equilibrium, we mean an equilibrium in which $s_t^i > 0$ for all i and all t.

$$(1)\qquad q_t = 1 - \frac{p_t^H - p_t^L}{\mu - 1}, \qquad \bar{q}_t = 1 - \frac{p_t^L - \delta p_{t+1}^H}{1 - \delta\mu}, \qquad t \geq 0.$$

Observe that by varying p_t^i locally around the equilibrium value $\bar{p}_t^i$, firm i (i = H, L) affects only its current profits, π_t^i. The candidate equilibrium can thus be found by maximizing each firm's current profits, given the opponent's equilibrium strategy. Optimizing π_t^L with respect to p_t^L produces the first order conditions:

$$(2)\qquad (1 - \delta\mu)\bar{p}_t^H - 2\mu(1 - \delta)\bar{p}_t^L + \delta(\mu - 1)\bar{p}_{t+1}^H = 0, \quad t \geq 0.$$

Maximizing π_t^H with respect to p_t^H produces the following conditions:

$$(3)\qquad (\mu - 1)\bar{p}_{t-1}^L - 2(1 - \delta)\bar{p}_t^H + (1 - \delta\mu)\bar{p}_t^L = 0, \quad t \geq 1$$

$$(4)\qquad (\mu - 1) - 2\bar{p}_0^H + \bar{p}_0^L = 0, \quad t = 0.$$

We can use equation (2) to express $\bar{p}_t^L$ as a function of $\bar{p}_t^H$ and $\bar{p}_{t+1}^H$, and $\bar{p}_{t-1}^L$ as a function of $\bar{p}_{t-1}^H$ and $\bar{p}_t^H$. Substituting the resulting expressions in (3) yields a second-order difference equation in $\bar{p}_t^H$:

$$(5)\qquad (\mu - 1)(1 - \delta\mu)\bar{p}_{t-1}^H + \omega(1)\bar{p}_t^H + \delta(\mu - 1)(1 - \delta\mu)\bar{p}_{t+1}^H = 0, \quad t \geq 1,$$

where $\omega(1) \equiv \delta(\mu - 1)^2 - 4\mu(1 - \delta)^2 + (1 - \delta\mu)^2$.[11] The general solution to this linear homogenous difference equation is of the form $\bar{p}_t^H = c_1(1)\lambda_1^t(1) + c_2(1)\lambda_2^t(1)$, $t \geq 0$, where $\lambda_1(1)$ and $\lambda_2(1)$ are the roots of the characteristic equation:

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[11]The argument of ω as well as those of the other constants defined later in this section refers to the case n = 1.

(6) $$(\mu - 1)(1 - \delta\mu) + \omega(1)\lambda + \delta(\mu - 1)(1 - \delta\mu)\lambda^2 = 0.$$

We show in Appendix A that this equation has two real roots satisfying $0 < \lambda_1(1) < 1 < \lambda_2(1)$. Since $\{\bar{p}_t^H\}_{t=0}^{\infty}$ must remain bounded as $t \to \infty$, we must have $c_2(1) = 0$, and so:

(7) $$\bar{p}_t^H = c_1(1)\lambda_1^t(1), \quad t \geq 0.$$

From equation (2) we then obtain:

(8) $$\bar{p}_t^L = \theta(1)c_1(1)\lambda_1^t(1), \quad t \geq 0,$$

where $\theta(1) = [(1 - \delta\mu) + \delta(\mu - 1)\lambda_1(1)]/[2\mu(1 - \delta)]$. Substituting (7) and (8) into (4) finally yields $c_1(1) = (\mu - 1)/[2 - \theta(1)]$. Appendix A verifies that $c_1(1) > 0$.

In Appendix B, we argue that any equilibrium entails positive sales for each firm in every period; given that the profit functions π_t^i (i = H, L) are concave for all $t \geq 0$, equations (7) and (8) describe the unique candidate for an equilibrium. The interiority proof can be summarized as follows: consumer optimization implies that in any candidate equilibrium sales occur according to the pattern sketched in Figure 1; all that could happen is that some of the sales in this pattern are zero (i.e., $\underline{q}^t = \bar{q}^t$ or $\bar{q}^t = \underline{q}_{t+1}$ for some $t \geq 0$). We show that if $s_t^i = 0$, then some firm (either i or its opponent) could profitably deviate by lowering its price in one of the periods in which it does not sell. Appendix B also proves that it is not an optimal response to the candidate equilibrium for some firm to wipe out the sales of its opponent in any particular period. We therefore have:

<u>Proposition 2</u>: There exists a unique open-loop equilibrium to the duopoly sales game with $\delta\mu < 1 < \mu$. This equilibrium is characterized by equations (7) and (8).

The equilibrium of the sales game displays several remarkable properties:

(1) Both prices are decreasing at a constant rate $\lambda_1(1)$, and converge to zero as t approaches infinity. This implies in particular that firms intertemporally price discriminate, and end up saturating the entire market. Note that both of these results differ from the monopoly case.

(2) Both firms' sales are decreasing at the rate $\lambda_1(1)$. This implies that market shares are constant for all $t \geq 1$, and that cumulative sales are concave. Consequently, the diffusion curves have no inflection points.

(3) Per period profits decline at the same rate $\lambda_1(1)^2$, for all $t \geq 1$, and converge to zero as t approaches infinity.

Since many of these and further properties of the solution are shared with the case $n > 1$, we proceed with the derivation for that case. Additional properties will be discussed in Sections 5 and 6.

4.2 Case 2: $\delta^n\mu < 1 < \delta^{n-1}\mu$, for some $n > 1$.

As in case 1, the condition $\delta^n\mu < 1 < \delta^{n-1}\mu$ has precise implications for the time pattern of sales of high and low quality goods. Consider the premium that a consumer is willing to pay for the privilege to purchase the high quality good in period $t + (n - 1)$ rather than acquiring the low quality good in period t, $(\delta^{n-1}\mu - 1)f(q)$. Since this premium is positive and decreasing in q, there exists a consumer $\underline{q}_{t+n-1}$ who is just willing to pay the premium charged in the market, $\delta^{n-1}p^H_{t+n-1} - p^L_t$. Consumers $q < \underline{q}_{t+n-1}$ prefer waiting until period $t + (n - 1)$ to purchase H; consumers $q > \underline{q}_{t+n-1}$ prefer purchasing L in period t. Similarly, there exists a consumer $\bar{q}_t$ for which $f(q) - \delta^n\mu f(q) = (1 - \delta^n\mu)f(q) = p^L_t - \delta^n p^H_{t+n}$. Consumers $q < \bar{q}_t$ prefer purchasing L in period t to waiting for H in period $t + n$; consumers $q > \bar{q}_t$ would rather delay their purchases.

Note in particular that all consumers $q < \underline{q}_{n-1}$ prefer purchasing H in period $(n - 1)$ to purchasing H in any later period or purchasing L at any date. Consequently all $q < \underline{q}_{n-1}$ necessarily purchase H in one of the periods $0 \leq s \leq n - 1$.

Consumer self-selection thus results in a sales pattern with several remarkable features. First, the

high quality firm has no immediate competition from the low quality firm during the first n - 1 sales periods. In other words, H's market areas do not start touching those of L until period n - 1. Second, by the time period n - 1 comes around, the low quality firm has carved out a sequence of non-overlapping intervals of customers. After period (n - 1) firm L continues to pick out pockets of customers in this manner; firm H then serves the customers that are left behind.

Appendix B (Proof of Proposition 3, Lemma 2) shows that in any equilibrium firm H necessarily sets prices $\{p_0^H, p_1^H, \ldots, p_{n-1}^H\}$ such that $s_1^H = s_2^H = \ldots = s_{n-2}^H = 0$. Hence any equilibrium which otherwise remains interior must exhibit the sales pattern depicted in Figure 2.

<Insert Figure 2 about here>

Sales in period t are given by $s_t^L = \bar{q}_t - \underline{q}_{t+n-1}$ (for $t \geq 0$), $s_t^H = \underline{q}_t - \bar{q}_{t-n}$ (for $t \geq n$), $s_{n-1}^H = \underline{q}_{n-1} - q_0$, and $s_n^H = \underline{q}_0$ where

$$\underline{q}_{t+n-1} = 1 - \frac{\delta^{n-1} p_{t+n-1}^H - p_t^L}{\delta^{n-1}\mu - 1}, \quad t \geq 0$$

(9)
$$\underline{q}_0 = 1 - \frac{p_0^H - \delta^{n-1} p_{n-1}^H}{\mu(1 - \delta^{n-1})}$$

$$\bar{q}_t = 1 - \frac{p_t^L - \delta^n p_{t+n}^H}{1 - \delta^n \mu}, \quad t \geq 0.$$

As in the case n = 1, L optimizes Π^L by maximizing each π_t^L with respect to p_t^L. This yields:

(10)
$$(1 - \delta^n \mu)\bar{p}_{t+n-1}^H - 2\mu(1 - \delta)\bar{p}_t^L + \delta(\delta^{n-1}\mu - 1)\bar{p}_{t+n}^H = 0, \quad t \geq 0.$$

For $t \geq n$, the optimal p_t^H maximizes π_t^H, and so:

$$(11)\qquad (\delta^{n-1}\mu - 1)\bar{p}^L_{t-n} - 2\delta^{n-1}(1-\delta)\bar{p}^H_t + (1-\delta^n\mu)\bar{p}^L_{t-(n-1)} = 0, \quad t \geq n.$$

The optimal p^H_{n-1} maximizes $\pi^H_0 + \delta^{n-1}\pi^H_{n-1}$, so that:

$$(12)\qquad 2(\delta^{n-1}\mu - 1)\bar{p}^H_0 - 2\delta^{n-1}(\mu - 1)\bar{p}^H_{n-1} + \mu(1-\delta^{n-1})\bar{p}^L_0 = 0.$$

Finally, p^H_0 must minimize $\pi^H_0 + \delta^{n-1}\pi^H_{n-1}$:

$$(13)\qquad \mu(1-\delta^{n-1}) - 2\bar{p}^H_o + 2\delta^{n-1}\bar{p}^H_{n-1} = 0.$$

Using equation (10) to express $\bar{p}^L_{t-1}$ in function of $\bar{p}^H_{t-1}$ and $\bar{p}^H_t$, and $\bar{p}^L_{t-(n-1)}$ in function of $\bar{p}^H_t$ and $\bar{p}^H_{t+1}$, and substituting in (11), again yields a second order difference equation in $\bar{p}^H_t$:

$$(14)\qquad (\delta^{n-1}\mu - 1)(1-\delta^n\mu)\bar{p}^H_{t-1} + \omega(n)\bar{p}^H_t + \delta(\delta^{n-1}\mu - 1)(1-\delta^n\mu)\bar{p}^H_{t+1} = 0, \quad t \geq n,$$

where $\omega(n) \equiv \delta(\delta^{n-1}\mu - 1)^2 - 4\mu\delta^{n-1}(1-\delta)^2 + (1-\delta^n\mu)^2$. Similar reasoning to the case $n = 1$ establishes that the solution to (14) must satisfy:

$$(15)\qquad \bar{p}^H_t = c_1(n)\lambda_1^{t-(n-1)}(n), \quad t \geq n - 1,$$

where $\lambda_1(n)$ is the smallest root (see Appendix A) of the characteristic equation:

$$(16)\qquad (\delta^{n-1}\mu - 1)(1-\delta^n\mu) + \omega(n)\lambda + \delta(\delta^{n-1}\mu - 1)(1-\delta^n\mu)\lambda^2 = 0.$$

Substituting (15) into (10) yields:

$$\bar{p}_t^L = \theta(n)c_1(n)\lambda_1^t(n), \quad t \geq 0, \tag{17}$$

where $\theta(n) = [(1 - \delta^n\mu) + \delta(\delta^{n-1}\mu - 1)\lambda_1(n)]/[2\mu(1 - \delta)]$. To determine $c_1(n)$, we multiply equation (13) by $(\delta^{n-1}\mu - 1)$ and add the resulting expression to equation (12). This gives $c_1(n) = (\delta^{n-1}\mu - 1)/(2\delta^{n-1} - \theta(n)) > 0$. Finally, we may calculate $\bar{p}_0^H$ from (13):

$$\bar{p}_0^H = \mu(1 - \delta^{n-1})/2 + \delta^{n-1}c_1(n). \tag{18}$$

As in case 1, we argue in Appendix B that the candidate solution is unique and constitutes an actual equilibrium of the game. Hence:

<u>Proposition 3</u>: There exists a unique open-loop equilibrium to the duopoly game with $\delta^n\mu < 1 < \delta^{n-1}\mu$. This equilibrium is characaterized by equations (15), (17) and (18).

