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Continuous Time Research and Development Investment and Innovation: Effects on Price and Dividend Paths*

by

Thomas A. Rietz**

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Abstract

Here, I solve a general equilibrium, stochastic, dynamic control problem. In it, an agent who owns a productive asset decides how much of a non-storable good to consume and how much to invest in research and development. Combined, two features distinguish this from previous work. First, the agent maximizes lifetime expected utility (instead of profits or income). Second, the investment level affects the probability of a research and development innovation which would make future dividends jump. Dividend evolution is represented by a continuous time Poisson process with the jump probability depending on the investment level. In equilibrium, the agent chooses the investment level to give an optimal expected innovation rate. This results in endogenously chosen, stationary growth rates in asset dividends and prices. These non-stationary price and dividend paths are of a type that Marsh and Merton (1986) predict will violate variance bounds tests such as Shiller's (1981). However, they are not subject to Shiller's (1986) criticism of Marsh and Merton, because they result from a general equilibrium with all agents behaving optimally and rationally.

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**Department of Finance, Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60208-2006

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I. Introduction

Productive asset owners usually have some control over the income stream resulting from their assets. This control may take many forms including investment in more productive assets or in research and development. Research and development investment may, or may not, result in innovations that increase the productivity of both current and new assets. More intensive research and development efforts will likely result in more frequent or more significant innovations. Finally, innovations may be sequentially more difficult to realize as the most evident and easily implemented ideas are likely the first utilized. Here, I study how such research and development opportunities effect asset owners, prices and income streams.

Two features make the problem faced by the asset owners here unique. First, risk averse asset owners maximize lifetime expected utility (instead of expected income or final wealth) in a general equilibrium setting. Second, asset owners directly affect the probability of a research and development innovation through their investment levels. The innovations are the stochastic shocks that the owners face here. Thus, through their investment levels, the agents in this economy have some control over the probability structure they face.[1]

The model here is a general equilibrium, real asset pricing model with non-storable, consumption good dividends similar to Lucas (1978) and Brock (1982). The dividends follow a continuous time jump process. Merton (1971) models jump processes that represent defaults in a partial equilibrium analysis. Here, the jumps represent research and development innovations that increase dividends. The agent invests the consumption good directly and irreversibly in research and development which increases the jump probability. Optimal investment results in a stationary stochastic dividend growth rate. As in Mehra and Prescott (1984 and 1985) and Brock (1982), this allows the model to explain non-stationary price and dividend paths that have constant expected growth rates. Here however, the innovation process and, hence, the dividend growth rate, are partially endogenous because the agent selects the research and development investment level.

I find a closed form solution showing that optimization problems with such partially endogenous jump process can be solved in closed form. The solution shows that the agent invests a constant fraction

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[1]The works mentioned below and other typical asset pricing models (i.e. Cox, Ingersoll and Ross (1985)) assume utility maximizing agents face entirely exogenous stochastic processes which govern returns. While search, exploration and innovation examples sometimes include endogenous probabilities, they usually assume income or wealth maximization instead of utility maximization. See Malliaris (1982) for examples. Here, I have both utility maximization and a partially endogenous process governing returns.

of the dividend in research and development, even when investment affects the innovation probability. This extends Merton's (1971) solutions and investment results to endogenous jump processes in general equilibrium.

I also show that the agent chooses an investment level that attains optimal dividend and price growth rates given the agent's preferences and the investment characteristics.[2] The expected growth rates do not change as long as the underlying preference and research and development parameters do not change. Thus, in this economy, we will observe constant expected dividend and price growth rates and we can use Mehra and Prescott's (1985) calibration method for testing so long as preferences and asset characteristics remain unchanged.

The stationary growth rates chosen by the agent lead to non-stationary price and dividend paths. These paths do not conform to the stationarity assumptions that Grossman and Shiller (1982) and LeRoy and Porter (1981) use when they conclude that prices are too volatile to be explained by the present value relationship. Instead, they are non-stationary in precisely the way that Marsh and Merton (1986) predict will *reverse* the inequality of the variance bounds test. Prices here are *expected* to vary more than the present value relationship predicts. Further, the paths here result from a fully rational, optimizing equilibrium. Thus, this model is not subject to Shiller's (1986) criticism of Marsh and Merton's example that generates a similar dividend path.

