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Strategy-Proof Exchange

by

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Abstract

We consider the allocation of goods in exchange economies with a finite number of agents who may have private information about their preferences. In such a setting, standard allocation rules such as Walrasian equilibria or rational expectations equilibria are not compatible with individual incentives. We characterize the set of allocations rules which are incentive compatible, or in other words, the set of strategy-proof social choice functions. The social choice functions which are strategy-proof are those which can be obtained from trading according to pre-specified proportions. The number of proportions which can be accommodated is proportional to the number of agents. Such rules are necessarily inefficient, even in the limit as the economy grows.

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1. Introduction

Allocations which can be decentralized in exchange economies with complete information or large numbers of agents are well known. That is, if the members of an economy know all the parameters of the economy, then there exist institutions which will lead to Walrasian equilbria. [See Moore (1991) for a recent survey of implementation with complete information.] If there is incomplete information, then it has been shown that the Walrasian correspondence is almost non–manipulable for large numbers of agents. That is, Postlewaite and Roberts (1976) have shown that for large enough replications of the economy gains from manipulation can be limited to any given level.

Unfortunately, the Walrasian correspondence cannot be achieved when information is incomplete and the economy is not large. Hurwicz (1972) showed that Walrasian allocations are manipulable when there is a finite number of agents. Agents are better off not acting in accordance with their (competitive) demands, but instead should take into account the influence they have on the price. These difficulties make it impossible to decentralize the Walrasian correspondence among a finite number of agents when information is incomplete. As Hurwicz showed, for certain utility profiles of other agents, the Walrasian allocation associated with an agent's true utility is worse for that agent than the Walrasian allocation associated with some alternative utility. This means that Walrasian allocations are manipulable.

In spite of the Hurwicz results, we might hope that competitive allocations could be achieved as the Bayesian equilibria of some mechanism. Palfrey and Srivastava (1987) have shown, however, that the Walrasian correspondence also fails to satisfy Bayesian incentive compatibility. Thus, the Walrasian correspondence cannot be achieved as the (Bayesian) Nash equilibria of any game form when there is incomplete information and finite numbers of agents.[1] Moreover, even if we consider rational expectations allocations instead of Walrasian ones, these difficulties cannot be overcome. Bayesian incentive compatibility is also violated

[1] An exception to this applies if information is non–exclusive, as examined by Postlewaite and Schmeidler (1986) and Blume and Easley (1990). Information is non–exclusive if the information of any one agent is known collectively by the other agents. Such an information structure makes the implementation problem similar to complete information implementation. However, many relevant situations will not have such an information structure. For example, agents may know more about their own preferences than about the preferences of other agents.

by the rational expectations equilibrium correspondence [Palfrey and Srivastava (1987)].

Given that the classic competitive allocations cannot be obtained with a finite number of agents and incomplete information,[2] we should try to discover what can be obtained. The question of what is implementable with incomplete information and a finite number of agents has been answered if agents are Bayesians and we know their prior distributions. A characterization of allocations which can be implemented in Bayesian equilibria is given in Jackson (1991). Abreu and Matsushima (1991) present results for virtual implementation via an iterative elimination of dominated strategies with incomplete information, and Palfrey and Srivastava (1989) examine implementation in undominated Bayesian equilibria. Palfrey (1990) and Palfrey and Srivastava (1990) provide surveys of the incomplete information implementation literature.

A shortcoming of the incomplete information implementation literature is that mechanisms provided in constructive proofs depend in key ways on exact knowledge of agents' prior distributions. If the priors are not those anticipated by the mechanism designer, then undesired equilibria can arise in the mechanisms constructed. So unless we are sure of the priors, mechanisms demonstrated in this theory are not sure to implement the desired social choice correspondence . These mechanisms, however, were developed to prove existence results; not with this sort of robustness in mind. One would like to find simple mechanisms which are immune to changes in the structure of the environment.

To this end, we characterize the class of strategy–proof social choice functions in classic exchange economies.[3] Strategy–proofness implies that regardless of the preferences of the other agents, an agent is best off with the allocation associated with his/her true preferences. One way to think of strategy–proofness is as the requirement that Bayesian incentive compatibility hold for all possible priors. This strong property means that a social choice

[2] Even when one considers equilibrium refinements, the Bayesian incentive compatibility condition is necessary for implementation. This means that refinements will not help us to implement competitive allocations.

[3] For a look at production economies, see Shenker (1992). He examines strategy–proof social choice functions which satisfy differentiability conditions, extending earlier results of Satterthwaite and Sonnenschein (1981) on differentiable and strategy–proof social choice functions . The Satterthwaite and Sonneschein results provide only local implications of strategy–proofness, and Shenker accomplishes the formidable task of obtaining a global characterization. In the pure exchange case, however, the differentiablility condition is quite strong and excludes almost all of the rules which we identify. Thus, there is no clear relation between Shenker's work and ours.

function can be decentralized regardless of the information structure in the economy. Furthermore, as it turns out, strategy–proof social choice functions correspond to the outcomes of very simple and natural trading mechanisms.

In many situations, when designing institutions or markets, one may have some idea of what information agents are likely to have, without knowing their exact priors. The relevant incentive compatibility constraint may lie between a Bayesian one and strategy–proofness. A full characterization of strategy-proof allocation rules will provide an important benchmark and help shed light on what can be achieved in more structured situations.

The cost of strategy–proofness is efficiency. It has been shown that in exchange economies strategy–proof social choice functions which are efficient are also dictatorial. The first work in this direction was by Hurwicz (1972), who obtained such a result for two agents and two goods, given individual rationality with respect to an endowment. This result was extended to many goods by Dasgupta, Hammond, and Maskin (1979), and Hurwicz and Walker (1990). Recently, Zhou (1991) has shown that the result holds for two people without requiring individual rationality or any continuity properties.[4] Such negative results, however, do not give us an idea of how ineffіcent such social choice functions are, or how the inefficiency depends on the size of the economy. By offering a complete characterization of the strategy–proof social choice functions , we provide a deeper understanding of the tradeoffs between incentives and efficiency.

Strategy–proof social choice functions are those which can be obtained from trading according to pre–specified proportions. The number of proportions which can be accommodated is proportional to the number of agents. This means that strategy–proof rules become more "flexible" in choosing the direction of trade as the economy grows. However, due to the incentive constraints the particular way in which the proportion is chosen cannot depend on the excess demand of the economy, but instead only on the numbers of agents on either side of the market. As a result, the inefficiency in the market does not disappear

[4] There are strategy–proof, efficient and non–dictatorial social choice functions for three or more agents. Satterthwaite and Sonnenschein (1981) give an example such that the total endowment is given to one of two agents, depending on the shape of the preferences of a third agent. However, this example is not individually rational and also fails to satisfy other conditions which we will consider, such as non–bossiness. Our characterization result will show that there are no strategy–proof and efficient social choice functions which satisfy an anonymity condition and non–bossiness.

in the limit.

This provides a different insight from the results of Roberts and Postlewaite (1976). They showed that Walrasian allocations were almost incentive compatible as the economy is replicated. This means that if one insists on efficiency, then one can find allocations which are almost incentive compatible. However, this allows for small gains from manipulation, and leaves unclear which allocations will arise if agents act in their own interest and take advantage of the gains from manipulation.[5][6] Here we take as given that agents will act in their own interest, and characterize the set of allocation rules which are incentive compatible (strategy–proof). As it turns out, the rules which are incentive compatible are not close to being efficient, even for very large economies.

2. Definitions

Consider a classical exchange economy with n agents and l goods, where both n and l are finite. The endowment of goods in the economy is $e \in \mathbb{R}_+^{nl}$. An allocation is a list of the goods given to each agent and the set of (balanced) allocations is

$$A = \{\ x \in \mathbb{R}_+^{nl} \mid \ \sum_i x^i = \sum_i e^i \}.$$

For $x \in \mathbb{R}_+^{nl}$, we use the notation x^i to denote the l dimensional allocation of goods to agent i, and x_k^i to denote the allocation of the k-th good given to agent i. It is assumed that $\sum_i e_k^i > 0$ for each k.

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[5] Jackson (1992) provides a partial answer to this question by showing that for large enough economies agents will find it optimal to act in accordance with demands arbitrarily close to their competitive demands. This is only a partial answer since it may be that if each agent only deviates slightly, the aggregated demands and resulting prices may deviate substantially.

[6] Roberts and Postlewaite (1976) and Jackson (1992) both consider incentive compatibility for a given economy, as opposed to across all possible realizations. Their results are easily extended to situations where there is a finite set of possible realizations of the economy, but not to infinite ones. Recent work by Gul and Postlewaite (1992) shows that for large enough replications of the economy, there exist interesting allocation rules (not necessarily Walrasian) which are Bayesian incentive compatible and almost efficient. However, their work also relies on a finite type space, and it is not clear how it extends to infinite type spaces. Mas-Colell and Vives (1988) consider implementation when an infinite state space is allowed and get a convergence to Walrasian equilibrium, but assuming that the distribution of types is known to the mechanism designer.

Agent i's preferences are represented by a utility function $u^i : R^l \to I\!R$. U denotes the set of all u^i which are continuous, strictly quasi–concave, and increasing.[7] Let u denote the vector $(u^1, u^2, \ldots, u^n)$ and $u^{-i}, \widehat{u}^i$ denote the vector $(u^1, \ldots, u^{i-1}, \widehat{u}^i, u^{i+1}, \ldots, u^n)$

A *social choice function* is a map from utilities into allocations, $f : U^n \to A$. A_f represents the range of f and $f^i(u)$ represents the allocation given to individual i at u.

A social choice function f is *strategy–proof* if

$$u^i(f^i(u)) \geq u^i(f^i(\widehat{u}^i, u^{-i}))$$

for all $i \in \{1, \ldots, n\}$, $u \in U^n$, and $\widehat{u}^i \in U$.

A social choice function f is *individually rational* (with respect to an endowment e) if

$$u^i(f^i(u)) \geq u^i(e^i)$$

for all i and $u \in U^n$.

A social choice function f is *anonymous* if for all u, i and j

$$f^i(u^{-i,j}, u^i, \widetilde{u}^j) = f^j(u^{-i,j}, \widetilde{u}^i, u^j)$$

whenever $\widetilde{u}^i = \widetilde{u}^j$ and $u^i = u^j$.

3. Two Agents

We start with a characterization for two people, and then consider the case of three or more people in the next section. We first consider the case of two people trading two goods, since this is the easiest to illustrate.

3.1 Two People and Two Goods

The social choice functions which are strategy–proof in this case are those which can be derived from the following mechanism, which is referred to as *fixed–price trading*. The

[7] A function u^i is increasing if $x^i \geq y^i$ and $x^i \neq y^i$ implies that $u^i(x^i) > u^i(y^i)$, where $\geq$ indicates greater than or equal to in all coordinates. The convexity and monotonicity of preferences are important to our analysis. If the domain of preferences includes all non–monotonic and non–convex ones, then a strategy–proof social choice function is dictatorial [see Barbera and Peleg (1990) and Moreno (1991)].

mechanism is not a direct mechanism, however it is dominant strategy incentive compatible. Thus a strategy–proof social choice function is defined by finding the outcome associated with the dominant strategies of the agents as a function of their utilities. An agent is selected, which for convenience is referred to as agent 1. Two prices are selected for units of the first good in terms of units of the second good. The first price indicates the price at which agent 1 can offer to sell the first good, and the second price indicates the price at which agent 1 can offer to buy the first good. Prespecified limits indicate the most agent 1 can offer to buy or sell.[8] The selling price is no more than the buying price, and the limits are such that any final allocation is nonnegative. This mechanism is set and fixed before agents appear with their preferences. Once preferences are realized, agent 1 chooses either to offer to buy or to offer to sell good one, and declares up to how much she is willing to buy or sell. At the same time, agent 2 indicates how much of the first good he is willing to sell, and how much of the first good he is willing to buy. [Given the quasi–concavity of preferences, agent 1 will only be willing to either buy or sell; but not both since the selling price is no more than the buying price. Agent 2, however, may be willing to do both if the prices are not the same.] If agent 1 has offered to buy, then goods are exchanged in the amount of the minimum of what agent 1 declared she is willing to buy and what agent 2 declared he is willing to sell. If agent 1 has offered to sell, then goods are exchanged in the amount of the minimum of what agent 1 declared she is willing to sell and what agent 2 declared he is willing to buy.

EXAMPLE 1. *A Non–Anonymous Rule.*

Agent 1 is endowed with ten units of each of the two goods and agent 2 is endowed with five units of each of the two goods. The prices at which agent 1 may offer to buy or sell are different. Agent 1 may offer to buy good one at a price of 2 (units of good two per unit of good one) and sell good one at a price of 1. Notice, that given the shape of the range, agent 1 has a unique best element in the range. If for instance, agent 1 finds buying 3 units of good one most preferred, then she will offer to buy up to 3 units of good one. If agent 2 has the preferences pictured below, then he will offer to sell up to 2 units of good one (at a price of 2) and buy up to 1 unit of good one (at a price of 1). In this case, agent

[8] The set of all possible trades is closed, but it may be that the range is not connected. For instance, it is possible that agents are restricted to trade only whole units.

2 sells 2 units of good one to agent 1 at a price of 2. The final allocation is then (12,6) for agent 1 and (3,9) for agent 2.

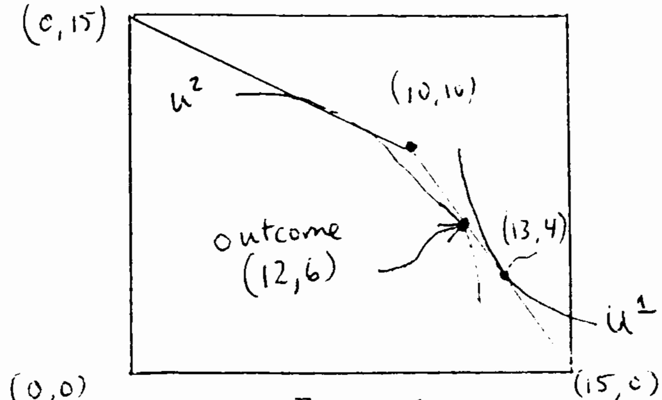

FIGURE 1

If instead, agent 2 has the preferences pictured below, then he will not offer to sell good one, but will offer to buy up to 2 units of good one. In this case, no goods are exchanged and the final allocation is the initial endowment.

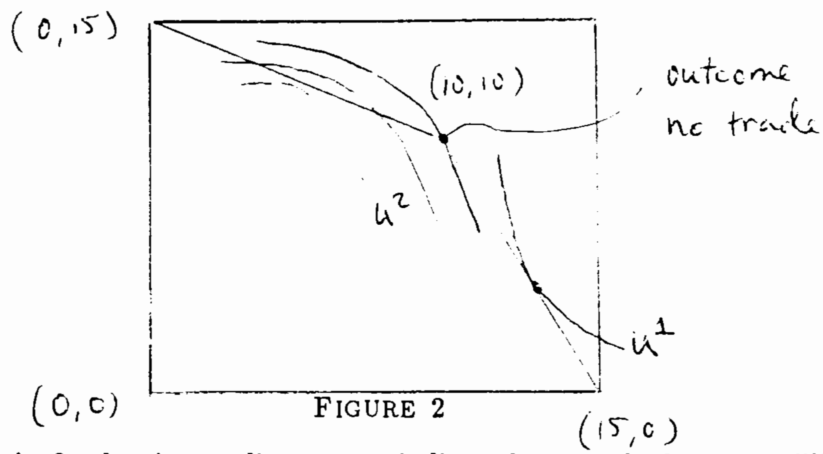

FIGURE 2

Essentially, in fixed–price trading agents indicate how much they are willing to buy or sell according to two fixed prices, and then the short side of the market is rationed. It is easily checked that the outcomes of such a procedure (associated with the dominant strategies as a function of utilities) define a strategy–proof social choice function which is individually rational with respect to the endowment point. It is interesting that these are the only social choice functions which are strategy–proof and individually rational.

THEOREM 1. *A social choice function defined on a two person, two good exchange economy is strategy–proof and individually rational with respect to an endowment point if, and only if, it is the result of fixed–price trading.*

The statement of Theorem 1 is somewhat loose, since we have not yet precisely defined fixed–price trading. It is possible that the range of a strategy–proof social choice function is closed but not connected, in which case one must worry about tie breaking rules when agents are indifferent between two allocations. This is made precise in Theorem 2, of which Theorem 1 is a special case.

A remark is in order, relating Theorem 1 to the work of Satterthwaite and Sonnenschein (1981), which also considers strategy–proof social choice functions for exchange economies (among other settings). [9] Satterthwaite and Sonnenschein (1981) showed that if a social choice function is strategy–proof and everywhere total, then it must be dictatorial. A dictatorial social choice function in an exchange economy is one which always gives the same agent the entire endowment. As Satterthwaite and Sonnenschein acknowledged, the everywhere total condition used in their theorem is not well understood, and may be rather strong. Of course, given their result, the everywhere total condition must rule out all the non–dictatorial social choice functions described in the above theorem. One implication of the everywhere total condition is that at every preference profile, at least one of the agents must be able to change the allocation by some small change in utility. On the surface, this seems to be a rather innocuous condition. It is however, violated by the social choice functions derived from fixed–price trading rules. For instance, if both agents wish to buy good one and sell good two at the given prices, then no trade occurs. This will remain true for small changes in utilities, since both agents will still want to buy good one and sell good two.

3.2 Two Agents and More than Two Goods

We now turn to strategy–proof rules for two agents who are exchanging $l \geq 2$ goods. An additional definition and some notation will be helpful in describing strategy–proof rules.