Note that the solution defined by (15)-(18) for the case $n > 1$, when evaluated at $n = 1$, reduces to the solution defined by (6)-(8) for the case $n = 1$. In particular, for a given value of δ, the constants λ_1, $(\mu\theta)$, and (c_1/μ) depend only on $\alpha \equiv \delta^{n-1}\mu$. Thus, for a given value of δ, models with two different values of μ which result in the same value of α, will exhibit the same speed of convergence and the same relative prices after period $n(\mu)$.[12] This property has an important consequence: the speed of convergence and the relative prices (after period $n(\mu)$) must therefore be nonmonotonic in μ. This point will be further elaborated on in Section 5. Finally it is worth remarking that the properties (1)-(3) mentioned at the end of Section 4.1 also hold when $n > 1$, for $t \geq n - 1$.

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[12]For a given δ, define $A(n) = \{\mu: \delta^n\mu < 1 < \delta^{n-1}\mu\}$. Also let $\alpha(\mu) = \delta^{n(\mu)-1}\mu$, where $n(\mu)$ is the unique value of n such that $\delta^n\mu < 1 < \delta^{n-1}\mu$, i.e., $n(\mu) = [-\log \mu/\log \delta] + 1$. By [x] we mean the largest integer smaller than x. Observe that, for each integer $n \geq 1$, the image of the set $A(n)$ under α is equal to the interval $(\delta, 1)$. Consequently, for each $\hat{\alpha} \in (\delta, 1)$ and each $n \geq 1$, there exists a unique value of μ such that $\alpha(\mu) = \hat{\alpha}$.

Table 1 provides a summary of all the equilibrium variables: prices (p_t^i), sales (s_t^i), profits (π_t^i), net present value of profits (Π^i), cumulative sales (S_t^i), and market shares (defined as $m_t^L = s_t^L/s_t^H$).

<Insert Table 1 about here>

5. Qualitative Implications

In the previous section, we demonstrated that the intertemporal linkages which are present in a market for durable goods dramatically modify the pattern of sales that would be observed in a static (or rental) context. This suggests that the intertemporal substitutability in demand present in such markets may have an important impact on the nature of competition. We pursue this question here by studying how an increase in the relative quality of good H, μ, affects the outcome of the game. To economize on space, we summarize the comparative dynamics as μ varies in the interval $(\delta^{-n+1}, \delta^{-n})$ in Table 2.

<Insert Table 2 about here>

Table 3 collects the limiting values of the main equilibrium variables as μ approaches either of the end points of the interval $(\delta^{-n+1}, \delta^{-n})$. In our discussion below, we will frequently refer to these tables.

<Insert Table 3 about here>

When $\mu = 1$, firms are selling perfect substitutes, and hence the Bertrand outcome obtains. Consequently, the market is fully penetrated in time period zero, with firm H selling 2/3 and firm L selling 1/3 (see Table 3).[13] As μ increases (but remains less than $1/\sqrt{\delta}$), both $\bar{p}_0^H$ and $\bar{p}_0^L$ increase, but

[13]This division of the market is really obtained by considering the limit of the solution as μ decreases to one.

$\bar{p}_0^H$ increases faster than $\bar{p}_0^L$ (see Results 2, 4 and 5 in Table 2). Furthermore, $\lambda_1(1)$ becomes positive so that all prices $\bar{p}_t^H$ and $\bar{p}_t^L$ ($t \geq 0$) become positive. In fact, over the considered range for μ, $\bar{p}_t^H$ and $\bar{p}_t^L$ are increasing in μ, for all $t \geq 0$. Since $\lambda_1(1)$ is increasing for $\mu < 1/\sqrt{\delta}$, the rate of price decrease slows down as μ increases (see Result 1 in Table 2). Thus, higher values of μ are associated with higher price paths. Sales in the initial period, s_0^L and s_0^H, decrease, but the higher relative prices cause firm H to lose market share. Firm H's market share grows in every period $t \geq 1$, now exceeding the value of 2/3, despite the fact that relative prices in these periods increase equally much as in period zero (see Result 5 in Table 2). Total market penetration for firm H, $S_\infty^H = \Sigma_{t=0}^\infty s_t^H$, remains constant at 2/3. Consistent with the constant market penetration and the decrease in initial sales, sales in all subsequent periods, s_t^H and s_t^L ($t \geq 1$), increase. The impact of an increase in μ ($1 \leq \mu < 1/\sqrt{\delta}$) is also to increase profits of both firms in every period $t \geq 0$. However the high quality firm is able to improve its relative profit position, π_t^H/π_t^L, in every period $t \geq 0$ (Table 2, Result 7 shows this for $t \geq 1$). Sales occur more slowly over time but the net present value of both firms' profits increases (see Results 8 and 9 in Table 2). While our model is dynamic and sales occur over an infinite number of time periods, many of the above results are consistent with the comparative statics in the rental model.[14] As μ increases, firm H improves its relative quality position, so that competition is relaxed.

However, when μ reaches the value $1/\sqrt{\delta}$, further increases in μ have dramatically different consequences. While $\bar{p}_0^H$ continues to increase, all other prices $\bar{p}_t^H$, $t \geq 1$ and $\bar{p}_t^L$, $t \geq 0$ start to decrease. Furthermore, $\lambda_1(1)$ decreases for $1/\sqrt{\delta} < \mu < 1/\delta$, so that the rate of price decrease picks up as μ increases. The entire price path for the low quality firm is thus lowered; the same is true for the high quality firm except in the first period where price increases. Relative prices $\bar{p}_t^L/\bar{p}_t^H$, $t \geq 0$, decrease, so that $\bar{p}_t^L$ decreases faster than $\bar{p}_t^H$ for $t \geq 1$. The impact on sales is as follows: firm H loses sales in period zero, but firm L gains sales. In period 1, firm H gains sales, but firm L loses sales. In period 2 and beyond both firms lose sales. Despite this, total market penetration for each firm remains constant.

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[14] In the rental model, as μ increases, $\bar{r}^H$, $\bar{r}^L$ and π^H increase, while $\bar{r}^L/\bar{r}^H$, π^L/π^H, s^L and s^H decrease. The ratio s^L/s^H remains constant at 1/2.

Also, firm H is able to obtain a higher market share in all periods $t \geq 1$.

The impact on profits of an increase in μ in the interval $(1/\sqrt{\delta}, 1/\delta)$ is as follows. Firm L's profits decline in every single period. Firm H earns greater profits in period zero. Sooner or later, each π_t^H $(t \geq 1)$ switches from being increasing to becoming decreasing. The switch point occurs for μ values closer to $1/\sqrt{\delta}$ when t is larger. On balance, however, firm H is able to improve its net present value of profits, Π^H.

The effects we just described culminate when μ approaches $1/\delta$. All prices but $\bar{p}_0^H$ converge to marginal cost and consequently π_t^H, $t \geq 1$ and π_t^L, $t \geq 0$ converge to zero. Thus, the case $\mu = 1/\delta$ differs from the case $\mu = 1$ only in that $\bar{p}_0^H$ and π_0^H are positive. Diffusion is almost instantaneous; in period zero, firm H sells a quantity of 1/2 and firm L a quantity of 1/3. The remaining customers, those in the interval [5/6, 1], buy from firm H in period 1.

At this point the reader may be a little perplexed as to why most of the comparative dynamics effects change direction when μ crosses the value $1/\sqrt{\delta}$, and why an almost perfectly competitive outcome obtains when μ approaches $1/\delta$. We may explain this reversal as the outcome of two counteracting forces: the immediate competition effect and the deferred competition effect. When μ increases, good H in period t becomes of relatively higher quality than good L in period t. As in the static model, this relaxes the competition between contemporaneous goods. We refer to this relaxing of competition as the *immediate competition effect*. However, when the quality of good H increases, waiting to purchase H one period later becomes an increasingly attractive option relative to purchasing L in the current period. Good H in period $t + 1$ therefore becomes an increasingly better--though still inferior--substitute to good L in period t. As in the static model, the reduced quality differential resulting from an increase in μ intensifies competition between these two products. We refer to this increased competition as the *deferred competition effect*. When $\mu = 1$, the immediate competition is maximal, while the deferred competition is minimal. When μ approaches $1/\delta$, the immediate competition is minimal, but the deferred competition is maximal. On balance, the least competitive outcome obtains when the quality advantage of firm H is neither too small nor too large, i.e., when $\mu = 1/\sqrt{\delta}$. The interaction between the immediate and the deferred competition effect therefore explains the quasiconcave shape and limiting values of the variables $\bar{p}_t^H$ $(t \geq 1)$, $\bar{p}_t^L$ $(t \geq 0)$, s_t^H $(t \geq 2)$, s_t^L $(t \geq 0)$, π_t^H $(t \geq 1)$, π_t^L $(t \geq 0)$ and λ_1 (see Table 2). It should also be observed that the deferred competition effect is always absent for good H in period zero; consequently both $\bar{p}_0^H$ and π_0^H are everywhere increasing in μ.