Finally, I show that dividends and prices may explode, converge to zero or remain constant as time passes. Thus, actual and predicted levels of dividends and prices may, or may not, converge to stationary distributions over time. Thus, depending on the selected growth rates, we may, or may not, be able to construct appropriate conditional variance bounds tests as Gilles and LeRoy (1988) suggest.

Though I present this model in the context of asset pricing, it has natural applications in a variety of areas including research and development, education, investment, capital accumulation and growth.[3] In the next section, I present a framework for these types of problems. Then, I add slightly more structure and illustrate how to find closed form solutions. Finally, I discuss the properties and implications of the solution.

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[2]The investment characteristics will include how fast dividends decay, how much investment increases the innovation probability and how much an innovation increases dividends.

[3]For instance, a firm must decide how much to spend for research and development and an innovation will suddenly increase the firm's returns; the larger the investment, the more likely an innovation. An inventor must decide how much effort to expend developing new inventions in hopes of increasing future income; the more effort expended, the more likely a profitable invention will be found. Similarly, a worker must decide how much time to spend working and how much to spend learning new skills that will increase future productivity and income; the more time spent learning, the more likely a useful skill will be learned.

II. The Investment Scenario

An agent lives alone in an economy forever and owns an asset ("grove of trees") that provides a continuous stream of non-storable, consumption good dividends ("fruit"). The grove's fruit output tends to decrease over time. However, the agent has some control over this decay. Suppose the agent invests some of the fruit in research and development. Successful research and development results in a proportional, positive jump in the grove's dividend stream. Suppose the more invested, the higher the likelihood that a jump will occur, but higher current output makes it less likely an innovation will occur for a given investment level. The agent must decide how much fruit to eat and how much to invest in research and development.

You may interpret research and development as the agent "planting" fruit and an innovation as a new tree growing. The more fruit planted, the more likely a new tree will grow. Due to positive external effects (e.g., cross pollination), each new tree leads to a proportional increase in total fruit output. However, due to negative external effects (e.g., crowding), the higher the current fruit output is (i.e., the more trees there are), the harder it is for a new tree to grow.

III. The Agent's Investment Problem

The current dividend (the grove's output level) is the state of the world. The dividend decays over time as the asset becomes less productive and takes positive jumps when an innovation occurs as given by the state evolution equations:

$$\mathrm{d}\delta(t) = -\alpha(\delta,t)\delta(t)\mathrm{d}t + f(\delta,t)\mathrm{d}q \tag{1}$$

with:

$$\mathrm{d}q = \begin{cases} 1 \text{ with probability } \lambda(\delta,t)i(t)\mathrm{d}t \\ 0 \text{ with probability } 1-\lambda(\delta,t)i(t)\mathrm{d}t \end{cases} \tag{2}$$

where $\delta(t)$ is the asset's dividend per unit of time at time t, $\alpha(\delta,t)>0$ represents the dividend's natural decay, $f(\delta,t)$ represents the return to an innovation and $\mathrm{d}q$ is an innovation-generating Poisson process with parameter $\lambda(\delta,t)i(t)$. The investment level per unit of time at time t as a fraction of the dividend is given by $i(t)$ and $\lambda(\delta,t)$ gives the increase in the innovation probability resulting from a unit of investment. Thus, the amount of investment directly increases the chances of an innovation, but higher dividends decrease the probability of innovation for given absolute investment levels.[4]

The agent has an infinite horizon.[5] With discounting, the agent solves:

$$J(\delta(t),\infty) = \max_{c(s),i(s)\geq 0} E_t \int_{s=t}^{\infty} e^{-\rho s} U(c(s),i(s))\mathrm{d}s \tag{3}$$

subject to the budget constraint:

$$c(t) \leq \delta(t)(1-i(t)), \tag{4}$$

where $J(\cdot,\cdot)$ is the agent's indirect utility function, $c(t)$ is consumption per unit of time at time t, $E_t(\cdot)$ is the expectation operator given the information available at time t, $U(\cdot,\cdot)$ is the agent's von Neumann-Morgenstern utility function and ρ is the continuous time discount factor. Again, $i(t)$ represents the fraction of the dividend invested. The agent's utility function will display non-satiation in the consumption good. Since the consumption good can not be stored, the budget constraint will hold with equality.

