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"Knowledge, Discovery, and Growth"

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Knowledge, Discovery and Growth

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By

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Section 1 Introduction

This paper presents a model of knowledge, especially technological knowledge, with which technological change can be studied. Economists and others who study economic growth generally agree that technological change is perhaps the most important driver of economic development.[1] Technological change comes about by invention, or discovery. It proceeds by finding new ways of doing things, by finding new things to do, or new uses for things already known. Something previously unknown or nonexistent becomes known or comes into existence. More generally, knowledge is created by acts of discovery or invention.

A formal model of the growth of technology or knowledge must therefore accommodate creative acts, but do so without predicting specific creations, inventions or discoveries; to predict them would come very close to making them. A model that could predict specific discoveries in detail would itself be an engine for making creative acts--for short, a Promethean machine.[2]

This paper does not contain a theory of economic development. Rather a partial model is presented, intended, like a modular component in a stereo sound system, to fit within or be linked to a more general economic model. The partial

* I am indebted to Sondra Fargo for editorial help.

1 Lucas [1988] calls for the formulation of formal models of economic development addressing what he calls the mechanics of the subject. The model presented in this paper is a step in that direction.

2 This matter is discussed further in this Introduction, and in Section 4 below. That section contains a discussion of creative process(es) that serves as the basis for the

model presented here contains functions that can serve as connectors to a broader economic model, in which they could be endogenously determined; in the present model they are exogenous parameters.

In economics technology is modeled in a number of different ways. Almost all of them use the concept of *commodity*—for example, the production function, a relation between input commodities and output commodities expressed mathematically by a function. In some cases processes of production are explicitly introduced through variables called *activity levels* or *process intensities*.[3] Then production possibilities are expressed by a mapping, which may be linear or nonlinear, between the space of activity levels and the commodity space. Although there are several different models of technology, designed to fit different purposes, the simplest and perhaps most general is the *production set* model. The production set, a subset of the commodity space, consists of all input-output combinations that economic agents know how to produce. *Commodity* and *production set*, primitive concepts in economics, are usually taken as exogenously given.

Part 2 of this paper presents a model of technology that, as is conventional is, views technology as knowledge of how to produce things, yet, unlike conventional models, this model explicitly formalizes knowledge of how to produce things. Technology is therefore a special category of knowledge, and in this paper, commodities and production sets are derived from technological knowledge.

Because technological knowledge depends on other knowledge—such as scientific knowledge—technology is embedded in a more general model of knowledge. That model is presented in Section 3—even though it should logically

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model of the growth of knowledge (discovery) in Section 5.

[3] See for example Koopmans, T.C., [1957].

precede the model of technology—because the intuitive, informal discussion of modeling technology in Section 2 serves to motivate and clarify the model of knowledge presented in Section 3.

Knowledge, which resides in the mind of a person, may have several representations. Here it is modeled as a finite subset of the set of all possible statements in a natural language, say, English, based on a finite alphabet, augmented by a finite number of other symbols. From now on, English means augmented English. The collection of sentences of finite length, (i.e., finite strings of symbols from the alphabet, well-formed according to the rules or usages of the language) is countably infinite. It contains a representation of every item of knowledge that can conceivably be expressed in (augmented) English. This formulation avoids on the one hand, models that implicitly bound what might be known in the future, and on the other, introducing aggregates that are not well-defined sets.

The *knowledge of person i at time t* is represented as a finite subset of the set of possible English sentences. Formally, that set is a primitive of the model. Given that no unique representation of that knowledge can be expected, the applier of the model must represent what is in the mind of the person whose knowledge is being modeled. In these circumstances an economical or parsimonious representation best serves the way in which the model in this paper is to be used. Therefore, an equivalence relation on the set of sentences is introduced—in Section 3—which structures the knowledge set so it consists of *equivalence classes* of sentences. Each such class can be represented by one *representative sentence.* A representative sentence can be interpreted as an "idea."

A knowledge set is also given additional structure. For instance, different *areas of knowledge* may be distinguished formally. This model of knowledge is presented and discussed further in Section 3.[4]

This investigation focuses mainly on the growth of knowledge, which takes place by two processes, by learning from others, and by the discovery (creation or invention), of new knowledge. Acquisition of knowledge by one person does not diminish what is available for others to acquire—a property sometimes extended to the idea that knowledge once in existence is a public good, freely available to all. But that is clearly not true. One can borrow Von Neuman's book on quantum mechanics from a library without charge, but that does not automatically transfer the knowledge represented in it into the borrower's mind. Acquiring knowledge, even if already known to others, entails expending effort and resources, and also entails private acts of discovery—not so different in this respect from creating new knowledge—a matter discussed further in Section 4.

Discovery occurs when we add new ideas (representative sentences) to our stock of knowledge. And discovery, whether of socially new or of privately new knowledge, typically involves a creative act on the part of the discoverer. Hence, a

[4] The word 'knowledge' is used in this paper to refer to statements that the person whose knowledge is being modeled is aware of and believes to be valid. Thus, someone who believes he has seen a flying saucer would have in his knowledge set the sentence," I saw a flying saucer." Another usage common among those working in a Bayesian framework, and among those who study knowledge as an operator in a modal logic, would regard this as a statement of belief, and reserve the term "knowledge" for statements that are absolutely true. Knowledge in the latter sense is hard for human beings to come by. Even mathematical theorems, which provide perhaps the best candidates for absolutely true statements, are proved by human beings, whose operations are subject to error, (or by machines whose operations are also subject to error). There does not seem to be a better term for the conglomeration of beliefs, convictions and structures of information that make up what, in common usage, is what someone knows. These and other distinctions applying to knowledge are discussed in Section 3.

model of the growth of knowledge must rest ultimately on a theory of the process(es) of creation. The view of acts of creation presented in Section 4 provides the basis for the model of the growth of knowledge in Section 5.[5] Briefly summarized, that view is the following:

i) Discovery results from interaction between an individual mind and a body of knowledge;

ii) More knowledge facilitates discovery, but does not guarantee it; (Discoveries are almost always made by people with knowledge of the area in which the discovery is made, e.g., discoveries in game theory are usually made by people who know a lot of game theory, in chemistry, by chemists);

iii) Having more (adequately structured) knowledge does not inhibit discovery;

iv) Individual differences in cognitive and other skills and abilities are relevant to making discoveries;

v) Discovery is purposeful—intensity of effort and resources devoted to that activity have an effect;

vi) Creative acts typically involve bringing together disparate ideas, often from different frames of reference.

Clearly, to understand the process of coming to a discovery or creation, we must understand the ideas that an inventor worked with while making it. Books on the history of science or technology are well stocked with descriptions of that process for particular inventions—a retrospective sort of understanding.

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5 The terms discovery, invention, creative act are used interchangably in this paper. This seems natural in a model in which new knowledge is a sentence (as yet unknown) waiting to be found in the set of sentences complementary to the current knowledge of the discoverer. Fuller discussion of these matters can be found in Section 4.

But, to recapitulate an earlier point, a theory of discovery or creation cannot reasonably try to predict what particular creation(s) will arise when a specific set of ideas is subjected to the effort of a particular person, (who may not be the theorist). If there were such a theory, the theory would itself make the discoveries.[6] Furthermore it would make them without the abilities, skills, effort and resources of *any* person other than the theorist.

Fortunately, to analyze the economic role of discovery, we don't need a model that predicts or explains specific discoveries. It is enough to explain:

i) a measure of the *flow of discoveries*,

ii) the dependence of that flow on economic and social parameters, and,

iii) the economic effects of a flow of discoveries, measured in some average sense.

Also, a model can provide *ex ante* for discoveries whose specific nature is unknown—for example, new commodities or of production processes—to be

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[6] Some work in Artificial Intelligence goes in this direction. For example, Newell and Simon [1956], and Newell, Shaw and Simon [1957], wrote a computer program designed to prove given theorems in a given mathematical setting. (There is also a program, called BACON, that "inferred" certain physical laws, e.g., Boyle's Law, from given data.) A program that successfully came up with proofs of theorems would seem to constitute a Promethean machine, at least in a limited area of operation. Whether Simon's ideas about creative thinking generally, or computer programs embodying them, constitute a theory of discovery that is capable of predicting discoveries not yet made is a question that goes well beyond the scope of this paper. Whatever the answer turns out to be, it is fair to say that in the present state of knowledge there is no theory capable of being used to predict specific discoveries relevant to economic growth.

It may be possible to predict a direction in which discovery may be expected, without thereby making the discoveries themselves. A decision by an investigator to devote effort and resources to an area of research is often based in part on the conviction that the area is important and researchable, (i.e., that significant results are likely).

included *ex post*, after their specific nature is known, thus, permitting *ex ante* analysis of economic consequences of discoveries.

To model the growth of knowledge without attempting to make what I have called a Promethean machine, some concept of *amount of knowledge* is useful. In this model the amount of knowledge is a measure of the set of sentences that constitutes knowledge. Such a measure would generally be multidimensional, corresponding to distinctions among different kinds of knowledge. The simplest case is that of a one dimensional measure. While this may seem a gross oversimplification, it is one that connects naturally with existing models in which technological knowledge is represented by a (real) parameter—as, for example, a coefficient multiplying a Cobb-Douglas or CES production function—and technological change is expressed as a change, usually an increase, in the value of that parameter. The one-dimensional measure used here *is the number of representative sentences in the knowledge set*, and its growth is the change over time of that number. This is analogous to the one-dimensional measure of a heterogeneous collection of objects used in production called "amount of capital." Such a measure can be a useful starting point for developing a more sophisticated model.

Section 5 presents a mathematical model of the growth of knowledge based on the view summarized. A "prepared mind" is stimulated by a combination of ideas to conceive a new idea. In terms of sets of sentences, several subsets of knowledge come together to stimulate the conjecture and exploration of a candidate new sentence. Thus, the possibilities of cross-fertilization are given by the power set of the knowledge set being considered. But, because a subset of sentences can be construed as a compound sentence--and as such an element of the knowledge set—the size of the knowledge set itself measures the number of potential cross-fertilizings.

These ideas can be expressed in different mathematical structures. At present we don't have a clear basis for choosing among them. To avoid merely mathematical complexities, I give these ideas a very simple mathematical expression. Starting with an isolated person, this leads to a first order linear difference equation in the size of the set representing the knowledge of that person at time t. For mathematical convenience this difference equation is replaced by the analogous differential equation, in continous time and with the amount of knowledge treated as a continuous variable—a real number. Because knowledge sets are typically very large, and the changes relatively small, this does not appear to be an unwarranted simplification.

Next I consider a community of persons engaged in research or R&D activities—in the attempt to produce new knowledge. Such a community consists of persons who communicate with one another through various means. I extend the model of an isolated person to one of an interacting community. The result is a system of differential equations that characterizes the simultaneous growth of the knowledge of each person in the community—a system of equations that is the basis for analyzing the growth of the subset of knowledge that is technology.

The growth of knowledge, and of technology, can be exponential, depending on the values of the parameters that represent ability, skill, resources and effort applied to discovery and learning. This means that there can be (exponentially) increasing returns in the production of knowledge, and exponential growth in the amount of knowledge. Is that consistent with what can be observed?

Section 5 presents data on the growth of knowledge in computer science from 1958 to 1990. The body of knowledge in that field is measured here in two ways: first, by the number of pages published each year from 1958 to 1990 in the field of computer science, and second, the number of articles published in those years. Of course, the underlying assumption is that on average, the number of ideas per paper, or per sentence, isn't too far from constant. The results are that knowledge so measured grew exponentially at about 10 % per year over that period. That rate of growth is probably somewhat understated, because papers that properly belong in computer science were not found. Some were published in journals not primarily devoted to computer science, and this is more likely to have been the case in later than in earlier years.

By introducing of a (time varying) commodity space and a (time varying) attainable production set--both defined in terms of the underlying technology—we can analyze the growth of the attainable production set and its dependence on the underlying parameters.[7] This is done in Section 6, in the context of a Leontief model of production. Exponentially increasing returns in knowledge and technology translate into exponentially increasing returns in the production set. In examples analyzed in Section 6 as knowledge grows, a fixed amount of the primary resource yields an exponentially increasing amount of outputs over time, because the output coefficients grow exponentially.[8] Can this result be consistent with the physical laws governing matter and energy?

First, growth of knowledge may lead not to growth of a given set of output coefficients, but to new ways of satisfying economic wants via substances and

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[7] Unlike some models of technology used in economics, the production set is not a primitive of the model, but a defined entity.

[8] This is the case under the assumptions about how knowledge translates into technology and production. If decreasing returns of a sufficiently (but implausibly) high order were introduced at this step, exponential growth in knowledge might be cancelled in the translation into production.

processes not yet known, in a way that remains well inside the fundamental constraints imposed by physical laws.[9]

Learning curves in manufacturing are a well-observed phenomenon usually attributed to increase of knowledge among people engaged in production. Hence, we should be able to derive learning curves from a model of the growth of knowledge. We use the model of the growth of knowledge in Section 7 to derive two forms of learning curves.

Insofar as application to economic development is concerned, this paper is an attempt to provide a model of the growth of knowledge and technology that can be connected to and support such analyses.[10] I hope that it will eventually be linked with or embedded in a model in which a general dynamic economic analysis of its implications for economic growth or development can be carried out, and the effects of instruments of social and economic policy studied.

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[9] This matter is discussed more fully in Section 6, especially in footnote 79.

[10] Literature on the subjects of creativity, discovery, invention and learning is diverse and large. I do not claim to provide complete references to it, but I have tried to indicate relationships of ideas underlying this model to that literature. Because these ideas have been the subject of studies in economic history, in the history of science, in individual and social psychology, in philosophy, logic and computer science, and my knowledge of these areas is limited, I have no doubt overlooked things I should have known and acknowledged.

Section 2. A Model of Technology

2.1 Technology and Knowledge

The effects of discovery or invention on economic life come mainly through changes in technology and ultimately changes in production of goods and services. Among the many ways of representing technology used in economic models, the production set model is perhaps the simplest and most general. In that representation we postulate a commodity space, usually a Euclidean space, and a subset of it, called the *production set*, which contains all the input-output vectors that are technologically feasible, (those for which there is knowledge of a process, or processes, which if carried out would produce the specified outputs from the specified inputs).[11] An act of production is conventionally thought of in economics as a choice of a point from the production set; its physical execution does not appear separately in the theory.

Here I present a somewhat different model of technology, in which the commodity space and the production set are constructed from technological knowledge, rather than given exogenously. To motivate this model and to clarify its interpretation it is useful to have an example of a technology in mind. A wide choice of examples is available—for instance, in agriculture, steelmaking, aircraft manufacturing, pharmaceuticals—but, unfortunately, these examples are complicated. Knowledge about them is described in an

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[11] In some models , such as infinite horizon models or models with uncertainty, and infinitely many states, the commodity space is infinite dimensional. There are also (hedonic) models in which the commodity space is replaced by a Euclidean space whose coordinates measure qualities or properties. In that case a commodity is regarded as a bundle of qualities. In such models the list of qualities is fixed, and the commodity space is again Euclidean, but possibly of different dimension than the space of qualities. While there is provision for new bundles of qualities, there is no provision for new qualities.

extensive and complex literature not readily accessible to non-specialists. An example drawn from these unfamiliar areas of technology might be more confusing than illuminating.

One example, however, probably familiar to most people—the art of cooking—is the knowledge of how to produce edible dishes from raw materials. To use this example in the model proposed here, I take this body of knowledge, as of today, to be what is recorded in cookbooks now in existence. Typically, a cookbook contains

(i) recipes,

and

(ii) discussions of matters relevant to the preparation of food that may or may not be part of a recipe.