When μ crosses δ^{-1}, the value of $n(\mu)$ jumps from 1 to 2. The solution for this case was computed in Section 4.2. As can be seen from Figure 2, firm H now has such a great quality advantage that good H in period 1 becomes a superior substitute to good L in period zero. Good H in period zero therefore has neither immediate nor deferred competition from good L. However, good L in period zero faces immediate competition from good H in period 1 and deferred competition from good H in period 2. More generally, good L in period t has good H in period $t + 1$ as an immediate competitor, and good H in period $t + 2$ as a deferred competitor. Except for the initial period, the comparative dynamics effects as μ traverses the interval $(\delta^{-1}, \delta^{-2})$ are therefore similar to those for $n = 1$. The variables $\bar{p}_t^L$ $(t \geq 0)$, s_t^L $(t \geq 0)$, π_t^L $(t \geq 0)$ and $\lambda_1(2)$ are all quasiconcave functions attaining the value zero at $\mu = \delta^{-1}$ and $\mu = \delta^{-2}$, with a unique maximum at $\mu = \delta^{-3/2}$. The lack of competition in the first two periods allows firm H to increase both $\bar{p}_0^H$ and $\bar{p}_1^H$. It does this in such a way as to keep s_0^H constant at the level 1/2. Interestingly, s_1^H continually decreases from a value of 1/6 when $\mu = \delta^{-1}$ to a value of zero when $\mu = \delta^{-2}$. In the meantime, s_2^H increases from zero at $\mu = \delta^{-1}$ to 1/6 at $\mu = \delta^{-2}$; this ensures continuity of the sales pattern as μ crosses δ^{-2}. It can also be shown (as was the case when $n = 1$) that s_0^L is a quasiconcave function of μ, attaining the value 1/3 at $\mu = \delta^{-1}$ and $\mu = \delta^{-2}$, with a unique maximum at $\mu = \delta^{-3/2}$. Surprisingly, the total market penetration achieved by firm H remains constant at 2/3. Finally, as before, Π^H is increasing in μ and Π^L is quasiconcave in μ.

Further transitions of μ through the intervals $(\delta^{-n+1}, \delta^{-n})$ for $n > 2$ produce the same comparative dynamics effects. Again, these are explained by the immediate competition between good L in period t and good H in period $t + (n - 1)$, and the deferred competition between good L in period t and good H in period $t + n$, for all $t \geq 0$. We may thus conclude:

Proposition 4: An increase in the relative quality of good H has two opposing effects. It differentiates good H from good L in the periods in which they are in immediate competition; however, it also makes waiting an extra period to purchase good H a relatively more attractive option, thereby increasing deferred competition. The net effect on competition is ambiguous.

It should further be observed that if firm L has a lead time of n periods in bringing its product to market, then in each period $t \geq n$ firm H faces competition from firm L on both of its market boundaries: on the left, it competes with good L at time $(t - n)$, and on the right it competes with good L at time $(t - n + 1)$. Consequently, as μ increases towards δ^{-n}, the deferred competition for each good becomes maximal, so that all prices (and hence profits) converge to zero. When firm L has a timing advantage, an increase in the quality of good H may therefore leave both firms worse off.

6. The Continuous Time Limit

In this section we investigate the behavior of the solution to our model as the time interval between successive periods is allowed to become vanishingly small.

Let r be the interest rate at which consumers and firms discount surplus and profits over a given amount of real time (say one year). Also, let z denote the length of the time period over which prices remain constant, measured in terms of the unit of real time. In our example, if the time period is one day, then $z = 1/365$; if the time period is one month, then $z = 1/12$. We assume throughout that interest is compounded continuously, so that the discount factor between successive periods is given by $\delta = e^{-rz}$. We would like to understand the behavior of our model as z becomes arbitrarily small, or equivalently as δ approaches one. Since the solution developed in Section 4 (equations (6)-(8) and (15)-(18)) is mainly driven by the behavior of λ_1, we will study how λ_1 depends on the discount factor δ.

When $\delta = 0$, there is no intertemporal competition, and the solution degenerates to that of the static model (see Section 2). From equation (6), at $\delta = 0$ we have $\lambda_1 = (\mu - 1)/(4\mu - 1)$. When δ increases in the interval $(0, 1/\mu)$, waiting for good H in period $t + 1$ becomes an increasingly attractive option relative to purchasing good L in period t, and deferred competition intensifies. At $\delta = 1/\mu$, deferred competition culminates so that $\lambda_1 = 0$. As δ traverses the value $1/\mu$, good H in period t + 1 and good L in period t become immediate competitors; the deferred competition for good L in period t now comes from good H in period t + 2. On the interval $(1/\mu, 1/\sqrt{\mu})$, immediate competition is thus maximal and deferred competition minimal at $\delta = 1/\mu$. As δ increases, immediate competition relaxes but deferred competition intensifies. When δ approaches $1/\sqrt{\mu}$, once more the immediate competition is minimal and the deferred competition maximal. This results in λ_1 having a quasiconcave shape on the interval $(1/\mu, 1/\sqrt{\mu})$. Since the same economic forces are at work any time δ traverses intervals of the form $(\mu^{-1/n}, \mu^{-1/(n+1)})$, λ_1 has a quasiconcave shape on these intervals as well (see Figure 3).

<Insert Figure 3 about here>

We conclude that λ_1 oscillates with increasing frequency as δ approaches one. We will now show that the amplitude of this oscillation remains uniformly bounded away from one. From (16), λ_1 is the smallest root of

$$(19) \qquad \lambda_1^2 + \psi\lambda_1 + \delta^{-1} = 0$$

where $\psi \equiv \psi(\alpha,\delta) = \dfrac{\delta(\alpha - 1)^2 - 4\alpha(1 - \delta)^2 + (1 - \delta\alpha)^2}{\delta(1 - \delta\alpha)(\alpha - 1)}$ and $\alpha \equiv \alpha(\delta) = \mu\delta^{n-1}$. It is easy to show that λ_1 is a strictly decreasing function on the interval $[0, 1/\mu]$. For n > 1, we can bound λ_1 as follows: $\lambda_1(\alpha(\delta), \delta) \leq \max_{\alpha\in[1,\delta^{-1}]}\lambda_1(\alpha, \delta) = \lambda_1(\delta^{-1/2}, \delta)$. Indeed, for fixed δ, λ_1 is maximized at $\mu = \delta^{1/2-n}$, i.e., $\alpha = \delta^{-1/2}$ (see Result 1, Table 2). Note that $\psi(\delta^{-1/2}, \delta) = -2(2 + 3\delta^{1/2} + 2\delta)$. Substituting this expression into (19), it can be shown that $\lambda_1(\delta^{-1/2}, \delta)$ is a decreasing function of δ, with $\lim_{\delta\to 1}\lambda_1(\delta^{-1/2}, \delta) = 7 - \sqrt{48} < 1$. Consequently, $\lambda_1(\alpha(\delta), \delta)$ oscillates with a maximum amplitude $\lambda_1(\delta^{-1/2}, \delta)$ which decreases to $7 - \sqrt{48}$ as $\delta \to 1$.

As z approaches zero, the number of time periods contained in any small but positive interval of real time approaches infinity. Since the rate of price decrease per period, $(1 - \lambda_1)$, is bounded away from zero, the rate of price decrease on any positive real time interval approaches 100% as z goes to zero. This immediately implies that the entire price path of the low quality firm converges uniformly to zero as $z \to 0$. The properties of firm H's price path can be derived as follows. Since $n - 1 = [-\log \mu/\log \delta]$ and since $\delta = e^{-rz}$, the real time before firm H cuts its price below $\bar{p}_0^H$, $(n - 1)z$, converges to $\bar{\tau} = \log \mu/r$. Thus, $e^{-r\bar{\tau}}\mu = 1$. We know that for $\tau < \bar{\tau}$, the high quality firm's price path $\bar{p}^H(\tau)$ is constant at a level $\bar{p}^H(0)$.[15] Since for any $\epsilon > 0$ the rate of price decrease over any real time interval $(\tau - \epsilon, \tau + \epsilon)$ approaches 100% as $z \to 0$, $\bar{p}^H(\tau) = 0$ for $\tau > \bar{\tau}$. Also, since $\bar{p}_0^H = (1 - \delta^{n-1})(\mu/2) + \delta^{n-1}\bar{p}_{n-1}^H$ (see Table 1), $\delta^{n-1} \to e^{-r\bar{\tau}}$ and $\bar{p}_{n-1}^H \to 0$, $\bar{p}_0^H$ converges to $(1 - e^{-r\bar{\tau}})(\mu/2) = \bar{p}^H(0)$.

Limiting sales can also be easily computed. First, since total sales for firm L, S_∞^L, are equal to 1/3 in every equilibrium and since s_t^L decreases at a rate $(1 - \lambda_1)$, limiting sales must satisfy $s^L(0) = 1/3$ and $s^L(\tau) = 0$ for all $\tau > 0$. In addition, $s_0^H = 1/2$ and $s_t^H = 0$ $(0 < t < n - 1)$ for all $\delta > 1/\mu$ (see Table 1), implying $s^H(0) = 1/2$ and $s^H(\tau) = 0$ $(0 < \tau < \bar{\tau})$. Using the same argument as for firm L, $s^H(\tau) = 0$ for $\tau > \bar{\tau}$; consequently firm H makes all of its remaining sales at time $\bar{\tau}$, $s^H(\bar{\tau}) = 1/6$. We conclude that the limiting profits satisfy $\Pi^H = (1 - e^{-r\bar{\tau}})(\mu/4)$ and $\Pi^L = 0$.

The extreme competition that arises in the limit as $z \to 0$ is remarkable and is in fact somewhat reminiscent of the Coase conjecture (Coase, 1972). The latter proposition says that a durable good monopolist will introduce his product to the market at a price which approaches marginal cost, in the limit as $z \to 0$. This occurs because the monopolist has no commitment power, and hence faces unbridled competition from his future selves. Here, we assume that firms have commitment power, so our outcome obtains for entirely different reasons. In our model, unbridled competition arises in the limit as $z \to 0$ because good L at time τ and good H at time $(\tau + \bar{\tau})$ become perfect substitutes for all $\tau \geq 0$.[16]

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[15] By $\bar{p}^H(\tau)$, $\tau \in [0, \infty)$, we mean the right continuous extension of the pointwise limit of the discrete time price path $\{\bar{p}_t^H\}$ as $z \to 0$.

[16] In the limit one can view the firms as offering a product line with a continuum of goods, indexed by τ, which are pairwise perfect substitutes, sold and bought at discounted prices $e^{-r\tau} p^H(\tau)$ and $e^{-r\tau}p^L(\tau)$. It is immediate that with zero costs the "Bertrand" outcome $p^L \equiv 0$ and $p^H(\tau) = 0$ for $\tau \geq \bar{\tau}$ is *an* equilibrium outcome of the limit game, since no firm can unilaterally deviate and improve profits. See

Meanwhile, the most direct competition for good H at time zero comes from firm H's own sales at time τ ($0 < \tau < \bar{\tau}$). Firm H is therefore able to keep its price constant over the interval $[0, \bar{\tau})$ and earn positive profits. However, competition drives all prices $\bar{p}^H(\tau)$, $\tau \geq \bar{\tau}$ down to zero. The premium any customer q is willing to pay to purchase good H at time zero rather than waiting to obtain good H at time $\bar{\tau}$ at marginal cost, is thus limited to $(1 - e^{-r\bar{\tau}})\mu f(q)$. Note that firm H obtains monopoly profits relative to this rescaled demand function. This is because, analogous to Stokey's (1979) result, the optimally intertemporally discriminating price path over a bounded time interval $[0, \bar{\tau}]$ involves charging the monopoly price at time zero and never cutting the price thereafter. Unfortunately, when the time grid becomes arbitrarily fine the disadvantages to firm L of producing a lower quality good become extreme; while it is still able to secure some sales, those sales must occur at a price close to marginal cost, leaving no profits in the limit.