[9] Theorem 1 is also related to Hagerty and Rogerson (1987). Hagerty and Rogerson examine strategy–proof mechanisms in a bilateral trade setting in which one agent is a buyer and one agent is a seller with a single unit of an indivisible good. Although the settings are not comparable, the Hagerty and Rogerson mechanisms are intuitively the mechanisms one obtains from the above theorem when additional restrictions are placed on possible trades. If one agent is declared seller and permitted only to sell one (indivisible) unit of good one, then the above theorem yields mechanisms similar to those of Hagerty and Rogerson (1987).

We say that a set $B \subset A$ is *diagonal* if for each agent i and for all x and y in B $(x \neq y)$, $x^i \not\geq y^i$ and $y^i \not\geq x^i$.[10]

Given points a, b, and c in A, we write ab to denote the set of points lying on the line segment with endpoints a and b, so $ab = \{x \mid \exists\gamma \in [0,1] \text{ such that } x = \gamma a + (1-\gamma)b\}$. Then $c \in ab$ indicates that c lies on the line segment connecting a and b. We write $c \succeq_i ab$ if $c^i \geq \gamma a^i + (1-\gamma)b^i$ for some $\gamma \in [0,1]$. Thus, $c \succeq_i ab$ indicates that c lies above the segment ab from agent i's perspective.

Fixed–price trading has an extension to the case of more than two goods called *fixed–proportion trading*, which is informally described as follows. Trade may occur along a number of line segments emanating from the endowment, instead of just 2. As it was for the two good case, strategy–proofness will imply that one agent always has a uniquely defined most preferred direction of trade. In particular, this implies that trade occurs along $k \leq l$ diagonal line segments are selected, each having the endowment as an endpoint. The line segments are chosen such that for one of the agents, say agent 1, the following holds. Choose any x from one segment and y from another segment, then $e \succeq_1 xy$. [11] This condition assures that for each utility function of agent 1, points on at most one segment are individually rational, which turns out to be a consequence of strategy–proofness. The range A_f then consists of a closed set of points lying on these segments, including the endowment point e. Agent 1 selects a point in A_f on one of the segments, and simultaneously agent 2 declares demands along each of the segments. Essentially, the trading procedure is a simple extension of the fixed price trading for two goods. Agents indicate their desired trades in $k \leq l$ different fixed proportions, and then the short side of the market is rationed.

EXAMPLE 2.

There are two agents and three goods. Endowments are $e^1 = e^2 = (5,5,5)$ and the total endowment is $(10,10,10)$. Agent 1 can buy units of any good from agent 2, but at a price of one unit of *each* of the other goods to do so. Thus agent 1 can offer multiples

[10] We say $x^i \geq y^i$ if $x^i_k \geq y^i_k$ for each k.

[11] It is this condition which implies that $k \leq l$. Notice that $e \succeq_1 xy$ implies that $y^1_j - e^1_j < 0$ whenever $x^1_j - e^1_j > 0$. Since neither trade is less than zero, it follows that they are positive in different coordinates. We can find at most l such trades which are each positive in some direction, but never positive in a component for which another is positive.

(but not combinations) of the trades $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$. The range of f in terms of agent 1's final allocations is thus

$A_f = \{\ x \mid \exists\gamma \in [0,1]$ s.t. either

$$x^1 = \gamma(5,5,5) + (1-\gamma)(10,0,0), \text{ or}$$

$$x^1 = \gamma(5,5,5) + (1-\gamma)(0,10,0), \text{ or}$$

$$x^1 = \gamma(5,5,5) + (1-\gamma)(0,0,10), \}$$

Agent 2's allocation, x^2, is simply (10,10,10) – x^1.

If agent 1's most preferred point in the range is, say, (7,3,3), then the allocation is agent 2's most preferred point from the set of convex combinations of (5,5,5) and (7,3,3). [Again, allocations are written in terms of agent 1's final allocation.]

Notice that given the structure of the range, agent 1 always has a unique most preferred point. Also notice that there is no point on any other segment which agent 1 prefers to any point which lies between her top and e. For instance, consider the case when her most preferred point in the range is (7,3,3). Take any point on another segment, such as one which lies on the segment between (5,5,5) and (0,0,10). By strict quasi–concavity and the fact that (7,3,3) is most preferred, it follows that $\frac{5}{7}$(7,3,3) + $\frac{2}{7}$(0,0,10) = (5,$\frac{15}{7}$,5) is preferred to (0,0,10). By monotonicity (5,5,5) is preferred to (5,$\frac{15}{7}$,5), and so (5,5,5) is preferred to (0,0,10). Since (7,3,3) is preferred to (5,5,5), it follows from strict quasi–concavity that (7,3,3) is preferred to any convex combination of (5,5,5) and (0,0,10).◇

It is this property that one of the agents will only want to trade along one of the segments that makes the fixed–proportion trading strategy–proof. As it turns out, the converse is also true: any strategy–proof trading rule must have this property. This is formally stated in the next theorem after we give a precise definition of fixed–proportion trading.

Given a set $B \subset A$ and a utility profile $u \in U^2$, let $T^i(B,u)$ denote the set of allocations in B which maximize u^i. This set is nonempty if B is closed. In such a case, let $t^i(B;u)$ be a selection from $T^i(B,u)$.

A social choice function f defined on a two person exchange economy is the result of *fixed–proportion trading* if the following hold:

(1) A_f is closed and diagonal and contains e. There exists an agent i such that for all

x and y in A_f either $x \in ey$, $y \in ex$, or $e \succeq_i xy$.

(2) t^i and t^j are such that $t^i(A_f; u) \neq t^i(A_f; u^i, v^j)$ only if $v^j(t^i(A_f; v^j, u^i)) \geq v^j(t^i(A_f; u))$, and for any $a \in A_f$ $t^j(ea \cap A_f; u) \neq t^j(ea \cap A_f; v^i, u^j)$ only if $v^i(t^j(ea \cap A_f; v^i, u^j)) \geq v^i(t^j(ea \cap A_f; u))$.

(3) $f(u) = t^j(ea \cap A_f, u)$, where $a = t^i(A_f, u)$

Condition (1) assures that A_f lies along $k \leq l$ diagonal line segments, each having the endowment as an endpoint. If one chooses any x from one segment and y from another segment, then $e \succeq_i xy$.

Condition (2) states that t^i and t^j are choices among i and j's most preferred points which are either constant, or choose in favor of the other agent. This defines the strategy–proof tie breaking rules which come into play if the range is not connected. Ties will only ever have to be broken over at most two points, given the shape of the range from (1). In the case where A_f is connected (for instance if f is continuous), then there will never be any need to break ties and condition (2) does not bind.

Condition (3) states the outcome of f is agent j's most preferred point in the range which lies between the endowment and agent i's most preferred point in the range.

THEOREM 2. *A two person social choice function is strategy–proof and individually rational with respect to an endowment point if, and only if, it is is the result of fixed–proportion trading.*

3.3 Anonymous Rules for 2 Agents

The descriptions of fixed–price and fixed–proportion trading seem asymmetric since one of the agents can only offer to either buy or sell, while the other agent can offer to do both. There are, however, anonymous social choice functions generated by the types of mechanisms described in the first two theorems. The anonymous rules are those for which the prices or proportions are the same for both agents. In this case the range of the social choice function lies on a line segment and each agent has a single most preferred point. Thus even though one of the agents can offer to both buy and sell, he will not choose to do so.

EXAMPLE 3. *An Anonymous Rule with Two Goods.*

Each agent is endowed with ten units of each of the two goods. The same price is chosen for both buying and selling. For example suppose that the price is 2 (units of good two exchanged for each unit of good one). Agent 1 declares whether she wants to buy or sell good one and up how many units of good one she is willing to trade. At the same time agent 2 does the same. Given the convexity of preferences and the shape of the range, neither agent will be willing to offer to both buy and sell. If both agents wish to buy or sell, then no trade occurs. If one agent wishes to buy and one wishes to sell, then trade occurs. The size of the trade is equal to the smaller of the two declarations. This is illustrated below.

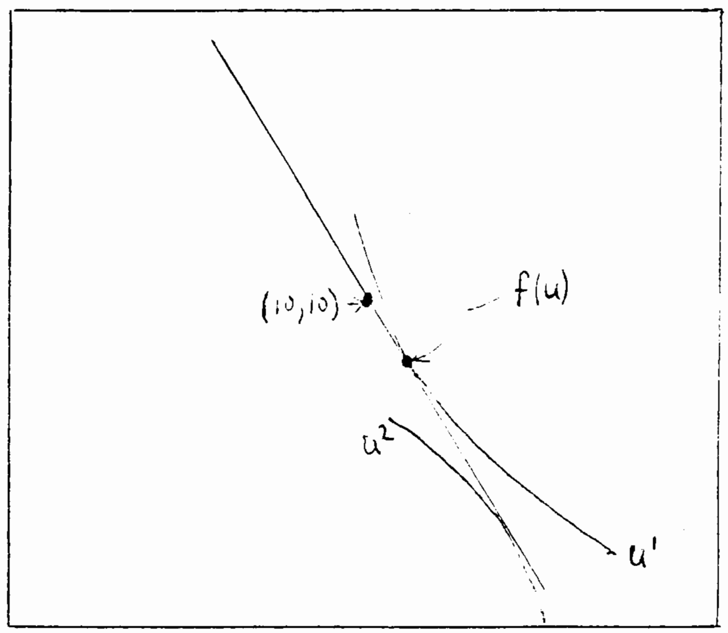

FIGURE 3

COROLLARY 1. *A two person social choice function is strategy–proof and anonymous if and only if it is the result of fixed–proportion trading along a line segment centered at the equal split of the total endowment.*

Notice that anonymity reduces the dimension of the range of a two person strategy–proof social choice function when there are three or more goods. That is, there exist non–anonymous rules which trade along $k \leq l$ segments emanating from the endowment. However, for k to be larger than 2 it must be that the agents are treated asymmetrically as required by condition (1). The only way condition (1) can hold simultaneously for both agents (anonymity) is for all points to lie on a single line.

3.4 Dropping Individual Rationality for 2 Agents

The previous theorems treat cases in which some sort of individual rationality with respect to an endowment is satisfied. The next theorem characterizes strategy–proof social choice functions without the individual rationality constraint. Intuitively, it is a simple extension of the previous results. Essentially the strategy–proof trading rules are similar to fixed–proportion trading, except that one of the agents can dictate over a part of the range which replaces the endowment.

We state the result for the two person, two good case as this is much easier to state than the general case. It extends to the l good case in the obvious way.

A strategy–proof social choice function , f, with range A_f satisfies the following conditions.

(1) A_f is a diagonal set and there exist points a, b, c, and d such that

(i) $a_1^1 \geq x_1^1 \geq d_1^1$ for all $x \in A_f$.

(ii) $x_1^1 \geq b_1^1 \Rightarrow x \in ab$ and $x_1^1 \leq c_1^1 \Rightarrow x \in cd$

(iii) let g be the point of intersection of the line containing a and b and the line containing c and d. Then there exists i such that for each $x \in A_f$ either $x \leq_i ag$ or $x \leq_i gd$.

(2) t^i and t^j are such that $t^i(B;u) \neq t^i(B;u^i,v^j)$ implies $v^j(t^i(B;v^j,u^i)) \geq v^j(t^i(B;u))$, $t^j(B;u) \neq t^j(B;v^i,u^j)$ implies $v^i(t^j(B;v^i,u^j)) \geq v^i(t^j(B;u))$.

(3) if $t^i(A_f,u) \in ab$, then $f(u) = t^j(xb \cap A_f,u)$ where $x = t^i(A_f,u)$, and if $t^i(A_f,u) \in cd$, then $f(u) = t^j(cx \cap A_f,u)$ where $x = t^i(A_f,u)$,

(4) otherwise $f(u) = t^i(A_f,u)$.

Examples of sets A_f which satisfy condition (1) are given in the following figures. Agent i dictates if her most preferred point in the range is in the region between b and c, and otherwise the function behaves much like the fixed–price trading defined earlier.

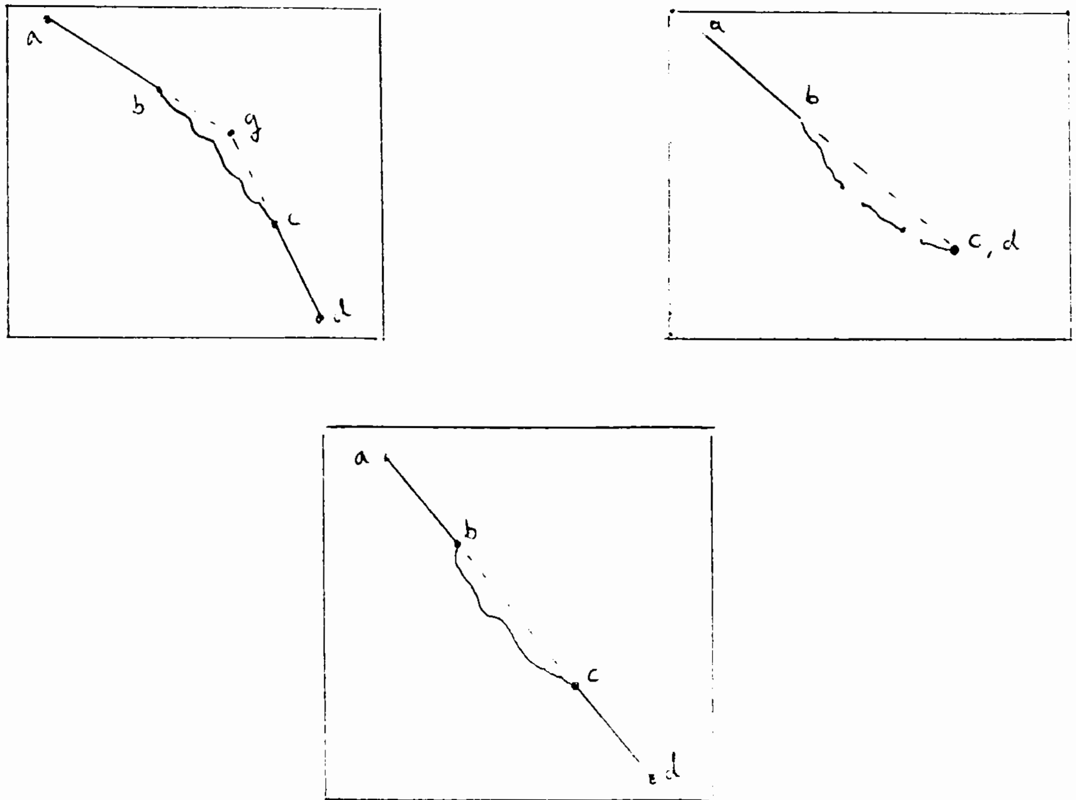

FIGURE 4

THEOREM 3. *A two person, two good social choice function is strategy–proof if, and only if, it satisfies conditions (1) → (4), above.*

4. Three or More Agents

With three or more agents, the class of strategy–proof social choice functions is substantially richer than with only two agents. In order to simplify the analysis, we impose two conditions. The first condition is really only a simplifying one which eliminates the necessity of worrying about the tie–breaking rules for situations where the range is finite or disconnected. We will only consider social choice functions which satisfy:

Consider i, $u \in U^n$, $\overline{u}^i$, and $\widetilde{u}^i$ such that there exists $a \in \mathbb{R}^l$, $\lambda \in \mathbb{R}_+$ and $\lambda' \in \mathbb{R}_+$, and $\gamma \in (0,1)$, such that $f^i(u) = e^i + \lambda a$ and $f^i(u^{-i},\overline{u}^i) = e^i + \lambda' a$, $\widetilde{u}^i(\gamma f^i(u) + (1 - \gamma) f^i(u^{-i},\overline{u}^i)) > \widetilde{u}^i(f^i(u))$ and $\widetilde{u}^i(\gamma f^i(u) + (1-\gamma) f^i(u^{-i},\overline{u}^i)) > \widetilde{u}^i(f^i(u^{-i},\overline{u}^i))$. A social choice function f is *tie–free* if $f^i(u) \neq f^i(u^{-i},\widetilde{u}^i) \neq f^i(u^{-i},\overline{u}^i)$ for all such i, u, $\overline{u}^i$, and $\widetilde{u}^i$.

This condition says that given a fixed utility profile for other agents, if an agent can obtain two different trades in the same proportion away from the endowment, and at a given utility prefers some convex combination of those trades to either trade, then the agent is not forced to choose one of those trades. In the proof of Theorem 4, this condition is only used to avoid tie–breaking when identifying the rationing schemes. Extensions to allow for the possiblity of ties have been illustrated in Theorems 2 and 3.

The second condition we impose is more substantial, but still quite agreeable. It is the non–bossy property defined by Sattherthwaite and Sonnenschein (1981). A social choice function f is *non–bossy* if for any i, u, and $\widetilde{u}^i$, $f^i(u) = f^i(u^{-i}, \widetilde{u}^i)$ implies $f(u) = f(u^{-i}, \widetilde{u}^i)$.

Non–bossiness states that if only i changes preferences and i's outcome is not changed, then the outcome of other agents are not changed. It rules out a series of social choice functions which are not dictatorial but are degenerate in other ways. Two examples ruled out by this condition are [see Sattherthwaite and Sonnenschein (1981)]: (i) the entire endowment is given to either agent 2 or agent 3, depending on the shape of agent 1's preferences, and (ii) agents 1 and 2 choose allocations from prespecified sets and agent 3 obtains whatever is left.