A recipe consists of four parts:

First, a description of what will be produced if the recipe is carried out, including not only the name of the product, e.g., Cheese Bites, but perhaps also information relevant to the use of the product—in this case, that it is "interesting, attractive and not messy," and therefore good to serve at cocktail parties;

Second, a list of ingredients and outputs, with all relevant quantities specified;

Third, a list of statements describing the actions to take in order to execute the recipe, (i.e., to carry out the act of production specified by the recipe);

Fourth, a statement to the effect that the recipe works, like the claim often found in the Introduction to the cookbook, for example: "All the recipes in this book have been tested under a variety of relevant conditions." This can be thought of as a statement attached to each recipe. Lacking an explicit claim, I take it that publication of the cookbook vouches for the reliability of its contents. In economics the standard models of technology focus on the list of ingredients and products, which are modeled as input-output vectors in the commodity space, and abstract from everything else.[12]

The model I use here incorporates *all* the elements of a recipe. Knowledge of how to produce a product or service, not restricted to preparation of food, is thought of as embodied in ***recipes***. Each recipe is a list of statements in a natural language, say English, that describe what is to be done, including at each step how much of each substance, equipment or labor is involved in that step.[13] Ingredients, substances and procedures involved are described and identified—named—in the natural language. Whatever details are necessary to identify the elements required are given. Using descriptions in the natural language permits denumerably many distinctions among entities (substances, objects, laborious procedures) to be made, without specifying them once and for all in advance.

To constitute a recipe a list of statements must also contain a description of what *uses* it fulfills,[14] and a statement certifying that *it works*. Thus, all four parts of a recipe are modeled as a list of English sentences.[15]

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[12] Some of these standard models introduce assumptions about the structure of the collection of recipes, e.g., constant returns to scale.

[13] Statements describing the actions to be taken are typically imperatives, not propositions in the sense of formal logic, that is, not either true or false. If it is desirable to describe the actions in the form of logical propositions, rather than merely natural language sentences, the list of steps that constitute the third part of a recipe can be replaced by a compound conditional statement of the form: If the following actions are taken (a list of actions), then the outcome will be (a statement describing the outcome). In this form, the third part of a recipe is just a (compound) statement.

[14] A patent application is required to contain a description of the use of the

This model of technology does not use the concept of a commodity. The objects, substances or other entities that appear in a recipe can be anything that can be given a name in the natural language. Although the collection of names of entities in the language at a given time is a finite set, a new name can be coined at any time to refer to a substance or entity hitherto unknown, or to make new distinctions among substances or objects. Furthermore, there can be more than one name for the same thing, and there is no assumption that the equivalence of these names is recognized.

For each named substance or object, there is also some way of representing its quantity, for instance by elements of an additive group, like the integers, the rationals or the real numbers.

Different cookbooks may each contain a recipe for the same dish, and these recipes may have different descriptions. For example, one can find recipes for beef stew in many cookbooks. These recipes may not prescribe exactly the same steps in preparation, may not specify exactly the same

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technology for which a patent is sought.

Two otherwise identical recipes that differ only in the use part of the recipe are different. For example, consider recipes that describe medical treatments. One is the prescription of a daily dose of aspirin to relieve symptoms of arthritis. The "use" part of this recipe is "reduce arthritic pain." Another recipe is the prescription of a daily dose of aspirin to prevent heart attack. The "use" portion of this recipe is different and the technological knowledge expressed in these recipes is also different.

[15] In practice, a recipe is typically an incomplete specification of what must be done to produce the intended result. A recipe typically relies on information to be supplied by those who execute it to resolve ambiguities or otherwise complete it. Computer software manuals provide many examples of this. This practice arises in part from the difficulty of writing down a full description of any complicated process. This kind of ambiguity seems to be inherent in natural language, as modern philosophers have pointed out. This phenomenon makes transfer of technology difficult and costly even between groups that have a common background, and even more so between groups in different cultures.

ingredients, or the same quantities. They may give different names to the product, or describe slightly different uses for the product. These recipes would be different. Yet if the differences are small enough, these recipes can be considered minor variations of the "same recipe." It is natural in a case like this to regard all these descriptions as equivalent.

As already mentioned, a cookbook, and more generally a body of technological knowledge, may also contain statements of a general nature embodying knowledge. For example, some cookbooks discuss the chemical composition and modes of action of various baking powders.[16] While statements about the action of baking powders may be common to cooking and to chemistry, it would not be hard to find statements about chemistry that are never found in cookbooks, and to find a graduated collection of statements that are 'between' them in the sense that they go from statements more relevant to cooking to statements increasingly remote from it. This illustrates a situation in which the line separating technological knowledge from other areas of knowledge is somewhat arbitrary. In the present model, the *certification of tested recipes* makes an unambiguous distinction between technology and other knowledge.

I embed the model of technology in a larger model of a *body of knowledge*, and where necessary make distinctions among *fields of knowledge.* The larger model includes partially specified recipes, as well as tested recipes. For example, it might include recipes that have the first three parts, but lack the fourth. These are conjectured, but as yet untested recipes. It could also include statements about properties of entities that appear in recipes, or about entities

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[16] In a book dealing with large scale preparation of foods, such as commercial baking, the role of chemistry would be even more obvious.

related to such entities, or about statements about such statements, and so on. And it could include the contents of books and journal articles on food chemistry, human physiology, and more remote areas of knowledge relevant to the model's use.

Thus, technological knowledge is a special case of knowledge. The discussion of modeling technology presented in this section motivates the model of knowledge in Section 3.

Section 3 A Model of Knowledge

To begin with let the set of persons at any time, t, be a given finite set, denoted

$\mathbf{A} = \{1,2,...,N\}$.[17]

Knowledge resides in the mind of a person. What a person 'knows' can be described by a finite collection of sentences in a natural language—English, for us.[18] Let **E** denote the set of English sentence, (i.e., finite strings of symbols from the English alphabet augmented with a finite number of other symbols).[19]

The availability of quantifiers, such as 'for all', and 'for some', which are, of course, English words, makes the language rich enough to include mathematics. It also includes self-referential sentences. Thus, the set **E** includes undecidable propositions (Godel's theorem). I do not require that this

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[17] The set of persons might vary over time with the growth of population, or as a result of decisions that result in persons entering or leaving the community of those seeking to discover. Here it is assumed that the set is constant over time.

[18] Consider the statement "Adam knows how to ride a bicycle." Although the act of riding a bicycle can be described, it is clear that knowing the description may not in itself constitute being able to ride a bicycle. In modeling technology 'knowing how' to produce something was equated with knowing the recipe for it. But to carry out a recipe may require the kind of knowledge that might better be called skill. Polanyi ([1958] p. 49.) has made this distinction.

When it is desired to maintain a distinction between 'knowing' and 'knowing how' in the sense of skill, an appropriate modification of the model must be made. This distinction is expressed in the formal model in Section 4.

[19] More generally, the set **E** may change over time, if new symbols are added to the alphabet.

model include only statements that are either "true" or "false," or that it be free of the problems of self-reference.[20]

For i in **A,** let K'(i,t) be a subset of **E**.

The set K'(i,t) is interpreted as person i's knowledge at time t. It includes any recipes that person i knows at t, together with any other knowledge that i has. Thus,

$$K'(i,t \supseteq K'^*(i,t)),$$

where K'*(i,t) is the set of recipes known by i at t.

The set K'(i,t) may include the rules of logic, the rules of the calculus of probability, or of other systems of thought. However, it is possible for person i to know some sentences, and to know the rules of logic, and yet not to know a sentence that is implied by what he or she does know, or one that is logically equivalent to what he knows.

Here, in contrast to other usage, the word 'knowledge' refers to sentences (or structures of them) that may or may not be true, or whose truth may be unknown, as well as to sentences that person i is aware of, and to which he may attach some degree of belief. Because the credence that person i attaches to a sentence can be expressed by a sentence; such beliefs can be expressed this in the model.[21]

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[20] The set of strings over a finite alphabet is countably infinite. But this is not essential to the analysis carried out with the model presented here. (See Section 6.) Therefore, augmenting the alphabet, for instance with symbols that permit quantifiers is not troublesome.

[21] The following are examples of such statements. "I think that the sentence "The moon is made of green cheese." is false." "The probability that Fermat had a proof for his famous theorem is less than one half."

Let I(i,t) be an equivalence relation on the set K'(i,t). Sentences are equivalent according to I(i,t) if person i at t regards them as expressing the same thing. In particular, the relation I(i,t) expresses the equivalence of recipes that would be so considered in the discussion of equivalent recipes in Section 2.

The relation I(i,t) can be extended from K'(i,t) to all of **E**. One such extension results from letting the equivalence classes be singletons on E \ K'(i,t), but others are also possible.

Now, let

$$K(i,t) = K'(i,t)/I(i,t),$$

namely, the quotient set of K'(i,t) with respect to the relation I(i,t). The elements of K(i,t) are representatives of the equivalence classes of I(i,t), and may be called *ideas*. (Here the term 'ideas' must be understood broadly, for instance, to include the basic recipes discussed in Section 2.) Each of these may be represented by a canonical sentence that expresses the idea. From now on, an element of the knowledge set K(i,t) is understood to be a canonical sentence— a representative of an equivalence class of sentences that express the same idea or entity.

As in the case of technology in Section 2, it is useful to have an example of knowledge in mind. Examples in Economics, or Game Theory (or a subfield of specialization, such as Economic Theory are plentiful. In these fields existing knowledge is for the most part written in the form of books, journal articles, or preprints. However, the set K(i,t) is interpreted as including only what i has in his/her head at time t. Thus, if person i bought Myerson's book on game theory at time t-1 and has it on his shelf unread, its contents are not included in K(i,t), unless i has acquired this knowledge in some other way.

The sets K(i,t)[22] have the possibility of being organized with different structures. In the example of economic theory, general equilibrium theory might be distinguished from principle-agent theory. Or, a sentence that states, say, that one theorem is a special case of another, might be included in a

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[22] It is sometimes convenient to have a notation for the knowledge held by a group, though this is used in this paper only occasionally in Sections 6, and 7 and could be dispensed with there.

If A is a subset of **A**(t), then what the persons in A together know at time t may be denoted,

$$\bigcup_{A} K(i,t) = K(A,t).$$

The knowledge that constitutes the technology known by the group A(t) is known by somebody. A recipe may be known by some person, or a group of people together may know it. Let

$$K^*(A,t)$$

denote the set of recipes (the technology) known by A at time t. Then,

$$K(A,t) \supseteq K^*(A,t).$$

Furthermore, if r is a recipe in K*(A,t) then there is a subset $\mathcal{A}(r)$ of A who together know r. I.e.,

$$r \in \bigcup_{i \in \mathcal{A}(r)} K(i,t).$$

And,

$$\bigcap_{A} K(i,t)$$

is the *common specialization* of the group A.

structure consisting of relations defined on a subset of sentences in K(i,t), expressing the depth of knowledge in an area.

We introduce a measure of the amount of i's knowledge at t, denoted,

$$k(i,t)$$

called the *size* of K(i,t).

A natural candidate is the measure of K(i,t), which, since K(i,t) is finite, is the number of elements in K(i,t) .[23] [24]

Then,

$$k(i,t) = |K(i,t)|$$ [25]

Distinctions of substance among different ideas or knowledge can be introduced formally. Let

$$\mathbf{L} = \{L_q, q = 1,2,...\}$$

be a partition of **E** / I(i,t)[26] and let

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[23] The possibility that the cardinality of K(i,t) exaggerates person i's knowledge at t may arise if the set contains several statements that describe "the same knowledge," while person i knows that to be the case. The sets K'(i,t) and the relation I(i,t) are primitives of the model. Together they determine K(i,t). The interpretation given above and illustrated in Section 2 makes clear that in any interpretation of the formal model, elements of K(i,t) are to be distinct ideas. Double counting should not arise.

[24] The motivation and justification for introducing this measure of knowledge is indicated in Section 1, the Introduction, and is discussed more fully in Section 4.

[25] The knowledge of i at t can include sentences to which i attaches no credibility. "The moon is made of green cheese.", as well as the sentence "I think the sentence, "The moon is made of green cheese." is false." can both be in i's knowledge set, K'(i,t). The number of such sentences can be arbitrarily increased without any real change in i's knowledge. In order to avoid notational complexity, the set K' (i,t) will be understood not to contain sentences to which i gives no credibility, and therefore the sets K(i,t) will not contain ideas which i regards as incredible.

$$K_q(i,t) = L_q \subseteq K(i,t),$$

and let

$$k_q(i,t) = |K_q(i,t)|, \quad \text{for } q = 1,2,\ldots \ .$$

The partition **L** classifies knowledge, the elements in Lq are considered to be of the same kind. The sets L_q can be called *areas of knowledge*, or *subfields*. (Since K(i,t) is finite, only a finite number of the sets $K_q(i,t)$ can be nonempty.)

If the partition **L** is the finest possible, then to know that a statement (idea, or recipe) is in the set Lq' for some q' identifies that statement up to the equivalence relation I(i,t); the statement is uniquely identified as a particular idea.

In this model new knowledge consists of new statements appended to an existing knowledge set. This involves a creative act. The next section, Section 4, discusses the creative process.

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[26] The partition L may depend on i and t, in which case it is interpreted as person i's classification of knowledge at time t. The partition is restricted to consist of a finite number of sets. The notation suppresses this dependence.

Section 4 Creativity and Discovery

4.1 Creative Power and Knowledge

Person i makes a *discovery* (or an *invention*) when he adds a new sentence or list of sentences to his knowledge, perhaps describing a new product, a new process of production, or a new theorem.[27] Discovery or invention involves an act if creation— something that *was not* before a certain time *is* afterwards. Section 4.1 aims to develop an understanding of the act of discovery sufficient to justify and support the model presented in Section 5. That model is not a model of the creative act itself, but *of the growth of knowledge as a result of discovery*—a model that can be used to analyze the economic significance of discoveries.

Plato, speaking through the mouth of Socrates, states that poets create by"...divine power ....",[28] a power that people have been trying to understand ever since. Some have studied the lives of 'obviously' gifted creators, such as Einstein, or Picasso, looking to detect how they differ from people who have not made such remarkable discoveries or creations. Although individuals differ in intellectual abilities, and in creative powers, whatever they may be, it is also clear that human beings have created, discovered and invented in every time and place where human beings lived. The divine power seems to have

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[27] The word"discover" is sometimes used to refer to the bringing to light of something that already exists, while "invent" is used to refer to bringing into existence something that did not previously exist. (See Hadamard [1945], p. xi, for instance.) "Columbus discovered America." "Edison invented the electric light bulb." I prefer a usage that permits us to say "Arrow discovered the Impossibility Theorem of Social Welfare." without affirming the "existence" of that theorem before Arrow's statement of it, even though it is in the list of statements that constitute **E**.

been broadly distributed to humanity. Whatever the abilities are that facilitate creative achievements, originality, discovery and invention, those abilities are part of the human genetic endowment.[29] [30]

Most discoveries are made by people who have knowledge of the field in which the discovery is made—in mathematics by mathematicians, in chemistry by chemists, and so on. In earlier times, when technology was further removed from science, discoveries and inventions were made by people without academic or professional credentials, but rarely by those with little knowledge of the field to which the inventions belong. Even today many 'small'

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[28] "For not by art do they speak these things, but by divine power...." [Plato , Ion] p. 19.

[29] This does not exclude the possibility that individuals with the same genetic endowment end up with different creative abilities because of different developmental histories. The point here is that there does not seem to be a special faculty of "creativeness."

The view that creative faculties are part of the human endowment is supported by the existence of continuity between man and our evolutionary ancestors, or even by the existence of some rudimentary form of creative powers in creatures not directly in our evolutionary line. Such a continuity is suggested by the following observation. A colony of macaque monkeys was established by a Japanese laboratory on an island, where they could be observed from concealment. The monkeys were fed potatoes, which 'appeared' from time to time on the sandy beach where they lived. The monkeys spent considerable time in brushing the sand off the potatoes before eating them. One young female monkey 'discovered' that sandy potatoes could be cleaned by washing them in a stream, and later in the salt water that lapped on their beach.

The practice of washing potatoes did not spread immediately to all of the colony. Older monkeys, no doubt set in their ways, were highly resistant to the new way, while young ones were quicker to adopt it.

The macaque genius who discovered washing potatoes also discovered swimming in the water, a practice hitherto unknown among the monkeys. Again some old macaques refused to venture into the water, while most of the young ones swam and played with evident enthusiasm. See Itani and Nishimara [1973].