Translate the agent's problem into present value terms by letting $W(\delta(t)) = e^{\rho t} J(\delta(t),\infty)$. To find a solution, I will assume that $W(\delta(t))$ has continuous derivatives of all orders less than two in δ. The solution I find does indeed satisfy this assumption. In this case, the optimal investment level solves the present value version of the Hamilton-Jacobi-Bellman equation:

$$0 = \max_{i(t),\,c(t)} \left\{ \begin{aligned} & U(c(t),i(t)) - \rho W(\delta(t)) - \alpha(\delta,t)\delta(t) W_\delta(\delta(t)) \\ & \qquad + \lambda(\delta,t) i(t) [W(\delta(t) + f(\delta,t)) - W(\delta(t))] \end{aligned} \right\} \tag{5}$$

where $W_\delta(\delta(t))$ is the partial derivative of W with respect to δ.[6]

The agent solves this (no trade) problem by deciding how much to invest and consuming the remaining dividend. Using the budget constraint to substitute the consumption level into (5) leaves investment as the only important decision variable. Differentiating (5) with respect to investment gives the first order conditions for an interior investment solution. Solving the first order conditions and

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[4] This problem is easily extended to include a white noise term (random events affecting dividends), a Poisson process with a negative jump that occurs with a probability that is either independent of the investment level or decreases linearly with higher investment (natural disasters that the agent may or may not be able to guard against with investment), or a positive jump that is independent of investment (natural innovation).

[5] While finite horizon versions can be solved using the same techniques, the time remaining in the horizon is an important factor that complicates the solutions considerably while giving nearly identical results.

[6] See Malliaris (1982, pp. 108-113 and 121-124) to see how Bellman's Principal of Optimality and a Taylor's Series expansion (hence the need for continuous derivatives) are used to derive equation (5).

substituting this investment level into (5) gives a stochastic differential equation in the indirect utility function, $W(\delta(t))$.[7] Finally, finding a closed form solution for $W(\cdot)$ gives the agent's investment function and shows how the agent's indirect utility and optimal investment level varies with the state of the world and the parameters of the problem.

To solve the agent's problem, suppose the agent displays constant relative risk averse preferences as given by the CRRA utility function:

$$U(c(t),i(t)) = \frac{c(t)^{1-\gamma}}{1-\gamma}. \tag{6}$$

For convenience, I will assume $1-\gamma > 0$.[8]

Recall that the budget constraint, $c(t) \leq \delta(t)(1-i(t))$, will hold with equality and substitute (4) and (6) into (5) to find that the optimal investment level solves the Hamilton-Jacobi-Bellman equation:

$$0 = \max_{i(t)} \left\{ \begin{aligned} &(1-\gamma)^{-1}\delta(t)^{1-\gamma}(1-i(t))^{1-\gamma} - \rho W(\delta(t)) - \alpha(\delta,t)\delta(t)W_{\delta}(\delta(t)) \\ &\qquad + \lambda(\delta,t)i(t)[W(\delta(t)+f(\delta,t))-W(\delta(t))] \end{aligned} \right\} \tag{7}$$

Denote the optimal investment level at time t by $i^*(t)$ and differentiate (7) to find that the first order condition for an interior solution is:

$$-\delta(t)^{1-\gamma}(1-i^*(t))^{-\gamma} + \lambda(\delta,t)[W(\delta(t)+f(\delta,t))-W(\delta(t))] = 0 \tag{8}$$

or:

$$i^*(t) = 1 - \lambda(\delta,t)^{-\frac{1}{\gamma}}[W(\delta(t)+f(\delta,t))-W(\delta(t))]^{-\frac{1}{\gamma}}\delta(t)^{\frac{1-\gamma}{\gamma}} \tag{9}$$

provided that the right hand side of this equation is positive.[9] Substituting this investment level into (7) gives a stochastic differential equation in $W(\cdot)$. The solution to this equation gives the agent's indirect utility and optimal investment levels as functions of the current dividend level.

Substituting (9) into (7) and suppressing the explicit dependence on time and dividend levels gives:

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[7]Hereafter, I will call $W(\cdot)$ the indirect utility function though, strictly speaking, it is the present value of the indirect utility function.

[8]This implies that agent is not highly risk averse or will substitute intertemporally relatively freely. The problem can be solved with $1-\gamma < 0$, but keeping track of the sign of $1-\gamma$ complicates the analysis of the solutions unnecessarily.