The interesting thing we have found for three or more agents, is that it is possible to incorporate a number of prices or proportions. To illustrate, consider the following example for the n agent, 2 good case. For each integer from $n-1$ to $n/2$, choose two possible prices for trade. Each price is in terms of units of good 2 per unit of good 1. One price will be called the buying price (of good 1) and the other, the selling price (of good 1). So for instance, for $n-1$ we might assign the prices of 2 and 1/2. In this case the agents will be considering possible allocations of the form $x^i = e^i + \alpha(1,-2)$ and $x^i = e^i + \alpha(-1, 1/2)$. These prices must be selected so that the buying price is at least as large as the selling price. Each pair of these is called a trade proposal. Trade proposals must be selected so that the buying prices form a non–increasing sequence, and the selling prices form a non–decreasing sequence. Agents simultaneously declare their demands for each of the prices of trade in each trade proposal.[12] Begin with the n-1 trade proposal. If exactly n-1 agents

[12] These must be consistent with some utility. For example an agent cannot demand to buy at one buying price and then to sell at a lower buying price. Notice also, that since the buying price is at least as large as the selling price, an agent cannot request to buy at the buying price and sell at the selling price, since both cannot be improving trades for a

desire to buy at the buying price, or n-1 agents desire to sell at the selling price, then the particular price is said to be matched. If not, proceed to the n-2 proposal. If exactly (n-2) agents desire to sell at the selling price, or buy at the buying price then good then trade is conducted at that price. The procedure is continued until a price is found so that the rank of the trade proposal matches the number of agents agreeing to buy (sell). If no such proposal is found then no trade occurs. At most one price will be selected by this procedure. If a price is selected, then trade occurs at that price. So for instance, if the n-3 buying price is selected, then n-3 agents will buy good 1 at that price, and the remaining 3 agents will sell good 1 at that price. The side of the market which has declared demands in excess of that declared by the other side are rationed according to the uniform rationing rule. This is the rule which chooses a size of trade such that if each agent is given the minimum of his or her demand and that size trade, then the market clears.

The social choice function derived from this rule is strategy–proof since it is a dominant strategy to declare truthful demands. This is seen as follows. Suppose that through honest declarations some price has been chosen. Suppose that it is a buying price. An agent who requested to buy at that price could only benefit by being a buyer at a lower price, or a seller at a higher price. No lower price can be selected with this agent as a buyer, since all the other agents who are buyers at the original price will also wish to buy at lower prices, and so the lower price will not be matched. Similarly no higher selling price can be matched.

For large numbers of agents, the above rule is fairly flexible, since it permits trade in a multitude of different directions. The price is chosen depending on the numbers of agents lining up on each side of the market. Such rules are not efficient since the selection of the trading price is not responsive to the exact demands of the agents, but only to they are positive or negative. However, this is the cost to the incentive compatibility requirements. We will return to discuss this in more detail.

We now turn to the general characterization theorem. This requires a definition of fixed proportion trading for a general n–person, l–good exchange economy. The rule consists of three basic parts. (1) Trade can only occur in one proportion which is selected from an a priori fixed set of proportions which satisfy some additional restrictions. The proportions are grouped into subsets which we call "trade proposals". Each trade proposal has a shape

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single utility function.

similar to that of the range discussed in Theorem 2, which assures that each agent has a unique most preferred trade from a given proposal. Trade proposals have relative positions so that they are what we term "nested". (2) The proportion according to which trade occurs is selected by examining the demands of agents. Each trade proposal is assigned a number, and a proportion in that proposal is "matched" if exactly that number of agents demand trades which are positive multiples of that proportion. The nesting of trade proposals assures that at most one proportion can be matched at any given preference profile (See Lemma 1). (3) Agents are given trades in the direction of their demanded trade in the selected proportion. No one's trade is larger than their demanded trade; but one or both sides of the market may be rationed. If rationing occurs, then it is done "uniformly", so that the agents who are rationed are rationed equally and no agent receives a trade which is larger than the rationed amount.

Let us be more specific.

A *trade proposal* $P \subset I\!R^l$ is a set of feasible trade proportions which satisfies the following: (i) if $a \in P$, then $a \not\geq 0$ and $a \not\leq 0$, and (ii) if $b \in P$ and $b \neq a$, then there exists $\gamma \in (0,1)$ such that $\gamma a + (1-\gamma)b \leq 0$.

The shape of a trade proposal is similar to the restrictions placed on the range in the fixed proportion trading we saw for two agents (Theorem 2). Similarly, there can be at most l different proportions in any given trade proposal. The difference is that now there may be more than one trade proposal. Which trade proposal is chosen, depends on the signs of the demands of each agent in the given proportions through what we call a matching process (to be defined shortly). The fact that such a matching process is well defined has implications for the interrelations of trade proposals which is captured in the following property.

A collection of trade proposals $\{P(k) \mid n > k \geq n/2\}$ are *nested* if for each $k' < k$ and $a \in P(k')$ and $b \in P(k)$, either there exists $\gamma > 0$ such that $\gamma b \leq a$ or there exists $\gamma > 0 \in (0,1)$ such that $\gamma a + (1-\gamma)b \leq 0$.

For any k and $a \in P(k)$ let $\alpha(u^i, a, e^i)$ be the $\alpha \in I\!R$ which maximizes $u^i(e^i + \alpha a)$ subject to $e^i + \alpha a \in I\!R^l_+$. Thus, $\alpha(u^i, a, e^i)$ is the scalar such that $\alpha(u^i, a, e^i)a$ is i's most preferred trade among those in proportion a. Since the endowment is fixed in our analysis, we will simplify the notation and use $\alpha(u^i, a)$.

Given a set of nested trade proposals $\{P(k) \mid n > k \geq n/2\}$, we say that u *matches* $a \in P(k)$ if either

(i) there exists $C \subset \{1, 2, \ldots, n\}$ with $\#C = k$ such that $\alpha(u^i, a) > 0$ for each $i \in C$ and $\alpha(u^i, a) \leq 0$ for each $i \notin C$, or

(ii) not (i) and either $P(\underline{k}) = \{a\}$ or $P(\underline{k}) = \{a, -\gamma a\}$, where $\underline{k}$ is the smallest integer which is at least as large as $n/2$.

A proposal is matched if exactly k agents desire "positive" trades in a given proportion. The exceptional case is the one where there are too few agents for any particular proposal, and the upper most proposal only has one possible proportion. In that case, one does not have to worry about incentives in selecting a proportion in which to trade and so any u matches.

Since the proportion along which trade occurs is selected by the signs of demands, rather than their sizes, it will often be necessary to ration. It turns out that the strategy–proofness condition has strong implications as to which sorts of rationing rules can be used. In particular, the rationing is done uniformly. [13]

Consider u which matches $a \in P(k)$ and such that for each i there exists $r^i \in [0, 1]$ such that $f^i(u) = e^i + r^i\alpha(u^i, a)a$. f satisfies *uniform rationing* at u if

(a) $\text{sign}[\alpha(u^j, a)] = \text{sign}[\alpha(u^i, a)]$, and $|\alpha(u^j, a)| \geq r^i|\alpha(u^i, a)|$ and $r^i < 1$, imply that $f^i(u) = f^j(u)$.

(b) $\text{sign}[\alpha(\tilde{u}^i, a)] = \text{sign}[\alpha(u^i, a)]$, and either $|\alpha(\tilde{u}^i, a)| \geq r^i|\alpha(u^i, a)|$ and $r^i < 1$, or $|\alpha(\tilde{u}^i, a)| = |\alpha(u^i, a)|$ imply that $f(u^{-i}, \tilde{u}^i) = f(u)$.

If trade occurs, then according to the uniform rationing all those who are rationed on a given side are rationed to the same trade. If some individual is rationed when announcing a given utility, then the outcome is the same when that individual announces any utility which requests a trade as large or larger along that same proportion.

The social choice function f is the result of *fixed proportion trading* on an n person, l good exchange economy if:

[13] For more on strategy–proofness and the uniform rule, see Sprumont (1991). The uniform rule arises in other contexts and has other nice properties [see Aumann and Dreze (1986) and Thomson(1990)].

(1) For each integer k, $n > k \geq n/2$, there exists a trade proposal $P(k)$. If n is even, then for any a and b in $P(n/2)$ there exists $\gamma \in I\!R$ such that $b = \gamma a$.

(2) The trade proposals are nested.

(3) If u matches $a \in P(k)$, then for each i there exists $r^i \in [0,1]$ such that $f^i(u) = e^i + r^i \alpha(u^i, a)a$, where $e^i + r^i \alpha(u^i, a)a \in I\!R^l_+$. If no match occurs, then $f(u) = e$.

(4) If trade occurs in proportion a at u and agents on either side are rationed, then they are rationed uniformly. Finally, if $f^i(u) = e^i = f^i(u^{-i}, \widetilde{u}^i)$, then $f(u^{-i}, \widetilde{u}^i) = f(u)$.

Conditions (1) and (2) lay out the structure of the range. Condition (3) states that trade only occurs if some proportion is matched and then no one receives more than their demand. (4) indicates that rationing is done uniformly, and that bystanders cannot affect the trades of others. First we verify that fixed proportion trading is well defined.

LEMMA 1. *If $b \in P(k)$ is matched and $a \neq b$ is in $P(k')$ for some k', then a is not matched (unless $k' = k = n/2$ and $a = -b$).*

PROOF: Suppose the contrary, so that some $a \in P(k')$ is also matched. Without loss of generality, assume that $k' \leq k$. From (1), we need only consider $k > n/2$. First consider the case where there exists $\gamma > 0$ such that $\gamma b \leq a$. It follows that any agent who has $\alpha(u^i, b) > 0$ also has $\alpha(u^i, a) > 0$. This means that a cannot be matched. The other case to consider is where there exists $\gamma \in [0,1]$ such that $\gamma a + (1-\gamma)b \leq 0$ (from the definition of $P(k)$ if $k' = k$ or from the definition of nested if $k' < k$). Since $k > n/2$, there must exist some i such that both $\alpha(u^i, b) > 0$ and $\alpha(u^i, a) > 0$. Let $\overline{\alpha}^i = \min\{\alpha(u^i, b), \alpha(u^i, a)\}$. Then by the strict quasi-concavity of utility $u^i(e^i + \overline{\alpha}^i b) > u^i(e^i)$ and $u^i(e^i + \overline{\alpha}^i a) > u^i(e^i)$. Then, again by strict quasi-concavity, $u^i[\gamma(e^i + \overline{\alpha}^i a) + (1-\gamma)(e^i + \overline{\alpha}^i b)] > u^i(e^i)$. Simplifying provides $u^i[e^i + \overline{\alpha}^i(\gamma a + (1-\gamma)b)] > u^i(e^i)$. This contradicts the fact that utility is increasing and that $\gamma a + (1-\gamma)b \leq 0$. ∎

Fixed proportion trading is illustrated in the following example.

EXAMPLE 4.

There are two goods and five agents who each have an endowment of 10 units of each good. There are two trade proposals. The trade proposals are $P(4) = \{(1,-2),(-2,1)\}$

and $P(3) = \{(1,-1),(-1,1)\}$. Thus any agent views the trade proposals as pictured below.

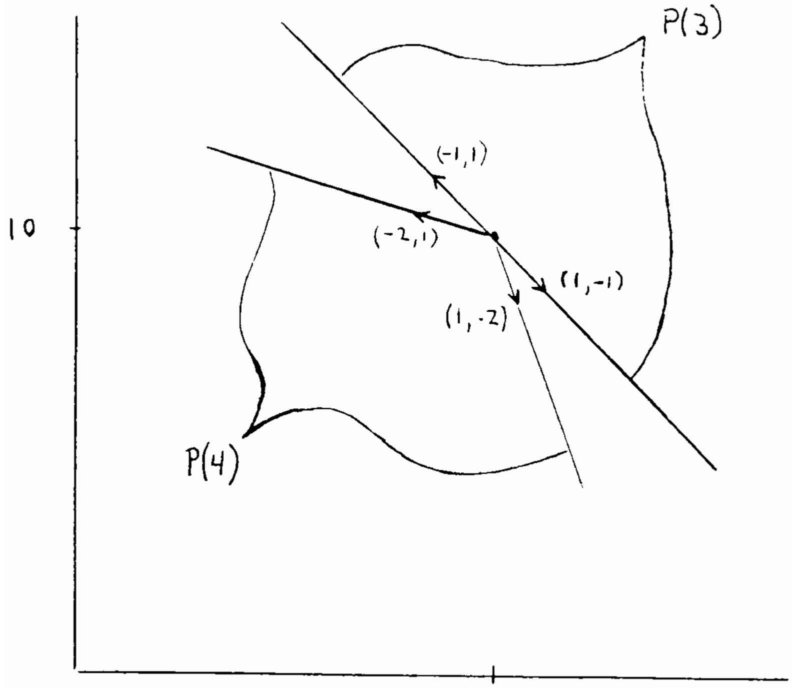

FIGURE 5

Notice that the nesting condition (2) requires that the proposal $P(4)$ "lie below" the proposal $P(3)$, and that the proposals lie in the same (lower) half space.

Given the utilities pictured below, the proportion (-2,1) is "matched" and so trade occurs in that proportion. Given the utilities of agents 1 and 2, it is clear that agents 3,4, and 5 will have to be rationed.

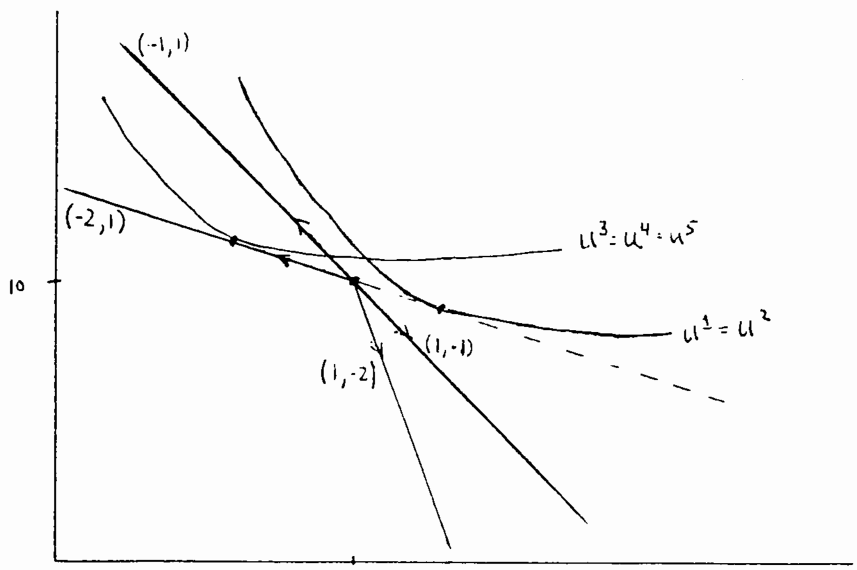

FIGURE 6

THEOREM 4. *A social choice function f is strategy-proof, anonymous, and non-bossy if, and only if, it is the result of fixed proportion trading away from the equal split point.*

Fixed proportion trading is in fact immune to coalitional manipulations. That observation (See Lemma 4 in the appendix) plays a key role in the proof of Theorem 4.

The anonymity condition is a bit stronger than one might like, since it requires that the fixed proportion trading be centered at the equal split of the total endowment. This is easily modified, by instead requiring that anonymity only be satisified in net trades. It is clear that if we simply move the endowment, that the characterization is essentially unchanged.[14]

COROLLARY 2. *A social choice function f is strategy–proof, anonymous with respect to net trades, and non–bossy if, and only if, it is the result of fixed proportion trading away from the endowment.*

5. Concluding Remarks.

The Size of Message Spaces

As many authors have noted, if there is any cost to transmitting information, then one will wish limit the size of messages which agents need to communicate in order to operate a mechanism. One appealing aspect of the strategy–proof social choice functions is that they can be decentralized through fixed–price and fixed–proportion trading mechanisms, which are very economical in the amount of information which needs to be transferred. The most any agent needs to communicate is a finite number of points in R^l_+. [Recall that the mechanisms only need to know each agent's most preferred trade according to each fixed price or proportion.] In contrast, to find a Walrasian allocation one must know the demand function of each agent, which indicates a point for each possible vector of prices.[15]

Efficiency and Large Economies

Fixed–proportion trading satisfies a number of desirable conditions. In addition to strategy–proofness, it satisfies coalitional strategy–proofness, an anonymity condition, in-

[14] To be careful about the details, we say that f is anonymous with respect to net trades if if for all u, i and j $f^i(u^{-i,j}, \overline{u}^i, u^j) = f^j(u^{-i,j}, u^i, \overline{u}^j)$ whenever $u^i(x^i) = u^j(y^j)$ and $\overline{u}^i(x^i) = \overline{u}^j(y^j)$ for all x and y such that $x^i - e^i = y^j - e^j$. [Recall that u^i is defined on all of $\mathbb{R}^l$.] There are no other modification necessary, since our definition of fixed price trading was deliberately general.

[15] See Hurwicz (1986) for a recent survey of work on informationally efficient mechanisms.

dividual rationality, envy–freeness,[16] and depends only on the ordinal representation of preferences. The real cost of strategy–proofness is the loss of efficiency. Fixed proportion trading has a number of sources of inefficiency. First, choosing trade from a fixed menu of proportions will clearly lead to inefficiencies. It is only by chance that that a proportion will be chosen which clears the market, without any rationing. Second, if no proportion is matched, then no trading takes place at all. This leads to a tradeoff. With only one proportion there is no difficulty in matching, but there may be a large degree of rationing. With more proportions there may be less rationing in general, but there is also a chance of not matching. Third, the matching process requires that the majority of agents obtain trades which are nonnegative multiples of the matched proportion. Since the proposals are nested, this implies that the majority always ends up in the same half–space. Fourth, when there are more than two goods, trading is done according to set proportions. Each trade proposal contains at most l proportions (where l is the number of goods), and so there are many directions in which trade can never occur. These last two restrictions on strategy–proof rules make it clear that there is no hope for any sort of approximate efficiency result in the limit (as the economy grows).