[30] Campbell, D., [1974] Suggests that the ways humans have of learning and knowing are a result of evolution. This influential paper has engendered a

discoveries that lead to improved technology are made by those who carry out production. Of course, what matters is not the persons credentials, but the fact of working intensively in that field. Many medical discoveries were once made by practicing physicians. These days medical discoveries are made by researchers—biologists, biochemists and the like, or M.D.'s, who are focused on research, and more rarely by physicians who are exclusively in clinical practice, or whose activity and experience are removed from the areas that now spawn discoveries.

This suggests that the *knowledge of* and *attention to* a field play important roles in discovery in that field.

Consider how a graduate student and an experienced faculty member (in economics) read and understand a new paper. The student who usually must work hard to follow the paper's contents, can probably summarize it in a limited way, and perhaps supply missing steps in the arguments, but will often not be able to see and evaluate its contents in the context of the broader literature.[31] The faculty member can usually absorb the contents more quickly, relate them to other work in the field and judge the paper's significance in light of her knowledge of the literature. The faculty member is also more likely to see other applications of the methods used, or think of other methods that could be used either with or instead of the ones in the paper.

A few years later when the student has become an experienced faculty member herself, she will perform much as the faculty member now. On the other hand, if the faculty member should choose to read a paper in an

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substantial literature exploring evolutionary theories of mind, rationality and related matters.

unfamiliar field he is likely to have the same difficulties as the student, perhaps mitigated by his knowledge and experience in those aspects of his field that relate to the material he is reading.

One important difference between the student and the faculty member is the difference in their knowledge. The knowledge of a novice most typically comprises relatively isolated pieces rather than of a richly integrated structure of thought.[32]

The task of understanding the paper involves making what are subjectively discoveries, a task not essentially different from making discoveries.[33]

It is important to note that commanding a large body of knowledge does not in itself lead to difficulties that offset the benefits of knowledge, provided that knowledge about the structure of knowledge is also large.[34] This is typically the case for an experienced specialist.

Another aspect of the facility that goes with expertise and experience involves the distinction between 'knowing' and 'knowing how,' mentioned in footnote 16, Section 3. One aspect of knowing how to produce something is

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[31] This observation must be qualified. Matters of ability aside, the student may bring with him knowledge from another relevant field of which the faculty member knows little.

[32] This point is related to how knowledge is stored and accessed. This is discussed further in footnote 35.

[33] Hadamard [1945] p.75, observes that there is no essential difference for him between trying to build up a mathematical argument or trying to comprehend a given one.

[34] "Contrary to popular belief it is not possible to have too much knowledge about a task domain." The author, Amabile [1990] p. 82, comments that what is important is "...the way in which that knowledge is stored, and the ease with which it can be accessed [If information is stored] in wide categories with easy access to association, increased information should only lead to increased creativity." See also footnote 35.

captured by the description of the process necessary to produce it, as in the representation of technology by recipes. But a second aspect of knowing how is not so convincingly captured by verbal representation of knowledge. Such knowledge, which might better be called skill, may not involve language at all.[35] The student may also have lesser skills than the more experienced faculty member. The model presented in Section 5 deals with both aspects of knowledge.

Although knowledge exists in the mind of an individual, a single person rarely operates in isolation. Rather an individual typically works in a community of people whose interests overlap, and who can share their knowledge through formal and informal communication. It seems clear that knowledge of persons in a community grows more rapidly than does that of persons working in isolation, whose product often shows signs of that isolation.[36]

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[35] Polanyi, M. [1958] emphasized this distinction. Recent research on brain functioning strongly suggests that the brain uses a variety of different mechanisms, which together interact to carry out certain complex functions. One recent example reported in the New York Times Science Pages, Sept. 13, 1992, may serve to illustrate. A brain-injured person could not say whether an elephant is bigger than a dog, but had no difficulty in making the comparison when shown a picture of an elephant and of a dog. Since the pictures are the same size, the comparison is made by access to knowledge of the relative sizes of elephants and dogs, and not by direct perception of the pictures.

[36] However, it should be noted that one of the 'signs of isolation' is the persistent pursuit of an idea outside the conventional mainstream. There are examples where this has resulted in path-breaking discoveries.

4.2 The Act of Discovery

Hadamard states, "Indeed, it is obvious that invention or discovery, be it in mathematics or anywhere else, takes place by combining ideas."[37] This conception, not unique to Hadamard, is frequently mentioned by students of creativity. Koestler, in his book *The Act of Creation*[38] makes it the central element in his understanding of creation. In Koestler's view, normal thought takes place within a frame of reference, or an associative context, a type of logic, universe of discourse, or a particular code or matrix. In ordinary life we may use many different frames of reference, usually one at a time, switching from one to another as the situation requires, but ordinarily we treat them as independent. According to Koestler, creating involves bringing together otherwise independent frames of reference. Koestler gives this process the name *bisociation*.[39]

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[37] Hadamard [1945], p 29. He quotes Paul Valery: "It takes two to invent anything. The one makes up combinations; the other chooses, recognizes what is important to him in the mass of things which the former has imparted to him.

"What we call genius is much less the work of the first than the readiness of the second one to grasp the value of what has been laid before him and to choose it." p. 30.

[38] Koestler, [1964].

[39] Science News, Vol.141, June 6 1992 reports that two British scientists discovered a new way of shaping ceramic materials. They brought the knowledge of how cars are now painted into a new context, that of shaping ceramic materials. Car painting involves lowering the alkalinity of the car surface so that polymers settle and coat the car, where they are cured to form a permanent paint. "It suddenly struck me that if we can generate a base at an electrode, then we could precipitate [ceramic] materials from solution," recalls Philip J Mitchell, an electrochemist at Loughborough (England) University of Technology. In the June 4 NATURE, he and Loughborough University materials scientist Geoffrey D. Wilcox describe an electrochemical process that creates such a basic environment. They report that they have used this approach to make a variety of ceramic films in different shapes, including hair-width ceramic tubes." The story includes a statement by a materials

Bisociation may be understood to identify a property either of the *process* of creation, or of the creative *product*. It is possible to argue, as Perkins did, that the creative process does involve the joining of different frames of reference, yet maintain that ordinary mental processes can and do accomplish the work of bisociation, that there is no special process of bisociation.[40]

This gathering of ideas from different domains, or combining different frames of reference, suggests that the potential for discovery or for creative acts afforded by a given knowledge base is related to the set of *combinations* of ideas or of frames of reference in that knowledge base. Thus, a state of knowledge can be said to generate a set of potential discoveries—discoveries waiting to happen. Indeed, the history of science provides many examples of roughly simultaneous and independent discovery of the same new knowledge, or creation.[41]

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scientist, James H. Adair, at the University of Florida to the effect that "It [the discovery] could really have an impact on how we make complex ceramics."

[40] Perkins [1981], p. 96, goes on to point out the following. "Besides such considerations, there is another sense in which ordinary thinking contains bisociative potential. Thinking within a frame of reference requires sensitivity to the rules of the game, and events may occur that challenge the rules. Just by functioning within a frame, you are in a position to notice or more generally recognize the unexpected. Time and again in the history of science, investigators have accidentally encountered phenomena that should not have occurred, recognized them as anomalies, and gone on to revise or devise frames of reference to accommodate them. Of course, such recognition does only half the work of bisociation, challenging the established frame of reference but not relating it to another one. Nonetheless, it's important to grasp that the work of bisociation—if bisociation is the ultimate outcome—has in a sense begun already when an anomaly in the prevailing frame of reference is observed."

[41] One such example is the independent discovery of the idea of evolution by Darwin and Wallace. Darwin [1911], p. 68, described his moment of insight as follows.

It isn't necessary to go deeply here into the processes that produce an act of creation or discovery, (even if those processes were fully known, as they are

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"In October 1838, that is, fifteen months after I had begun my systematic inquiry, I happened to read for amusement "Malthus on Population," and being well prepared to appreciate the struggle for existence which everywhere goes on from long-continued observation of the habits of animals and plants, it at once struck me that under these circumstances favorable variations would tend to be preserved, and unfavorable ones to be destroyed."

Alfred Russell Wallace [1905] pp. 361-362, describes his arrival at the theory thusly.

"One day something brought to my recollection Malthus' "Principles of Population" which I had read about twelve years before. I thought of his clear exposition of " the positive checks to increase" - - disease, accidents, war, famine -- which keep down the population of savage races to so much lower an average than that of more civilized peoples. It then occurred to me that these causes or their equivalents are continually acting in the case of animals also; and as animals usually breed much more rapidly than does mankind, the destruction every year from these causes must be enormous in order to keep down the numbers of each species, since they evidently do not increase regularly from year to year, as otherwise the world would long ago have been densely crowded with those that breed most quickly. Vaguely thinking over the enormous and constant destruction which this implied, it occurred to me to ask the question, Why do some die and some live? And the answer was clearly, that on the whole the best fitted live. From the effects of disease the most healthy escaped; from enemies, the strongest, the swiftest, or the most cunning; from famine, the best hunters or those with the best digestion; and so on. Then it suddenly flashed upon me that this self-acting process would necessarily *improve the race*, because in every generation the inferior would inevitable be killed off and the superior would remain--that is, the *fittest would survive*."
In addition to being an example of independent arrival at the same discovery, this provides an example of an insight generated by the "cross-fertilization" accompanying the combining of ideas from different frames of reference. One might think after the fact that human populations and animal populations are not such different frames of reference, but it is clear from the quoted passages that they were so regarded by both Darwin and Wallace.

It should also be noted that the existence of simultaneous and independent discoveries has been disputed. Smith [1981], p. 384, argues that nearly all cases of apparently independent invention can be explained by communication. (Cited in Mokyr 1990].)

surely not), because this investigation is not intended to model those processes as such. Rather my aim is, first, to focus on certain salient characteristics of then that provide the basis for a quantitative model of the growth of knowledge, including the knowledge we call technology, and, second, to enable us to assess the model's economic significance.

One salient idea is that the knowledge base itself contains the seeds of discoveries "waiting to happen." These seeds are combinations of ideas which will sooner or later stimulate a *prepared mind* to conceive the new idea and make the new discovery.

In this context a prepared mind has four properties:
command of the relevant knowledge base;
sufficient command of the skills relevant to the task at hand;
intense focus on the specific knowledge in question; and,
suitable disposition to the discovery to be made.[42]

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[42] Commenting on Einstein's early work, Clark [1971] p. 52, writes "Thus even at this early stage, when dealing with a subject far removed from the new concept of space and time to be embodied in relativity, Einstein revealed two aspects of his approach to science which became keys to his work: the search for a unity behind disparate phenomena, and the acceptance of a reality " apart from the direct visible truth." Clark in discussing one of the famous four 1905 papers, "On a Heuristic Viewpoint Concerning the Production and Transformation of Light," says " It contained Einstein's first implied admission of the duality of nature which was to haunt his life and an early hint of the indeterminacy problem which drove him, as de Broglie has put it, " to end his scientific life in sad isolation and --paradoxically enough--apparently far behind the ideas of his time." (Clark [1971] p.63.)

Einstein's belief in the underlying unity of nature, which may have been a motivating force behind the undoubted intensity and persistence of his thought , was well-suited to the "discoveries waiting to be made" in relativity and allowed him to be the one who made them. These same values and beliefs were a handicap to scientific discovery later in his scientific life. Einstein was in this respect not unique. There are numerous examples of this kind.

The first three properties do not require further comment, but the fourth does. An act of discovery is an interaction between elements of knowledge of different kinds. Some are of those elements make up knowledge in a field of specialization, and others are more general ideas that can be seen as integral and perhaps unchangeable aspects of the investigator's personality.[43] " Thus, discovery or creation of new knowledge doesn't arise just from "ripeness"—discoveries waiting to happen—but also entails a contribution from the discoverer(s) that goes beyond knowledge and effort.[44]

There are cases of important discoveries made by an "outsider" whose knowledge of the field was limited relative to that of the established authorities. One can argue that ignorance of established ideas tends to free the mind to see from a different perspective, or to make connections that others

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[43] Examples of the latter include Einstein's often quoted conviction that "God does not play at dice," or Kepler's view of the planetary system, which he attributed to "...physical reasons or, if you prefer, metaphysical reasons." Those reasons are exposed in the following statement by Kepler. "My ceaseless search concerned primarily three problems, namely, the number, size, and motion of the planets--why they are just as they are and not otherwise arranged. I was encouraged in my daring inquiry by that beautiful analogy between stationary objects, namely, the sun, the fixed stars, and the space between them, with God the Father, the Son, and the Holy Ghost. I shall pursue this analogy in my future cosmographical work." Quoted in [Koestler 1971 p. 125].

[44] It is often pointed out that important discoveries in science have been made by people who did not have "too much knowledge." Having a fresh point of view uncontaminated by conventional ideas is thought to make it easier to come up with something new. In the present model, discoveries are the result of an interaction between a person and a body of knowledge. It may be that ability to bring together ideas from different fields is facilitated by ignorance of one of the fields, but it seems more likely that the abilities of the discoverer, the intensity of her interest in the problem and the existence in her knowledge of the relevant ideas are more important than whether those ideas are concealed in a body of other ideas. In the present model, the ability of the investigator plays an explicit role, while the possibility of confusion resulting from knowing too much does not.

wouldn't see. One can also argue that what is crucial here it is the personality of the discoverer that, not ignorance of the field. In any field at any time some people see things from one unconventional viewpoint or another, and not necessarily due to ignorance. In most cases the product of the unique perspective is dismissed, and does not contribute to knowledge in a significant way. Sometimes the investigators are dismissed as cranks, often correctly. But in rare cases valuable discoveries are made. These cases, whether of deep insight or crankish error, exemplify the role played by the personal characteristics of the investigator. They underlie what in this paper is meant by a "prepared mind," (i.e., a property of the individual distinct from, but interacting with, the body of knowledge she commands).

Because our focus here is not on the individual discoverer, the model in Section 5 can attend to these phenomena without detailed examination of the mental processes involved.

A related question concerns the role of chance in discovery. It is evident that chance plays a role in individual discoveries. This topic has been much discussed both in general terms,[45] and with reference to specific examples of

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45 Hadamard [1945] mentions the view of the French psychologist Souriau that invention occurs by pure chance and cites Souriau **Theorie de l'Invention** (Paris 1881). Hadamard also mentions the opinion of the biologist Nicolle**,Biologie de l'Invention** pp. 5-7 to the effect that "The act of discovery is an accident." Hadamard rejects these views of the role of chance in discovery on philosophical grounds, and on the basis that it flies in the face of experience. How can that view explain the fact that some individuals, such as Poincare, make a stream of important discoveries over a lifetime, while others do not?

More recently Donald Campbell in a well known paper [1960], put forward the idea that discovery proceeds by selection from randomly generated ideas. This view of discovery and invention as an evolutionary process driven by blind chance is interesting as an attempt to see how widely an evolutionary model can be applied, but in my view it is not a satisfactory explanation of discovery and invention, or creative activity. To elaborate on Hadamard's cogent objections, one serendipitous discoveries, such as Fleming's discovery of penicillin. However, the model presented in Section 5 is not stochastic. It could be modified, as indicated in Section 5, so as to make it so. But, here the model is not so modified, mainly to avoid technical complexities associated with a stochastic model. Also, while chance doubtless plays a role in discovery, it isn't so clear what that role is and how it should enter a formal analysis. Is chance more significant than 'noise' in determining *whether* a particular discovery is made, or *when* it is made, or *who* makes it?[46] The model presented in Section 5 does not address these questions, but rather is meant to facilitate analysis of the economic consequences of an ongoing process of discovery. In any case, that model of discovery can be modified to include stochastic elements.

However discoveries emerge, they also differ in importance. Some, like the theory of relativity, or the discovery of the genetic code, fundamentally change existing knowledge, while others appear to be small additions or modifications. The model in Section 5 makes no formal distinction between 'big' and 'small' discoveries. The need for that distinction depends on how the model is to be used. For some purposes the model will require more structure,

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has only to consider the experience of those who engage in research and write papers. One wonders whether Campbell's paper was the outcome of a process in which Campbell's mind was bombarded by a rain of randomly generated ideas which he screened to select the idea of evolution as an explanation of creativity.

Mokyr [1990] pp. 273-299, discusses an analogy between technological change and evolution.