[9]As I will show later, an interior solution exists ($i(t) > 0$) if the expected return on investment is "big enough".

$$0 = (1-\gamma)^{-1}\delta^{1-\gamma}\left\{\lambda^{-\frac{1}{\gamma}}[W(\delta+f)-W(\delta)]^{-\frac{1}{\gamma}}\delta^{\frac{1-\gamma}{\gamma}}\right\}^{1-\gamma} - \rho W(\delta) - \alpha\delta W_\delta(\delta)$$
$$+ \lambda\left\{1-\lambda^{-\frac{1}{\gamma}}[W(\delta+f)-W(\delta)]^{-\frac{1}{\gamma}}\delta^{\frac{1-\gamma}{\gamma}}\right\}[W(\delta+f)-W(\delta)]. \tag{10}$$

To solve (10), conjecture that the indirect utility function takes the form $W(\delta(t))=g\delta^{\epsilon}$. Suppose $f=h\delta$ with $h>0$. This implies that an innovation multiplies current dividends by $(1+h)>1$. Thus, $W(\delta+f)-W(\delta)=W((1+h)\delta)-W(\delta)=[(1+h)^{\epsilon}-1]g\delta^{\epsilon}:=jg\delta^{\epsilon}$ where j is defined to equal $(1+h)^{\epsilon}-1$ for notational simplicity. Finally, let $\lambda(\delta,t)=l\in(0,1)$ and $\alpha(\delta,t)=\alpha\in(0,1)$, where l and α are independent of δ and t. Substituting into (10) and simplifying gives:

$$0 = (1-\gamma)^{-1}l^{\frac{\gamma-1}{\gamma}}j^{\frac{\gamma-1}{\gamma}}g^{\frac{\gamma-1}{\gamma}}\delta^{\frac{\varepsilon(\gamma-1)+1-\gamma}{\gamma}} - \rho g\delta^{\varepsilon} - \alpha g\varepsilon\delta^{\varepsilon} + ljg\delta^{\varepsilon}$$
$$- l^{\frac{\gamma-1}{\gamma}}j^{\frac{\gamma-1}{\gamma}}g^{\frac{\gamma-1}{\gamma}}\delta^{\frac{\varepsilon(\gamma-1)+1-\gamma}{\gamma}}. \tag{11}$$

Equation (11) will hold for all δ if the exponents on δ in all terms are equal and the coefficients on each term add to zero. Thus, determine ϵ by equating the exponents on the δ's in (11). This gives $\epsilon=\epsilon(\gamma-1)\gamma^{-1}+(1-\gamma)\gamma^{-1}$, which will hold if we set $\epsilon=1-\gamma$. Thus, the exponent on the agent's indirect utility function is the same as the exponent on the agent's direct utility function. Further, since $1-\gamma>0$, this implies $j>0$.

Substituting this into (11) and dividing through by $\delta^{1-\gamma}$ gives:

$$0 = (1-\gamma)^{-1}l^{\frac{\gamma-1}{\gamma}}j^{\frac{\gamma-1}{\gamma}}g^{\frac{\gamma-1}{\gamma}} - \rho g - \alpha(1-\gamma)g + ljg - l^{\frac{\gamma-1}{\gamma}}j^{\frac{\gamma-1}{\gamma}}g^{\frac{\gamma-1}{\gamma}}. \tag{12}$$

To solve this equation, write it as:

$$Ag^{\frac{\gamma-1}{\gamma}} + Bg = 0 \tag{13}$$

where $A=\gamma(1-\gamma)^{-1}l^{\frac{\gamma-1}{\gamma}}j^{\frac{\gamma-1}{\gamma}}$ and $B=-\rho-\alpha(1-\gamma)+lj$. Solving for g gives two possibilities: $g=0$ or $g=A^{\gamma}(-B)^{-\gamma}$. The solution $g=0$ implies that the agent invests the entire dividend and receives zero lifetime utility. Under the right conditions, the agent can clearly do better. In particular, if we assume:

$$(1-\gamma)[\rho+\alpha(1-\gamma)-lj] > 0, \tag{14}$$

substituting $g=A^{\gamma}(-B)^{-\gamma}$ into the conjectured form for W gives:

$$W(\delta(t)) = \gamma^{\gamma} l^{\gamma-1} j^{\gamma-1} \{(1-\gamma)[\rho+\alpha(1-\gamma)-lj]\}^{-\gamma} \delta(t)^{1-\gamma} \tag{15}$$

which will be greater than zero.[10,11]

A. Conditions for an Interior Investment Solution

This solution assumes that the optimal investment level is strictly greater than zero and less than one. Because marginal utility approaches infinity as consumption falls to zero, the agent will never invest the entire dividend ($i(t) < 1 \ \forall \ t$). Assuming (14) holds assures this. Then it is obvious from Equation (9) since the indirect utility of an innovation is positive and λ is assumed positive. Alternatively, notice that the first order condition can never hold with equality when investment is one.