[16] An allocation is envy–free if no agent prefers the trade received by some other agent to his or her own. This is a straightforward consequence of Lemma 2 in the appendix and anonymity.

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Appendix: Proofs of the Theorems

We begin by stating definitions and lemmas which will be helpful in the proofs of the theorems.

The notation $C(x, u^i)$ denotes the upper contour set of u^i at x:

$$C(x, u^i) = \{z \in A \mid u^i(z^i) \geq u^i(x^i)\}.$$

We say that $\widetilde{u}^i$ is a *concavification of* u^i *through* x if
(i) $C(x, \widetilde{u}^i) \subset C(x, u^i)$ and
(ii) $z \in C(x, \widetilde{u}^i)$ and $z^i \neq x^i \Rightarrow u^i(z^i) > u^i(x^i)$,

Notice that by the strict quasi–concavity of utility, given any $u^i \in U^i$ and $x \in A$ we can find a $\widetilde{u}^i \in U^i$ which is a concavification of u^i through x.

A set $B \subset A$ is *diagonal* if for each agent i and for all x and y in B $(x \neq y)$, $x^i \not\geq y^i$ and $y^i \not\geq x^i$.

LEMMA 2. *If f is strategy–proof and non–bossy, $f(u) = x$ and $\widetilde{u}^i$ is a concavification of u^i through x for each $i \in C \subset \{1, 2, \ldots, n\}$, then $f(u^{-C}, \widetilde{u}^C) = x$.*

PROOF: Pick $i \in C$. Suppose that $f(u^{-i}, \widetilde{u}^i) = z \neq x$. Since f is non–bossy, $z^i \neq x^i$. By strategy–proofness $\widetilde{u}^i(z^i) \geq \widetilde{u}^i(x^i)$. By (ii), this implies that $u^i(z^i) > u^i(x^i)$, which contradicts the strategy–proofness of f at u. Thus $f(u^{-i}, \widetilde{u}^i) = x$. Repeat this argument for each $j \in C$. ∎

LEMMA 3. *If $n = 2$, then the range of a strategy–proof social choice function is diagonal.*

PROOF: Suppose the contrary. Then there exist preference profiles u and $\overline{u}$ such that $f(u) = x \neq y = f(\overline{u})$, and (without loss of generality) $x^1 \geq y^1$. This implies that and $y^2 \geq x^2$. For each agent $i = 1, 2$ choose preference $\widetilde{u}^i \in U^i$ which is simultaneously a concavification of u^i through x and a concavification of $\overline{u}^i$ through y. The existence of such a $\widetilde{u}^i$ is assured by the fact that either $x^i \geq y^i$ or $y^i \geq x^i$. Since f is non–bossy[17] and strategy–proof, it follows from Lemma 2 that $f(\widetilde{u}) = x$. Likewise, $f(\widetilde{u}) = y$, which contradicts the fact that f is single valued. ∎

Remark that the above is not true for $n \geq 3$ as is clear from the general definition of fixed price trading.

DEFINITION. *A social choice function f is coalitionally strategy–proof if for all $C \subset \{1, 2, \ldots, n\}$, u, $\overline{u}^C$, there exists $i \in C$ such that $u^i(f(u)) \geq u^i(f(u^{-C}, \overline{u}^C))$ for each $i \in C$.*

This is a weak version of coalitional strategy–proofness. It requires that no coalition can deviate and make all of its members strictly better off.

LEMMA 4. *If f is strategy–proof and non–bossy, then it is coalitionally strategy–proof.*

PROOF: Suppose that for some coalition C there exist u and $\overline{u}^C$ such that $u^i(f(u^{-C}, \overline{u}^C)) > u^i(f(u))$ for each $i \in C$. For each i in C, let $\widetilde{u}^i$ simultaneously be a concavification of u^i

[17] Since $n = 2$, $x^1 \neq y^1$ if and only if $x^2 \neq y^2$ and so any f is necessarily non–bossy.

through $f(u)$ and a concavification of $\overline{u}^i$ through $f(u^{-C}, \overline{u}^C)$.[18] It follows from Lemma 2 that $f(u^{-C}, \widetilde{u}^C) = f(u)$ and that $f(u^{-C}, \widetilde{u}^C) = f(u^{-C}, \overline{u}^C)$ which contradicts the fact that f is single valued. ▮

PROOF OF THEOREM 2.

It is straightforward to check that a social choice function which is the result of fixed–proportion trading is strategy–proof and individually rational with respect to an endowment point. We show the converse. Let f be a strategy–proof social choice function which is individually rational with respect to an endowment point e. We show that f must be defined according to fixed proportion trading. If A_f has less than three points, then the result is straightforward. Therefore, suppose that $\#A_f \geq 3$.

We let $O_{-i}(u^i) = \{z \mid \exists u^{-i} \text{ s.t. } z = f(u^{-i}, u^i)\}$ denote the option set which agent i offers to the other agent(s).

LEMMA A. *If $x \in O_{-i}(u^i)$ and $y \in O_{-i}(u^i)$, where $u^i(x^i) > u^i(y^i)$, then either $y \succeq_i ex$ or $yx \succeq_i e$.*

PROOF: Without loss of generality, let $i = 1$. Suppose the contrary so that $x \in O_2(u^1)$, $y \in O_2(u^1)$, and $u^1(x^1) > u^1(y^1)$; but $y \not\succeq_1 ex$ and $yx \not\succeq_1 e$.

Let u^2 be such that $f(u^1, u^2) = y$. For each $\epsilon \in (0,1)$, choose $\widetilde{u}^2_\epsilon$ a concavification of u^2 through y such that $\widetilde{u}^2_\epsilon(z^2) \geq \widetilde{u}^2_\epsilon(x^2) \Rightarrow yx \succeq_1 (1-\epsilon)z$. It follows from Lemma 2 that

$$f(u^1, \widetilde{u}^2_\epsilon) = y \ \forall \epsilon \in (0,1) \tag{1}$$

For each $\epsilon \in (0,1)$, choose $\widetilde{u}^1_\epsilon$ a concavification of u^1 through x such that $\widetilde{u}^1_\epsilon(z^1) \geq \widetilde{u}^1_\epsilon(e^1) \Rightarrow (1+\epsilon)z \succeq_1 xe$. From Lemma 2, it follows that $x \in O_2(\widetilde{u}^1_\epsilon)$ for all $\epsilon \in (0,1)$.

Individual rationality for agent 1 implies that $(1+\epsilon)f(\widetilde{u}^1_\epsilon, \widetilde{u}^2_\epsilon) \succeq_1 xe$. Strategy–proofness for agent 2 and the fact that $x \in O_2(\widetilde{u}^1_\epsilon)$ implies that $yx \succeq_1 (1-\epsilon)f(\widetilde{u}^1_\epsilon, \widetilde{u}^2_\epsilon)$. Combining these observations,

$$f(\widetilde{u}^1_\epsilon, \widetilde{u}^2_\epsilon) \in \{z \mid yx \succeq_1 (1-\epsilon)z \text{ and } (1+\epsilon)z \succeq_1 xe\}. \tag{2}$$

By the continuity of u^1 we can find a neighborhood B of x such that $u^1(z^1) > u^1(y^1)$ for all $z \in B$. For some small enough ϵ, $\{z \mid yx \succeq_1 (1-\epsilon)z \text{ and } (1+\epsilon)z \succeq_1 xe\} \subset B$.[19] From (1) and (2) it follows that for small enough ϵ $u^1(f^1(\widetilde{u}^1_\epsilon, \widetilde{u}^2_\epsilon)) > u^1(f^1(u^1, \widetilde{u}^2_\epsilon))$, which contradicts the strategy–proofness of f. ▮

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[18] Since $u^i(f(u^{-C}, \overline{u}^C)) > u^i(f(u))$, it is possible to find $\widehat{u}^i$ which is a concavification of u^i through both $f(u)$ and $f(u^{-C}, \overline{u}^C)$. Let u'^i be a concavification of $\overline{u}^i$ through $f(u^{-C}, \overline{u}^C)$. Choose $\widetilde{u}^i$ so that $C(\widetilde{u}^i, f(u)) = C(\widehat{u}^i, f(u))$ and $C(\widetilde{u}^i, f(u^{-C}, u^C)) = C(\widehat{u}^i, f(u^{-C}, u^C)) \cap C(u'^i, f(u^{-C}, u^C))$.

[19] Our supposition that $y \not\succeq_1 ex$ and $yx \not\succeq_1 e$ implies $\{x\} = \{z \mid yx \succeq_1 z \text{ and } z \succeq_1 xe\}$. [To see this, consider z in this set. There exist γ and α in [0,1] such that $\gamma y^1 + (1-\gamma)x^1 \geq z^1 \geq \alpha e^1 + (1-\alpha)x^1$. If both γ and α are 0, then $z = x$. Otherwise, if $0 \neq \gamma \geq \alpha$ then $y^1 \geq \frac{\alpha}{\gamma}e^1 + \frac{(\gamma-\alpha)}{\gamma}x^1$ which implies $y \succeq_1 ex$ contradicting our supposition. If $\alpha > \gamma$ then $\frac{\gamma}{\alpha}y^1 + \frac{(\alpha-\gamma)}{\alpha}x^1 \geq e^1$ which implies $yx \succeq_1 e$ contradicting our supposition.] The set $\{z \mid yx \succeq_1 (1-\epsilon)z \text{ and } (1+\epsilon)z \succeq_1 xe\}$ is closed and shrinks uniformly around x as a function of ϵ. Thus for any neighborhood B of x, there exists a small enough ϵ so that $\{z \mid yx \succeq_1 (1-\epsilon)z \text{ and } (1+\epsilon)z \succeq_1 xe\} \subset B$.

LEMMA B. *If x and y are points in A_f, and $y \succeq_i ex$, then $y \in ex$.*

PROOF: Without loss of generality let $i = 2$. Suppose that $y \notin ex$. This implies that $y \neq x \neq e \neq y$. By the diagonality of A_f, there exists $p \in R^l_{++}$ such that $p \cdot e^1 = p \cdot x^1$ (and so $p \cdot e^2 = p \cdot x^2$).

We can find u^1 such that $u^1(x^1) > u^1(y^1) > u^1(e^1)$, $u^1(x^1) > u^1(z^1)$ for all z such that $z \neq x$ and $p \cdot z^1 \leq p \cdot x^1$, and such that $x \in O_2(u^1)$.[20] Thus by Lemma A it follows that $z \in O_2(u^1)$ implies that $p \cdot z^1 \geq p \cdot x^1$. We can also find u^2 such that $u^2(y^1) > u^2(e^1)$ and $u^2(e^1) > u^2(z^1)$ for all $z \neq e$ such that $p \cdot z^1 \geq p \cdot x^1$.[21] Individual rationality for agent 2 implies that $f(u) = e$. This contradicts coalitional strategy-proofness (Lemma 4 and notice non-bossiness is always satisfied when $n = 2$), since $u^1(y^1) > u^1(e^1)$ and $u^2(y^1) > u^2(e^1)$. ∎

LEMMA C. *If x and y are points in A_f, then at least one of the following holds: $e \succeq_1 xy$; $e \succeq_2 xy$; x, y, and e are collinear.*

PROOF: If x, y, and e are not all distinct, then it is clear that they are collinear. So consider the case in which $x \neq y \neq e \neq x$. If $y \succeq_1 ex$, $y \succeq_2 ex$, $x \succeq_1 ey$, or $x \succeq_2 ey$, then the Lemma follows from Lemma B. So we are left to consider the case in which $y \not\succeq_1 ex$, $y \not\succeq_2 ex$, $x \not\succeq_1 ey$, and $x \not\succeq_2 ey$. Suppose that the statement in the lemma does not hold so that $e \not\succeq_1 xy$ and $e \not\succeq_2 xy$. The fact that no point is greater than any convex combination of the other two ($y \not\succeq_1 ex$, $y \not\succeq_2 ex$, $x \not\succeq_1 ey$, $x \not\succeq_2 ey$, $e \not\succeq_1 xy$ and $e \not\succeq_2 xy$) implies that $l \geq 3$ and that there exists $p \in I\!R^l_{++}$ such that $p \cdot e^2 = p \cdot x^2 = p \cdot y^2$. There exists $p' \in I\!R^l_{++}$ close to p such that $p' \cdot y^2 > p' \cdot e^2 = p' \cdot x^2$. Now repeat the argument from Lemma B (using p' in the place of p). ∎

LEMMA D. *If x, y, and z are distinct points in A_f such that no two are collinear with e, then there exists i such that $e \succeq_i xy$, $e \succeq_i yz$ and $e \succeq_i xz$.*

PROOF: Suppose the contrary. Since no two of the points are collinear with e, we know from Lemma C that it must be that $e \succeq_i$ two of the pairs, while $e \succeq_j$ the third where $j \neq i$. Without loss of generality, suppose that $e \succeq_1 xy$, $e \succeq_1 yz$, and $e \succeq_2 xz$.

By diagonality $y^1_k > e^1_k$ for some k. Then $e \succeq_1 xy$ and $e \succeq_1 yz$ implies that $z^1_k < e^1_k$ and $x^1_k < e^1_k$. But this implies that $z^2_k > e^2_k$ and $x^2_k > e^2_k$, which contradicts the fact that $e \succeq_2 xz$. ∎

LEMMA E. *There exists i such that for any x and y in A_f either $e \succeq_i xy$, $x \in ey$, or $y \in ex$.*

PROOF: If there exist x, y, and z, distinct points in A_f such that no two are collinear with e, then the result follows from Lemma D. Otherwise, all points lie on at most two line segments which have e as endpoints. The result then follows from Lemma C. ∎

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[20] First, find $\overline{u}^1$ such that $\overline{u}^1(x^1) > \overline{u}^1(y^1) > \overline{u}^1(e^1)$, which is possible since by the diagonality of A_f and the fact that $y \succeq_2 ex$ implies that $e \not\succeq_1 xy$. Concavify $\overline{u}$ through x so that $\overline{u}^1(x^1) > \overline{u}^1(z^1)$ for all z such that $z \neq x$ and $p \cdot z^1 \leq p \cdot x^1$. Next find $\widehat{u}^1$ such that $x \in O_2(\widehat{u}^1)$. Let $\widetilde{u}^1$ be a concavification of $\widehat{u}^1$ through x. Let u^1 have the same upper contour sets as $\overline{u}^1$ through e and y, and have upper contour set $C(\overline{u}^1, x) \cap C(\widetilde{u}^1, x)$ through x.

[21] Let the upper contour set of u^2 through e intersect the hyperplane $\{z \mid p \cdot z^2 = p \cdot e^2\}$ only at e, and be close enough to the hyperplane so that $u^2(y^2) > u^2(e^2)$.

Let i be the agent identified in Lemma E. Lemma E implies that A_f lies on $k \leq l$ line segments emanating from e. Pick one of these line segments and let a denote its endpoint (recall that A_f is closed). Let U_a denote the set of u's such that $T^i(A_f, u) \subset ea$. It follows from individual rationality[22] that $f(u) \in ea$. It follows from Lemma 4 (which implies unanimity) that the range of f restricted to U_a is $A_f \cap ea$. It is also true that U_a is the set of all utilities which are single peaked over $A_f \cap ea$. The characterization of f over $ea \cap A_f$ then follows from Theorem 3 in Barbera and Jackson (1991). Then notice that U^i is partitioned by the selection of the k different a's.

PROOF OF THEOREM 1

Theorems 1 is a special case of Theorem 2.

PROOF OF THEOREM 3.

We show that if f is strategy–proof then it satisfies (1)–(4). Again, the converse is easily checked.

We accomplish this by extending the domain of f to include extra utility functions which are single peaked over the range, but are not in U^i. We extend f so that it is still strategy–proof over the extended domain. We then show that the range of the extended function just over the new utility functions satsifies the candidate range condition and includes the entire original range. We then characterize f over the new part of the domain, and from that derive the characterization over the original domain.

Define the binary relation $\succ$ over A_f by $x \succ y$ if $x_1^1 > y_1^1$. By Lemma 3, A_f is diagonal and so $\succeq$ is complete and transitive when restricted to points in A_f. Let $\overline{A}_f$ denote the closure of A_f. It follows that $\succeq$ is complete and transitive when restricted to points in $\overline{A}_f$.

The utility function u^i is *single peaked* on $\overline{A}_f$ if $T^i(A_f, u)$ is a single point and for each x and y in $\overline{A}_f$: $x \succ y \succeq t^i(\overline{A}_f, u)$ or $t^i(\overline{A}_f, u) \succeq y \succ x$ implies that $u^i(x^i) < u^i(y^i) \leq u^i[t^i(\overline{A}_f, u)]$. Let S_f^i denote the set of all continuous utility functions defined over A which depend only on i's allocation and which are single peaked on $\overline{A}_f$.

We extend f to the domain $(U^1 \cup S_f^1) \times (U^2 \cup S_f^2)$ as follows. The extension is denoted as $\overline{f}$.

(1) If $u \in U^1 \times U^2$ then $\overline{f}(u) = f(u)$

(2) If $u \in S_f^1 \times U^2$ and $u^1 \notin U^1$, then $\overline{f}(u)$ is the x in the closure of $\{x \mid \exists v^1 \in U^1,\ f(v^1, u^2) = x\}$ which maximizes u^1. If there is more than one such x choose the one of those which maximizes u^2. [Since u^1 is single peaked on $\overline{A}_f$, it follows that there are at most two such points.] If both agents are indifferent, then choose the smallest one according to $\succeq$.

(3) If $u \in U^1 \cup S_f^1 \times S_f^2$ and $u^2 \notin U^2$, then $\overline{f}(u)$ is the x in the closure of $\{x \mid \exists v^2 \in U^2,\ \overline{f}(u^1, v^2) = x\}$ which maximizes u^2. If there is more than one such x choose the one of those which maximizes u^1. If both agents are indifferent, then choose the smallest one according to $\succeq$.