The idea that random search can be a useful procedure in certain types of problems or situations is no doubt valuable. (See for example, Reiter, S. and G. Sherman, [1965].)

[46] Mokyr gives examples of a few inventions made during the 19th century that might have appeared at any time, He argues that the rare ability of the discoverer played a role in determining when they appeared, and that this is a matter of chance.

but for present purposes one can assume that the flow of discoveries is a 'typical' mixture of big and small ones.[47]

The model presented in Section 5 formalizes what is discussed in this section. To summarize:

1. Discovery grows out of the existing structure of knowledge;

2. It does so as the result of individuals applying effort, resources, skill and talent to their existing structure of knowledge;

3) Individuals operate not in isolation, but in communities. They learn from each other, and their interaction facilitates the growth of knowledge of all of them;

4) Discovery proceeds by combining ideas.

The main aim of Section 5 is to use these propositions, and the insights they summarize, the basis for a dynamic model of the growth of knowledge, of technology and of production possibilities.

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[47] In a stochastic version of the model in Section 5, the distribution of the significance of discoveries might be a primitive of the model.

Section 5 Growth of Knowledge

5.1 Remarks on Modeling Growth of Knowledge

We begin with a class of models of the growth of knowledge in the case of an isolated person, i, as follows.

Recall from Section 3 the partition

$$\mathbf{L} = \{L_q, q = 1,2,...\}$$

of **E** \ I(i,t), and recall that

$$K_q(i,t) = L_q \cap K(i,t)$$

and

$$k_q(i,t) = |K_q(i,t)|.$$

The partition **L** classifies knowledge, the elements in Lq being of the same kind. The sets L_q can be called *areas of knowledge*, or *sub-fields*. Since K(i,t) is finite, only a finite number of the sets $K_q(i,t)$ can be nonempty. Then,

$$k_q(i,t+1) - k_q(i,t)$$

gives the change in the (amount of) knowledge of person i in the q^{th} subject area in the period t to t + 1.

A model that at time t determined the values of the variables $k_q(i,t+1)$ would thereby predict the number of discoveries in the q^{th} sub-field. If the partition **L** of **E** / I(t) were the finest possible, then the model would predict the discoveries up to I-equivalence.

If the sets L_q were fine enough, say, in the extreme case singletons, the determination of $k_q(i,t+1)$ amounts to predicting a specific discovery—because the one sentence that defines the equivalence class also defines the discovery. There is a gradation of specificity from this Promethean theory, to the other extreme at which Q = 1, discoveries are counted without making any distinctions based on substance.

For the uses to which the model is put in this paper, Q = 1, or 2 is sufficient. Furthermore, discoveries are treated as deterministic. The situation may be likened to modeling the occurrence of fires. There are fires of many kinds. It may be for certain purposes necessary to distinguish chemical fires from forest fires, and both from house fires. Nevertheless, for other purposes it is desirable to ignore distinctions among types of fires, and to treat the total number of undifferentiated fires per year as if it were deterministic, perhaps interpreting that number as the expected number of fires that would come out of a stochastic model. One could relate that number usefully to characteristics of the pre-existing situations that tend to give rise to fires. The result can be useful in spite of the impossibility of predicting individual fires.

The motivation for the choices made here is twofold. First, to explore a model that gives rise to a one dimensional measure of the growth of knowledge, and second, to keep the technicalities of the model as simple as possible, while allowing the effect of the growth of knowledge on technology and production possibilities to be analyzed.

We turn to the formal model of the growth of knowledge.

5.2 Individual Characteristics

The discussion in Section 4 sees discovery as the result of an interaction between a person, the agent of discovery, and a body of knowledge s/he commands. A person's characteristics play a role in facilitating or impeding discovery. These characteristics are in part innate, and in part the result of decisions made by the agent himself, such as decisions to invest in the acquisition of skills, say through education or apprenticeship, and social decisions, such as how much society invests in its educational system, and its research institutions, and how it allocates effort and resources to particular areas of knowledge. These decisions would be endogenous in a more complete model; in this model, they are formalized by exogenously determined functions of time.

The first step in the construction of the model is to introduce four functions that characterize the persons in the set **A**.[48] These are:

1) $$a(i,t) = (a_1(i,t), a_2(i,t)) \quad \text{for } i \in \mathbf{A}, \quad t = 1,2, \ldots.$$

Here $a_1(i,t)$ is interpreted as a measure of person i's endowment of creative abilities. These include the qualities of mind or personality that might predispose someone to make a particular kind of discovery as discussed in Section 3.2. The parameter $a_2(i,t)$ represents the level of skill that person i has

48 These parameters are initially presented as functions of time. We will usually assume that they are constant. The model presented here is intended ultimately to be embedded in a general equilibrium model in which the values of many of these parameters are determined endogenously, by investment decisions such as those required to acquire skills, or those that allocate resources to certain activities. Allowing these parameters to be functions of time still permits them to be exogenous in the present model.

attained at time t. Creative abilities may, of course, be no different from cognitive abilities generally;[49]

2) $e(i,t)$ for i Î **A**, $t = 1,2,...,$

to be interpreted as the intensity of effort put forth by person i at time t;

3) $r(i,t)$ for $i \in \mathbf{A}$, $t = 1,2,...,$

to be interpreted as the resources i has available to use at t;

and finally,

4) $d(i,t) = f(e(i,t), r(i,t), a(i,t))$ for $t = 1,2,...,$

where d(i,t) is to be interpreted as a measure of i's *fertility* or *productivity* in discovering.

I assume that the function f is nonnegative and is increasing in all its arguments. That is, more effort increases i's fertility in discovery, more resources, and more skill and ability do too.[50]

Choice of the functions e(i,t) and r(i,t) in general depend on person i’s incentives of to expend effort and resources to increase his or her knowledge. These incentives in turn depend at least in part on the economic returns to such knowledge, such as reputation and consequent salary and working conditions in the case of an academic, or profit in the case of an investor in R&D. In general, these functions would be *strategies*, or *decision functions*. But, to keep the model simple and focused on knowledge and technology, the demand

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[49] D. N. Perkins , among others, would hold this view. [Perkins (1981)].
[50] This is in keeping with the idea that creative powers are to be found in all persons.

side is omitted. Therefore, the elements necessary to make decisions about how to allocate effort and resources to discovery and learning are not endogenous; they are treated as exogenous parameters. The dependence of the growth of knowledge, and of technology and production on these parameters can be studied in a kind of 'comparative dynamics' analysis. This aspect of the analysis is discussed further in Section 6.[51]

To simplify notation, define

$$e(i,t) = (e(i,t), r(i,t)),$$

and write

4') $$d(i,t) = f(e(i,t), a(i,t)).$$

5.3 Growth of Knowledge of an Isolated Person

Hadamard's observation, cited above, that "it is obvious that invention or discovery ...takes place by combining ideas," is one among many expressions of the same notion. Indeed, if there is anything universally agreed upon by students of creative activity, it is this idea. Valery's statement of the idea seems particularly apt.[52] According to Valery, the discoverer plays two roles, one as the knower of her body of knowledge, who makes up combinations of ideas that she presents to herself in her second role, as the one who recognizes the value of what is before her.

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[51] If person i's knowledge at t is partitioned, as in Sections 2, and 5.1, so that

$$K(i,t) = K_1(i,t) \cup K_2(i,t) \cup \ldots \cup K_Q(i,t),$$

then the parameters $\varepsilon(i,t)$, $\rho(i,t)$, and perhaps also $\alpha(i,t)$, and hence $\delta(i,t)$ should all be made functions of q = 1,2,...,Q, so that $\varepsilon(i,t)$ is replaced by $e_q(i,t)$, etc. In this case constraints on the total effort and resources available apply.

[52] See footnote 37.

This idea suggests that products of discovery or creative activity result from an interaction between the potential discoverer and his knowledge, an interaction in which combinations of existing ideas generate potential discoveries in the mind of the discoverer. In this view, the body of knowledge is a population of ideas that breeds new ideas, and requires the intervention of a discoverer to bring them to awareness.

Every combination of elements of K(i,t) is a potential stimulant for, or seed of, a new idea in the mind of i. Therefore the number of subsets of K(i,t) is the number of opportunities for discovery presented to person i at time t. Then, person i's fertility parameter, itself determined by the intensity of effort and resources devoted to discovery, and by i's skill and ability, determines the yield of discoveries from the potential ones.

Modeling this might seem to require that *combinations of subsets* make up the breeding population, i.e., subsets of the power set of K(i,t). However, the power set of K(i,t) is determined by K(i,t) itself, and so is its size. A subset of K(i,t) can be uniquely described by a compound sentence—an element of K(i,t). Thus, the set K(i,t) itself can be used in place of its power set, and considered to be the "breeding population." The use of the size of K(i,t) instead of that of its power set amounts to a (nonlinear) change of units. The larger the set K(i,t) the more combinations of statements it permits, and therefore the greater the opportunity for a combination to stimulate a new idea in the mind of person i.[53] The more person i's mind is a 'prepared mind', the

[53] One might think that the task of sorting through a large body of knowledge would be overwhelming. Casual observation of the functioning of those who command a large body of knowledge suggests the contrary of this idea; psychological research referred to above suggests that current understanding of how the brain stores knowledge and has access to it seem to be consistent with the view that more knowledge does not inhibit discovery, and that combinatorial

more will be his productivity in discovery from a given body of existing knowledge.

Because, as discussed in Section 4.1, there is no inhibitory effect on discovery from more knowledge, the relationship between knowledge and discovery is one increasing in knowledge. And because more effort, or more resources, skill or ability produces more discovery from a given body of knowledge, the relationship between discovery and the parameters that represent these characteristics is also increasing in the parameter d(i,t).[54]

Perhaps the simplest mathematical expression of these observations is the following.

---

complexity may not cause overwhelming difficulties. Two recent summaries of the present state of brain research suggest that the acquisition of knowledge changes the structure of the brain. "One way of looking at tuning, adaptive filtering, and associative processes is that they provide means of incorporating "knowledge" into the structure of the brain. ... Sustained activation provides a means for working with his incorporated knowledge, when the original information is no longer present." (Desimone [1992] p 246.) The paper by Kandel and O'Dell [1992] describes functional and anatomical changes in the brain that result from learning and storing memories. The paper describes how mechanisms used for neural development are also used to make activity-dependent changes in the brain.

If knowledge is incorporated into the structure and functioning of the brain, then perhaps the image of shuffling through a stack of cards on which ideas are written is not a useful metaphor for the mental processes of associating combinations of ideas, especially when deep and persistent thought is involved.

Minsky [1985] presents a related view of *mind* in a cognitive-artificial intelligence framework.

54 It is also assumed that knowledge once acquired is never forgotten. But, apparently the recipe for making Stradivarius violins has been lost to us, though Stradivarius himself and his proteges no doubt did not forget it while they lived. Similarly, some technologies of pre-literate cultures have to be rediscovered, if they are not to be lost. Nevertheless, in light of the availability and cost of memory devices nowadays, this seems not too restrictive an assumption. It should be recalled that the existence of a record of a piece of knowledge does not in itself constitute knowledge. Someone must acquire that information to make it his or her knowledge. The deciphering of Mayan gliphs is one of a number of examples that come to mind.

Let K(t) denote a body of knowledge and d(t) the parameter measuring the fertility or productivity of a potential discoverer who commands the knowledge K(t) at t. Let k(t) be the size of K(t). Then, the equation

(5) $$k(t+1) - k(t) = d(t)k(t), \qquad t= 1,2,...,$$

describes the growth of knowledge over time that the process of discovery described above would generate.

Thus, the growth of knowledge is the result of two factors, one is the number of 'breeding parent' statements in the population of statements that constitutes the relevant body of knowledge at t, and the other is a measure, namely d(t), of the discoverer's fertility or productivity in finding something new and interesting among the potentially new statements or ideas "waiting to be discovered."[55]

Equation (5) is a difference equation in discrete variables. The set K(t) would typically be very large, and the difference in the size over a short interval of time comparatively small. It is a convenient idealization to replace (5) by its continuous analog, in which the variables k(t) and t are continuous.[56] Here k(t) is a continuous measure of the size of K(t). Then, the differential equation analogous to (5) is

---

[55] If one of the roads mentioned in 5.1 above, but not taken, namely a stochastic model, were to be followed, it would seem natural here to say that the interaction between a person of given abilities, who is making a given effort, and a body of knowledge that allows certain combinations of ideas would result in a *probability* of discovering something new.

[56] With this transition the set K(i,t) may as well be a subset of an uncountable set, such as a set indexed by the real numbers, or a Euclidean space. In such a case the sets K(i,t) would be required to be measurable and the size of K(i,t) might be, in the case of a subset of a Euclidean space, its Lebesgue measure.

(6) $\dot{k}(t) = \delta k(t)$ $t \geq 0$.

where

$$\dot{k}(t) = \frac{dk(t)}{dt}.$$

This is an ordinary differential equation with variable coefficient. The focus here is on the basic ideas underlying the model, hence, it is desirable to carry out the analysis in the simplest mathematical setting in which it makes sense. For this reason, suppose that for all t,

d(t) = d(0) = d.

Then equation (6) has the form

(6') $\dot{k}(t) = \delta k(t)$

This familiar differential equation for k(t) has the solution

(7) $k(t) = C \exp(\delta t)$.

In the continuous model with Q >1, the time derivative

$\partial k_q(i,t)/\partial t$

replaces the time difference $k_q(i,t+1) - k_q(i,t)$, and gives the change in the knowledge (in isolation) of person i in the q^{th} subject area in the period t to t + 1.

It will now be assumed that for each i Î A and all t, the partition **L,** = **L**(i,t), of E/I(i,t) contains only a finite number, Q, of subsets. Then

$K(i,t) = K_1(i,t) \cup K_2(i,t) \cup \ldots \cup K_Q(i,t)$, and, because the sets Kq(i,t) are pairwise disjoint,

(8) k(i,t) = åk_q(i,t).

The counterpart of equation (6) applied to these knowledge sets is,

$$(9) \qquad \begin{pmatrix} \dot{k}_1(i,t) \\ \vdots \\ \dot{k}_Q(i,t) \end{pmatrix} = \begin{pmatrix} \delta_{11}(i,t) & \dots & \delta_{1Q}(i,t) \\ \vdots & \vdots & \vdots \\ \delta_Q(i,t) & \dots & \delta_{QQ}(i,t) \end{pmatrix} \begin{pmatrix} k(i,t) \\ \vdots \\ k(i,t) \end{pmatrix},$$

where the coefficient d_{qr} (i,t) represents, for r = q, i's fertility in discovery arising from allocating effort, resources and skills relevant to the sub-field q, and innate ability, and , for r ≠ q, d_{qr}(i,t) represents serendipitous discovery arising from effort devoted to r that results in finding something in q. The same k(i,t) appears on the right in every row, because any ideas that i knows at t can combine to suggest a discovery in any subfield. The effect of i's interests and the allocation of effort and resources that i makes among the fields of knowledge 1, ..., Q are expressed by the matrix of coefficients in equation (9).

For some purposes it would be desirable to preserve the distinctions among subfields when there is more than one person. But, because of (8), equation (9) implies

$$(10) \qquad \dot{k}(i,t) = \delta(i,t)k(i,t),$$

where

d(i,t) = åd_{qr} (i,t),

the sum being taken over all q and r. Therefore, the distinctions among subfields disappear, and, when it is assumed that the coefficients are constants, (10) is the same as (6') with k(t) = k(i,t) and d = d(i,0) = d(i).

For simplicity, unless otherwise stated, I assume that Q = 1. This amounts to making no distinctions among the discoveries the model attempts to predict. If Q > 1 is assumed, then the functions k(i,t) must be replaced by the vector of functions (k_1(i,t),...,k_Q(i,t)), and the coefficients d(i,t) by d_{qr}(i,t).