Intuitively, the agent will invest a positive amount ($i(t) > 0$) if the expected returns to investing are "big enough" to entice the agent to give up current consumption. For the interior investment solution to exist, from (8), the agent must have:

$$0 < 1 - \lambda(\delta,t)^{-\frac{1}{\gamma}} [W(\delta(t)+f(\delta,t)) - W(\delta(t))]^{-\frac{1}{\gamma}} \delta^{\frac{1-\gamma}{\gamma}}. \tag{16}$$

Recall that $W(\delta(t)+f(\delta,t))-W(\delta(t))=jg\delta^{1-\gamma}$. Substituting g into this expression, then substituting the result into (16) and solving shows how big "big enough" is. In particular, the return to investment must be such that:

$$lj > (1-\gamma)[\rho+\alpha(1-\gamma)]. \tag{17}$$

This says that the expected return to research and development investment must exceed the social interest rate implied by discounting and consumption smoothing (caused by risk aversion).[12]

Thus, returns to investment must be of the right size, small enough that the agent's discounted expected utility does not explode, but large enough to exceed the internal rate of interest implied by discounting and dividend decay. Combining Equation (17) and Equation (14), we must have:

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[10]The condition $(1-\gamma)[\rho+\alpha(1-\gamma)-lj]>0$ serves as a transversality condition in the sense that it insures that the dividend level grows slow enough that the agent's discounted expected future utility converges.

[11]This is not the most general form of the problem that can be solved. It can be solved when α, γ, λ, j and ρ depend on time. However the exponents on δ from λ and $[W(\delta+f)-W(\delta)]$ must add to $1-\gamma$ to find a closed form solution. In this case, I assume that λ and j are independent of δ. Another possibility is if $\lambda(\delta)=c_1\delta^{\tau}$ and $j(\delta)=c_2\delta^{-\tau}$, where τ, c_1 and c_2 are constants.

[12]This condition is analogous to the condition that leads to capital stock accumulation in Koopmans' optimal growth model (that the returns to capital investment exceed the social interest rate implied by discounting and population growth.) See Nagatani (1981, pp. 18-30) for a discussion of this similar condition. Here, instead of population growth, I have dividend decay which decreases the per capita dividend over time.

$$(1-\gamma)lj < (1-\gamma)\left[\rho+\alpha(1-\gamma)\right] < lj. \tag{18}$$

B. The Optimal Investment Level

Substituting g into Equation (9) gives the optimal investment level:

$$i^*(t) = 1 - \gamma^{-1}l^{-1}j^{-1}(1-\gamma)\left[\rho+\alpha(1-\gamma)-lj\right]. \tag{19}$$

Recall that $i(t)$ is the fraction of the current dividend invested, so investment is a constant fraction of current "wealth". The agent's CRRA utility function and the nature of investment leads to this result. To find a solution using the conjectured form of the indirect utility function, each term in Equation (7) must be exponential in δ. Further, all these exponents must be equal. Thus, $1-i^*(t)$ (from the first term) and $i^*(t)$ (from the last term) must both be exponential functions of δ. This is only possible if investment (as a fraction of dividends) is independent of dividends (i.e the exponent on δ in both is zero).

C. The Dividend Path Under Optimal Investment

Suppose the agent invests the optimal amount, $i^*(t)$, in the time period $(t,t+\Delta)$. Let $\eta^*\Delta$ denote the probability that an innovation occurs in this time period. It is given by:

$$\begin{aligned}\eta^*(t)\Delta := li^*(t)\Delta &= l\left\{1-\gamma^{-1}l^{-1}j^{-1}(1-\gamma)\left[\rho+\alpha(1-\gamma)-lj\right]\right\}\Delta \\ &= \left\{\gamma^{-1}l-\gamma^{-1}j(1-\gamma)\left[\rho+\alpha(1-\gamma)\right]\right\}\Delta.\end{aligned} \tag{20}$$

Since the optimal investment level does not depend on time or the dividend level, neither does η^*. This implies that the agent chooses a constant expected rate of innovation. With optimal investment, the dividend evolves according to:

$$d\delta(t) = -\alpha(t)\delta(t)dt + f(\delta,t)dq^*, \tag{21}$$

with the probability that $dq^*=1$ equal to η^*dt plus terms of order less than dt. Thus, the agent's preferences determine the innovation and expected dividend growth rates independent of time or the current dividend level.