LEMMA A'. *$\overline{f}$ is strategy–proof over $(U^1 \cup S_f^1) \times (U^2 \cup S_f^2)$.*

PROOF: First we show that $\overline{f}$ is strategy–proof over $(U^1 \cup S_f^1) \times U^2$. Then we show that $\overline{f}$ is strategy–proof over $(U^1 \cup S_f^1) \times (U^2 \cup S_f^2)$.

[22] The strict quasi–concavity of u^i and Lemma E imply that $u^i(e^i) > u^i(b^i)$ for any $b \notin ea$.

Suppose that $\overline{f}$ is manipulable at $u \in (U^1 \cup S^1_f) \times U^2$ by one of the two agents.

Case 1: There exists $\widehat{u}^1 \in (U^1 \cup S^1_f)$ such that $u^1(\overline{f}^1(\widehat{u}^1, u^2)) > u^1(\overline{f}^1(u))$.

By the definition of $\overline{f}$ (2), agent 1 cannot manipulate $\overline{f}$ at $u^1 \in S^1_f$, $u^1 \notin U^1$. Thus $u^1 \in U^1$. Also by (2) $\overline{f}(\widehat{u}^1, u^2)$ is in the closure of $\{x \mid \exists v^1 \in U^1,\ \overline{f}(v^1, u^2) = x\}$. Find $a \in \{x \mid \exists v^1 \in U^1,\ \overline{f}(v^1, u^2) = x\}$. close enough to $\overline{f}(\widehat{u}^1, u^2)$ so that $u^1(a^1) > u^1(\overline{f}^1(u))$. (Recall that u^1 is continuous.) This contradicts the strategy–proofness of f.

Case 2: There exists $\widehat{u}^2 \in U^2$ such that $u^2(\overline{f}^2(u^1, \widehat{u}^2)) > u^2(\overline{f}^2(u))$.

Since f is strategy–proof, it must be that $u^1 \in S^1_f$, $u^1 \notin U^1$. Let $a = \overline{f}(u^1, \widehat{u}^2)$, $b = \overline{f}(u)$, $\sigma^1(u^2) = \{x \mid \exists v^1 \in U^1,\ f(v^1, u^2) = x\}$, and $\sigma^1(\widehat{u}^2) = \{x \mid \exists v^1 \in U^1,\ f(v^1, \widehat{u}^2) = x\}$. Since u^2 is continuous, there are neighborhoods $B(a)$ and $B(b)$ such that $u^2(a'^2) > u^2(b'^2)$ for all $a' \in B(a)$ and $b' \in B(b)$. Assume that $u^1(a^1) \geq u^1(b^1)$ [an analagous argument handles the other case]. Since a is not in the closure of $\sigma^1(u^2)$ [otherwise by (2) it would be selected over b], there exists some neighborhood $B'(a) \subset B(a)$ such that $B'(a) \cap \sigma^1(u^2) = \emptyset$. Notice also that since a is in the closure of $\sigma^1(\widehat{u}^2)$, then any neighborhood of a must intersect $\sigma^1(\widehat{u}^2)$. Find $v^1 \in U^1$ which has peak in $B'(a) \cap \sigma^1(\widehat{u}^2)$, and such that there is some $b' \in B(b)$ such that $v^1(b') > v^1(c)$ for all $c \in \sigma^1(u^2) \cup \sigma^1(\widehat{u}^2)$, $c \notin B(b) \cup B'(a)$.[23] It follows that $f(v^1, \widehat{u}^2) \in B'(a)$ while $f(v^1, u^2) \in B(b)$. Thus $u^2(f^2(v^1, \widehat{u}^2)) > u^2(f^2(v^1, u^2))$, which contradicts the strategy–proofness of f.

We have shown that $\overline{f}$ is strategy–proof over $(U^1 \cup S^1_f) \times U^2$. The argument to show that $\overline{f}$ is strategy–proof over $(U^1 \cup S^1_f) \times (U^2 \cup S^2_f)$ parallels cases 1 and 2 above. ∎

LEMMA B'. *The restriction of $\overline{f}$ to $S^1_f \times S^2_f$ can be written as*

$$\overline{f}(u) = min\{a, max[t^i(\overline{A}_f, u), b], max[t^j(\overline{A}_f, u), c], max[t^i(\overline{A}_f, u), t^j(\overline{A}_f, u), d]\},$$

where max and min are defined relative to $\succeq$, t^i and t^j are strategy–proof tie breaking rules,[24] *and $a \succeq b \succeq c \succeq d$ for $a, b, c, d \in \overline{A}_f$.*

PROOF: This follows from Theorem 3 in Barbera and Jackson (1990). The fact that S^i_f includes all the needed single peaked functions can be verified since A_f is diagonal and we can then use "V" shaped utility functions to get the necessary single peaked functions over $\overline{A}_f$. That is , in S^i_f it is no longer necessary that utility be increasing. Thus S^i_f includes utility functions whose upper contour sets are subsets of the upper contour sets of Leontief preferences. These preferences generate all the single peaked preferences over $\overline{A}_f$. ∎

LEMMA C'. *$\overline{A}_f$ satisfies (1) (of the definition preceding Theorem 3) with a, b, c, d corresponding to those given in Lemma B'.*

PROOF: Strategy–proofness implies that unanimity is satisfied for $u \in S^1_f \times S^2_f$. This implies that $\overline{f}$ restricted to $S^1_f \times S^2_f$ has range $\overline{A}_f$. From Lemma B' we know that (i) holds.

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[23] Without loss of generality, assume that $a \succeq b$. Notice that since $u^1 \in S^1_f$ and $u^1(a^1) \geq u^1(b^1)$, it must be that if $x \in \sigma^1(u^2)$, then either $x \succeq a$ or $b \succeq x$. [Otherwise 1 would choose x instead of b from $\sigma^1(u^2)$.] Given the diagonality of A_f and the fact that $\sigma^1(u^2) \cup \sigma^1(\widehat{u}^2)$ does not include any points between a and b, such a v^1 can be found.

[24] The rule $t^i(\overline{A}_f, u)$ is strategy–proof if and only if $t^i(\overline{A}_f, u) \neq t^i(\overline{A}_f, u^i, v^j)$ implies $v^j(t^i(\overline{A}_f, v^j, u^i)) \geq v^j(t^i(\overline{A}_f, u))$.

Next we verify (ii). Suppose that $x \in \overline{A}_f$, $x \succeq b$, but that x, a, and b are not collinear. One of the two agents, say agent 1, has a utility function $u^1 \in U^1$, with a as the most preferred point (or else some a' close to a), which is preferred to b which is preferred to x. Let agent 2 have a single peaked preference u^2 with x as peak. By strategy–proofness and the form of Lemma B', we know that $\overline{f}(u) = y$, where $x \succ y$. [It is not $d \succ x$ since then if agent 1 had a single peaked preference with peak at a, the outcome of f by Lemma B' would be x, and agent 1 would be better off announcing u^1 and getting y. It is not x, since then agent 1 could benefit from from deviating from u^1 and announcing a single peaked preference with peak at b, thus obtaining b which is preferred to x under u^1.] Since the only assumption made about u^2 is that it is single peaked at x, the above argument still holds if $u^2(a^2) > u^2(y^2)$, which is a feasible preference. But this contradicts strategy–proofness, since agent 2 can then deviate and announce a single peaked preference with peak at a thus improving the outcome from y to a. Hence our assumption was wrong. Parallel arguments establish that $c \succeq x \succeq d$ are collinear. Thus (ii) holds.

Next we verify (iii). Consider i, the agent defined in Lemma B'. Notice that when $u^j \in S^j_f$, that by the structure of the expression in Lemma B', agent i can always obtain any outcome between c and b. Define g as in (iii). We show that all points in $\overline{A}_f$ are "below" the line segment connecting a and g, relative to i's preferences. More specifically, there is no point x and preference $u^i \in S^i_f$ such that $b \succeq x \succ c$ and $u^i(a^i) > u^i(x^i) > u^i(b^i)$ (where z is the peak of u^i over $\overline{A}_f$). Suppose the contrary. Let y be i's most preferred point between b and c. Let j have the single peaked preference with peak at c and $u^j(a^j) > u^j(y^j)$. By strategy–proofness and Lemma B', the outcome of $\overline{f}$ at u is y. [The point cannot be outside of b and c since then i would wish to manipulate $\overline{f}$ from some single peaked preference with peak outside b and c. And since by Lemma B i can obtain any point between b and c, it must be his best in this interval.] Now by deviating and announcing a single peaked preference with peak at a, j can change the outcome from y to a, an improvement. Thus our supposition was wrong. ∎

LEMMA D'. *f is obtained by the procedure* (2) $\rightarrow$ (4) *over* A_f.

PROOF: Combining Lemmas B' and C', we know that this is true on $S^1_f \times S^2_f$ over $\overline{A}_f$. This implies that if for some $u \in U^1 \times U^2$ agent i ever has a peak in the dictatorial region, then one of agent i's tops is chosen (according to a strategy–proof tie breaking rule). Otherwise, agent i's peak over $\overline{A}_f$ lies on one of the segments and behaves as a single peaked preference on the segment. It is then easily checked that the characterization from Lemma B' extends. ∎

PROOF OF THEOREM 4.

We maintain the assumption that the social choice function f is tie–free.

The proof proceeds as follows. Step 1 verifies that if a social choice function satisifies (1)–(4), then it is strategy–proof , anonymous, and non–bossy. Steps 2 through 5 establishe the converse. These rely on our knowledge of two person strategy–proof rules. Given coalitional strategy–proofness (Lemma 4 above), we know what an n person rule looks like when agents can be partitioned into two groups who have identical utility functions by applying Theorem 2. [There are some details to work out since the range of the n person rule lies in $\mathbb{R}^{nl}$ while the two person rules have range in $\mathbb{R}^{2l}$.] Steps 2 and 3 use this logic to define the shape of the range, and in particular to establish that (1) and (2) of the definition of fixed proportion trading are satisfied. Step 4 shows that (3) and (4) hold on the limited domain where agents can be partitioned into two groups who have identical utility functions. The rest of the proof consists of making sure that trade is still limited to these proportions when agents have more diversity in their preferences. Step 5 shows that if (3) and (4) hold when agents can be partitioned into m groups such that all the members of a given group have the same utility function, then (3) and (4) also hold when agents can be partitioned into $m+1$ groups such that all the members of a given group have the same utility function. Thus by induction, Steps 4 and 5 establish that (3) and (4) hold, generally.

STEP 1. *A social choice function satisfying (1)–(4) is strategy–proof , anonymous, and non–bossy.*

PROOF: Let us verify that f is strategy–proof : Let $f(u) = x$ and consider $f(u^{-i}, \overline{u}^i) = y$. Notice that by (3) $u^i(x^i) \geq u^i(e^i)$. Thus if $y^i = e^i$, then $u^i(x^i) \geq u^i(y^i)$. So consider the case in which $y^i \neq e^i$, which implies that (3) applies and a match occurs at $u^{-i}, \overline{u}^i$ and trade is in proportion a. If x^i is also a trade in proportion a , then it follows from (4) that $u^i(x^i) \geq u^i(y^i)$. If x^i is not a trade in proportion a, then either no match occurs at u or some match other than a occurs at u. Thus, $\text{sign}[\alpha(\overline{u}^i, a)] \neq \text{sign}[\alpha(u^i, a)]$ and so $u^i(e^i) > u^i(y^i)$. Since $u^i(x^i) \geq u^i(e^i)$, it follows that $u^i(x^i) > u^i(y^i)$.

Anonymity is easily checked: Let $f(u) = x$ and $u^i = u^j$. If $x = e$, then $x^i = x^j$. If trade occurs, then according to (3), $x^i - e^i$ and $x^j - e^j$ are in the same proportion. If both agents recieve their most preferred trades in that proportion then $x^i = x^j$. Otherwise, by (4) the agents are rationed equally and so $x^i = x^j$.

We now verify that f is non–bossy. Consider any i, u and $\overline{u}^i$, such that $f(u) = x$, $f(u^{-i}, \overline{u}^i) = y$, and $x^i = y^i$. We show that $x = y$. First consider the case where $x^i \neq e^i$. Then (3) applies, $x^i = e^i + r^i\alpha(u^i, a)a$ and $y^i = e^i + \overline{r}^i\alpha(\overline{u}^i, a)a$. Since $y^i = x^i \neq e^i$ it must be that $\text{sign}[\alpha(\overline{u}^i, a)] = \text{sign}[\alpha(u^i, a)]$, and $|\alpha(\overline{u}^i, a)| \geq r^i|\alpha(u^i, a)|$. If $1 > r^i$ or $r^i = 1$ and $|\alpha(\overline{u}^i, a)| = |\alpha(u^i, a)|$, then $x = y$ follows from (4) and the definition of uniform rationing. The only other possibility is that $r^i = 1$ and $|\alpha(\overline{u}^i, a)| > |\alpha(u^i, a)|$. Since $x^i = y^i$, this means that $1 > \overline{r}^i$ and $|\alpha(u^i, a)| \geq \overline{r}^i|\alpha(\overline{u}^i, a)|$. Applying (4) (and the definition of uniform rationing) implies that $x = y$. Next we consider the case where $x^i = e^i = y^i$. It follows directly from (4) that $x = y$. ∎

Next, we show that if a social choice function is strategy–proof , anonymous, and non–bossy, then it satisfies (1)–(4). For any integer k, where $n \geq k \geq n/2$ let

$$U_k = \{u \in U \mid \exists C \subset \{1, \ldots, n\} \text{ s.t. } \#C = k \text{ and } u^i = u^j \text{ if } i, j \in C \text{ or } i, j \notin C\}$$

U_k denotes the part of the domain in which agents can be partitioned into a group of size k and a group of size $n - k$, such that all agents of each group have the same utility. Notice that $\cup_{n/2 \leq k \leq n} U_k$ includes all profiles where agents can be partitioned into (at most) two groups such that all agents of each group have the same utility.

STEP 2. *(1) holds. That is, for each k, $n > k \geq n/2$, there exists a trade proposal $P(k)$, and if n is even, then for each a and b in $P(n/2)$ there exists $\gamma < 0$ such that $b = \gamma a$. Furthermore, if $u \in U_k$ then trade occurs only if some $a \in P(k)$ is matched and then trade occurs in that proportion.*

PROOF: By anonymity, we can assume without loss of generality that the partition for U_k is of the first k agents and the next $n - k$ agents.

Using coalitional strategy–proofness (Lemma 4) and anonymity we can then apply Theorem 2 to obtain a characterization for f on the part of the domain where u is in U_k for some $n \geq k \geq n/2$ (i.e. when agents can be partitioned into two groups whose members have identical utilities). From the application of Theorem 2,[25] it follows that for one of the two groups either $x \in ey$, $y \in ex$, or $e \succeq_i xy$ for all x and y in the range of f restricted to U_k and i in the group. Find a subset P' of the range of f restricted to U_k such that if $x \in P'$ then $x \neq e$ and if $x, y \in P'$ then $x \notin ey$ and $y \notin ex$, and also such that for all z in the range of f restricted to U_k there exists $x \in P'$ such that $z \in ex$ or $x \in ez$. [The condition from the two person characterization implies that the range lies on line segments emanating from e. P' is constructed by picking exactly one point from each segment.] Now let $P(k) = \{a \in \mathbb{R}^l \mid \exists x \in P' \ s.t. \ a = x^i - e^i\}$ (pick any i in the group since by anonymity they all get the same net trade). Notice then by the conditions on P' (condition (1) of theorem 2) that for all a and b in $P(k)$ there exists $\gamma \in (0,1)$ such that $\gamma a + (1-\gamma)b \leq 0$.

Next, we show that if n is even, then $P(n/2)$ has at most two elements, and then these must be opposites. Suppose that there exist $a \in P(n/2)$ and $b \in P(n/2)$ such that $a \neq b$. Find $u \in U_{n/2}$ such that $\alpha(u^i, a) > 0$ for $n/2 \geq i \geq 1$ and $\alpha(u^i, b) > 0$ for $n \geq i > n/2$. From our previous arguments we know that $\gamma a + (1-\gamma)b \leq 0$. This implies that $a \leq -\frac{(1-\gamma)}{\gamma} b$ and $b \leq -\frac{\gamma}{(1-\gamma)} a$. Therefore $\alpha(u^i, b) < 0$ for $n/2 \geq i \geq 1$ and $\alpha(u^i, a) < 0$ for $n \geq i > n/2$. Thus trade occurs in both proportions by Theorem 2 and anonymity. This means that there exists $\gamma' > 0$ such that $a = -\gamma' b$.