When d(i,t) is a step function, representing one or more changes in the allocation of effort and resources to discovery, perhaps as a result of policy decisions, the solution of (10) is a piecing together of exponentials at rates determined by the effort and resources i devotes to discovery, and on i's skill and ability. A change in the values of these parameters results in a change in the value of d(i,t), and therefore those changes have the effect of changing the rate at which i's knowledge grows. If, for example, i decided to reduce to zero the effort devoted to exploring K(i,t), then i's knowledge would cease to grow altogether.[57] If any of the parameters were to change from time to time, then so would d(i) and the solution would be a concatenation of exponentials with different rates of growth, corresponding to the changes in d(i,t).

---

[57] Viewing effort and ability as parameters whose values may be changed is a special case in which δ(i) is a step function, i.e., a function of time. E.g.,

$$\delta(i,t) = \begin{cases} \delta'(i) & if \;\; t \le t' \\ \delta''(i) & if \;\; t > t' \end{cases}$$

The variable coefficient case arises in an example in Section 7.

5.4 Growth of Knowledge in a Community

Persons who function in a community communicate their ideas and results among themselves by various means. The existence of the community enriches the knowledge of its members and accelerates the process of discovery[58].

Let **A** be the set of persons that form the community under consideration. Each person in **A** has access to the knowledge of others, through publications or other forms of communication. But knowledge whether in the minds of others, or written in papers, does not enter into the mind of person i without effort. Person i therefore must allocate his effort between acquiring knowledge from others, and working to discover new knowledge—between reading papers and writing them.

Let

$$e^1(ii)$$

denote the effort i devotes to discovery, and

$$e^2(ij)$$

the effort given to acquiring knowledge from j. Then, for i,j in **A**,

$$d(ij) = f(e^1(i,j), e^2(ij), r^1(ij), r^2(ij), a(i)),$$

is the parameter that measures i's productivity in discovery, when j = i, and in acquiring knowledge from j otherwise.

---

[58] See, for example, Mokyr's account of invention in the Industrial Revolution in Europe. Mokyr [1990].

It is assumed, similarly to what was assumed in the case of an isolated person, that [59] for all i and j in {1, ..., N}

(11) $d(ij) \geq 0,$

where not all d(ii) = 0.

It is sometimes convenient to assume that for all i, d(ii) > 0.

When Q > 1, i.e., when more that one sub-field of knowledge is distinguished, the coefficients d(ij) are replaced by

$$\delta_{q_1 q_2} \ldots q_0(ij)$$

Equation (10), characterizing the growth of i's knowledge in isolation, must be modified to take account of the fact that i's knowledge can grow in two ways, the first by discovery and the second by learning from others. Thus, for i = 1,2,...,N, where N is the number of people in **A**,[60]

$$\begin{aligned}
&\dot{k}(1,t) = \delta(11)k(1,t) + \ldots + \delta(1N)k(N,t) \\
&\dot{k}(2,t) = \delta(21)k(1,t) + \ldots + \delta(2N)k(N,t) \\
(12)\quad &\vdots \\
&\dot{k}(N,t) = \delta(N1)k(1,t) + \ldots + \delta(NN)k(N,t).
\end{aligned}$$

---

[59] This allows the possibility that δ(ij) = 0 for some i and j. This is consistent with cases in which knowledge may grow but without having any effect on technology, as would be the case in a society that channeled human creative powers exclusively into the study of religious texts, or certain types of philosophical investigations. However it is consistent with the view expressed in Section 4 that creative powers are ubiquitous.

[60] The multiplicity of intellectual communities corresponding to the partition of K(i,t) into sub fields and the links among them, can be represented in the structure of Δ.

Equation (12) can be written more compactly in matrix form as[61]

(13) $\dot{k}(t) = \Delta k(t)$,

where

$$\dot{k}(t) = \begin{pmatrix} \dot{k}(i,t) \\ \vdots \\ \dot{k}(N,t) \end{pmatrix},$$
$$\Delta = \left((\delta(ij))\right), i, j = 1, \dots, N,$$
$$k(t) = \begin{pmatrix} k(i,t) \\ \vdots \\ k(N,t) \end{pmatrix}.$$

There are several processes of interaction among people in the community that are represented by equations (12) or (13). These include the following special cases.

First, suppose that each member of the community publishes or otherwise communicates to the others knowledge that s/he regards as new. We may suppose that $\partial k(i,t)/\partial t$ is the amount of new knowledge person i chooses to add to his knowledge set at time t. If person i should encounter at t some piece of knowledge already in K(i,t), then it would not be added to K(i,t). If a person's judgement of what is new is competent, then, at least to a first approximation, these time derivatives measure what is put into the communication network among the agents at time t. Each agent extracts from

---

61 In the variable coefficients case, where Δ is a function of time, the solution may be found in Gantmacher, F.R., [1959], Vol. II pp.133-136.

those inputs knowledge added to her own knowledge set at time t. This leads to the following equations.

$$
\begin{aligned}
&\dot{k}(1,t) = \delta'(11)k(1,t) + \delta'(12)\dot{k}(2,t)\cdots + \delta'(1N)\dot{k}(N,t) \\
&\dot{k}(2,t) = \delta'(21)\dot{k}(1,t) + \delta'(22)k(2,t)\cdots + \delta'(2N)\dot{k}(N,t) \\
&\vdots \\
&\dot{k}(N,t) = \delta'(N1)\dot{k}(1,t) + \ldots + \delta'(NN-1)\dot{k}(N-1,t) \\
&\qquad\qquad + \delta'(NN)k(N,t).
\end{aligned}
$$

This is a linear system which in vector form is

$$\Delta' \, \dot{k}(t) = \delta k(t) \, .$$

where

$$\Delta' \;=\; \begin{pmatrix} 1 & -\delta_{12} & \cdots & -\delta_{1N} \\ -\delta_{21} & 1 & \cdots & -\delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_{N1} & -\delta_{N2} & \cdots & 1 \end{pmatrix},$$

and

$$\delta \;=\; \begin{pmatrix} \delta_{11} \\ \delta_{22} \\ \vdots \\ \delta_{NN} \end{pmatrix}.$$

In general, Δ' is invertible. Hence, the system reduces to

(13a) $$\dot{k}(t) = \hat{\Delta}^{-1} \; k(t)$$

where

$$\hat{\Delta}^{-1} = \Delta'^{-1}\delta$$

is the inverse of $\hat{\Delta}$, with the i^{th} row multiplied by d(ii), i = 1, ..., N. The entries in $\hat{\Delta}^{-1}$ are functions of the d(ij) defined at the beginning of this section, and hence are functions of the parameters representing ability, skill, resources and effort.

Another special case of equations system (12) or (13) of interest is obtained if for each i and j ≠ i,

d(ij) = d(i).

Still another special case particularly relevant to economic development is that in which the knowledge of the agents is nested. That is, everything that is known to the least knowledgeable agent is also known to the next least knowledgeable one and so on to the agent whose knowledge set at time t includes all the others. Without loss of generality let them be

K(1,t) , K(2,t) , ... , K(N,t).

In this case, one view of the process of transfer of knowledge leads to the conclusion that the matrix D in equation (13) is a (lower) triangular matrix, all of whose entries are nonnegative, and whose diagonal elements are d(ii), i = 1,...,N, which may be assumed to be positive and distinct. In this view i's access to j's knowledge leads to discoveries just as i's access to i's knowledge set does, but perhaps with different yields.

A refinement of this view is that person i only works on that part of K(j,t) that is not already in K(i,t).

Then, under the assumption that knowledge is nested,

$$\begin{aligned}
\dot{k}(1,t) &= \delta(11)k(1,t) \\
\dot{k}(2,t) &= \delta(21)k(1,t) - k(2,t) + \delta(22)k(2,t) \\
&\vdots \\
\dot{k}(N,t) &= \quad \delta(N1)k(1,t) - k(2,t) + \\
&\quad\ \delta(22)k(2,t) - k(3,t) + \cdots + \\
&\quad\ \delta(NN)(k(N-1,t) - k(N,t)),
\end{aligned}$$

Which can also be written as,

(14) $$\dot{k}(t) = \hat{\Delta}''\, k(t),$$

where

$$\Delta'' =$$

$$\begin{pmatrix}
\delta(11) & 0 & 0 & \vdots & 0 \\
\delta(21) & \delta(22) - \delta(21) & 0 & \vdots & 0 \\
\delta(31) & \delta(32) - \delta(31) & \delta(33) - \delta(32) & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\delta(N1) & \delta(N2) - \delta(N1) & \cdots & \delta(NN) - & \delta(NN-1)
\end{pmatrix}.$$

Here there is a boundary condition, namely that

$$k(j+1,t) - k(j,t) \geq 0.$$

The set K(i,t)\K(i+1,t) contains what i knows at t, but i+1 does not know at t.

If we assume that effort, and resources of i devoted to learning from person 1 is always at least as productive as the same effort and resources allocated to learning from any other person, where $i \neq 1$, then

$$\Delta'' =$$

(15)
$$\begin{pmatrix} \delta(11) & 0 & \cdots & 0 \\ \delta(21) & \delta(22) - \delta(21) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \delta(N1) & 0 & 0 & \delta(NN) - \delta(N1) \end{pmatrix}.$$

In this case i does not allocate different amounts of resources and effort to acquiring knowledge from different people. Since everything that i can learn from others i can learn from person 1, it is equivalent under the stated assumption for i to learn everything from person 1 or to learn separately items that persons i -1, i-2, ..., 1, know that i does not. As above, these equations are valid for $k(1,t) - k(j,t) \geq 0$ for all $j = 2, ..., N$.

Note that if

$$d(j\ j) \leq d(j\ 1)$$

then agent j can use d(j 1) on the entire set K(1,t), including K(j,t). We suppose therefore that for all $j = 2, ... ,N$,

$$d(j\ j) > d(j\ 1).$$

If the dynamics are described by equation (14), and if d(11) > 0, then there is at least one eigenvalue that is real and positive.

We now examine the solutions of (13) more explicitly, under the assumption that *D has N linearly independent eigenvectors*.
Under that assumption, the solutions to (13), or (14), or (14) when (15) holds, consist of linear combinations of pure exponentials.

Let the (N linearly independent) eigenvectors of D be

$$s = (s_1, \cdots, s_N).$$

Let S be the matrix whose columns are the eigenvectors of D. Then the matrix L, given by .

(16) $$S^{-1} \Delta S = \Lambda,$$

where,

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix}$$

is the diagonal matrix whose (diagonal) elements are the eigenvalues of D.[62]

Under these conditions, i.e., when (16) is valid, the vector differential equation (13) has the following general solution.

---

[62] It is often assumed that the eigenvalues of Δ are distinct, since this is a sufficient condition for Δ to be diagonizable. This condition is not necessary, and the condition given is both sufficient and more natural in this problem.

$$
(17) \qquad \begin{aligned} k(t) &= \exp(\Delta t)k(0) \\ &= S\exp(\Lambda t)S^{-1}k(0) \\ \\ &= c_1 \exp(\lambda_1 t)s_1 + \cdots + c_N \exp(\lambda_N t)s_N . \end{aligned}
$$

Thus, the general solution is a linear combination of pure exponentials,[63] and the constants c_i are determined by the initial conditions, i.e.,

$$
c_i = S^{-1}k(0).
$$

Note that when distinctions among sub-fields are made equation (12) takes the form given in equation

$$
(18) \qquad \begin{aligned}
\dot{k}_1(1,t) &= \delta_{11}(11)k_1(1,t) + \cdots + \delta_{1Q}(11)k_Q(1,t) + \cdots + \delta_{1Q}(1N)k_Q(N,t) \\
&\vdots \\
\dot{k}_Q(1,t) &= \delta_{Q1}(11)k_1(1,t) + \cdots + \delta_{QQ}(11)k_Q(1,t) + \cdots + \delta_{QQ}(1N)k_Q(N,t) \\
&\vdots \\
\dot{k}_Q(N,t) &= \delta_{Q1}(N1)k_1(1,t) + \cdots + \delta_{QQ}(N1)k_Q(1,t) + \cdots + \delta_{QQ}(NN)k_Q(N,t)
\end{aligned}
$$

which can be written

$$
\dot{k}(t) = \tilde{\Delta}k(t),
$$

---

63 Under the assumption expressed in (11) some growth of knowledge is assured. Nevertheless, this model could apply to a society in which there is growth of knowledge, but not growth of technology, and therefore no economic growth derived from technology. A culture that channeled all creative activity into religious or abstract philosophical study cut off from the economic sphere would provide an example. (See Mokyr [1990] for a discussion of such societies in a global

where $\tilde{\Delta}$ is an NQ ´ NQ matrix and $\dot{k}(t)$ and $k(t)$ $\dot{k}(t)$ and $k(t)$ are NQ vectors. Assumptions about the structure of $\tilde{\Delta}$ can be imposed in order to address specific questions about the interaction among different sub-fields.

5.5 Aggregation of Knowledge

Consider an economy in which the research sector consists of a set A of people partitioned into two groups, the first denoted A_1, consisting of N_1 people, and a second denoted A_2, consisting of N_2 people. For the entire group

$$A = A_1 \cup A_2,$$

consisting of

$$N = N_1 + N_2$$

people, when no distinctions are made among sub-fields, equation (13) of Section 5 is the system

$$\begin{aligned}
\dot{k}(1,t) &= \delta_1' \sum_{j=1}^{N_1} k(j,t) + \delta_1'' \sum_{j=N1+1}^{N_1} k(j,t) \\
\dot{k}(2,t) &= \delta_2' \sum_{j=1}^{N_1} k(j,t) + \delta_2'' \sum_{j=N1+1}^{N_1} k(j,t) \\
\dot{k}_1(N_{1+1},t) &= \delta(N_{1+1}1)k(1,t) + \cdots + \delta(N_{1+1}N)k(N,t) \\
&\vdots \\
\dot{k}(N,t) &= \delta(N1)k(1,t) + \cdots + \delta(NN)k(N,t).
\end{aligned} \tag{19}$$

historical context.)

of differential equations governing the growth of knowledge.

Under certain assumptions stated below, this system can be put in the form

$$
\begin{aligned}
&\begin{pmatrix} \dot{k}_1(t) \\ \dot{k}_2(t) \end{pmatrix} = \Delta \begin{pmatrix} \bar{k}_1(t) \\ \bar{k}_2(t) \end{pmatrix}, \\
&\bar{k}_1(t) = \left(\sum_{i=1}^{N_2} k(i,t) \right) \\
&\bar{k}_2(t) = \left(\sum_{i=N_1+1}^{N_2} k(i,t) \right)
\end{aligned} \tag{20}
$$

where

$$
\Delta = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}.
$$

That is, equations (20) describe the behavior of the first sector and the second sector as aggregates. This equation system is derived from (19) as follows.

Suppose that

$$
\delta(ij) = \begin{cases} \delta'_i \text{ for } i, j \in A_1, \\ \delta''_i \text{ for } i \in A_1, j \in A_2, \\ \delta''_i \text{ for } i \in A_2, j \in A_1, \\ \delta'_i \text{ for } i, j \in A_2. \end{cases}
$$

Then,

$$
\begin{aligned}
&\dot{k}(1,t) = \delta_1' \sum_{j=1}^{N_1} k(j,t) + \delta_1'' \sum_{j=N_1+1}^{N} k(j,t) \\
&\dot{k}(2,t) = \delta_2' \sum_{j=1}^{N_1} k(j,t) + \delta_2'' \sum_{j=N_1+1}^{N} k(j,t) \\
&\qquad\qquad\qquad \vdots \\
(21) \qquad &\dot{k}(N_1,t) = \delta_N' \sum_{j=1}^{N_1} k(j,t) + \delta_{N_1}'' \sum_{j=N_1+1}^{N} k(j,t) \\
&\dot{k}(N_{1+1},t) = \delta_{N_1+1}'' \sum_{j=1}^{N_1} k(j,t) + \delta_{N_{1+1}}' \sum_{j=N_1+1}^{N} k(j,t) \\
&\qquad\qquad\qquad \vdots \\
&\dot{k}(N,t) = \delta_N'' \sum_{j=1}^{N_1} k(j,t) + \delta_N' \sum_{j=N_1+1}^{N} k(j,t)
\end{aligned}
$$

That is to say, each person learns equally effectively from the others in each group, though perhaps differently from those in the other group than from those in his own group.