D. Asset Prices

Solving the general equilibrium problem for the investment and quantity decisions simultaneously leads to an arbitrage restriction on asset prices. Let $Q(\delta,t)$ represent the number of units of the asset the agent holds at time t while $\delta(t)$ represents the dividends per unit of the asset. In equilibrium, $Q(\delta(t),t)=1$.

Substitute $Q(\delta(t),t)\delta(t)$ into the budget constraint above and derive the indirect utility function assuming no trade as $W(Q\delta)=g(Q\delta)^{1-\gamma}$.[13]

Suppose now the agent can trade the asset at the instantaneous price $P(\delta(t))$. Let $\iota(t)$ represent the fraction of the current dividend that agent uses to purchase new units of the asset. (When $\iota(t)<0$, the agent has sold units of the asset.) The rate at which the agent exchanges $\iota(t)$ and trees is δQP^{-1}. If the non-innovation depreciation in technology and the after-innovation jump in technology both apply to newly acquired trees, the rate at which ι is transformed into expected future utility under the optimal investment level is:

$$\lim_{\Delta\to 0} E_t \left\{ \frac{\partial W(t+\Delta)}{\partial Q(t+\Delta)} \frac{\partial Q(t+\Delta)}{\partial \iota(t)} \right\} = (1-\gamma)(1-\alpha)g(Q\delta)^{1-\gamma}\delta P(\delta)^{-1} + (1-\gamma)lji^{*}g(Q\delta)^{1-\gamma}\delta P(\delta)^{-1}. \quad \textbf{(22)}$$

Recall that, in equilibrium prices which imply no trade, the agent transforms research and development investment into expected future utility according to:

$$\lim_{\Delta\to 0} E_t \left\{ \frac{\partial W(t+\Delta)}{\partial i} \right\} = ljg(Q\delta)^{1-\gamma}. \quad \textbf{(23)}$$

The agent now can increase expected future indirect utility in two ways, by investing in research and development or by purchasing new trees. Both mean giving up (identical) consumption today. In equilibrium, the asset price must equate the increase in expected future utility from purchasing more trees and the increase in expected future utility from investing the same amount in more research and development. Equating Equations (22) and (23) and solving for price gives:

$$P(\delta(t)) = (1-\gamma)\times\left[(1-\alpha)l^{-1}j^{-1}+i^{*}\right]\delta = (1-\gamma)l^{-1}j^{-1}\times\left[(1-\alpha)+lji^{*}\right]\delta. \quad \textbf{(24)}$$

The two terms inside the brackets represent the instantaneous rate of dividend decay and the effects of optimal investment on the innovation rate. This implies that prices are homogeneous of degree one in dividends. Further, expectations about possible future innovations (good, or bad had the model included negative shocks) play an important role in determining prices.

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[13]The same prices can easily be found directly from an optimization problem that allows both trade and research and development investment. Simply set the state equal to $Q\delta$, conjecture that $P(\delta)$ takes the form $\rho\delta$, let the state evolve according to $dQ\delta=[(1-\alpha)\iota\rho^{-1}-\alpha]Q\delta dt+fdq$ where $f=\left\{\left[j(1+\iota\rho^{-1})^{1-\gamma}+1\right]^{\frac{1}{1-\gamma}}-1\right\}Q\delta$ and optimize the utility function subject to the budget constraint $c\leq(1-i-\iota)Q\delta$ where ι is the fraction of current wealth invested in new assets. Equilibrium occurs when ρ is such that $Q=1$ for all t. These dynamics imply that technical depreciation applies to old and newly acquired assets and that innovations apply to both as well.

IV. Implications

A. Investment Levels

The agent invests by giving up a portion of current consumption for a proportional increase in expected future consumption. With CRRA utility, the current consumption level does not affect how the agent views such a proportional tradeoff. So, the agent invests a constant fraction of the dividend. This extends Merton's (1971) result to the case of partially endogenous jump processes in general equilibrium.