We may summarize the results of this section as follows:

Proposition 5: The limit as time becomes continuous of the unique open-loop equilibrium to the duopoly sales game (where profits and surplus are discounted at a rate r per unit of real time) has the following prices and sales:

(i) $\bar{p}^L(\tau) \equiv 0$; $\bar{p}^H(\tau) = (\mu - 1)/2$ $(0 \leq \tau < \bar{\tau})$ and $\bar{p}^H(\tau) = 0$ $(\tau \geq \bar{\tau})$,

(ii) $s^L(0) = 1/3$ and $s^L(\tau) = 0$, $(\tau > 0)$; $s^H(0) = 1/2$, $s^H(\bar{\tau}) = 1/6$ and $s^H(\tau) = 0$ $(\tau \neq 0, \bar{\tau})$,

where $\bar{\tau}$ is defined implicitly from $e^{-r\bar{\tau}}\mu = 1$.

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footnote 18, below, for an analysis of the case of positive (or even distinct) marginal cost levels. The usual Bertrand argument applied to each pair of perfect substitutes does not suffice to show that this is the unique possible equilibrium outcome. Indeed, lowering the price of an individual good in order to steal away all the consumers of the competing good results in cannibalization of the rest of the firm's product line (i.e., lowering the price at time τ may induce the firm's customers in an interval around τ to switch their purchase to time τ). A more sophisticated argument is needed in order to establish uniqueness in the limit. What we have shown above is that the "Bertrand" outcome is the unique limit of the equilibria of the discrete approximating games.

7. Endogenous Quality Choice

Consider a two-stage game in which two firms (i = 1, 2) choose the quality of their product, before playing the price-setting game described in Section 2. Assume that the choice of quality is costless, and that the quality levels μ_i must belong to a bounded interval $[0, \hat{\mu}]$. Consumers' reservation prices for quality level μ_i are given by $\mu_i(1 - q)$.

Shaked and Sutton (1982) consider a static version of this two-stage game, and argue that the unique subgame perfect equilibrium outcome in pure strategies is asymmetric. One firm chooses the highest quality level available and the other firm chooses a significantly lower quality level. It is as if both firms try to relax price competition by choosing to serve distinct market niches (selling low quality to the "poor" and high quality to the "rich"). Shaked and Sutton's result can be illustrated in our model by considering the case where $\delta = 0$.

As was argued in Section 3, when firms choose quality levels (μ^L, μ^H), prices in the resulting subgame are the net present value of the duopoly rentals:

$$(20) \qquad \bar{p}^H = \frac{2\mu(\mu - 1)}{(4\mu - 1)}\mu^L, \qquad \bar{p}^L = \frac{(\mu - 1)}{(4\mu - 1)}\mu^L,$$

where $\mu = \mu^H/\mu^L > 1$ (see also footnote 2). It is then easy to compute the respective profits:

$$(21) \qquad \Pi^H = \frac{4\mu^2(\mu - 1)}{(4\mu - 1)^2}\mu^L, \qquad \Pi^L = \frac{(\mu - 1)}{(4\mu - 1)^2}\mu^H.$$

Since $\partial\Pi^H/\partial\mu > 0$ for all μ^L, one of the firms necessarily chooses $\mu^H = \hat{\mu}$. The lower quality level must satisfy $\partial\Pi^L/\partial\mu = (4\mu - 7)\mu^H/(4\mu - 1)^2 = 0$, and so $\mu^L = (4/7)\mu^H$. This confirms the Shaked and Sutton result.

In a dynamic context, when $\delta > 0$, it remains true that firms will necessarily choose different quality levels in any (pure strategy) subgame perfect equilibrium; otherwise, pure Bertrand competition would result. However, as we observed in Section 4, in a dynamic context it is no longer true that firms can specialize in serving different market niches. Consequently, it is no longer clear that for reasonable values of the discount factor (say $\delta \geq .9$) firms will end up producing significantly different quality levels.

To investigate this interesting issue, recall that for $\delta > 0$, the equilibrium profits in the subgame where firms choose quality levels μ^H and μ^L are given by:

$$\text{(22)} \qquad \Pi^H(\mu^H, \mu^L) = \mu^L \Pi^H(\mu), \quad \Pi^L(\mu^H, \mu^L) = \mu^L \Pi^L(\mu),$$

where $\Pi^H(\mu)$ and $\Pi^L(\mu)$ are tabulated in Table 1, and $\mu = \mu^H/\mu^L \geq 1$. From Table 2, Result 9, we know that $\Pi^H(\mu)$ is increasing in μ. Just as in the static model, therefore, we may therefore conclude that in equilibrium $\mu^H = \hat{\mu}$.

Matters are more complicated for firm L since, on each interval of the form $(\delta^{1-n}, \delta^{-n})$, $\Pi^L(\mu)$ is quasiconcave with $\lim_{\mu \downarrow \delta^{1-n}} \Pi^L(\mu) = \lim_{\mu \uparrow \delta^{-n}} \Pi^L(\mu) = 0$ (see Result 8 in Table 2). This immediately implies that the low quality firm must not only avoid the Bertrand competition which arises when selecting $\mu^L = \mu^H$, but also the extreme competition and the resulting zero profit level it would obtain when selecting $\mu^L = \delta^{n-1}\mu^H$ for some $n > 1$.

To understand the nature of firm L's best response, recall from Section 4.2 that, for given value of δ, the variables λ_1, $(\mu\theta)$, and (c_1/μ) depend only on $\alpha = \delta^{n-1}\mu$. From Table 1, we see that $\Pi^L = [\alpha(1 - \delta)(\theta c_1)^2]/[(1 - \delta\alpha)(\alpha - 1)(1 - \delta\lambda_1^2)]$, so that $\Pi^L(\mu)$ depends only on α as well. Consequently, the set of profit values attainable in the interval $(\delta^{1-n}, \delta^{-n})$, $\{\Pi^L(\mu), \mu\delta^n < 1 < \mu\delta^{n-1}\}$ is constant in n. Firm L seeks to maximize $\mu^L\Pi^L(\mu)$, so that for fixed value of $\Pi^L(\mu)$ higher values of μ^L are more desirable. Any maximizing value of μ^L must thus satisfy $n = 1$ or $\delta\mu^H < \mu^L < \mu^H$; in our model firm L chooses its quality level in the interval *closest* to μ^H. Furthermore, by Result 8, Table 2 the function $\Pi^L(\mu)$ attains its maximum on the interval $(1, \delta^{-1})$ at $\mu = \delta^{-1/2}$. Hence we may conclude that the optimal response μ^L lies in the interval $(\delta^{1/2}\mu^H, \mu^H)$. As δ converges to 1, we therefore find that $\mu^L \to \mu^H$. Summarizing:

Proposition 6: When the discount factor δ converges to 1, endogenous quality choice results in minimal product differentiation, i.e., $\mu^L \to \mu^H$.

It should be observed that for small δ, the inequality $\delta^{1/2}\mu^H < \mu^L < \mu^H$ provides essentially no restriction. For $\delta = 0$, the inequality is obviously satisfied at the static maximizer $\mu^L = (4/7)\mu^H$. Numerical computations show that for each value of δ, there is a unique optimal response to $\mu^H = \hat{\mu}$. As δ increases, μ^L/μ^H increases steadily from (4/7) to 1; for δ sufficiently close to 1 ($\delta \geq .9$), the optimal response is approximately given by $\mu^L = \sqrt{\delta}\,\mu^H$.

The reason we obtain such a dramatically different result from Shaked and Sutton is that when the discount factor is sufficiently high lowering μ_L no longer unambiguously relaxes price competition: it relaxes immediate competition, but also reinforces deferred competition.

8. Conclusion

In this paper, we have analyzed a vertically differentiated durable goods duopoly with linear demand and zero marginal costs of production, and have shown that there exists a unique equilibrium in open-loop strategies in which both firms engage in intertemporal price discrimination, so that the market eventually saturates. An exogenous increase in the quality of the most desirable product has ambiguous effects on competition and, except for the introductory price of the high quality good, may end up lowering all prices in the market. Shortening the time interval between successive periods (or, equivalently, increasing the discount factor) has nonmonotonic effects on prices, but in the limit, as the time interval shrinks to zero, all prices but the introductory price of the high quality good converge to marginal cost. Finally, in the game of endogenous quality choice firms select qualities which are arbitrarily close together when the length of the time period shrinks towards zero.

How robust are these conclusions to our modeling assumptions? First, it should be observed that all of our conclusions continue to hold in an k-firm oligopoly, with k > 2. After all but a finite number of periods,[17] all firms will attain positive sales in every period, so that there is intertemporal price discrimination, and the market is eventually saturated. Shortening the time period, and allowing quality to be chosen endogenously produces identical effects to the duopoly case.

Next, let us assume that firms have a strictly positive common marginal cost of production, denoted by c, and that consumers' reservation prices are given by $c + \mu_i(1 - q)$. All of our results then continue to hold, provided p_t^H and p_t^L are reinterpreted as markups above marginal cost. More interesting changes occur when high quality goods are more expensive to produce than low quality ones, so that the marginal cost c^H exceeds c^L, and reservation prices are given by $c_H + \mu_i(1 - q)$. As before, firms will intertemporally price discriminate, but because of its cost advantage, the low quality firm will now eventually set a price which the high quality firm does not want to undercut. Both firms' sales therefore extend only over a finite (though possibly large) number of time periods, and because the low quality firm enjoys a monopoly over the lowest valuation customers, the market does not fully saturate. An increase in the quality of good H again has ambiguous effects on competition, but tends to increase market penetration. Shortening the period length has a nonmonotonic effect on pricing, but in the limit the low quality firm will set a price strictly below c^H, so that all sales occur in the initial period.[18]

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[17]The time pattern of sales of the different quality levels μ_i is distributed across consumers' types according to the ranking of numbers $\delta^t\mu_i$ ($i = 1, ..., k$ and $t = 0, 1, ...$). Thus, if $\delta^r\mu_i < \delta^s\mu^j$, sales of good j in period s are to higher valuation consumers than sales of good i in period r. All sales levels will be positive, except possibly those for the highest quality firm during periods when it faces no direct competition. More precisely, if we let $\mu_1 > \mu_2 > \mu_k$, and define n to be the smallest integer such that $\delta^n\mu_1 < \mu_2$, then $s_t^H = 0$ for $t = 1, ..., (n - 2)$ if $n > 2$, and $s_t^H > 0$ otherwise.