It follows that,

$$
(22) \qquad
\begin{aligned}
&\sum_{j=1}^{N_1} \dot{k}(i,t) = \sum_{i=1}^{N_1} \delta_i' \left(\sum_{j=1}^{N_1} k(j,t) \right) + \sum_{i=1}^{N_1} \delta_i'' \sum_{j=N_1+1}^{N} k(j,t) \\
&\sum_{i=N_1+1}^{N} \dot{k}(i,t) = \sum_{i=N_1+1}^{N} \delta_i'' \left(\sum_{j=1}^{N_1} k(j,t) \right) + \sum_{i=N_1+1}^{N} \delta_i' \sum_{j=N_1+1}^{N} k(j,t).
\end{aligned}
$$

Let

(23)
$$\bar{\delta}_{11} = \frac{1}{N_1}\sum_{i=1}^{N_1}\delta'_i, \quad \bar{\delta}_{12} = \frac{1}{N_1}\sum_{i=1}^{N_1}\delta''_i,$$
$$\bar{\delta}_{21} = \frac{1}{N_2}\sum_{i=N_1+1}^{N}\delta''_i, \quad \bar{\delta}_{22} = \frac{1}{N_2}\sum_{i=N_1+1}^{N}\delta'_i.$$

Then, for

(24)
$$N_1\bar{\delta}_{11} = \delta_{11}, \quad N_1\bar{\delta}_{12} = \delta_{12}$$
$$N_2\bar{\delta}_{21} = \delta_{21}, \quad N_2\bar{\delta}_{22} = \delta_{22},$$

the system of equations (21) can be written (in vector form) as

(25)
$$\begin{pmatrix}\dot{k}_1(t)\\ \dot{k}_2(t)\end{pmatrix} = \Delta\begin{pmatrix}\bar{k}_1(t)\\ \bar{k}_2(t)\end{pmatrix},$$
$$\bar{k}_1(t) = \left(\sum_{i=1}^{N_1}k(i,t)\right),$$
$$\bar{k}_2(t) = \left(\sum_{i=N_1+1}^{N}k(i,t)\right)$$

where

$$\Delta = \begin{pmatrix}\delta_{11} & \delta_{12}\\ \delta_{21} & \delta_{22}\end{pmatrix}.$$

The effort, resources, skill and ability that an individual devotes to learning and discovery are likely to vary over her lifetime. Most likely the resulting time path of d(i j,t) is one that is concave, perhaps an parabola opening downward. However, it is plausible that the *average* fertility of a community of researchers is constant over time as young ones enter and old ones retire, leaving the total number constant. Therefore, the aggregate equations (25) lend themselves to a more meaningful analysis of the implications of policies that effect the parameters, in equation (24), namely:

(i) policies that effect the skills with which persons enter the research community;

(ii) policies effecting the allocation of resources and effort to learning and discovery; and

(iii) policies effecting the number of people in the research community.

Policies (i) and (iii) involve long-run considerations. They include the size and quality of educational institutions, and of research institutions. Policies (ii) are likely to have shorter horizons, involving current support for research efforts.

It is clear from (24) that analysis of policy instruments can proceed via the effects on the average level of productivity in the research community on the one hand, and the total number of researcher on the other.

It is also clear from (17), or the corresponding solution to (20), that resources devoted to increasing the elements of D yield increasing returns to scale in terms of knowledge. In Section 6, it will be clear that this implies increasing returns to scale in production.

5.6 Example I: An Example of the Growth of Knowledge in Two Fields.

Consider two fields, say, biology and pharmaceuticals, denoted 1 and 2 respectively.[64] People in biology do basic research; people in pharmeceuticals develop new drugs. Suppose there are N_1 people engaged in biological research and N_2 in developing drugs, where $N = N_1 + N_2$. On the one hand, suppose that the pharmacists use knowledge generated by the biologists, but that the biologists do not learn from the pharmacists. We will compare this with the situation in which neither the biologists nor the pharmacists learn from the other group. Then the aggregation procedure given by equations (20) through (25) applied to (26) yields

$$
\begin{aligned}
&\begin{pmatrix} \dot{k}_1(t) \\ \dot{k}_2(t) \end{pmatrix} = \Delta \begin{pmatrix} \bar{k}_1(t) \\ \bar{k}_2(t) \end{pmatrix}, \\
(28) \qquad &\bar{k}_1(t) = \left(\sum_{i=1}^{N_1} k_1(i,t) \right) \\
&\bar{k}_2(t) = \left(\sum_{i=N_1+1}^{N} k_2(i,t) \right)
\end{aligned}
$$

where

$$\Delta = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix},$$

where

$$\delta_{ij} = N_i \bar{\delta}_{ij}.$$

---

[64] This example is further developed in Section 6. It has several interpretations.

with

$$d_{12} = 0,$$

and

$$d_{11}-d_{22} > 0, \text{ and } d_{12} > 0.$$

Then,

$$D = \begin{pmatrix} \delta_{11} & 0 \\ \delta_{21} & \delta_{22} \end{pmatrix}$$

has eigenvalues

$$l_1 = d_{11}, \quad l_2 = d_{22}.$$

The corresponding eigenvectors are

$$s_1 = \begin{pmatrix} 1 \\ \dfrac{\delta_{21}}{\delta_{11} - \delta_{22}} \end{pmatrix}, s_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix S of eigenvectors is

$$S = \begin{pmatrix} 1 & 0 \\ \dfrac{\delta_{21}}{\delta_{11} - \delta_{22}} & 1 \end{pmatrix},$$

and hence

$$S^{-1} = \begin{pmatrix} 1 & 0 \\ \dfrac{-\delta_{21}}{\delta_{11} - \delta_{22}} & 1 \end{pmatrix}.$$

Therefore, the solution is given by

$$\bar{k}_1(t) = C_1 e^{\delta_{11}t} s_{11} + C_2 e^{\delta_{22}t} s_{21}$$
$$\bar{k}_2(t) = C_1 e^{\delta_{11}t} s_{12} + C_2 e^{\delta_{22}t} s_{22},$$

where

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{-\delta_{21}}{\delta_{11} - \delta_{22}} & 1 \end{pmatrix} \begin{pmatrix} \bar{k}_1(0) \\ \bar{k}_2(0) \end{pmatrix}$$
$$= \begin{pmatrix} \bar{k}_1(0) \\ -\bar{k}_1(0)\dfrac{\delta_{21}}{\delta_{11} - \delta_{22}} + \bar{k}_2(0) \end{pmatrix}.$$

Therefore

$$\bar{k}_1(t) = \bar{k}_1(0)e^{\delta_{11}t}$$

(29) $$\bar{k}_2(t) = \bar{k}_1(0)e^{\delta_{11}t}\frac{\delta_{21}}{\delta_{11} - \delta_{22}} + \left(\bar{k}_2(0) - \bar{k}_1(0)\frac{\delta_{21}}{\delta_{11} - \delta_{22}}\right)e^{\delta_{11}t},$$

These equations can be used to compare the effects on the growth of pharmaceutical knowledge of varying the parameters d_{ij}. It is interesting that the parameter d_{21} enters linearly, while d_{11} and d_{22} enter exponentially. It is evident that increasing d_{11} or d_{22} increases the growth rates of $\bar{k}_1(t)$ and $\bar{k}_2(t)$, respectively. Since d_{21} enters linearly into the equation for $\bar{k}_2(t)$, it may not be so evident that increasing d_{21} from the value 0 has the effect of increasing the growth of $\bar{k}_2(t)$ exponentially. The second case holds when $d_{21} = 0$.

In the second case, where the pharmacists do not learn from the biologists,

$d'_{12} = d'_{21} = 0;$

here primes denote the corresponding variables in the second case. It is immediately evident that

$$\bar{k}_1'(t) = \bar{k}_1'(0)e^{\delta_{11}'t}$$
$$\bar{k}_2'(t) = \bar{k}_2'(0)e^{\delta_{22}'t}.$$

If we assume that $d'_{ii} = d_{ii}$ for i = 1,2, then

$$\frac{\bar{k}_1(t)}{\bar{k}_2'(t)} = \frac{\bar{k}_1(0)}{\bar{k}_2'(0)},$$

and,

$$\frac{\bar{k}_2(t)}{\bar{k}_2'(t)} = \frac{\bar{k}_1(0)e^{\delta_{11}t}\rho + \left(\bar{k}_2(0) - \bar{k}_1(0)\rho\right)e^{\delta_{11}t}}{\bar{k}_2'(0)e^{\delta_{22}t}}$$

(30)

$$= \frac{\bar{k}_1(0)}{\bar{k}_2'(0)}\rho e^{(\delta_{11}-\delta_{22})t} + \frac{\left(\bar{k}_2(0) - \bar{k}_1(0)\rho\right)}{\bar{k}_2'(0)}.$$

Since the relevant quantities in this expression are positive, the ratio grows exponentially at the rate $d_{11} - d_{22}$. Thus, while both $\bar{k}_1(t)$ and $\bar{k}_2(t)$ increase exponentially in isolation, $\bar{k}_2(t)$ increases much faster when transfer of knowledge from 1 to 2 is possible than it does in isolation.

5.7 Evidence of exponential growth of knowledge

The model presented above predicts that in fields in which positive effort, resources, skill and ability are applied, knowledge grows exponentially. Is this at all consistent with what can be observed?

The following data were collected for the field of computer science, broadly defined. The number of pages published annually in a sample of twenty one journals of computer science, and separately, the number of articles published annually in those journals were collected for the period 1958 to 1991. If the number of ideas per article, or the number of journal pages per idea, is not too variable, these quantities would be good approximations to the measure $\overline{k}_q(t)$, where q labels the field of computer science.

Table I shows the data collected, and Table 2 exhibits the regression of the natural logarithm of number of pages and number of articles, respectively, on time.[65] These regressions show that the number of pages grew exponentially at the rate of about 9% per annum, and the number of articles at about 11% per annum. These regressions seem to be very close fits.

---

[65] Tables 1 and 2 were prepared by Sangeeta Kasturia.

Table I
The number of pages and the number of articles published annually in a sample of Computer Science journals from 1954 to 1991.

YEAR (TIME)	**PAGES (P)**	**ARTICLES (A)**
1954	380	48
1955	558	42
1956	750	72
1957	1038	96
1958	1080	154
1959	1490	186
1960	1298	182
1961	1992	204
1962	1692	190
1963	1960	242
1964	1628	220
1965	1956	236
1966	2844	316
1967	3158	322
1968	3440	352
1968	2862	274
1970	3414	324
1971	3094	356
1972	5014	428
1973	4700	426
1974	5678	498

1975	7302	688
1976	7302	632
1977	8630	720
1978	8560	722
1979	10036	1188
1980	11400	880
1981	12338	994
1982	15126	1108
1983	18376	1260
1984	23042	1508
1985	21982	1636
1986	24572	1732
1987	24956	1672
1988	24498	1644
1989	27234	1768
1990	27218	1888
1991	29686	1958

Table II

Regressions of the Natural Logarithms of the number of pages (LP) and the number of articles (LA) on time (TIME).

(1.) REGRESSION OF THE LOG OF THE NUMBER OF PAGES IN COMPUTER SCIENCE (LP) ON TIME:

$$LP = 6.361 + 0.11\ TIME$$
$$(107.65) \qquad (41.64)$$

R SQUARE = .979 RBAR SQUARE = .979

(2.) REGRESSION OF THE LOG OF THE NUMBER OF ARTICLES IN COMPUTER SCIENCE (LA) ON TIME:

$$LA = 4.301 + 0.09\ TIME$$
$$(56.21) \qquad (27.24)$$

R SQUARE = .954 RBAR SQUARE = .952

NOTE: THE NUMBERS IN PARENTHESIS ARE T-STATISTICS

Section 6 Growth of Technology and the Production Set

The technological significance of the growth of knowledge derives ultimately from its effect on production. The next step in the analysis is to use the dynamics of discovery and growth of knowledge to derive the growth of production possibilities. This might be carried out without introducing the concept of commodity into the model of technology. As the number of recipes grows, the number of different products, or useful substances would in general grow, or new recipes might give more efficient ways of producing existing things.

In order to connect the model with standard models of production the commodity and production set are introduced, and the analysis carried out in that framework. In the interest of clarity and simplicity, this is done in a simple Leontief model of production.

Research and Development (R&D) refers to the step that converts knowledge, including conjectured recipes, into tested recipes. In the present model there is so far nothing that distinguishes this step from any other act of discovery. However, it is plausible in this connection to think of an attempt by an individual to develop a new recipe, (which includes the invention of a new product, or a new use for an existing product as well as a new way of making something from existing materials), as primarily motivated by the anticipation of an economic return. Therefore, analysis of decisions to invest in R & D projects would require that the problem be embedded in a model in which anticipated returns could be expressed. This remains to be done. For the present, the model incorporates effort and resources devoted to R&D as parameters. It therefore allows analysis of the effect of R& D decisions on the technology K*. This is done in Section 6.3.

General knowledge leads to new technology through the application of effort, and resources to R & D. In this process a potential innovator finds a project that seems sufficiently promising in technical and economic terms, and commits resources to its development into a tested recipe and ultimately into production. The model as it stands does not include the economic structures, such as, markets and prices and consumers' demand, or other institutions, on which decisions to invest in the development of a particular area or project turn. Hence the analysis of expected or potential returns to investment in an R & D prospect is not endogenous. Suppose instead that this economic analysis is done in a different model (ultimately to be integrated with this one) and leads to the selection of a certain fraction of the prospects for investment and to the allocation of effort and resources to their development. The part of this process dealt with in the present model is as follows.

Suppose to begin with that every person in **A** is a potential innovator or participant in R&D. The knowledge K(i,t) of person i at time t consists of two parts, one containing tested recipes and the other containing the rest, so that

(1) $$K(i,t) = K^*(i,t) \cup K(i,t)\backslash K^*(i,t).$$

Then

(2) $$k(i,t) = k^*(i,t) + k^{**}(i,t),$$

where $k^*(i,t)$ is the size of $K^*(i,t)$ and $k^{**}(i,t)$ is the size of $K(i,t)\backslash K^*(i,t)$.

Suppose that the yield of new recipes from the knowledge of person i at t depends on the size of the knowledge base, and the effort and resources put into development by person i. Then, let

(3) $$w(i) = x(e^*(i), a(i)),$$

where, for all values of a(i),

$$x(0,a(i)) = 0.$$

Then, the yield of new recipes from the knowledge base K(i,t) given (3) is

$$k^*(i,t) = H(w(i),k(i,t))$$

The simplest form of H is that given in (4), where the coefficient w(i) is the *productivity of i in R&D*. (If person i is not engaged in R&D, then e*(i) = 0.) Then, for all i and t,[66]

(4) $$k^*(i,t) = w(i)k(i,t).$$

Let $$W = I\, w,$$

where w =(w(i),...,w(N)), and I is the identity matrix. It follows from (17) of Section 5 that

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66 This equation can be interpreted as saying that a tested recipe appears in K*(i,t) when all development work has been completed. That is, a tested recipe is transferred from K(i,t)\K*(it) to K*(i,t) when it is completed.

$$k^*(t) = Wk(t) = W \exp(\Delta t) S^{-1} k(0)$$
$$= W \exp(\Lambda t) S^{-1} k(0)$$

(5)

$$k^*(t) = \begin{pmatrix} w(1)\left(c_1 e^{\lambda_1 t} s_{11} + c_2 e^{\lambda_2 t} s_{21} + \cdots + c_N e^{\lambda_N t} s_{N1}\right) \\ \vdots \\ \vdots \\ w(N)\left(c_1 e^{\lambda_1 t} s_{1N} + c_2 e^{\lambda_2 t} s_{2N} + \cdots + c_N e^{\lambda_N t} s_{NN}\right) \end{pmatrix}.$$

Section 6.2 Growth of the Production Set in the Commodity Space

6.2.1 Knowledge, Technology and the Production Set

The classical economic model of production (the production set in the commodity space) can be derived from the model of technology presented in Section 2 as follows.

Given a collection of recipes, a commodity space can be defined. Let

K* = K*(t) = ÈK*(i,t)
i ∈A

denote the recipes known by the persons i in A at time t, and let

M(K*)

denote the list of names of entities that appear in K*. I.e., a name is in M(K*) if and only if there is some recipe in K* in which that name appears.[67] The list M(K*) is well-ordered in some arbitrary way, from 1 to m(K*), the index of the last item on the list.