We have identified $P(k)$ and shown that it is a trade proposal. We now show that trade occurs at u in U_k only if $a \in P(k)$ is matched and then occurs in proportion a. In this case it must be that either agents $1 \to k$ or else agents $k+1 \to n$ who play the role of the i from the two person characterization. We show that it is the larger coalition $1 \to k$ which plays the role of i in the two person analogy. This means that trade only occurs if $a \in P(k)$ is matched. If $k = n/2$ then the coalitions are of identical size and so by anonymity it must be that either coalition can match the proposal. If $k = n$ then by anonymity there is no trade. So consider the case where $n > k > n/2$ and suppose that it is the smaller coalition (agents $k+1$ to n) which plays the role of i in determining the trade. Also suppose that $P(k)$ contains at least two proportions a and b such that $b \neq \gamma a$ for all $\gamma \in \mathbb{R}$. [Otherwise trade only occurs when one group demands positive multiples of a and the other group demands postive multiples of $-a$ and so we can define $P(k)$ so that trade only occurs when the larger coalition matches a trade in $P(k)$.] Notice that if agents $i > k$ get the allocation $e^i + b$, then agents $i \leq k$ must receive $e^i - \frac{n-k}{k} b$. Find u^1 such that $u^1(e^1 - \frac{n-k}{k} b) > u^1(e^1 - \frac{n-k}{k} a) > u^1(e^1)$ and so that $(e^1 - \frac{n-k}{k} b)$ is most preferred over the line in proportion b and $(e^1 - \frac{n-k}{k} a)$ is most preferred on the line in proportion a. [By the definition of $P(k)$ there exists $\gamma' \in (0,1)$ such that $\gamma' a + (1-\gamma')b \leq 0$. By our supposition $\gamma b \neq a$ for all $\gamma \in \mathbb{R}$ and so $\gamma' a + (1-\gamma')b \neq 0$. This means that for some $\gamma \in (0,1)$ $-\gamma \frac{n-k}{k} a - (1-\gamma)\frac{n-k}{k} b \geq 0$ and $\neq 0$. It follows that we can find $p \in \mathbb{R}^l_{++}$ such that $p \cdot (e^1 - \frac{n-k}{k} a) = p \cdot e^1 < p \cdot (e^1 - \frac{n-k}{k} b)$. We can find u^1 to have $e^1 - \frac{n-k}{k}$ the most

[25] To be careful about the application, for $u \in U_k$ let $f_k(\hat{u}^1, \hat{u}^2) = [kf^1(u), (n-k)f^n(u)]$, $\hat{u}^1(x^1) = u^1(x^1/k)$ and $\hat{u}^2(x^2) = u^n(x^2/(n-k))$. It follows from Lemma 4 that f_k is a strategy–proof two person social choice function . It is also individually rational with respect to the equal split endowment. [If it was not individually rational at some $u \in U_k$, then it would not be individually rational for all agents in at least one of the two groups. If all these agents switched an announced the same utility as the other agents, then by anonymity there would be an equal split of the total endowment. This would provide a coalitional manipulation contradicting Lemma 4.] Thus, we can apply Theorem 2 to characterize f_k, which in turn provides restrictions on f over the domain U_k.

preferred point from the plane $\{z \in A \mid p \cdot z^1 = p \cdot e^1\}$ and upper contour set close enough to the plane so that it falls below $e^1 - \frac{k}{n-k} b$.] Let $\widehat{u}^1$ be a concavification of u^1 through $(e^1 - \frac{n-k}{k} a)$ so that $(e^1 - \frac{n-k}{k} a)$ is most preferred on the line in proportion a and $e^1 + b$ is the most preferred point on the line in proportion b. [Notice that by the construction of u^1, $u^1(e^1) > u^1(e^1 + b)$ and so $u^1(e^1 - \frac{n-k}{k} a) > u^1(e^1 + b)$. Thus such a concavification is possible.] Find u^n so that $(e^n + a)$ is most preferred on the line with proportion a and $(e^n - \frac{n-k}{k} b)$ is the most preferred point on the line through proportion b. Let u denote the profile so that $u^i = u^1$ if $i \leq k$ and $u^i = u^n$ if $i > k$. Then $f(u) = x$, where $x^i = e^i - \frac{n-k}{k} a$ for $i \leq k$. Let $\overline{u}$ denote the profile so that $\overline{u}^i = \widehat{u}^1$ if $i \leq n-k$ and $\overline{u}^i = u^1$ if $k \geq i > n-k$ and $\overline{u}^i = u^n$ if $i > k$. It follows from Lemma 2 that $f(\overline{u}) = x$. Let $\widetilde{u}$ denote the profile so that $\widetilde{u}^i = \widehat{u}^1$ if $i \leq n-k$ and $\widetilde{u}^i = u^n$ if $i > n-k$. Then $f(\widetilde{u}) = y$, where $y^i = e^i - \frac{n-k}{k} b$ for $i > n-k$. This contradicts coalitional strategy–proofness (Lemma 4) since for $k \geq i > n-k$ the outcome $e^i - \frac{n-k}{k} b$ is preferred to $e^i - \frac{n-k}{k} a$ at the utility $\overline{u}$ and so these agents can all announce $\widetilde{u}^i = u^n$ to manipulate the outcome. ∎

STEP 3. *(2) holds. That is, the trade proposals are nested: for each $k' < k$ and $a \in P(k')$ and $b \in P(k)$, either there exists $\gamma > 0$ such that $\gamma b \leq a$ or there exists $\gamma \in (0,1)$ such that $\gamma a + (1-\alpha) b \leq 0$ (where $0 \in I\!R^l$).*

PROOF: Suppose the contrary for some $k' < k$ and $a \in P(k')$ and $b \in P(k)$. By the definition of $P(\cdot)$ [see Step 2] there exist x and y such that for $i \leq k'$ $x^i = e^i + a$ and for $i > k'$ $x^i = e^i - \frac{k'}{n-k'} a$, and for for $j \leq n-k$ $y^j = e^j - \frac{k}{n-k} b$, while for $i > n-k$ $y^j = e^j + b$.
Case 1. There exists $p \in I\!R^l_{++}$ such that $p \cdot a > p \cdot b = 0$.

Let u^1 be such that and so that $e^1 + a$ is the most preferred trade on the line $e^1 + \gamma a$ and $e^1 + b$ is the most preferred trade on the line $e^1 + \gamma b$ and such that $u^1(e^1 + b) > u^1(e^1 + a)$.[26] Similarly, find u^n so that $u^1(e^1 + b) > u^1(e^1 - \frac{k'}{n-k'} a)$ and so that $e^1 + b$ is the most preferred trade on the line $e^1 + \gamma b$ and $e^1 - \frac{k'}{n-k'} a$ is the most preferred trade on the line $e^1 + \gamma a$. It follows that if $u^i = u^1$ for $i \leq k'$ and $u^i = u^n$ for $i > k'$, then $f(u) = x$.

Find $\widehat{u}^1$ such that $\widehat{u}^1(e^1 + a) > \widehat{u}^1(e^1 - \frac{k}{n-k} b)$ and so that $e^1 - \frac{k}{n-k} b$ is the most preferred trade on the line $e^1 + \gamma b$, $e^1 + a$ is the most preferred trade on the line $e^1 + \gamma a$ and $\widehat{u}^1$ concavifies u^1 through $e^1 + a$.[27] It follows from Lemma 2 that if $\widehat{u}$ is such that $\widehat{u}^i = \widehat{u}^1$ for $i \leq n-k$, $\widehat{u}^i = u^1$ for $n-k < i \leq k'$, and $\widehat{u}^i = u^n$ for $i > k'$, then $f(\widehat{u}) = x$. However, if $\overline{u}^i = \widehat{u}^1$ for $i \leq n-k$ and $\overline{u}^i = u^n$ for $i > n-k$, then $f(\overline{u}) = y$. This contradicts coalitional strategy–proofness since if $n-k < i \leq k'$ switch from u^1 to u^n, their outcome changes from $e^1 + a$ to $e^1 + b$, and $u^1(e^1 + b) > u^1(e^1 + a)$.
Case 2. There is no $p \in I\!R^l_{++}$ such that $p \cdot a > p \cdot b = 0$.

Let u^n be such that $e^1 + b$ is the most preferred trade on the line $e^1 + \gamma b$ and $e^1 - \frac{k'}{n-k'} a$ is the most preferred trade on the line $e^1 + \gamma a$ and such that $u^n(e^1 - \frac{k'}{n-k'} a) > u^n(e^1 + b)$.[28]

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[26] By our supposition $\gamma b \not\leq a$ for all $\gamma > 0$ and $\gamma a + (1-\gamma) b \not\leq 0$ for all $\gamma \in [0,1]$. It follows that we can find $p' \in I\!R^l_{++}$ such that $p' \cdot b > 0 = p' \cdot a$. Take u^1 to have $e^1 + a$ the most preferred point from the plane $\{z \in A \mid p' \cdot z^1 = p' \cdot e^1\}$ and upper contour set close enough to the plane so that it falls below $e^1 + b$.

[27] Take $\widehat{u}^1$ to have $e^1 - \frac{k}{n-k} b$ the most preferred point from the plane $\{z \in A \mid p \cdot z^1 = p \cdot e^1\}$ and upper contour set close enough to the plane so that it falls below $e^1 + a$.

[28] In this case we can find $p' \in I\!R^l_{++}$ such that $p' \cdot -a > 0 = p' \cdot b$. Take u^n to have $e^1 + b$

Similarly, find u^1 so that $u^1(e^1 - \frac{k'}{n-k'}a) > u^1(e^1 - \frac{k}{n-k}b)$ and so that $e^1 - \frac{k}{n-k}b$ is the most preferred trade on the line $e^1 + \gamma b$ and $e^1 - \frac{k'}{n-k'}a$ is the most preferred trade on the line $e^1 + \gamma a$. It follows that if $u^i = u^1$ for $i \le n-k$ and $u^i = u^n$ for $i > n-k$, then $f(u) = y$.

Find $\widehat{u}^n$ such that $\widehat{u}^n(e^1 + b) > \widehat{u}^n(e^1 + a)$ and so that $e^1 + b$ is the most preferred trade on the line $e^1 + \gamma b$, $e^1 + a$ is the most preferred trade on the line $e^1 + \gamma a$ and $\widehat{u}^n$ concavifies u^n through $e^1 + b$.[29] It follows from Lemma 2 that if $\widehat{u}$ is such that $\widehat{u}^i = u^1$ for $i \le n-k$, $\widehat{u}^i = u^n$ for $n-k < i \le n-k'$, and $\widehat{u}^i = \widehat{u}^n$ for $i > n-k'$, then $f(\widehat{u}) = y$. However, if $u^i = u^1$ for $i \le n-k'$ and $u^i = u^n$ for $i > n-k'$, then $f^i(u) = e^i - \frac{k'}{n-k'}a$ for $i \le n-k'$ and $f^i(u) = e^i + a$ for $i > n-k'$. This contradicts coalitional strategy–proofness since if $n-k < i \le n-k'$ switch from u^n to u^1, their outcome changes from $e^1 + b$ to $e^1 - \frac{k'}{n-k'}a$, and $u^n(e^1 - \frac{k'}{n-k'}a) > u^n(e^1 + b)$. ▮

STEP 4. *(3)–(4) hold for the case when agents can be partitioned into two groups such that all the members of a given group have the same utility function.*

PROOF: We are necessarily in U_k for some $k \ge n/2$. If any proposal is matched at u, then since there exists k such that $u \in U_k$, it must be that we have matched $a \in P(k)$. In this case we know from Theorem 2 (see Step 2) that trade is in proportion a and the short side of the market is rationed [with the possibility of no trade]. This means that everyone receives a trade between 0 and their most desired trade in proportion a and so if a is matched then trade is in that proportion. Suppose next that no a is matched. We are either in U_n and so by anonymity there is no trade, or else $n > k \ge n/2$. In this second case, since there was no match, it must be that either $\alpha(u^i, a) > 0$ for some $a \in P(k)$ and for all i, or $\alpha(u^i, a) \le 0$ for all $a \in P(k)$ and $i \in C$ (where C is as used in the defiition of U_k). In either case, from Theorem 2 (again see Step 2) it follows that there is no trade. Thus (3) is satisfied.

If there is any trade, then from Theorem 2 it follows that one side gets their most preferred trade and the other side is rationed equally to some non–zero trade. Finally, it is clear that since all agents in each part of the partition receive the same trade, if $f^i(u) = e^i = f^i(u^{-i}, \widetilde{u}^i)$, then $f(u^{-i}, \widetilde{u}^i) = f(u) = e$. Thus (4) is satisfied. ▮

STEP 5. *If (3)–(4) hold when agents can be partitioned into $m \ge 2$ groups such that all the members of a given group have the same utility function, then (3)–(4) also hold when agents can be partitioned into $m+1$ groups such that all the members of a given group have the same utility function.*

PROOF: The verification of (3) and (4) hold is divided into several parts.

5.1 If there exists $a \in P(n-1)$ such that $\alpha(u^i, a) > 0$ for all i, then $f(u) = e$.

5.2 If $a \in P(n-1)$ is matched, then trade is in proportion a. Any rationing is done uniformly.

5.3 If for some k, $n-1 > k \ge n/2$, 5.3.1 and 5.3.2 (below) hold for all $\widetilde{k} > k$, then 5.3.1 and 5.3.2 hold for k.

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the most preferred point from the plane $\{z \in A \mid p' \cdot z^1 = p' \cdot e^1\}$ and upper contour set close enough to the plane so that it falls below $e^1 - \frac{k'}{n-k'}a$).

[29] Take $\widehat{u}^n$ to have $e^1 + a$ the most preferred point from the plane $\{z \in A \mid p' \cdot z^1 = p' \cdot e^1\}$ and upper contour set close enough to the plane so that it falls below $e^1 + b$.

5.3.1 If $\#i$ such that $\alpha(u^i, a) > 0$ is larger than k for some $a \in P(k)$ and $\#i$ such that $\alpha(u^i, b) > 0$ is less than k' for all $k' > k$ and $b \in P(k')$, then $f(u) = e$.

5.3.2 If for some $a \in P(k)$, $\#i$ such that $\alpha(u^i, a) > 0$ is equal to k, then trade is in proportion a and any rationing is done uniformly.

5.4 If $\#i$ such that $\alpha(u^i, a) > 0$ is less than k for all k and $a \in P(k)$, then either $f(u) = e$ or there exists a such that trade is in proportion a and $P(\underline{k}) = \{a\}$ or $P(\underline{k}) = \{a, -\gamma a\}$ for some $\gamma > 0$, where $\underline{k}$ is the smallest integer greater than or equal to $\frac{n}{2}$.

Induction on 5.3, given 5.1 and 5.2; combined with 5.4 cover all the possible situations which can occur when agents can be partitioned into $m + 1$ groups such that all members of each group have identical preferences.

We use A, B, and C to represent elements of the partition of agents. x^A denotes the allocation which each agent in A receives. n^A denotes the number of agents in group A.

Proof of 5.1. Suppose the contrary. By non–bossiness, it must be that all allocations are different from the endowment. [If some group C changed utilities to match some other group, then all agents get their endowment since there would be only m groups. So if C get their endowments at u, then non–bossiness implies that all agents get their endowments at u.] Pick A. For each $B \neq A$ there exists $\gamma^B \in (0,1)$ and $\lambda^B \geq 0$ such that

$$(1 - \gamma^B)(x^A - e^A) + \gamma^B(x^B - e^B) \leq -\lambda^B a. \tag{I}$$

[If not, then we can find $\widehat{u}$ such that x^A and x^B are both preferred to the endowment and $\alpha(\widehat{u}, a) > 0$. If both A and B had utility $\widehat{u}$ there would be only m groups and no trade. They could manipulate via u^A and u^B to get x^A and x^B.]

If $\lambda^B = 0$ for each $B \neq A$, then it must be that all allocations are collinear. Then find two groups B and C such that x^B and x^C lie on the same side of the endowment and $\widehat{u}$ such that x^B and x^C are both preferred to the endowment and $\alpha(\widehat{u}, a) > 0$.[30] If both B and C had utility $\widehat{u}$ there would be only m groups and no trade. They could manipulate via u^B and u^C to get x^B and x^C.

So consider the case $\lambda^B > 0$ for some B. Summing (I) over $B \neq A$

$$(x^A - e^A)\sum_{B \neq A} \frac{n^B(1 - \gamma^B)}{\gamma^B} + \sum_{B \neq A} n^B(x^B - e^B) \leq -\sum_{B \neq A} \frac{n^B \lambda^B}{\gamma^B} a.$$

Since $\sum_{B \neq A} n^B(x^B - e^B) = -n^A(x^A - e^A)$, it follows that

$$(x^A - e^A)[\sum_{B \neq A} \frac{n^B(1 - \gamma^B)}{\gamma^B} - n^A] \leq -\sum_{B \neq A} \frac{n^B \lambda^B}{\gamma^B} a.$$

[30] Notice that the allocations must be individually rational for each agent. [Any group can get e by announcing the same utility as some other group.] Thus neither $x^B \leq -\delta a$ nor $x^C \leq -\delta a$. Since x^B and x^C lie on the same side of the endowment, we can find such a $\widehat{u}$.

Since $\sum_{B\neq A}\frac{n^B\lambda^B}{\gamma^B}>0$ it follows that either $\sum_{B\neq A}\frac{n^B(1-\gamma^B)}{\gamma^B}-n^A>0$ and so $(x^A-e^A)\leq -\alpha a$ for some $\alpha>0$, or $\sum_{B\neq A}\frac{n^B(1-\gamma^B)}{\gamma^B}-n^A<0$ and so $(x^A-e^A)\geq \alpha a$ for some $\alpha>0$.[31] By individual rationality [again, any group can get e by announcing the same utility as some other group], it follows that $(x^A-e^A)\geq \alpha a$ for some $\alpha>0$. Since A was arbitrary, the same holds for all groups, which contracts the fact that the sum of the allocations is equal to the sum of the endowments.

Proof of 5.2. Suppose that $a\in P(n-1)$ is matched but that trade is not in proportion a. Then there exists $p\in I\!R^l_{++}$ such that $p\cdot a=0$ and some group of agents A such that $p\cdot x^i>p\cdot(e^i+a)$ for all $i\in A$.

First notice that $\alpha(u^A,a)>0$. [Otherwise, find $\widehat{u}^A$ with $\widehat{u}^A(x^A)>\widehat{u}^A(e^A)$ and $\alpha(\widehat{u}^A,a)>0$. By step 5.1, $f(u^{-A},\widehat{u}^A)=e$ and A can manipulate at $\widehat{u}^A$ via u^A.]

Case 1. There exists a group of agents $B\neq A$ such that $p\cdot x^B=p\cdot(e^B+a)$.