[18]Interestingly, the limiting outcome is somewhat different in the model where the low quality firm continues to have a cost advantage, but where reservation prices are given by $\mu_i(1 - q)$, as in the zero marginal cost case analyzed in the main body of the paper. The important comparison then concerns the *discounted cost* of firm H versus the cost of firm L, i.e., $e^{-r\bar{\tau}}c_H = c_H/\mu$ versus c_L. If $c^H\mu > c^L$, then in the limit firm L sets an introductory price $\bar{p}^L(0)$ which barely undercuts c^H/μ. All consumers will then prefer to purchase good L at time 0, when offered at the price $\bar{p}^L(0)$, to waiting to purchase good H at time $\bar{\tau}$, when offered at the price c^H. Consequently, all sales will occur at time zero. Using reasoning analagous to the zero cost case, it can be shown that $s^H(0) = s^L(0) = (1 - (c^H/\mu))/2$ and $\bar{p}^H(0) = (c^H/\mu) + [(\mu - 1)(1 + (c^H/\mu))/2]$. On the other hand, when $c^H/\mu < c^L$, firm H sets $\bar{p}^H(\tau) = \mu C_L - \epsilon$ for all τ

If marginal production costs were identical, but strictly increasing, firms would sell in every period beyond their initial period of sales, due to the desire to smooth production over time. If, furthermore, those costs were independent of the length of the time period, the equilibrium would display the same limiting features as in the constant marginal cost case, with the initial price of the low quality firm converging to the (common) minimum marginal cost level.

Our demand formulation embedded two assumptions: perfect correlation between the reservation prices for the high and the low quality good, and linearity of demand, i.e., $f(q) = 1 - q$. The latter assumption can easily be relaxed, for as long as an interior equilibrium exists, our qualitative results will continue to hold. Without the perfect correlation, it will still be true that the equilibrium exhibits intertemporal price discrimination, and that the market is fully penetrated. Increasing the quality of the superior product will a priori have ambiguous effects, for both the immediate and the deferred competition effect will still be present. However, it will no longer be true that extreme competition arises when the length of the time interval is shrunk towards zero: both firms may now have limiting price paths which smoothly converge towards zero. Nevertheless, a continuity result holds: when the correlation between valuations is strong, the equilibrium outcome will be close to the one characterized in this paper. Consequently, the game of endogenous quality choice will still yield qualities which are close (though not arbitrarily close) together.

The model analyzed in this paper extends the marketing literature on the diffusion of consumer durables (see, e.g., Eliasberg and Jeuland (1986)) by allowing consumers to anticipate further price decreases rather than having them purchase myopically. However, unlike those marketing models, we assume that consumers are perfectly informed about the existence and desirability of all the goods right from the start of the game. We believe this is the main reason our model produces a concave diffusion

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$\geq \bar{\tau}$. All consumers will therefore prefer to purchase good H at time $\bar{\tau}$ to purchasing good L at time zero, when offered at the price c^L, so that firm L is priced entirely out of the market. The reason that firm L is unable to compete--despite its cost advantage--is that consumers compare discounted prices when choosing between H at time $\bar{\tau}$ and L at time 0. This in effect gives the high quality firm a cost advantage proportional to $(1/\mu)$.

curve, rather than the S–shaped curve that appears often in empirical studies of the diffusion process. An interestion extension of our work would be to model the process of information diffusion in the context of a fully optimizing framework.

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Appendix A

Proof that $0 < \lambda_1(n) < 1 < \lambda_2(n)$: Let $a_1 = \delta(\delta^{n-1}\mu - 1)^2$, $a_2 = (1 - \delta^n\mu)^2$ and $a_3 = 4\mu\delta^{n-1}(1 - \delta)^2$. Then the discriminant of the quadratic equation (16) is:

$$D = (a_1 + a_2 - a_3)^2 - 4a_1a_2 = (a_1 - a_2)^2 + a_3(a_3 - 2(a_1 + a_2)).$$

We will first show that $D > 0$, implying that both roots of the characteristic equation are real. Define $\alpha = \delta^{n-1}\mu$; by assumption $1 < \alpha < \delta^{-1}$. To argue that $D > 0$, it will suffice to show that $\phi(\alpha) \equiv 2(a_1 + a_2) - a_3 = \alpha^2\delta(1 + \delta) - 2\alpha(1 + \delta^2) + (1 + \delta) < 0$. Observe that the quadratic $\phi(\alpha)$ has $\phi(1) = -(1 - \delta)^2 < 0$ and $\phi(\delta^{-1}) = -(1 - \delta)^2\delta^{-1} < 0$. Since $\phi(\alpha) \to \infty$ as $\alpha \to \pm\infty$, we conclude that $\phi(\alpha) < 0$ everywhere on $[1, \delta^{-1}]$.

Now $\lambda_1(n)\lambda_2(n) = \delta^{-1}$ and $\lambda_1(n) + \lambda_2(n) = -\omega(n)/[2\delta(\delta^{n-1}\mu - 1)(1 - \delta^n\mu)]$. Since $\lambda_1(n)\lambda_2(n) > 0$, both roots must have the same sign; to show that $\lambda_1(n) > 0$ it suffices to observe that $\omega(n) = a_1 + a_2 - a_3 = \phi(\alpha) - (a_1 + a_2) < 0$. Moreover, $\lambda_1(n)\lambda_2(n) = \delta^{-1} > 1$ and $\lambda_1(n) < \lambda_2(n)$ imply that $\lambda_2(n) > 1$. Denote the quadratic in (16) as $h(\lambda)$. Since the coefficient of λ^2 in $h(\lambda)$ is positive, if both $\lambda_1(n)$ and $\lambda_2(n)$ exceeded 1, we would have $h(1) > 0$. However, $h(1) = (\delta^{n-1}\mu - 1)(1 - \delta^n\mu) + \omega(n) + \delta(\delta^{n-1}\mu - 1)(1 - \delta^n\mu) = -3\alpha(1 - \delta)^2 < 0$. We conclude that $\lambda_1(n) < 1$. ■

Proof that $c_1(n) > 0$: Since $\lambda_1(n) < 1$, $\theta(n) = \dfrac{(1 - \delta^n\mu) + \delta(\delta^{n-1}\mu - 1)\lambda_1}{2\mu(1 - \delta)} < \dfrac{1}{2\mu}$. Consequently, $2\delta^{n-1} - \theta(n) > \{4\mu\delta^{n-1} - 1\}/2\mu > (3/2)\mu > 0$, implying $c_1(n) > 0$. ■

Appendix B

Proof of Proposition 2: We have already shown that in any equilibrium, consumer self-selection implies that the pattern of sales over time is as depicted in Figure 1. The next lemma shows that all sales in this sequence must be strictly positive; consequently all equilibria must be interior.

Lemma 1: In any equilibrium, $s_t^H > 0$ and $s_t^L > 0$ for all $t \geq 0$.

Consider the set $Z_+ \times \{H, L\}$ and endow it with the following total order: $(t, i) < (t', j)$ if either $t < t'$ or $t = t'$ and $i = H$, $j = L$. This order represents the natural order of the sales pattern depicted in Figure 1. Note that in this order, no optimizing firm would want to be the first one to set its price equal to zero. Consequently, in equilibrium the market is never saturated in finite time.

Suppose now that, contrary to the statement of the lemma, $A = \{(\tau, j): s_\tau^j = 0\} \neq \phi$. Let $(t, i) = \min A$. Then if either $B = \{(\tau, k) > (t, i): s_\tau^k > 0\} = \phi$ or $\min B = (t', j)$ with $j \neq i$, firm i can lower its price in period t slightly and attract positive sales in that period without affecting its sales in any other period (only firm $j \neq i$ loses sales in period t' and in period t (if $i = L$) or period $(t - 1)$ (if $i = H$)). Suppose on the other hand that $\min B = (t', i)$. Then $s_\tau^k = 0$ for all $(t, i) \leq (\tau, k) < (t', i)$. We will now show that firm i could profitability lower its price in period t. Either way, we will have shown that the assumption $A \neq \phi$ yields a contradiction to equilibrium.

Let $\hat{q} < 1$ be the consumer who is indifferent between purchasing good i in period t' and good j in period t (if $i = L$) or period $(t - 1)$ (if $i = H$). Note that such a consumer exists, since all sales before (t, i) are assumed positive, and nobody buys between (t, i) and (t', i). Let us lower p_t^i to the point where $\hat{q}$ is indifferent between either of the above options and purchasing good i in period t; refer to this price as $\hat{p}_t^i$. Marginally lowering p_t^i below $\hat{p}_t^i$ will affect firm i's profit only in so far as it affects $\hat{\pi}^i = \pi_t^i + \delta^{(t'-t)}\pi_{t'}^i$. However, at $p_t^i = \hat{p}_t^i$, $\partial\hat{\pi}^i/\partial p_t^i = s_t^i + (\hat{p}_t^i)\partial s_t^i/\partial p_t^i + \delta^{t'-t}p_{t'}^i\partial s_{t'}^i/\partial p_t^i = (-\hat{p}_t^i + \delta^{t'-t}p_{t'}^i)\partial s_{t'}^i/\partial p_t^i + \hat{p}_t^i[\partial s_t^i/\partial p_t^i + \partial s_{t'}^i/\partial p_t^i]$. The term multiplying $\hat{p}_t^i$ is strictly negative, since it represents the impact upon the left market boundary of firm i in period t. Furthermore, since $\hat{q}$ is indifferent between i at t' and j at t, $(\hat{p}^i_t - \delta^{t'-t}p^i_{t'}) > 0$. We conclude that $\partial\hat{\pi}^i/\partial p^i_t < 0$, so that firm i can improve its profits by slightly lowering p^i_t. ■

Since any equilibrium must be interior, and since (7)-(8) is the unique solution to the interior first-order conditions, no equilibria other than (7)-(8) can exist. We now turn to the question of existence of equilibrium, i.e., whether the profiles (7)-(8) are optimal responses to each other. Lemmas 2 and 3 below establish that if firm H, in optimally responding to (8), decides to wipe out firm L's sales in a finite number of periods t ... t', then firm H's sales in periods (t + 1) ... t' are necessarily equal to zero. Similarly, if firm L, in optimally responding to (7), decides to wipe out firm H's sales in a finite number of periods t ... t', then firm L's sales in periods t ... (t' - 1) are necessarily equal to zero. In Lemma 4, we then use this information on the structure of optimal deviations to argue that no firm ever finds it optimal to wipe out its opponent's sales in a finite number of periods. Lemma 5 shows that wiping out an opponent's sales in an infinite number of periods is not profitable either.

Lemma 2: Consider any optimal response of firm i such that $s^i_t > 0$ and $s^i_{t'} > 0$, but $s^j_\tau = 0$ for all $(t, i) < (\tau, j) < (t', i)$ with $j \neq i$. Then $s^i_\tau = 0$ for all $t < \tau < t'$.