An *input-output array for a recipe r* is a function from M(K*) to Z, where Z is the set of possible measurements of substances on the list M(K*), say Z is an additive group. Let F(r) be a function whose value at the recipe r is the set of input-output arrays for r.

$$F(r) = \{f \in Z^{M(K^*)} \mid f \text{ is an input-output array for } r\}.$$

This function can be given the following representation. Each name n in M(K*) can be identified with its position on the list. A function

$$f \in F(r)$$

can be represented as the array

$$z = (z_1, \ldots z_{m(K^*)}),$$

where

$$z_j = 0$$

if the name in position j is not mentioned in recipe r.

Let Z* denote the space of m(K*)-tuples, z.

Commodities can now be defined.

---

[67] Suppose there is a recipe that calls for some substance that is not an instance of a commodity. Then either the recipe could not be recognized in the model, or the collection of sets corresponding to the list of commodities would not be a covering, still less a partition, of the set of existing names.

6.2.2 Commodities

A commodity is a set of names of objects from the list M(K*) that are regarded as equivalent in the model. Thus, let

$$\mathbf{P} = \{Mc\}_{c = 1, 2, \ldots, C},$$

be a partition of M(K*).

Each subset in the partition is a *commodity* and is given a name. For present purposes, the name of the commodity Mc can be identified with c.

For example, a set of names of objects consisting of "Taurus", "Oldsmobile" "Buick"... might be given the name "full sized American-made car," defined to be a commodity and identified as the third commodity. If Ford Motor Co. should introduce a newly designed full sized car next year, it could be included as an instance of the commodity "full sized American made car" next year.

For each commodity, c, let y_c denote the *quantity of c*, where y_c is an element of an additive group, G_c.[68] Then, the commodity space is

$$Sp(p) = G = \prod_{c=1}^{C} G_c$$

The elements of Sp(**P**) are vectors

$$y = (y_1, \ldots, y_C)$$

[68] E.g., y_c might be restricted to integer values, in which case c is considered to be indivisible, or a real number if c is regarded as divisible. Of course, other measures of 'quantity of c' are possible.

where the c^{th} component of y is a quantity of the c^{th} commodity. Also, there is a mapping, Y,

$$Y : Z^* \to S_p(P),$$

The function Y translates quantities of objects into quantities of commodities. Thus, let M_1 consist of the objects 1, 2, ..., p with measurements $z_1, z_2, ..., z_p$. Then, Y (z_1), $Y(z_2)$, ..., $Y(z_p)$ are the corresponding quantities of commodity 1, and the total amount of commodity 1 corresponding to $z_1, z_2, ..., z_p$ of the objects 1, ..., p is $\sum Y(z_j)$. The mapping Y therefore determines an aggregation rule for the partition **P**.

For example, suppose the objects in M_1 are sea-going vessels of different types and speeds. They appear on the list M(K*) as different objects, but the partition **P** defining commodities puts them in one class, M_1, called 'ship'. The measurements associated with the original objects might be linear dimensions of the vessel, parameters that characterize the shape, and the speed. These constitute a multidimensional quantity and would differ from vessel to vessel. The quantity of the commodity "ship" might be "ton-miles per hour of transport capacity," computed from the measurements z_j of the individual vessels via the function Y.

6.2.3 Production Set

The production set determined by K* is a subset, Y(K*), of the commodity space Sp(**P**).defined by the condition that

$$Y(K^*) = \{ y \in Sp(P) \,|.\, r \in K^* \text{ such that } z \in F(r) \text{ and } Y(z) = y \}$$

That is, the production set determined by K* consists of all commodity vectors y such that y corresponds to an input-output array produced by some recipe in K*.[69]

Assumptions can be imposed on K* that imply the familiar properties, (e.g., convexity, or linearity), assumed about the production set in models where it is a primitive.

If there is technological change, so that K*(t+1) …K*(t)[70], then the commodity space may change and the production set determined by K*(t+1) may be different from that determined by K*(t). If a new recipe produces a new product, then either that product can be classified as an instance of an existing commodity, or as a new commodity.

The commodity space associated with a given technology K*, is not unique. Different choices of **P** and of the mappings F and Y lead to different commodity spaces. The distinctions among substances and objects that are made for the purposes of knowledge may or may not be preserved in the distinctions that economic institutions make, and the distinctions made by economic institutions may not be preserved by economic models, each designed to serve a different purpose. However, in both the case of economic

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[69] It should be clear from the definition that the production set determined by K* also depends on the way commodities are defined and measured, i.e., on **P**, Φ and Ψ.

[70] Recall that that there is no forgetting of previously known technology.

institutions and models, the classification expressing the distinctions that are recognized is made with knowledge of the technology K*(t). Therefore, the commodity space can change after a new recipe enters K*.

While the description, properties and name of a new product cannot be known in advance of its invention or discovery, it can in advance be assigned the number m(K*) + 1 in the list of names M(K*(t+1))and, if the new product is to be classified as a new commodity, assigned the number C +1 in the list of commodities. A similar convention would apply to the case in which a new use for some object or substance is discovered, a use that leads to a distinction between objects or substances that were hitherto regarded as indistinguishable.[71]

6.2.4 The Attainable Production Set in the Commodity Space

Turning to the production set, (and suppressing the time index, t, temporarily), let the commodity space be (as in 6.2.3)

$$Sp(p) = G = \prod_{c=1}^{C} G_c$$

and suppose that Y, a subset of G, is the production set determined by the technology K*(t) = K*(1,t)x ... xK*(N,t), and the partition **P**. Suppose that elements $y \in G$,where

$$y = (y_1, ..., y_C).$$

[71] For example, the discovery of rH factors led to the introduction of more blood types than were recognized before. The discovery that aspirin is effective in preventing heart attacks is the basis of a new use for an existing substance. It constitutes a new recipe, but does not require a new commodity.

can be written in the form

$$y = (y^1, y^2),$$

where y^1 is the vector *of produced commodities*, and y^2 is the vector of *primary commodities* whose amounts are given from nature.[72] Suppose the produced commodities are $1,\ldots,C_1$ and the primary ones are $C_1 + 1,\ldots, C$.

Let

$$v \in G$$

denote the endowment vector. Only the components of v that are primary commodities can be different from 0. Then the set

$$\{y \in G \mid y \geq v\}$$

consists of commodity vectors that do not use more than the available amounts of unproduced commodities. Let

$$\hat{Y} = Y \subseteq \{y \in G \mid y \geq v\},$$

denote the set of *attainable productions*, where Y is the production set.[73] [74]

---

[72] Commodities are dated. Therefore the vector of produced commodities includes those currently produced, and y^2 includes those inherited from the past as well as those given from nature.

[73] A more explicit notation would show the dependence of Y and $\hat{Y}$ on K*, **P**, the functions Φ and Ψ, as well as v and t. Note that $y \geq v$ is equivalent to $-y \leq -v$.

[74] Under any of a number of standard assumptions about the technology, $\hat{Y}$ is compact.

The focus of this model is on the relationship between discovery and invention, and economic growth. Therefore, it is appropriate to assume that other possible causes of growth are absent. These include growth deriving from increasing returns and externalities, from trade, and from growth in the supply of primary resources. Therefore, the model of production considered is one with constant returns and no substitution possibilities, and the endowment of primary commodities is held constant. In that case, in the absence of technological change, $\hat{Y}$ would be constant over time.

Since Y, and therefore $\hat{Y}$, depend on the functions F and Y, analysis of the effect of new knowledge on production sets must involve properties of those functions. However, instead of specifying F and Y directly and deriving the production set through them, I consider a familiar simple model of production, and let F and Y be defined implicitly, determined by "reverse engineering".

Assume that Y is given by a Leontief input-output model with no joint production. Then,

$$Y = \{ y \in G \mid y = Ax, \; x \geq 0,$$

where A is an $L \times L_1$ matrix

and $x = (x_1, \ldots ,x_{C_1})\}$.

It is assumed that each produced commodity has one activity (industry) that produces only that commodity.[75]

---

[75] The structure underlying this production set can be sketched as follows. Each commodity has an equivalence class of objects involved in recipes in K*. I assume that for each such class of objects there is a recipe, called basic, such that whatever nonnegative multiple of its input-output array is specified, there is another recipe in K* whose input- output array is equivalent to it in the given commodity

Under standard conditions on the matrix A the efficient frontier of $\hat{Y}$ is the intersection of a hyperplane determined by the column vectors of A, and therefore by the coefficients of A, with the non-negative orthant of the commodity space. Then, the points at which the efficient frontier intersect the coordinate axes characterize the efficient frontier. Moreover, they also characterize the set $\hat{Y}$, since it is the convex hull of those points and the origin.

The coefficients of A depend on the recipes in K*, and hence as K* changes over time, so does the matrix A. The effect of discovery or invention on production possibilities is expressed in the change over time of the attainable production set $\hat{Y}$ and its efficient frontier. Since K* changes by including new recipes, we first model the relation between recipes and the coefficients of A.

Because of the underlying linearity it is plausible to suppose that the relation between recipes in K*(t) and the coefficients of Y is homogeneous linear. The coefficients of $\hat{Y}$ are the same as those of Y, and are denoted a_{pc} Thus, for p = 1, ... ,L_1, and c = 1, ..., L,

(2.1) $$a_{pc}(t) = b_{pc}^{1}k^{*}(1,t)+\cdots+b_{pc}^{N}k^{*}(N,t).$$

The coefficient b_{pc}^{i}, i= 1, ..., N, measures the effect of technological knowledge of person i on the coefficient a_{pc}. Since the sets K*(i,t) consist of tested recipes, the effect of knowledge in any one of them on the technical coefficients is self-contained. This justifies the assumption that the

classification. These basic recipes correspond to the industry input-output vectors in the commodity space. An additional linearity property assures the 'addition property' for commodities.

relation between knowledge of agents and technical coefficients is, to a first approximation, linear. Recipes that effect any particular coefficient can be in the knowledge set of any person. According to (2.1), the total effect is the sum of the effects from each person. Interactions among the knowledge sets of different persons are already captured in the underlying equations that determine the sets K*(i,t).

Substituting from (5), the solutions for the functions k*(i,t) in the case where D has N linearly independent eigenvectors[76], gives the following equation for $a_{pc}(t)$.

$$
\begin{aligned}
a_{pq}(t) &= b^1_{pc}k^*(1,t) + \cdots + b^N_{pc}k^*(N,t) \\
&= b^1_{pc}w(1)k(i,t) + \cdots + b^N_{pc}w(N)k(N,t) \\
&= b^1_{pc}w(1)\left(C_1e^{\lambda_1 t}s_{11} + C_2e^{\lambda_2 t}s_{21} + \cdots + C_Ne^{\lambda_N t}s_{N1}\right) \\
&\qquad + \cdots + \\
&\quad b^N_{pc}w(N)\left(C_1e^{\lambda_1 t}s_{1N} + C_2e^{\lambda_2 t}s_{2N} + \cdots + C_Ne^{\lambda_N t}s_{NN}\right) \\
&= C_1e^{\lambda_1 t}\left(b^1_{pc}w(1)s_{11} + \cdots + b^N_{pc}w(N)s_{N1}\right) \\
&\qquad + \cdots + \\
&\quad C_Ne^{\lambda_N t}\left(b^1_{pc}w(1)s_{N1} + \cdots + b^N_{pc}w(N)s_{NN}\right) \\
&= \sum_{j=1}^{N} C_je^{\lambda_j t}\sum_{i=1}^{N} b^i_{pc}w(i)s_{ji}\,.
\end{aligned}
\tag{2.1'}
$$

If the impact of recipes on a technical coefficient is independent of the source, then

---

[76] This assumption is made for simplicity. In the general case the solution would be a linear combination of functions $\exp(\lambda_i\ t)\ t^k$, which would further complicate the notation without changing the situation qualitatively.

(2.2) $$b^{i}_{pc} = b_{pc} ,$$

It follows that

(2.3) $$a_{pc} = b_{pc} \sum_{j=1}^{N} C_j e^{\lambda_j t} \sum_{i=1}^{N} w(i) s_{ji} .$$

If further, the productivity in R& D of agents is the same, then for all i,

w(i) = w,

and

(2.4) $$a_{pq} = b_{pq} w \sum_{j=1}^{N} c_j e^{\lambda_j t} \sum_{i=1}^{N} s_{ji} .$$

Let

$$\frac{1}{N} \sum_{i=1}^{N} s_{ji} = \overline{s}_j .$$

Then, (2.4) can be written

(2.5) $$a_{pc} = b_{pc} w N \sum_{j=1}^{N} C_j e^{\lambda_j t} \overline{s}_j ..$$

Notice that the same conditions yield the corresponding expression for the sum of the components of k*(t), namely, (2.6) $$\sum_{i=1}^{N} k^{*}(i,t) = wN\sum_{j=1}^{N} C_j e^{\lambda_j t}\overline{s}_j .$$

The next step in the analysis is to find the rate growth of the attainable production set $\hat{Y}$, and its efficient frontier. This is done in the context of a low dimensional example.

6.2.5 Growth of the Attainable Production Set

Consider an example in which there are 2 produced commodities and 1 primary commodity, i.e., $C = 3$, and $C_1 = 2$. Let the initial endowment vector be

v = (0, 0, 1),

and the matrix A be,

(2.7) $$A = \begin{pmatrix} a_{11} & -a_{21} \\ -a_{12} & a_{22} \\ -1 & -1 \end{pmatrix}.$$

Figure 6.2.1 shows the projection of the set ^Y into the plane given by y_3 = -1, and the points $\hat{y}_1$ and $\hat{y}_2$ at which the line determined by the two columns of A intersect the y_1 and y_2 axes respectively.[77]

[77] A necessary and sufficient condition that the line determined by the columns of

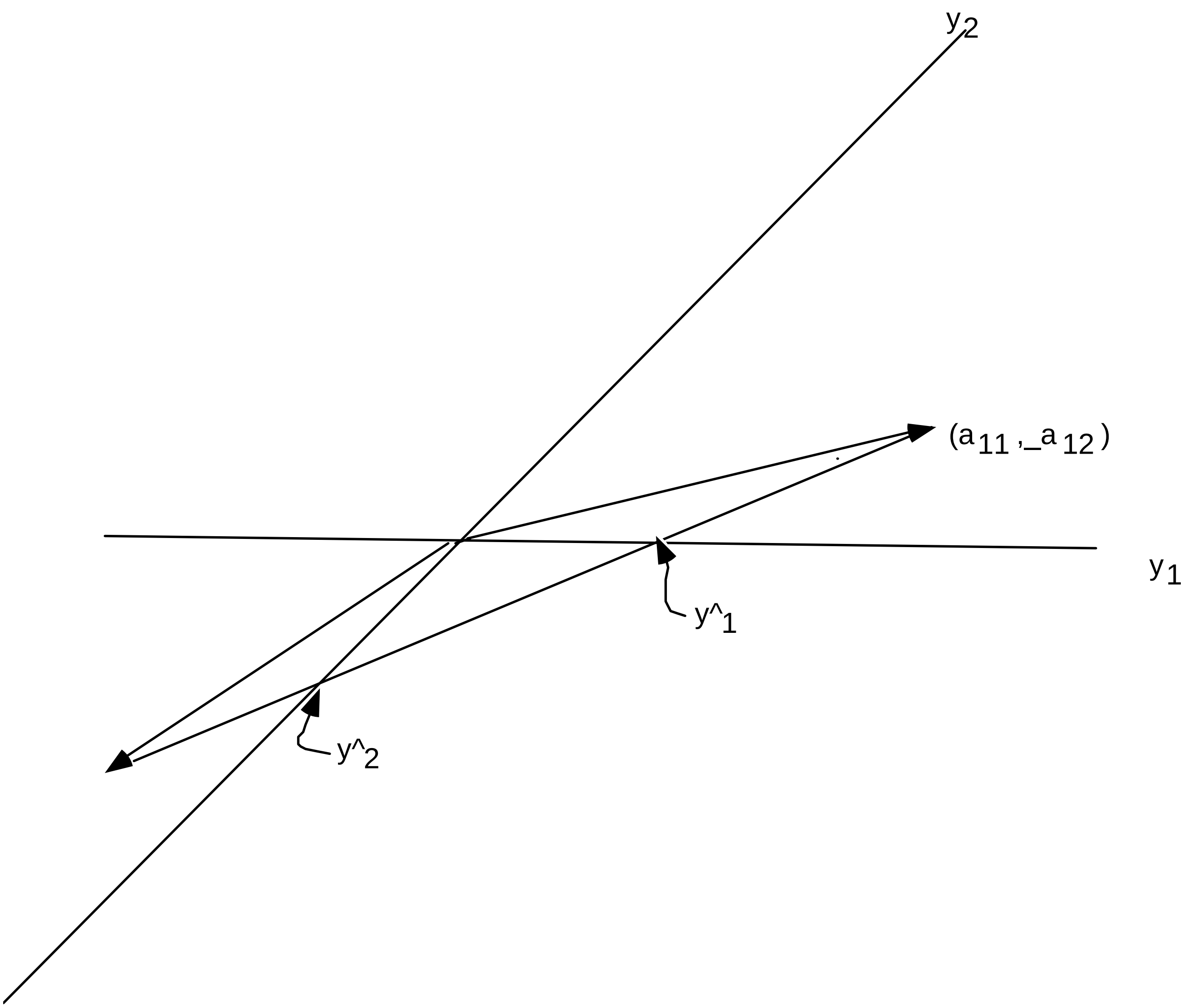

Figure 6.2.1

A intersect the non-negative quadrant is that $a_{11} \geq a_{21}$, and $a_{22} \geq a_{12}$.