B. Pricing Research and Development Characteristics

The last section shows how the agent chooses the investment level given the dividend level and the asset characteristics. The three asset characteristics are α, l and j. The first represents the rate at which the dividends (including those created by innovation) decay. The second represents the amount that investment increases the innovation probability (i.e. the amount of "jump likelihood" that a unit of investment purchases). The third represents the size of an innovation (i.e. the jump size in expected utility terms). These correspond roughly to research and development durability, efficiency and quality. In this section, I discuss how the agent responds to changes in these characteristics. Since the agent substitutes intertemporally relatively freely ($1-\gamma > 0$), all investment reactions are in the intuitive direction. However, price reactions include both substitution and income effects.

Recall, the agent gives up current consumption to invest and also gives up current consumption to purchase asset shares. Either act increases expected future consumption proportionally. Equation (19) shows how changes in asset characteristics affect the investment level. Note that higher investment leads to a higher innovation rate and a higher expected dividend growth rate. Given a dividend level, Equation (24) shows how the asset price varies with the asset characteristics.

With less durability (higher α), the benefits from an innovation do not last as long. This may reflect a higher physical capital depreciation rate or a faster rate of technical obsolescence. Increasing α decreases expected future consumption and increases its expected marginal utility. Differentiating Equation (19) shows that the investment level decreases when α increases. Substituting equation (19) into equation (24) and differentiating gives the following price reaction:

$$\frac{\partial P}{\partial \alpha} = -(1-\gamma) l^{-1} j^{-1} + (1-\gamma)\frac{\partial i^*}{\partial \alpha}. \tag{25}$$

The first term represents the effect of lower durability on current dividends. The second reflects the effect of the change in the investment level resulting from lower durability. Both terms are negative. Thus, the asset price falls in response to lower durability.

Greater efficiency (higher l) means that an innovation is more likely at any given investment level. This may result from better research and development management. In this case, there are price effects similar to both substitution and income effects. Equation (19) shows that investment increases when l increases. Substituting equation (19) into equation (24) and differentiating gives the following price reaction:

$$\begin{aligned} \frac{\partial P}{\partial l} &= -(1-\gamma)(1-\alpha)l^{-2}j^{-1} + (1-\gamma)\frac{\partial i^*}{\partial l} \\ &= l^{-2}j^{-1}(1-\gamma)\left\{ \gamma^{-1}(1-\gamma)[\rho+\alpha(1-\gamma)] - (1-\alpha) \right\} \end{aligned} \tag{26}$$

Here, the first term is negative while the second is positive. The second term reflects the effect of a higher investment level resulting from greater efficiency which makes assets more valuable. The first term is a substitution effect. As a means of generating expected future utility, research and development investment has become more attractive relative to purchasing more assets. Thus, there is a tendency for the price of assets to drop. Which term dominates ultimately depends on whether the weighted social interest rate implied by discounting and consumption smoothing, $\gamma^{-1}(1\text{-}\gamma)[\rho+\alpha(1\text{-}\gamma)]$, exceeds the gross technical discount rate implied by the depreciation alone, $1\text{-}\alpha$.

Higher quality or jump size (higher j), means that each innovation results in a greater proportional increase in dividends. This may result from better research and development skill. The results for quality are similar to the results for efficiency above. Investment increases with greater quality. There are both positive and negative price effects. Which dominates is determined by the relative social interest and technical depreciation rates.

C. Asset Dividend and Price Paths

The agent uses investment to attain optimal innovation and dividend growth rates. Dividend sample paths will evolve according to:

$$d\delta(t) = -\alpha(t)\delta(t)dt + f(\delta,t)dq^* \tag{27}$$

with the probability that $dq^*=1$ equal to $\eta^* dt$ plus terms of order less than dt. Present value prices are simple functions of the dividend.

Consider taking dividend and price samples in time increments of Δ where Δ is small enough that only one innovation can occur in a time increment. Number these observation by t, $t+1$, etc. The sample path of dividends will evolve according to:

$$\delta(t+1) = (1-\alpha+fq^*(t))\delta(t) \tag{28}$$

with the probability that $q^{*}(t)=1$ equal to η^{*}. Dividing (28) by $\delta(t)$, taking the natural log of both sides and rearranging gives:

$$\ln(\delta(t+1)) = \ln(\delta(t)) + \ln(1-\alpha+fq^{*}(t)). \tag{29}$$

Letting $\ln(1-\alpha+fq^{*}(t))$ be the error term, the log of dividends evolves according to a random walk with possible drift.[14] By a similar argument, the log of prices evolves according to a similar process. This is a form of price and dividend evolution that Marsh and Merton (1986) claim reverses the variance bounds test.[15] Thus, Shiller's (1981) findings of "excess" price volatility are exactly what we would expect in this economy.