Suppose to the contrary that there exists $t \leq t_1 < t_2 < t_3 \leq t'$ such that $s^i_\tau > 0$ for $\tau = t_1, t_2, t_3$ but $s^i_\tau = 0$ for $t_1 < \tau < t_2$ and $t_2 < \tau < t_3$. Let q_1 be the consumer who is indifferent between purchasing good i in periods t_1 and t_2, i.e. $f(q_1) = (p^i_{t_1} - \delta^\ell p^i_{t_2})/[\mu_i(1 - \delta^\ell)])$, where $\ell = t_2 - t_1$ and $\mu_i = \mu$ if i = H and $\mu_i = 1$ if i = L. Similarly, let q_2 be the consumer who is indifferent between purchasing i in periods t_2 and t_3, i.e. $f(q_2) = (p^i_{t_2} - \delta^r p^i_{t_3})/[\mu_i(1 - \delta^r)]$, where $r = t_3 - t_2$. Since $q_2 > q_1$, marginally increasing $p^i_{t_2}$ will affect firm i's profits only through $\hat{\pi}^i = p^i_{t_1}s^i_{t_1} + \delta^\ell p^i_{t_2}s^i_{t_2} + \delta^{\ell+r}p^i_{t_3}s^i_{t_3}$. Now:

$$\frac{\partial\hat{\pi}^i}{\partial p^i_{t_2}} = \frac{\delta^\ell v^i}{\mu_i(1 - \delta^\ell)(1 - \delta^r)} + \delta^\ell s^i_{t_2}$$

where $v^i = (1 - \delta^r)p^i_{t_1} - [(1 - \delta^\ell) + \delta^\ell(1 - \delta^r)]p^i_{t_2} + \delta^r(1 - \delta^\ell)p^i_{t_3}$. Note that $q_2 \gtrless q_1$ iff $v^i \gtrless 0$. Therefore, $\partial\hat{\pi}_i/\partial p^i_{t_2} > 0$ if $s^i_{t_2} > 0$, which implies firm i will not sell in period t_2. ∎

Lemma 3: Consider any optimal response of firm i such that *either* $s^j_t > 0$ and $s^j_{t'} > 0$, but $s^j_\tau = 0$ for all $t < \tau < t'$, *or* $s^j_\tau = 0$ for all $0 \leq \tau < t'$ and $s^j_{t'} > 0$. In the former case, define $C(i) = \{\tau: (t, j) < (\tau, i) < (t', j)\}$; in the latter case define $C(i) = \{\tau: (\tau, i) < (t', j)\}$. Let $t_1 = \min C(i)$ and $t_2 = \max C(i)$. Then $s^i_{t_1} > 0$ and $s^i_\tau = 0$ for all $t_1 < \tau < t_2$.

Clearly, $s^i_\tau = 0$ for all $t_1 \leq \tau \leq t_2$ is not optimal, for firm i could set $p^i_\tau = \bar{p}^i_\tau$ $(\tau = t_1, ..., t_2)$ and make positive sales without affecting the sales in any other period. By Lemma 2, there are at most two periods t_1' and t_2' such that $t_1 \leq t_1' \leq t_2' \leq t_2$ and such that $s^i_{t_1'} > 0$ and $s^i_{t_2'} > 0$. Suppose now that $t_1' \neq t_1$; define $\hat{p}^i_{t_1}$ to be the minimal price such that $s^i_{t_1} = 0$ (keeping all other prices constant). Then by marginally lowering $p^i_{t_1}$ below $\hat{p}^i_{t_1}$ firm i increases sales in period t_1; furthermore, any customers who switch from buying in period t_1' to period t_1 now buy earlier and at a higher price. Since this marginal deviation would be profitable, we conclude $s^i_{t_1} > 0$. Next, suppose $s^i_{t_2'} > 0$ for some $t_1 < t_2' < t_2$. Define $\hat{p}^i_{t_2}$ to be the minimal price such that $s^i_{t_2} = 0$ (holding all other prices fixed). By marginally increasing $p^i_{t_2'}$, firm i will have positive sales in periods t_1, t_2' and t_2, but not in any other intervening period. Lemma 2 now implies that it is optimal to keep on increasing $p^i_{t_2'}$ (until either $s^i_{t_2'} = 0$ or s^i_τ becomes positive for some $t < \tau < t'$). ∎

Lemma 4: Consider any response of firm i to the profile $\{\bar{p}^j_t\}^\infty_{t=0}$, $j \neq i$, such that *either* $s^j_t > 0$ and $s^j_{t'} > 0$ but $s^j_\tau = 0$ for all $t < \tau < t'$, *or* $s^j_\tau = 0$ for all $0 \leq \tau < t'$ and $s^j_{t'} > 0$. Then firm i can improve its profits by replacing p^i_τ with $\bar{p}^i_\tau$, for all $t_1 \leq \tau \leq t_2$.

By Lemma 3, we know that firm i cannot be optimizing unless $\pi^i_\tau = 0$ for all $t_1 < \tau < t_2$. Consequently, since $s^j_\tau = 0$ for all $t < \tau < t'$, changing p^i_τ to $\bar{p}^i_\tau$ for all $t_1 < \tau < t_2$ will not affect any of the π^i_τ, for $t_1 \le \tau \le t_2$. It will now suffice to show that changing $p^i_{t_1}$ to $\bar{p}^i_{t_1}$ and $p^i_{t_2}$ to $\bar{p}^i_{t_2}$ will increase $\pi^i_{t_1}$ and $\pi^i_{t_2}$.

Note that since $s^j_\tau = 0$ for $t < \tau < t'$, at least one of the inequalities $p^i_{t_1} < \bar{p}^i_{t_1}$ and $p^i_{t_2} < \bar{p}^i_{t_2}$ must hold. Suppose first that both inequalities hold. Observe that $\pi^i_{t_1}(p^i_{t_1}, p^i_{t_2}) \le \pi^i_{t_1}(p^i_{t_1}, \bar{p}^i_{t_2}) < \pi^i_{t_1}(\bar{p}^i_{t_1}, \bar{p}^i_{t_2})$, where the dependence on the prices $\bar{p}^i_\tau$, $t_1 < \tau < t_2$ and $\{\bar{p}^j_t\}^\infty_{t=0}$ has been suppressed in the notation. The first inequality holds because raising $p^i_{t_2}$ can only increase sales in period t_1. The second inequality holds because when in period t_1 firm i is competing against the upper envelope of the utility provided by its own prices in periods other than t_1 and by its opponent's prices, it can make no more profit than when just competing against the upper envelope of the utility provided by $\bar{p}^j_t$ and $\bar{p}^j_{t+1}$ (or, if $i = H$ and $t_1 = 0$, the envelope provided by $\bar{p}^j_{t_1}$), and because the profit function in competing against these prices is concave and uniquely maximized at $p^i_{t_1}$ (see Figure 4). Analogous reasoning shows that $\pi^i_{t_2}(p^i_{t_1}, p^i_{t_2}) \le \pi^i_{t_2}(\bar{p}^i_{t_1}, p^i_{t_2}) < \pi^i_{t_2}(\bar{p}^i_{t_1}, \bar{p}^i_{t_2})$.

Next, consider the case where $p^i_{t_1} < \bar{p}^i_{t_1}$ but $p^i_{t_2} \ge \bar{p}^i_{t_2}$. Let us first increase $p^i_{t_1}$ to $\bar{p}^i_{t_1}$. Since this can only increase sales in period t_2, we have $\pi^i_{t_2}(p^i_{t_1}, p^i_{t_2}) \le \pi^i_{t_2}(\bar{p}^i_{t_1}, p^i_{t_2})$. Furthermore, since $p^i_{t_2} \ge \bar{p}^i_{t_2}$, the profits to firm i in period t_1 from competing against the upper envelope of the utility provided by $\bar{p}^j_t$ and $\bar{p}^j_{t+1}$ exceed the profits from competing against the upper envelope of the utility provided by $(\bar{p}^j_t, \ldots, \bar{p}^j_{t'})$ and $(\bar{p}^i_{t_1+1}, \ldots, \bar{p}^i_{t_2-1}, \bar{p}^i_{t_2})$. Since the profits from competing against $\bar{p}^j_t$ and $\bar{p}^j_{t+1}$ are maximized at $\bar{p}^i_{t_1}$, we have $\pi^i_{t_1}(p^i_{t_1}, p^i_{t_2}) < \pi^i_{t_1}(\bar{p}^i_{t_1}, p^i_{t_2})$. Next let us lower $p^i_{t_2}$ to $\bar{p}^i_{t_2}$. Then, since $p^i_{t_2} \ge \bar{p}^i_{t_2}$, $\pi^i_{t_1}(\bar{p}^i_{t_1}, p^i_{t_2}) = \pi^i_{t_1}(\bar{p}^i_{t_1}, \bar{p}^i_{t_2})$, and as before $\pi^i_{t_2}(\bar{p}^i_{t_1}, p^i_{t_2}) < \pi^i_{t_2}(\bar{p}^i_{t_1}, \bar{p}^i_{t_2})$. The case where $p^i_{t_1} \ge \bar{p}^i_{t_2}$ and $p^i_{t_2} < \bar{p}^i_{t_2}$ is entirely analogous. ■

From Lemmas 3 and 4 we may conclude that if in any optimal response of firm i, $s^j_{t'} > 0$ for some $t' > 0$, then $p^i_t = \bar{p}^i_t$ for all $t \le t_2(t')$. Our final lemma deals with the remaining possible deviations: firm i wipes out all of firm j's sales beyond some finite period t.

Lemma 5: In responding to $\{\bar{p}^j_t\}_{t=0}^{\infty}$ firm i cannot increase profits above the candidate equilibrium level $\bar{\Pi}^i$ by choosing a price sequence $\{p^i_t\}_{t=0}^{\infty}$ such that $s^j_\tau = 0$ for all $\tau \geq t$, for some $0 \leq t < \infty$.

Suppose to the contrary that $s^j_\tau = 0$ for all $\tau \geq t$, but that either $s^j_{t-1} > 0$ or $t = 0$ in some strictly improving response. If $t > 0$ define t_3 such that $(t - 1, j) < (t_3, i) < (t, j)$; if $t = 0$ define $t_3 = 0$. Since we have shown that $p^i_\tau = \bar{p}^i_\tau$ for all $\tau < t_3$, necessarily

$$\text{(B.1)} \qquad \sum_{\ell=0}^{\infty} \delta^\ell \, \bar{\pi}^i_{t_3+\ell} < \sum_{\ell=0}^{\infty} \delta^\ell \, \pi^i_{t_3+\ell}.$$

Since $\pi^i_{t_3+\ell} \leq \mu$ for all $\ell \geq 0$, there therefore exists a $K < \infty$ such that:

$$\text{(B.2)} \qquad \sum_{\ell=0}^{\infty} \delta^\ell \, \bar{\pi}^i_{t_3+\ell} < \sum_{\ell=0}^{K} \delta^\ell \, \pi^i_{t_3+\ell}.$$

Now raise all prices in periods $\tau > t_3 + K$ to μ, and denote the resulting prices and profits by $\tilde{p}^i_\tau$ and $\tilde{\pi}^i_\tau$. Since sales in periods $\tau \leq t_3 + K$ can only increase, we have:

$$\text{(B.3)} \qquad \sum_{\ell=0}^{K} \delta^\ell \, \pi^i_{t_3+\ell} \leq \sum_{\ell=0}^{K} \delta^\ell \, \tilde{\pi}^i_{t_3+\ell}.$$

Observe that under $\tilde{p}^i_\tau$ there exists $t' \geq t_3 + K$ such that $s^j_\tau = 0$ for all $t < \tau < t'$, but $s^j_{t'} > 0$. Lemmas 3 and 4 now imply that

$$\text{(B.4)} \qquad \sum_{\ell=0}^{K} \delta^\ell \, \tilde{\pi}^i_{t_3+\ell} \leq \sum_{\ell=0}^{t'-t_3} \delta^\ell \tilde{\pi}^i_{t_3+\ell} < \sum_{\ell=0}^{t'-t_3} \delta^\ell \, \bar{\pi}^i_{t_3+\ell}.$$

It now suffices to observe that the sequence of inequalities (B.2)-(B.4) yields a contradiction. ■

Proof of Proposition 3: For n such that $\delta^{n-1}\mu < 1 < \delta^n\mu$, define the following order on the set $Z_+ \times$

{H, L}: (t, H) < (t + 1, H), for all $0 \leq t < n - 1$, and (t + n - 1, H) < (t, L) < (t + n, H), for all $t \geq 0$. Note that this order represents the natural order of the sales pattern implied by consumer self-selection. First, we note that Lemma 2 and Lemma 3 hold as stated, with identical proofs. This immediately implies that in any equilibrium $s_0^H > 0$ and $s_1^H = \ldots = s_{n-2}^H = 0$. Consequently, we may state our uniqueness result as:

<u>Lemma 1'</u>: In any equilibrium, $s_t^L > 0$ for all $t \geq 0$, $s_t^H > 0$ for $t = 0$ and all $t \geq n - 1$, and $s_1^H = \ldots = s_{n-2}^H = 0$.