Without loss of generality (by Lemma 2) assume that x^B is the most preferred point in the plane $\{z\mid p\cdot z=p\cdot(e^B+a)\}$ under u^B. Find $\widehat{u}$ which has the following properties: (i) x^B is the most preferred point under $\widehat{u}$ in the plane $\{z\mid p\cdot z=p\cdot(e^B+a)\}$, (ii) $\widehat{u}(x^A)>\widehat{u}(x^B)$, (iii) $\alpha(\widehat{u},b)>0$ for any k' and $b\in P(k')$ such that $p\cdot(e^j+b)>p\cdot(e^j+a)$, and (iv) $\alpha(\widehat{u},b)\leq 0$ for any k' and $b\in P(k')$ such that $p\cdot(e^j+b)<p\cdot(e^j+a)$. [This is possible by taking the upper contour set of $\widehat{u}^j$ through x tangent to the plane $\{z\mid p\cdot z^j=p\cdot e^j\}$ at x^j thus satisfying (i), and sufficiently close to the plane to satisfy (ii) and (iii). We can also make sure that the contour set through e is close enough to the plane to satisfy (iv).] Let $\overline{u}$ be such that $\overline{u}^i=\widehat{u}^j$ for all $i\in A\cup B$ and $\overline{u}^i=u^i$ for all $i\notin A\cup B$. At $\overline{u}$, individuals can be partitioned into m groups with identical utilities and so we know that (3) applies.

Then $f(\overline{u})=y$ where $p\cdot y^i\leq p\cdot e^i$ for all $i\in A\cup B$. The only way this might not happen is if $\overline{u}$ matched some $b\in P(k')$ with $b\neq a$. If $p\cdot(e^j+b)>p\cdot(e^j+a)$ then by (1) and (2) it must be that $k'<k$ and $b\geq\gamma a$ for some $\gamma\in I\!R$, $\gamma\neq 0$. (The other possibility is that there exists $\gamma\in(0,1)$ such that $\gamma a+(1-\gamma)b\leq 0$ which contradicts the fact that $p\cdot(e^j+b)>p\cdot(e^j+a)=p\cdot e^j$.) All agents $i\notin A\cup B$ with $\alpha(u^i,a)>0$ will have $\alpha(\overline{u}^i,b)>0$, as well as all agents in $A\cup B$ (by (iii)). So there are at least k agents with $\alpha(\overline{u}^i,b)>0$ and there cannot be a match. If $p\cdot(e^j+b)<p\cdot(e^j+a)$ then by (1) and (2) it must be that there exists $\gamma\in(0,1)$ such that $\gamma a+(1-\gamma)b\leq 0$. Then by (iv) at most the agents $i\notin A\cup B$ with $\alpha(u^i,a)\leq 0$ will have $\alpha(\overline{u}^i,b)>0$. There are less than k such agents, and so there cannot be a match.

Now concavify $\widehat{u}^j$ and u^j through both x and y to $\widetilde{u}^j$ for $j\in B$. [This is possible since both $\widehat{u}^j$ and u^j have x as the unique most preferred point in the plane $\{z\mid p\cdot z^j=p\cdot e^j\}$, and so both have the same ordering over x and y.] It follows from Lemma 2 that $f(\widehat{u}^A,\widetilde{u}^B,u^{-A,B})=y$ and that $f(u^A,\widetilde{u}^B,u^{-A,B})=x$. Agents in group A can thus manipulate f at $(\widehat{u}^A,\widetilde{u}^B,u^{-A,B})$, by announcing u^A.

Case 2. There exists a group of agents $B\neq A$ such that $p\cdot x^B>p\cdot(e^B+a)$.

Find $\widehat{u}$ which has the following properties: (i) e^B+a is the most preferred point under $\widehat{u}$ in the plane $\{z\mid p\cdot z=p\cdot(e^B+a)\}$ and (ii) $\widehat{u}(x^A)>\widehat{u}(e^A+a)$ and $\widehat{u}(x^A)>\widehat{u}(e^B+a)$.

It follows that the number of agents with $\alpha(u^i,a)>0$ is at least $n-1$. Thus, $f(\widehat{u}^A,\widehat{u}^B,u^{-A,B})$ is trade in proportion a, or is the endowment. Individuals in groups A and B can all be made better off by announcing u, which contradicts coalitional strategy-proofness.

[31] Since $0\nleq -a$, it is not possible that $\sum_{B\neq A}\frac{n^B(1-\gamma^B)}{\gamma^B}-n^A=0$.

Case 3. $p \cdot x^j < p \cdot (e^j + a)$ for all $j \notin A$.

3.1 There exists a group $B \neq A$ with $\alpha(u^B, a) > 0$ and such that if $\overline{u}^A = u^B$, then $f(u^{-A}, \overline{u}^A) = y$ where $u^B(y^B) > u^B(x^B)$.

Consider $\widehat{u}$ with $\alpha(\widehat{u}, a) = \alpha(u^B, a)$ and $\widehat{u}(x^A) > \widehat{u}(y^A)$. Then $f(u^{-A,B}, \widehat{u}^A, \widehat{u}^B) = y$ Let $\widetilde{u}^B$ be a concavification of $\widehat{u}$ and u^B through y^B and a concavification of u^B through x^B. It follows from Lemma 2 that $f(u^{-A,B}, \widetilde{u}^B, \widehat{u}^A) = y$ and $f(u^{-B}, \widetilde{u}^B) = x$. Group A can then manipulate via u^A.

3.2: Not 3.1. For any $B \neq A$ with $\alpha(u^B, a) > 0$, if $\overline{u}^A = u^B$, then $f(u^{-A}, \overline{u}^A) = y$ where $u^B(x^B) \geq u^B(y^B)$.

Let n be the agent with $\text{sign}[\alpha(u^i, a)] \leq 0$. If there exists any $B \neq A$ $(n \notin B)$ such that $x^B \leq e^B + \lambda a$ for $\lambda \geq 0$, then let B' denote the such B with $y^{B'}$ closest to e. For all B with $n \notin B$ (including A) such that $x^B \not\leq e^B + \lambda a$ for any $\lambda \geq 0$, concavify u^B through x^B and choose lower contour sets of u^B so that $\alpha(u^B, a) \geq \alpha(u^{B'}, a)$. If there does not exist any such B',[32] then choose all u^i $i \neq n$ to have the largest possible $\alpha(u^i, a) > 0$. In either case, if A announced u^B, for any $B \neq A$ $(n \notin B)$ then all agents with $\text{sign}[\alpha(u^j, a)] > 0$ receive the same trade $y^j - e^j = Ya$.

For each $i \notin A, \neq n$ there exists $\delta^i \leq 1$ and $0 < \lambda^i < 1$ such that

$$\lambda^i(x^A - e^A) + (1 - \lambda^i)(x^i - e^i) \leq \delta^i Y a. \tag{II}$$

If this were not true for all $\delta^i \leq 1$, then we could find a utility $\widehat{u}$ with both x^A and x^i prefered to $e + Ya$ and $\alpha(\widehat{u}, a) \geq \alpha(u^i, a)$. If A and B (such that $i \in B$) both had $\widehat{u}$, then the outcome would be y and they could manipulate f via u^A, u^B.

It follows from (II) that

$$\sum_{i \notin A, \neq n} \frac{n_i \lambda^i}{(1 - \lambda^i)}(x^A - e^A) + \sum_{i \notin A, \neq n} n_i(x^i - e^i) \leq \sum_{i \notin A, \neq n} \frac{n_i \delta^i}{(1 - \lambda^i)} Y a. \tag{III}$$

Noting that $\sum_{i \notin A, \neq n} n_i(x^i - e^i) = -n_A(x^A - e^A) - (x^n - e^n)$, (III) is rewritten as

$$\Big[\sum_{i \notin A, \neq n} \frac{n_i \lambda^i}{(1 - \lambda^i)} - n_A\Big](x^A - e^A) \leq \Big[\sum_{i \notin A, \neq n} \frac{\delta^i n_i}{(1 - \lambda^i)}\Big] Y a - (x^n - e^n). \tag{IV}$$

Since $x^n - e^n \leq -(n-1) - \epsilon Y a$ for some $\epsilon > 0$, we can rewrite (IV) [33]

$$\Big[\sum_{i \notin A, \neq n} \frac{n_i \lambda^i}{(1 - \lambda^i)} - n_A\Big](x^A - e^A) \leq \Big[\sum_{i \notin A, \neq n} \frac{\delta^i n_i}{(1 - \lambda^i)} - (n-1) - \epsilon\Big] Y a. \tag{V}$$

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[32] If $l = 2$ then it must be that all $B \neq A$ $(n \notin B)$ are such that $x^B \leq e^B + \lambda a$ for some $\lambda \geq 0$. The extra complication enters only when $l \geq 3$.

[33] It must be that $x^n - e^n \leq -\delta a$ for some $\delta \geq 0$, otherwise, we could concavify u^n through x^n and choose $\alpha(u^n, a) > 0$. The outcome would still be x contradicting 5.1. It must be that $\delta > 0$ since 5.1 implies that n can get at least e^n by matching any other agent's utility. Given that $x^n - e^n \leq -\delta a$ for some $\delta > 0$, by concavifying u^n through x^n, we can choose $\alpha(u^n, a)$ so that $y^n - e^n$ is shorter than the projection of $x^n - e^n$ onto a. Finally, notice that $y^n - e^n = -(n-1)Ya$.

Since $p \cdot (x^A - e^A) > p \cdot a = 0$, it follows from (V) that

$$\left[\sum_{i \notin A, \neq n} \frac{n_i \lambda^i}{(1-\lambda^i)} - n_A \right] \leq 0 \qquad (VI).$$

Since $\delta^i \leq 1$ for each $i > 1$, it follows that $\sum_{i \notin A, \neq n} n_i + \epsilon > \sum_{i \notin A, \neq n} n_i \frac{\delta^i - \lambda^i}{1-\lambda^i}$, which implies that

$$\sum_{i \notin A, \neq n} \frac{n_i \lambda^i}{1-\lambda^i} - n_A > \sum_{i \notin A, \neq n} \frac{n_i \delta^i}{1-\lambda^i} - (n-1) - \epsilon. \qquad (VII)$$

If (VI) holds with equality, then (V) and (VII) would imply that $0 \leq -a$ which is a contradiction. Thus (VI) holds with inequality, and so (V)–(VII) imply that $x^A - e^A \geq \gamma Y a$ for some $\gamma > 1$ (where γ is in fact $[\sum_{i \notin A, \neq n} \frac{\delta^i n_i}{(1-\lambda^i)} - (n-1) - \epsilon]/[\sum_{i \notin A, \neq n} \frac{n_i \lambda^i}{(1-\lambda^i)} - n_A]$).

The fact that $x^A - e^A \geq \gamma Y a$ for some $\gamma > 1$ coupled with (II) implies that $x^i - e^i \leq \delta Y a$ for $i \notin A, \neq n$ and some $\delta < 1$. This is impossible since we know that $u^i(x^i) \geq u^i(y^i)$ and $\alpha(u^i, a) \geq Y$. Thus our original supposition was incorrect.

To complete the proof of 5.2, we need to show that there exists $r^i \in [0,1]$ such that $f^i(u) = e^i + r^i \alpha(u^i, a) a$ for all i and that any rationing is done uniformly.

First, consider $i = n$. Agent n can force $f(u) = e$ by a change of u^n so that $\alpha(u^n, a) > 0$. This means that $f^n(u) = e^n + r^n \alpha(u^n, a) a$ for $r^n \geq 0$. Suppose that $r^n > 1$. Concavify u^n through x^n to $\tilde{u}^n$ such that $\alpha(\tilde{u}^n, a) = \alpha(u^n, a)$. Then $f^n(u^{-n}, \tilde{u}^n) = f^n(u)$, which contradicts the tie-free assumption, since at $\tilde{u}^n$ a convex combination of x^n and e^n is preferred to either. Hence, $f^n(u) = e^n + r^n \alpha(u^n, a) a$ for some $r^n \in [0,1]$. (Rationing on this side of the market is necessarily uniform since there is only one agent with $\alpha(u^n, a) \leq 0$.)

Next, consider B such that $\alpha(u^B, a) > 0$. If B matched some other B' with $\alpha(u^{B'}, a) > 0$, then trade would be in a nonnegative multiple of a. This (together with the tie-free condition) means that $f^B(u) = e^B + r^B \alpha(u^B, a) a$ for some $r^B \geq 0$. First, we show that $r^B \leq 1$. If B announced the same utility as n, then the outcome for B, y^B, would be such that $\gamma y^B + (1-\gamma) x^B \leq e^B$ for some $\gamma \in [0,1]$. [Otherwise find $\hat{u}^B$ with $\hat{u}^B(y^B) > \hat{u}^B(x^B)$ and $x^B = \alpha(\hat{u}^B, a) a + e^B$. $a \in P(k-1)$ would still be matched and so B's trade would have to be in proportion a, but then B could manipulate by matching u^n.] Suppose that $r^i > 1$. Then consider $\tilde{u}^B$ with $\alpha(\tilde{u}^B, a) = \alpha(u^B, a)$, and $\tilde{u}^B(\gamma y^B + (1-\gamma) x^B) > \tilde{u}^B(x^B) = \tilde{u}^B(y^B)$. The tie–free condition implies that that $f(u^{-B}, \tilde{u}^B) \neq x^B$. Since we know from our previous argument that $f^i(u^{-B}, \tilde{u}^B) = e^B + \tilde{r}^B \alpha(u^B, a) a$ for some $\tilde{r}^B \geq 0$. Strategy–proofness at $\tilde{u}^b$ implies that $\tilde{r}^B < r^B$, and strategy–proofness at u^b implies that $\tilde{r}^B < 1$. Concavify u^B through x^B to $\hat{u}^B$ such that $\alpha(\hat{u}^B, a) = \alpha(u^B, a)$. Then $f^B(u^{-B}, \hat{u}^B) = f^B(u)$, which contradicts the tie-free assumption. Hence, $r^B \in [0,1]$. Next we show that the rationing is done uniformly. Suppose the contrary, so that there exists $B \neq n$ and $A \neq n$ such that $r^B < 1$ and $r^A \alpha(u^A, a) > r^B \alpha(u^B, a)$. In this case it must be that if B matched A, then the outcome for A and B would be x^B. [By strategy–proofness it must be a trade at least as large as x^B. If it is larger, then by strategy–proofness at u^B it must be to a point beyond $\alpha(u^B, a)$. In that case, we could concavify u^B through x^B to $\hat{u}^B$ such that $\alpha(\hat{u}^B, a) = \alpha(u^B, a)$, and get a contradiction of the tie free assumption.] This implies that if A matched B, then the outcome for A and B would be x^B (according to uniform rationing for m groups). However, then at $\hat{u}^A = u^B$ group A can manipulate via u^A. Thus our supposition was wrong and so rationing is done uniformly.

Proof of 5.3.1 Without loss of generality, assume that there exists $b \in P(k+1)$ such that $\gamma b \leq a$ for some $\gamma > 0$, and $b \neq a$.[34] This implies that there exists some group A with $\alpha(u^A, a) > 0$ and $\alpha(u^A, b) \leq 0$.

Case 1. $\alpha(u^A, b) = 0$, and $\alpha(u^B, a) > 0$ implies $\alpha(u^B, b) > 0$ for all $B \neq A$.

In this case, it must be that $x^A = e^A$. [It cannot be that $p \cdot x^A > p \cdot e^A$ for any $p \in \mathbb{R}^l_{++}$ such that $p \cdot b = 0$, otherwise we can find $\widehat{u}$ with x^A preferred to any trade in proportion b and with $\alpha(\widehat{u}, b) > 0$, and $\alpha(\widehat{u}, c) \leq 0$ for any $c \neq b$, $c \leq \gamma b$ for some $\gamma > 0$. Then either b is matched at some $\widetilde{k} > k$ in which case $f^A(u^{-A}, \widehat{u}^A)$ is trade in direction b, or else 5.3.1 applies for some $\widetilde{k} > k$ in which case there is no trade. Either way, A can manipulate via u^A. Thus $p \cdot x^A \leq p \cdot e^A$ for all $p \in \mathbb{R}^l_{++}$ such that $p \cdot b = 0$. If $x^A \neq e^A$, then A could find a $\widehat{u}^A$ as described above so that the outcome is arbitrarily close to e^A (set $\alpha(\widehat{u}^A, b)$ close to 0) which is better for A than x^A. Thus, $x^A = e^A$.] It must also be that $x^B = e^B$ for all $B \neq A$. Suppose the contrary. Then there must exist some B and $p \in \mathbb{R}^l_{++}$ such that $p \cdot x^B > p \cdot e^B$ and either $p \cdot a = 0$ or $p \cdot b = 0$. Then find $\widehat{u}$ such that $\alpha(\widehat{u}, a) > 0$, $\alpha(\widehat{u}, b) \leq 0$, and $\widehat{u}(x^B) > \widehat{u}(e^B)$. It follows that $f(u^{-A,B}, \widehat{u}^A, \widehat{u}^B) = e$ (since 5.3 holds for m groups). Let $\widetilde{u}^A$ concavify both u^A and $\widehat{u}^A$ through e^A. Then $f(u^{-A,B}, \widetilde{u}^A, \widehat{u}^B) = e$ and $f(u^{-A}, \widetilde{u}^A) = x$. Then B can manipulate at $\widehat{u}^B$ via u^B.

Case 2. $\alpha(u^A, b) < 0$, and $\alpha(u^B, a) > 0$ implies $\alpha(u^B, b) > 0$ for all $B \neq A$.