From Equation (28), the time t conditional expected value of the dividend N periods in the future is:

$$E_t[\delta(t+N) \mid \delta(t)] = (1-\alpha+\eta^{*}f)^{N}\delta(t) \tag{30}$$

and the variance is:

$$VAR_t[\delta(t+N) \mid \delta(t)] = \left\{\left[(1-\alpha)^2+2(1-\alpha)f\eta^{*}+f^2\eta^{*}\right]^{N}-(1-\alpha+f\eta^{*})^{2N}\right\}\delta(t)^2. \tag{31}$$

As N increases, Equations (42) and (43) can behave in a number of ways. Both the conditional expected value and the conditional variance of the dividend could explode, both could go to zero or the expected value could go to zero while the variance explodes or remains constant.[16] Thus, the conditional forecasts of future dividend and price levels may converge as in Brock (1982). They may not.

In any case, the time series of dividends depends on the agent's preferences. This leads to three important implications for econometric testing. First, since dividends and prices are not stationary, stationarity assumptions (such as those used in variance bounds tests) may lead to faulty conclusions.[17] Second, since the conditional mean and variance of asset prices and dividends may not converge, we may

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[14]This random walk has positive drift if $E[\ln(1-\alpha+fq^{*}(t)]>0$, negative drift if $E[\ln(1-\alpha+fq^{*}(t)]<0$ and no drift if $E[\ln(1-\alpha+fq^{*}(t)]=0$. This depends on the agent's preferences.

[15]See Shiller (1986).

[16]Both the conditional mean and the conditional variance will explode if $\eta^{*}f>\alpha$. The mean will be constant at $\delta(t)$ and the variance will explode if $\eta^{*}f=\alpha$. The mean will go to if $\eta^{*}f<\alpha$. In this case, the variance will explode, go to a constant greater that zero or go to zero if $(1-\alpha)^2+2(1-\alpha)f\eta^{*}+f^2\eta^{*}>0$, $(1-\alpha)^2+2(1-\alpha)f\eta^{*}+f^2\eta^{*}=0$ or $(1-\alpha)^2+2(1-\alpha)f\eta^{*}+f^2\eta^{*}<0$ respectively. The same conditions determine whether actual dividend level and variances converge. The parameter restrictions assumed to solve the problem do not appear to rule out any of these possibilities.

[17]See Gilles and LeRoy (1988) for a discussion about how non-stationarity affects variance bounds tests.

not be able to construct variance bounds tests using conditional variances as Gilles and LeRoy (1988) suggest. Finally, the growth rate and covariance of dividends will change whenever preferences change. Thus, tests that assume constant growth rates must be modified if there is reason to believe that preferences or innovation characteristics have changed over the sample period.

V. Conclusions

In this paper, I describe a solution method for stochastic dynamic control problems in which the control affects the probability of a jump in the state variable. This method may be applied to a number of problems. I apply it to an investment problem in which the research and development investment level affects the probability of an innovation which causes a productive asset's dividend stream to jump.

Using CRRA utility and assuming that the representative agent invests dividends directly, I find that the agent always invests a constant fraction of the dividend. This extends Merton's (1971) result. The agent uses research and development investment to attain an optimal dividend growth rate which depends upon the agent's preferences and durability, efficiency and quality of research and development investment. This implies several things. First, economic growth, dividend growth or innovation rates will depend upon preferences and innovation characteristics, changing whenever preferences or characteristics change. Second, price and dividend paths are not stationary. Both the actual and conditionally forecasted price and dividend levels may not converge as the horizon grows. Third, the non-stationarity in prices and dividends is of the type that Marsh and Merton (1986) claim will reverse the conclusions from unconditional variance bounds tests such as Shiller's (1981). The log of dividends follows a random walk with a possible drift. Thus, variance bounds violations are expected in general equilibrium economies with research and development opportunities such as those modeled here. Finally, conditional moments of the dividend stream may or may not converge. Thus, we may not be able to construct appropriate variance bounds tests using conditional moments. It depends on preferences and the nature of the research and development process.

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