We have already argued that $s_0^H > 0$ and $s_1^H = \ldots = s_{n-2}^H = 0$. The proof that $s_t^H > 0$ for $t > n - 1$ and $s_t^L > 0$ for $t > 0$ now follows the proof of Lemma 1, regardless of whether $s_{n-1}^H > 0$ or $s_{n-1}^H = 0$ (if $s_{n-1}^H = 0$, just delete (n - 1, H) from the definition of the set A). It remains to be shown that $s_{n-1}^H > 0$. First, if it were optimal to have $s_{n-1}^H > 0$, then given any $p_0^L \geq 0$, equations (12) and (13) determine the unique interior solution: $p_0^H = ((\mu - 1) + p_0^L)/2$ and $p_{n-1}^H = ((\mu\delta^{n-1} - 1) + p_0^L)/(2\delta^{n-1})$. If, on the other hand, it were optimal to have $s_{n-1}^H = 0$, then good H at time zero competes directly with good L at time zero, so that $\pi_0^H = p_0^H[1 - (p_0^H - p_0^L)/(\mu - 1)]$. Maximizing the latter expression with respect to p_0^H yields, as before, $p_0^H = ((\mu - 1) + p_0^L)/2$. Since the optimal p_0^H is identical in both cases, and since $\pi_0^H + \delta^{n-1}\pi_{n-1}^H$ is strictly concave in p_{n-1}^H on the domain where $s_{n-1}^H > 0$, we conclude that profits at the interior solution dominate profits at the boundary solution. Consequently $s_{n-1}^H > 0$. ∎

Lemma 4 remains true, except when $t_1 = t_2$ (i.e., j = H, t = 0 and t' = n - 1), in which case no improvement is possible. The proof is identical, except for the fact that the upper envelope of utility provided by all prices except p_t^j is always replaced by the upper envelope of the utility of the two nearest neighbors in the initial sequence of Figure 2. Lemma 5 holds verbatim, which an identical proof; hence we conclude that (15) and (17) do indeed constitute an equilibrium.

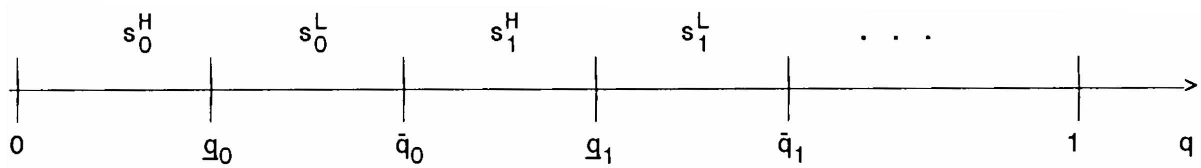


Figure 1

The Time Pattern of Sales for the Case $\delta\mu < 1 < \mu$
(Picture Not Drawn to Scale)

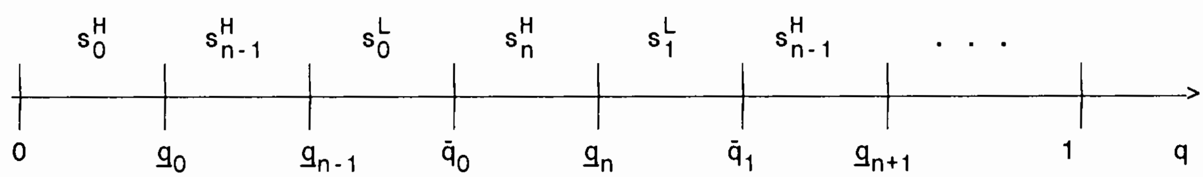


Figure 2

The Time Pattern of Sales for the Case $\delta^n\mu < 1 < \delta^{n-1}\mu$ (n > 1)
(Picture Not Drawn to Scale)

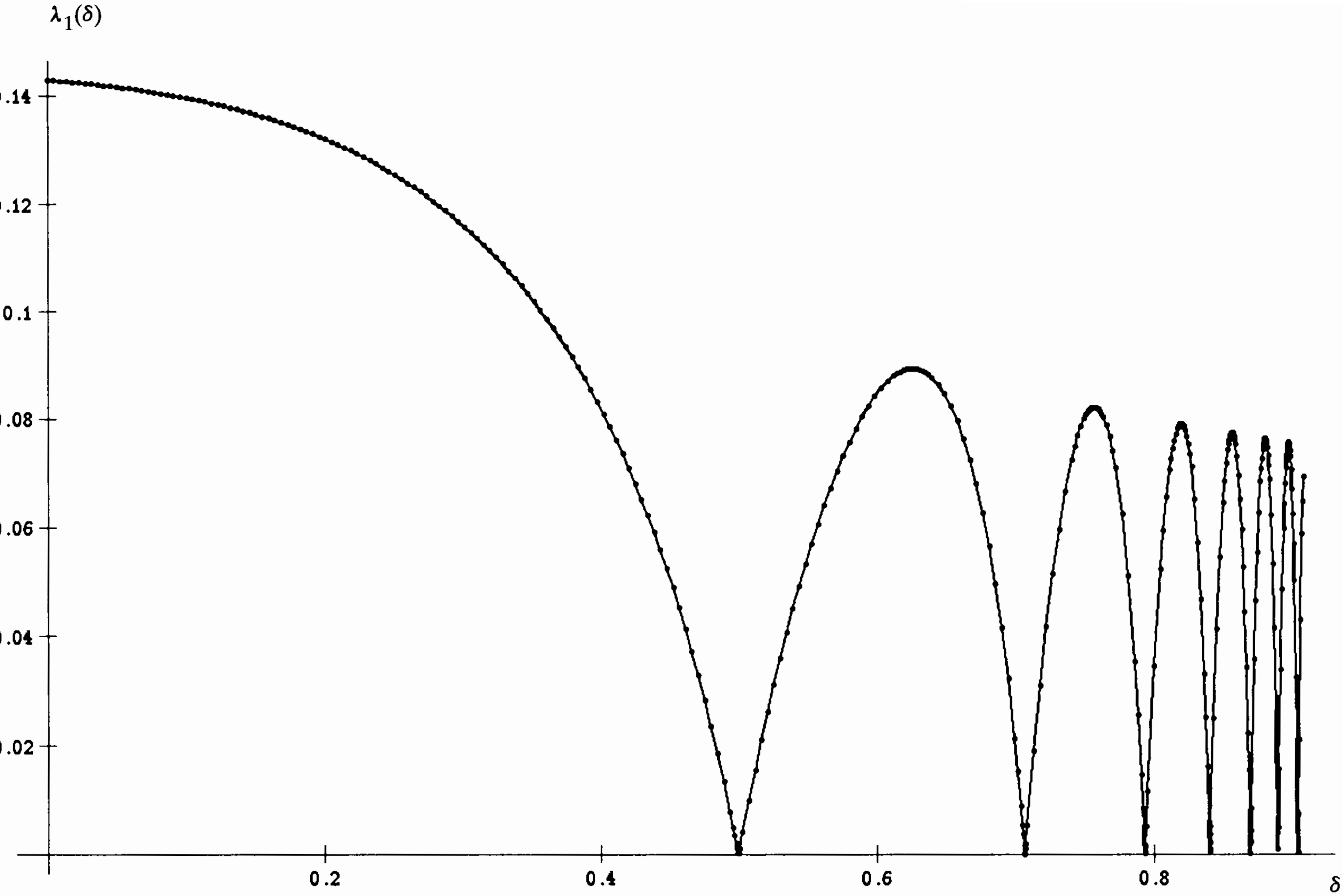
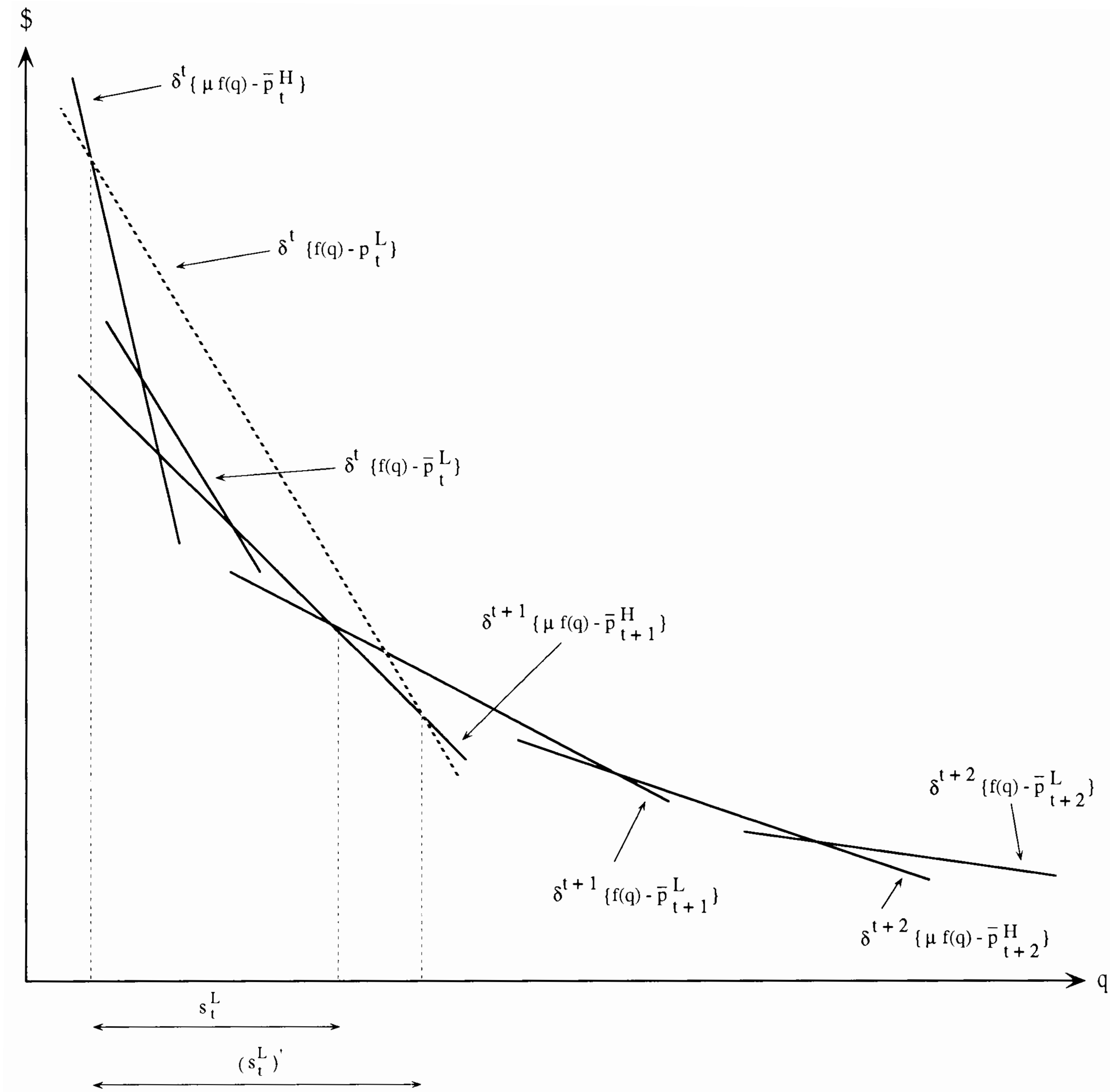


Figure 3

The Behaviour of λ_1 as a Function of the Discount Factor δ.