Suppose that $x^A \neq e^A$. It follows from case 1 that there exists some $\gamma > 0$ such that $x^A \leq e^A - \gamma b$. [Case 1 implies that e^A is available to A so the outcome must be individually rational for A. If $p \cdot x^A > p \cdot e^A$ for some $p \in \mathbb{R}^l_{++}$ such that $p \cdot b = 0$, then A with u^A as in case 1 could manipulate.] The facts that $\alpha(u^A, a) > 0$ and $u^A(x^A) \geq u^A(e^A)$ guarantee that $x^A \not\leq e^A - \gamma a$ for any $\gamma \geq 0$. Consider any $p \in \mathbb{R}^l_{++}$ such that $p \cdot x^A = p \cdot e^A$ and $p \cdot a > 0$. [Since $x^A \not\leq e^A - \gamma a$ for any $\gamma \geq 0$ and $x^A \leq e^A - \gamma b$ for some $\gamma > 0$, it follows that there exists such a p.] There is no $C \neq A$ such that $p \cdot x^C > p \cdot x^A$, otherwise find $\widehat{u}$ with both x^C and x^A preferred to e and $\alpha(\widehat{u}, a) > 0$ and $\alpha(\widehat{u}, b) \leq 0$. The outcome if both A and C announced $\widehat{u}$ would be e, and they could manipulate via u^A and u^C. Thus all B are such that $p \cdot x^B = p \cdot x^A$. Slightly altering p to $p' \in \mathbb{R}^l_{++}$ so that $p' \cdot x^A = p' \cdot e^A$ and $p' \cdot a > 0$ implies that all x^B are collinear with e and x^A. [If $l = 2$ then there is no need to alter p.] If there are two groups with nonzero trades in the same direction, then it is possible to find $\widehat{u}$ with both trades preferred to e, and either $\alpha(\widehat{u}, a) > 0$, and $\alpha(\widehat{u}, b) \leq 0$, or $\alpha(\widehat{u}, b) > 0$ (in which case both groups had $\alpha(u^B, b) > 0$ to begin with). The outcome if both groups announced $\widehat{u}$ would be e and so they could manipulate via u. Thus there must be some group C with $x^C = e^C$. Find $\widehat{u}$ with x^A preferred to e, $\alpha(\widehat{u}, a) > 0$, and $\alpha(\widehat{u}, b) < 0$. It follows that $f(u^{-A,C}, \widehat{u}^A, \widehat{u}^C) = e$. Let $\widetilde{u}^C$ concavify both u^C and $\widehat{u}^C$ through e^C. Then $f(u^{-A,C}, \widehat{u}^A, \widetilde{u}^C) = e$ and $f(u^{-C}, \widetilde{u}^C) = x$. Then A can manipulate at $\widehat{u}^A$ via u^A. Thus our supposition was wrong and so $x^A = e^A$. It then follows from case 1 and non-bossiness

[34] To see that this is without loss of generality, consider the construction of the proposals described in step 2. In step 2, alter the construction as follows: For any $\overline{k}$ and $a \in P(\overline{k} - 1)$ such that there is no $b \in P(\overline{k})$ such that $\gamma b \leq a$ for any $\gamma > 0$, let $a \in P(\overline{k})$. $P(\overline{k})$ will still meet the definition of trade proposal, and the trade proposals will still be nested. It will simply be the case that when $a \in P(\overline{k})$ is matched for u, then there is no trade and so $r^i = 0$ for each i where $f^i(u) = e^i + r^i \alpha(u^i, a) a$. Thus the previous steps of the proof still hold. Finally, notice that for the premise of 5.3.1 to hold, it must be that $a \notin P(k+1)$. By our new construction this means that there must exist some $b \in P(k+1)$ such that $\gamma b \leq a$ for some $\gamma > 0$.

that $f(u) = e$.

Case 3. $\alpha(u^A, b) \leq 0$, and $\alpha(u^B, a) > 0$ and $\alpha(u^B, b) \leq 0$ for some $B \neq A$.

First take the case that $\alpha(u^C, a) > 0$ implies $\alpha(u^C, b) > 0$ for all C such that $B \neq C \neq A$.

Neither x^B nor x^A can be such that $p \cdot x^A > p \cdot e^A$ for some $p \in I\!R^l_{++}$ such that $p \cdot b = 0$, otherwise they could manipulate from one of case 1 or 2. Since each can get e by matching the other, it must be that $x^A \leq e^A - \gamma b$ and $x^B \leq e^B - \gamma' b$ for some γ and γ' greater than or equal to 0. If either gets e then cases 1 and 2 and non–bossiness imply that the outcome is e. So consider the case where both are not the endowment. Repeating the argument from Case 2 leads to a contradiction. Thus, $f(u) = e$.

Iteration of the above argument covers the case where additional groups $B' \neq A$ are such that $\alpha(u^{B'}, a) > 0$ and $\alpha(u^{B'}, b) \leq 0$.

Proof of 5.3.2 This is analagous to the proof of 5.2, except that in Case 3 we must consider the possibility that $\alpha(u^i, a) \leq 0$ for all $i \notin A$. This means that $n_A = k$. In that case, first suppose that for all $B \neq A$, $x^B - e^B \leq -\lambda a$ for some $\lambda \geq 0$. Let $\widetilde{u}^A$ concavify u^A so that $\alpha(\widetilde{u}^A, a) = 0$. This implies that $\alpha(\widetilde{u}^A, b) \leq 0$ for all $b \in P(\overline{k})$ for all $\overline{k} \geq k$. Find $\widehat{u}$ with both x^B and x^C preferred to e where $B \neq A \neq C$. Since $n_A = k$ and $\alpha(\widetilde{u}^A, b) \leq 0$ for all $b \in P(\overline{k})$ and $\overline{k} \geq k$, it follows that $f(\widetilde{u}^A, \widehat{u}^B, \widehat{u}^C, u^{-A,B,C}) = e$. However, $f(\widetilde{u}^A, u^B, u^C, u^{-A,B,C}) = x$, and so groups B and C can manipulate f. Thus our supposition was wrong and so there exists some B such that $x^B - e^B \not\leq -\lambda a$ for all $\lambda \geq 0$. In this case, concavify u^B to $\widetilde{u}^B$ so that $\alpha(\widetilde{u}^B, a) > 0$. This means that either some $b \in P(\overline{k})$ is matched for $\overline{k} > k$ or else 5.3.1 applies. Since $x \neq e$, it must be that some $b \in P(\overline{k})$ was matched. However, then x^A should be a nonnegative trade in proportion b, which contradicts the fact that $p \cdot b \leq 0 < p \cdot (x^A - e^A)$, where p us as defined in the proof of 5.2..

The proof that there exists $r^i \in [0,1]$ such that $f^i(u) = e^i + r^i \alpha(u^i, a)a$ for all i and that any rationing is done uniformly is a straightforward extension of the proof of the same fact in step 5.2.

Proof of 5.4 Without loss of generality, whenever there exists $b \in P(k)$ for some $k > \underline{k}$ and no $a \in P(\widetilde{k})$ such that $b \leq \gamma a$ for some $\gamma > 0$ for any $\widetilde{k}$, $k > \widetilde{k} \geq \underline{k}$, then let $b \in P(\widetilde{k})$ for all $\widetilde{k}$, $k > \widetilde{k} \geq \underline{k}$. [We could make such a construction in Step 2. $P(\widetilde{k})$ will still meet the definition of trade proposal, and the trade proposals will still be nested. It will simply be the case that when $b \in P(\widetilde{k})$ is matched for u, then there is no trade and so $r^i = 0$ for each i where $f^i(u) = e^i + r^i \alpha(u^i, a)a$. Thus the previous steps of the proof still hold.]

First let us treat the situation in which there exist $a \in P(\underline{k})$ and $b \in P(\underline{k})$ such that $b \neq \gamma a$ for all $\gamma \neq 0$. We must show that $f(u) = e$.

Case 1. There exists A with $\alpha(u^A, b) = 0$, and such that if A had $\alpha(\widetilde{u}^A, b) > 0$ then there would be enough at least $\underline{k}$ agents with $\alpha(u^i, b) > 0$.

In this case, it must be that $x^A = e^A$. [See Case 1 of 5.3.1.] It must also be that $x^B = e^B$ for all $B \neq A$. Suppose the contrary. Choose $p \in I\!R^l_{++}$ such that $p \cdot a < 0$ for all a in all $P(\underline{k})$. [The fact that such a p exists follows from the fact that the fact that $P(\underline{k})$ is a trade proposal and it contains a and b with $b \neq \gamma a$ for all $\gamma \neq 0$.] Since $x \neq e$, there must exist $B \neq A$ and $x^B \neq e^B$ such that $p \cdot x^B \geq p \cdot e^B$. Then find $\widehat{u}$ such that $\alpha(\widehat{u}, a) \leq 0$ for all a in all $P(\underline{k})$ and $\widehat{u}(x^B) > \widehat{u}(e^B)$. It follows that $f(u^{-A,B}, \widehat{u}^A, \widehat{u}^B) = e$. Let $\widetilde{u}^A$ concavify both u^A and $\widehat{u}^A$ through e^A. Then $f(u^{-A,B}, \widetilde{u}^A, \widehat{u}^B) = e$ and $f(u^{-A}, \widetilde{u}^A) = x$. Then B can manipulate at $\widehat{u}^B$ via u^B.

Case 2. There exists A with $u^A(y^A) < u^A(e^A)$ for any $y \neq e$ such that $y^A \leq e^A + \sum_{a \in P(\underline{k})} \lambda_a a$ for some set of λ_a such that $\lambda_a \geq 0$ for each a. [y^A lies in the convex hull of $P(\underline{k})$.] Also, if A had $\alpha(\widetilde{u}^A, b) > 0$ then there would be enough at least $\underline{k}$ agents with $\alpha(u^i, b) > 0$.

Suppose that $x^A \neq e^A$. Case 1 implies that e^A is available to A so the outcome must be individually rational for A. It also follows from case 1 that $x^A \leq e^A - \gamma b$ for some $\gamma > 0$. [Otherwise some A from case 1 could manipulate.] Thus there exists $p \in \mathbb{R}^l_{++}$ such that $p \cdot x^A = p \cdot e^A$ and $p \cdot a < 0$ for all a in $P(\underline{k})$. There is no $C \neq A$ such that $p \cdot x^C > p \cdot x^A$, otherwise find $\widehat{u}$ with both x^C and x^A preferred to e and $\alpha(\widehat{u}, a) \leq 0$ for all a in $P(\underline{k})$. The outcome if both A and C announced $\widehat{u}$ would be e, and they could manipulate via u^A and u^C. Thus all C are such that $p \cdot x^C = p \cdot x^A$. Slightly altering p to $p' \in \mathbb{R}^l_{++}$ so that $p' \cdot x^A = p' \cdot e^A$ and $p' \cdot a < 0$ implies that all x^C are collinear with e and x^A. [If $l = 2$ then there is no need to alter p.] If there are two groups with nonzero trades in the same direction, then it is possible to find $\widehat{u}$ with both trades preferred to e, and $\alpha(\widehat{u}, a) \leq 0$ for all a in $P(\underline{k})$. The outcome if both groups announced $\widehat{u}$ would be e and so they could manipulate via u. Thus there must be some group C with $x^C = e^C$. Find $\widehat{u}$ with x^A preferred to e and $\alpha(\widehat{u}, a) \leq 0$ for all a in $P(\underline{k})$. It follows that $f(u^{-A,C}, \widehat{u}^A, \widehat{u}^C) = e$. Let $\widetilde{u}^C$ concavify both u^C and $\widehat{u}^C$ through e^C. Then $f(u^{-A,C}, \widehat{u}^A, \widetilde{u}^C) = e$ and $f(u^{-C}, \widetilde{u}^C) = x$. Then A can manipulate at $\widehat{u}^A$ via u^A. Thus our supposition was wrong and so $x^A = e^A$. It then follows from case 1 and non–bossiness that $f(u) = e$.

Case 3. There exists A with $u^A(y^A) < u^A(e^A)$ for any $y \neq e$ such that $y^A \leq e^A + \sum_{a \in P(\underline{k})} \lambda_a a$ for some set of λ_a such that $\lambda_a \geq 0$ for each a. [y^A lies in the convex hull of $P(\underline{k})$.] But not case 1 or 2.

First notice that x^A must be individually rational for A. [A can match any other group to get e since we are not in case 1 or 2.] If $x^A = e^A$ then follow the argument of case 1. If $x^B = e^B$ for any $B \neq A$ the same argument holds. So consider the possibility that $x^B \neq e^B$ for all B. If $x^A \leq e^A - \gamma b$ for some $b \in P(\underline{k})$ and $\gamma > 0$, then follow the argument of case 2. So suppose that $x^A \not\leq e^A - \gamma b$ for any $b \in P(\underline{k})$ and $\gamma > 0$. Consider any $B \neq A$. If $e \not\geq \gamma' x^A + (1-\gamma')x^B$ for any $\gamma' \in [0,1]$ then we can find $\widehat{u}$ with x^A and x^B both preferred to the endowment. If $x^B \not\leq e^B + \gamma a$ for any $a \in P(\underline{k})$ then we can choose $\widehat{u}$ so that $\alpha(\widehat{u}, a) \leq 0$ for all $a \in P(\underline{k})$. If $x^B \leq e^B + \gamma b$ for some $b \in P(\underline{k})$, then it must be that $\alpha(u^B, b) > 0$ and so choose $\widehat{u}$ so that $\alpha(\widehat{u}, b) > 0$. In either case, if both A and B announce $\widehat{u}$, then the outcome is e and they can jointly manipulate f via u^A and u^B. So $e \geq \gamma' x^A + (1-\gamma')x^B$ for some $\gamma' \in [0,1]$ for any $B \neq A$. This means that all x^B are in line and in the opposite direction from x^A. Find two such groups B and C and a common preference $\widehat{u}$ which prefers x^B and x^C to e. [If $x^B \leq e^B$ then it must be that $x^C \leq e^C$ and so both groups could jointly manipulate by announcing some $\overline{u}$ with $\alpha(\overline{u}, a) \leq 0$ for all $a \in P(\underline{k})$ to get e,] If $x^B \leq e^B + \gamma b$ for some $b \in P(\underline{k})$ and $\gamma > 0$, then it must be that $\alpha(u^B, b) > 0$ and $\alpha(u^C, b) > 0$ and so choose $\widehat{u}$ so that $\alpha(\widehat{u}, b) > 0$. Otherwise choose $\widehat{u}$ so that $\alpha(\widehat{u}, a) \leq 0$ for all $a \in P(\underline{k})$. In either case if both B and C announce $\widehat{u}$, then the outcome is e and they can jointly manipulate f via u^B and u^C.

Case 4. Not cases 1, 2 or 3.

Suppose $x \neq e$. Then there exists some A and $p \in \mathbb{R}^l_{++}$ such that $p \cdot x^A > p \cdot e^A$ and $p \cdot a < 0$ for all $a \in P(k)$ for any k. Then there exists $\widehat{u}^A$ such that $\widehat{u}^A(x^A) > \widehat{u}^A(e^A)$ and $\widehat{u}^A(y^A) < \widehat{u}^A(e^A)$ for any $y \neq e$ such that $y^A \leq e^A + \sum_{a \in \cup_k P(k)} \lambda_a a$ for some set of λ_a such that $\lambda_a \geq 0$ for each a. This contradicts Cases 1, 2 and 3. Thus $f(u) = e$.

Finally, let us treat the case where either $P(\underline{k}) = \{a\}$ or $P(\underline{k}) = \{a, -\gamma a\}$ for some $\gamma > 0$. We show that if there is any trade, then it must be in proportion a.

Without loss of generality we can treat the case $P(\underline{k}) = \{a\}$ as if it were $P(\underline{k}) = \{a, -\gamma a\}$ for some $\gamma > 0$. [We could make such a construction in Step 2. $P(\underline{k})$ will still meet the definition of trade proposal, and the trade proposals will still be nested. It will simply be the case that when $-\gamma a \in P(\underline{k})$ is matched for u, then there is no trade (unless $\underline{k} = n/2$ and a is also matched) and so $r^i = 0$ for each i where $f^i(u) = e^i + r^i \alpha(u^i, a)a$. Thus the previous steps of the proof will still hold.]

There must exist some groups of agents with $\alpha(u^i, a) = 0$, otherwise some other step applies. If there is only one such group A then $x^A = e^A$. [The proof of this parallels case 1 of 5.3.1.] It must also be that x^B is in proportion a for all $B \neq A$. Suppose the contrary. Then there must exist some B and $p \in \mathbb{R}^l_{++}$ such that $p \cdot x^B > p \cdot e^B$ and $p \cdot a = 0$. Then find $\widehat{u}$ such that $\alpha(\widehat{u}, a) = 0$, and $\widehat{u}(x^B) > \widehat{u}(e^B)$. It follows that $f^{A,B}(u^{-A,B}, \widehat{u}^A, \widehat{u}^B) = e^{A,B}$. Let $\widetilde{u}^A$ concavify both u^A and $\widehat{u}^A$ through e^A. Then $f^B(u^{-A,B}, \widetilde{u}^A, \widehat{u}^B) = e^B$ and $f(u^{-A}, \widetilde{u}^A) = x$. Then B can manipulate at $\widehat{u}^B$ via u^B.

If there are two groups of agents A and B with $\alpha(u^i, a) = 0$, then both get the endowment. [Neither x^B nor x^A can be such that $p \cdot x^A > p \cdot e^A$ for some $p \in \mathbb{R}^l_{++}$ such that $p \cdot a = 0$, otherwise that group could manipulate from the previous case or if a were matched. Since they can get e by matching the other, it must be that the outcome is e.] We can repeat the previous argument to establish that all other groups trade in proportion a.

Iteration of this reasoning allows for additional groups $B \neq A$ such that $\alpha(u^B, a) = 0$.