(the figure was drawn for the value $\mu = 2$)



<u>Figure 4</u>

When competing against the upper enveloppe of utility provided $\bar{p}^H_t$ and $\bar{p}^H_{t+1}$, firm L's sales, $(s^L_t)'$, exceed its sales from competing against the upper enveloppe of utility provided by all $\bar{p}^H_{t+l}$ and $\bar{p}^L_{t+l+1}$ for $l \geq 0$, s^L_t.

Table 1: The Equilibrium Variables in the Sales Model

$\hat{p}_0^H$	$(1-\delta^{n-1})(\mu/2)+\delta^{n-1}c_1(n)$
$\hat{p}_t^H,\ t \geq n-1$	$c_1(n)\lambda_1^{t-(n-1)}(n)$
$\hat{p}_t^L,\ t \geq 0$	$\theta(n)c_1(n)\lambda_1^t(n)$
s_0^H	$1/2$
s_{n-1}^H	$\frac{1}{2}-\frac{\delta^{n-1}-\theta(n)}{\mu\delta^{n-1}-1}c_1(n)=\frac{\theta(n)}{2(2\delta^{n-1}-\theta(n))}$
$s_t^H,\ t \geq n$	$\frac{\delta^{n-1}(1-\delta)}{(1-\delta^n\mu)(\delta^{n-1}\mu-1)}c_1(n)\lambda_1^{t-(n-1)}(n)$
$s_t^L,\ t \geq 0$	$\frac{\mu\delta^{n-1}(1-\delta)}{(1-\delta^n\mu)(\delta^{n-1}\mu-1)}\theta(n)c_1(n)\lambda_1^t(n)$
π_0^H	$(1-\delta^{n-1})\frac{\mu}{4}+\frac{\delta^{n-1}c_1(n)}{2}$
π_{n-1}^H	$\frac{\theta(n)c_1(n)}{2(2\delta^{n-1}-\theta(n))}$
$\pi_t^H,\ t \geq n$	$\frac{\delta^{n-1}(1-\delta)}{(1-\delta^n\mu)(\delta^{n-1}\mu-1)}\left[c_1(n)\lambda_1^{t-(n-1)}(n)\right]^2$
$\pi_t^L,\ t \geq 0$	$\frac{\mu\delta^{n-1}(1-\delta)}{(1-\delta^n\mu)(\delta^{n-1}\mu-1)}\left[\theta(n)c_1(n)\lambda_1^t(n)\right]^2$
Π^H	$\left\{(1-\delta^{n-1})\frac{\mu}{4}+\frac{\delta^{n-1}c_1(n)}{2}\right\}+\delta^{n-1}\left\{\frac{\theta(n)c_1(n)}{2(2\delta^{n-1}-\theta(n))}\right\}+\frac{\delta^{2n-1}(1-\delta)c_1^2(n)\lambda_1^2(n)}{(1-\delta^n\mu)(\delta^{n-1}\mu-1)(1-\delta\lambda_1^2(n))}$
Π^L	$\frac{\delta^{n-1}\mu(1-\delta)}{(1-\delta^n\mu)(\delta^{n-1}\mu-1)(1-\delta\lambda_1^2)}[\theta(n)c_1(n)]^2$
$S_t^H,\ t \geq n-1$	$\frac{\delta^{n-1}c_1(n)}{\delta^{n-1}\mu-1}\left\{1+\frac{(1-\delta)}{(1-\delta^n\mu)}\frac{\lambda_1(n)(1-\lambda_1^{t-(n-1)}(n))}{1-\lambda_1(n)}\right\}$
$S_t^L,\ t \geq 0$	$\frac{\delta^{n-1}\mu(1-\delta)\theta(n)c_1(n)}{(\delta^{n-1}\mu-1)(1-\delta^n\mu)}\frac{(1-\lambda_1^{t+1}(n))}{(1-\lambda_1(n))}$
m_0^L	$\frac{2\delta^{n-1}\mu(1-\delta)\theta(n)c_1(n)}{(\delta^{n-1}\mu-1)(1-\delta^n\mu)}$
$m_t^L,\ 1 \leq t \leq n-2$	∞
m_{n-1}^L	$\frac{2\delta^{n-1}\mu(1-\delta)}{(1-\delta^n\mu)}\lambda_1^{n-1}(n)$
$m_t^L,\ t \geq n$	$\mu\theta(n)\lambda_1^{n-1}(n)$

All formulae in the table are valid when $n = 1$, with the following exceptions: $s_0^H = (s_0^H + s_{n-1}^H)|_{n=1} = 1/(2-\theta)$ and $\pi_0^H = (\pi_0^H + \pi_{n-1}^H)|_{n=1} = c_1/(2-\theta)$.

Table 2: Comparative Dynamics Results for $\delta^n\mu < 1 < \delta^{n-1}\mu$

$\lambda_1(n) = \bar{p}^L_{t+1}/\bar{p}^L_t,\ t \geq 0, \quad = s^L_{t+1}/s^L_t,\ t \geq 0$ $= \bar{p}^H_{t+1}/\bar{p}^H_t,\ t \geq n-1, \quad = s^H_{t+1}/s^H_t,\ t \geq n.$ <u>Result 1</u>: $\lambda_1(\mu)$ is a quasiconcave function with $\lambda_1(\delta^{-n+1}) = \lambda_1(\delta^{-n}) = 0$, attaining a unique maximum at $\mu = \delta^{(1/2)-n}$.
$\theta(n)c_1(n) = \bar{p}^L_0.$ <u>Result 2</u>: $(\theta c_1)(\mu)$ is a quasiconcave function of μ with $(\theta c_1)(\delta^{-n+1}) = (\theta c_1)(\delta^{-n}) = 0$, attaining a unique maximum at $\mu = \delta^{(1/2)-n}$.
$c_1(n) = \bar{p}^H_{n-1}.$ <u>Result 3</u>: $c_1(\mu)$ is an increasing function of μ, with $\lim_{\mu\downarrow\delta^{-n+1}} c_1(\mu) = 0$, $\lim_{\mu\uparrow\delta^{-n}} c_1(\mu) = \delta^{-n}(1-\delta)/2.$
$(1-\delta^{n-1})(\mu/2) + \delta^{n-1}c_1(n) = \bar{p}^H_0.$ <u>Result 4</u>: $\bar{p}^H_0(\mu)$ is an increasing function of μ, with $\bar{p}^H_0(\delta^{-n+1}) = (\delta^{-n+1} - 1)/2$ and $\bar{p}^H_0(\delta^{-n}) = (\delta^{-n} - 1)/2.$
$\theta(n) = \bar{p}^L_t/\bar{p}^H_{t+(n-1)},\ t \geq 0.$ <u>Result 5</u>: $\theta(\mu)$ is a decreasing function of μ, with $\lim_{\mu\downarrow\delta^{-n+1}} \theta(\mu) = \delta^{n-1}/2$ and $\lim_{\mu\uparrow\delta^{-n}} \theta(\mu) = 0.$
$\mu\theta(n) = s^L_t/s^H_{t+(n-1)},\ t \geq 1.$ <u>Result 6</u>: $(\mu\theta)(\mu)$ is a decreasing function of μ, with $\lim_{\mu\downarrow\delta^{-n+1}} (\theta\mu)(\mu) = 1/2$ and $\lim_{\mu\uparrow\delta^{-n}} (\theta\mu)(\mu) = 0.$
$\mu\theta^2(n) = \pi^L_t/\pi^H_{t+(n-1)},\ t \geq 1.$ <u>Result 7</u>: $(\mu\theta^2)(\mu)$ is a decreasing function of μ, with $\lim_{\mu\downarrow\delta^{-n+1}} (\mu\theta^2)(\mu) = 4\delta^{1-n}$ and $\lim_{\mu\uparrow\delta^{-n}} (\mu\theta^2)(\mu) = 0.$
$\Pi^L.$ <u>Result 8</u>: $\Pi^L(\mu)$ is a quasiconcave function of μ, with $\Pi^L(\delta^{-n+1}) = \Pi^L(\delta^{-n}) = 0$, attaining a unique maximum at $\mu = \delta^{(1/2)-n}$.
$\Pi^H.$ <u>Result 9</u>: $\Pi^H(\mu)$ is an increasing function of μ, with $\Pi^H(\delta^{-n}) = (\delta^{-n+1} - 1)/4$ and $\Pi^H(\delta^{-n}) = (\delta^{-n} - 1)/4.$

Observe that on the interval $\mu \in (\delta^{1-n}, \delta^{-n})$ the function $n(\mu)$ is constant. We abuse notation somewhat by denoting the dependence of the solutions λ_1, c_1, etc., on the variable μ as $\lambda_1(\mu)$, $c_1(\mu)$, ...

Table 3: Limiting Values of Selected Equilibrium Variables ($\delta^n\mu < 1 < \delta^{n-1}\mu$)

Variables	Limit as $\mu \downarrow \delta^{-n+1}$	Limit as $\mu \uparrow \delta^{-n}$
$\bar{p}_0^H$	$(\delta^{-n+1} - 1)/2$	$(\delta^{-n} - 1)/2$
$\bar{p}_{n-1}^H$	0	$\delta^{-n}(1 - \delta)/2$
$\bar{p}_t^H$, $t \geq n - 1$	0	0
$\bar{p}_t^L$, $t \geq 0$	0	0
s_0^H	1/2	1/2
s_{n-1}^H	1/6	0
s_n^H	0	1/6
s_t^H, $t \geq n + 1$	0	0
s_0^L	1/3	1/3
s_t^L, $t \geq 1$	0	0
π_0^H	$(\delta^{-n+1} - 1)/4$	$(\delta^{-n} - 1)/4$
π_t^H, $t \geq n - 1$	0	0
π_t^L, $t \geq 0$	0	0

Table values are valid for $n \geq 1$, with one exception at $n = 1$: $s_0^H = (s_0^H + s_{n-1}^H)|_{n=1} = 2/3$.