Then the equations

$$y = Ax$$

are

(3.1) $$y_1 = a_{11}x_1 - a_{21}x_2$$

(3.2) $$y_2 = -a_{12}x_1 + a_{22}x_2$$

(3.3) $$-1 = -x_1 - x_2$$

From (3.3), $x_1 = 1 - x_2$.

Substituting in (3.1) and (3.2) gives

(3.5) $$y_1 = -x_2\left(a_{11} + a_{21}\right) + a_{11}$$

and

(3.6) $$y_2 = x_2\left(a_{12} + a_{22}\right) - a_{12}.$$

Solving (3.5) for x_2 in terms of y_1 yields

(3.7) $$x_2 = \frac{a_{11} - y_1}{a_{11} + a_{21}}.$$

Substituting in (3.6) gives,

(3.8) $$y_2 = \frac{a_{11}\left(a_{12} + a_{22}\right) - a_{12}\left(a_{11} + a_{21}\right)}{a_{11} + a_{21}} - y_1 \frac{a_{12} + a_{22}}{a_{11} + a_{21}}.$$

Let

$$B = \frac{a_{11}\left(a_{12} + a_{22}\right) - a_{12}\left(a_{11} + a_{21}\right)}{a_{11} + a_{21}},$$

and,

$$C = \frac{a_{12} + a_{22}}{a_{11} + a_{21}}.$$

Then equation (3.8) can be written as

$$y_2 = B - C\, y_1.$$

To find the points $\hat{Y}_1$ and $\hat{Y}_2$ where the efficient frontier of $\hat{Y}$ intersects the y_1-and y_2-axes respectively, set $y_2 = 0$ and solve (3.8) for $\hat{Y}_1$, and then carry out the corresponding procedure for $\hat{Y}_2$. Thus, setting $y_2 = 0$ gives,

$$\hat{y}_1 = \frac{B}{C}$$

(3.9)
$$= \frac{a_{11}\left(a_{12} + a_{22}\right) - a_{12}\left(a_{11} + a_{21}\right)}{a_{12} + a_{22}}.$$

$$= \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{12} + a_{22}}$$

Setting $y_1 = 0$, yields

$$\hat{y}_2 = B$$

(3.10)
$$= \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{21}}.$$

Substituting from (2.1)and (2.2) into equations (3.9) and (3.10) respectively yields equations (3.11) and(3.12) which give the time paths $\hat{Y}_1$(t) and $\hat{Y}_2$(t), in the special case when equation (2.2) is valid

(3.11)
$$\hat{y}_1(t) = \frac{b_{11}\overline{k}^*(t)b_{22}\overline{k}^*(t) - b_{12}\overline{k}^*(t)b_{21}\overline{k}^*(t)}{b_{21}\overline{k}^*(t) + b_{22}\overline{k}^*(t)}$$
$$= \frac{\overline{k}^*(t)\left(b_{11}b_{22} - b_{12}b_{21}\right)}{b_{21} + b_{22}},$$

and,

(3.12)
$$\hat{y}_2(t) = \frac{b_{11}\overline{k}^*(t)b_{22}\overline{k}^*(t) - b_{12}\overline{k}^*(t)b_{21}\overline{k}^*(t)}{b_{11}\overline{k}^*(t) + b_{12}\overline{k}^*(t)},$$
$$= \frac{\overline{k}^*(t)\left(b_{11}b_{22} - b_{12}b_{21}\right)}{b_{11} + b_{12}},$$

where

(3.13) $$\bar{k}^{*}(t) = \sum_{i \in A'} k^{*}(i,t).$$

It follows that

(3.14) $$\hat{y}_1(t) = wN\, B_1 \sum_{j=1}^{N} C_j e^{\lambda_j t} \bar{s}_j \ ,$$

where

$$B_1 = \frac{\left(b_{11}b_{22} - b_{12}b_{21}\right)}{b_{21} + b_{22}},$$

and

(3.15) $$\hat{y}_2(t) = wN\, B_2 \sum_{j=1}^{N} C_j e^{\lambda_j t} \bar{s}_j$$

where

$$B_2 = \frac{\left(b_{11}b_{22} - b_{12}b_{21}\right)}{b_{11} + b_{12}}.$$

The set of commodity vectors attainable with one unit of the third commodity, is the convex hull of the three points, 0, $\hat{Y}_1$(t), and $\hat{Y}_2$(t) Hence the growth of $\hat{Y}$ (t) is determined by $\hat{Y}_1$(t) and $\hat{Y}_2$(t). As equations (3.14)) and (3.15)) make clear, their growth is exponential in t (when there is a positive eigenvalue) at a rate that depends on the number of people, their abilities and the resources allocated to their respective areas, and to the effort expended on R &D in those areas, and on the pursuit of knowledge in the areas they rest on.

Figure 6.2.2 shows the attainable production set at two points in time under the assumption that B_1 is greater than B_2.[78]

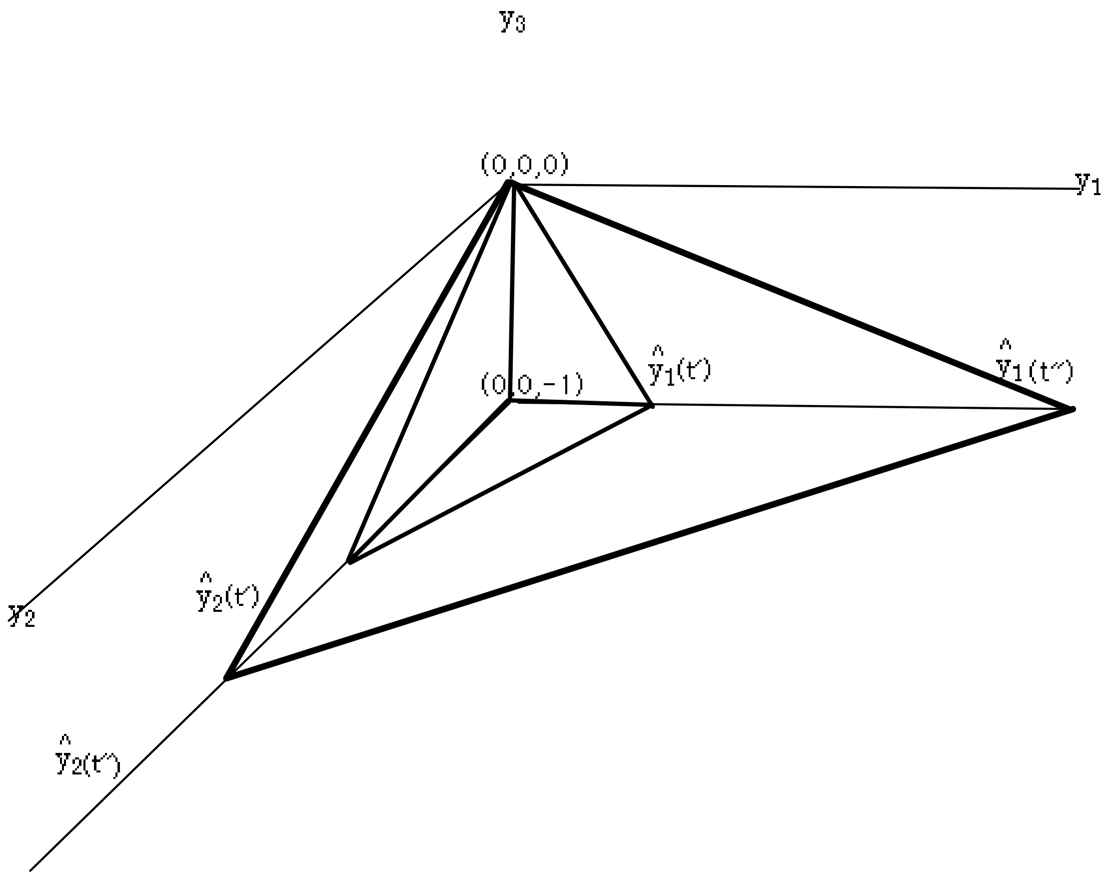

Figure 6.2.2

In this example, under the assumptions made, the set of attainable productions grows exponentially over time.[79] Therefore so would production.

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[78] In higher dimensions the changes in the attainable production set can be more complex.

[79] The existence of fixed factors of production is not incompatible with increasing returns to scale in the "knowledge sector," and therefore not with exponential growth of the production set over time. On the other hand, fundamental physical constraints, such as the laws of thermodynamics, together with finite bounds on the

If the model of production were, say a linear activity analysis model, the attainable production set could be characterized by the points at which the rays in the commodity space generated by the basic activities intersect the resource constraints. Of course, the analysis of the growth of the attainable set would be more complicated, because the direction of a ray determined by basic activities could change over time.

If a recipe involving a new product were discovered, then it is either included in the set Mc of some existing commodity c, or the classification **P** is changed so that a new commodity is introduced. In that case, it is commodity $C_1 + 1$, and the differential equation for the growth of that commodity determines its trajectory starting from the initial condition

$$y_{L_1+1} = 0.$$

If the set of potential developers is a subset A of the set **A** of persons, then the sum in equation (3.15) is taken over the subset A. If the set of developers is specialized by industry, so that there is a set, A_p' whose

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amount of matter or energy in the world ultimately set limits on growth of the attainable production set. However, the relevance of these bounds is unclear, because at any particular time knowledge of the possibilities of new substances or new forms of energy is limited. Therefore, calculations of ultimate limits of growth must be relative to knowledge existing at that time. Experience shows that those calculations can be far from the truth.

An example of this is provided by the calculation by Lord Kelvin of the age of the sun. Kelvin based his calculation on the idea of Helmholtz that the sun's energy came from gravitational contraction. Kelvin calculated that the age of the sun was at most 500 million years. This ws inconsistent with Darwin's theory of evolutional, something which both Darwin and Kelvin recognized. Neither man was willing to accept the conclusions unreservedly. Kelvin was aware that the calculation was based on the accuracy of Helmholtz's theory of gravitational contraction as the source of solar energy, and to his credit Kelvin stated that 'I do not say that there may not be laws which we have not discovered." See Ferris knowledge is relevant to the coefficients a_{pq} if an only if p = p', then the sum in equation (2.1) is taken over $i \in A_{p'}$.[80]

Section 6.3 Growth of the Attainable Production Set in Example I

6.3.1

Recall that in Example 1 of Section 5.6, in which biologists and pharmacists do research and R&D, the parameters d_{ij} depend on the skills and abilities of the members of the respective groups, and on the effort and resources they devote to discovery and to learning from others—on the parameters e(i) and a(i).}

Recalling equation (4), for all i and t,

$$k^*(i,t) = w(i)k(i,t),$$

where, from (3), w(i) depends on the effort and resources e*(i) that i devotes to R&D , and on i's ability and skill parameters a(i).

To compare the growth of attainable production sets of different economies, characterized by different complexes of parameter values, involves comparing the time paths of the attainable production sets. In this example, the attainable production set is determined by the points $\hat{y}_i(t)$ defined in 6.2.4. Using equations (3.11) and (3.12), we see that

$$\hat{y}_1(t) = w(2)B_1\overline{k}_2(t)$$

and

[1988], pp. 247-248.

80 Recall from footnote 22 that $A_{p'}$ denotes the set of agents whose knowledge is relevant to the coefficients where p = p'.

$$\hat{y}_2(t) = w(2)B_2\bar{k}_2(t),$$

since the biologists do not directly contribute any new recipes.

Substituting the solution for $\bar{k}_2(t)$ from equation (29) Section 5, yields,

$$\hat{y}_1(t) = w(2)B_1\left(\bar{k}_1(0)e^{\delta_{11}t}\frac{\delta_{21}}{\delta_{11}-\delta_{22}} + \left(\bar{k}_2(0) - \bar{k}_1(0)\frac{\delta_{21}}{\delta_{11}-\delta_{22}}\right)e^{\delta_{11}t}\right)$$

and,

$$\hat{y}_2(t) = w(2)B_2\left(\bar{k}_1(0)e^{\delta_{11}t}\frac{\delta_{21}}{\delta_{11}-\delta_{22}} + \left(\bar{k}_2(0) - \bar{k}_1(0)\frac{\delta_{21}}{\delta_{11}-\delta_{22}}\right)e^{\delta_{11}t}\right)$$

Comparing this situation with the one in which d21 = 0 = d12, we see that

$$\frac{\hat{y}_1(t)}{\hat{y}_1'(t)} = \frac{w(2)B_1\left(\bar{k}_1(0)e^{\delta_{11}t}\frac{\delta_{21}}{\delta_{11}-\delta_{22}} + \left(\bar{k}_2(0) - \bar{k}_1(0)\frac{\delta_{21}}{\delta_{11}-\delta_{22}}\right)e^{\delta_{22}t}\right)}{w(2)B_1\left(\bar{k}_2(0)e^{\delta_{22}t}\right)}$$

$$= \frac{\bar{k}_1(0)}{\bar{k}_2'(0)}\frac{\delta_{21}}{\delta_{11}-\delta_{22}}e^{(\delta_{11}-\delta_{22})t} + \frac{\bar{k}_2(0) - \bar{k}_1(0)\dfrac{\delta_{21}}{\delta_{11}-\delta_{22}}}{\bar{k}_2'(0)}.$$

This is the same expression as for the growth of $\bar{k}_2(t)$ relative to $\bar{k}_2'(t)$ as in equation (30) of section 5. The same result applies to the ratio

$\frac{\hat{y}_2(t)}{\hat{y}_2'(t)}$. Thus, the addition to the relative rate of growth of knowledge due to communication between the basic research community and the R & D community carries over to the relative rate of growth of the attainable production set for given resources.

6.3.2 Remarks on Development

Consider an example in which there are two countries each with a single sector that does both basic research and R&D. Suppose that Country 1 is underdeveloped compared to Country 2. That is, K(1,0) is very small compared to K(2,0), where K(i,0) is the present state of knowledge in Country 1. Then k(1,0) is much smaller than k(2,0). The disparity between the states of knowledge in the research sector between the two countries reflects the backwardness of the current technology in Country 1 and the lack of knowledge of more basic subjects as well. Example I can be reinterpreted to represent the growth of two countries, one of which, Country 1, is less developed than is Country 2. We may assume that everything known in country 1 is also known in country 2. In that case, the matrix D' has the form

$$\Delta' = \begin{pmatrix} \delta'_{11} & 0 \\ \delta'_{21} & \delta'_{22} - \delta'_{21} \end{pmatrix}.$$

Let

$$\begin{aligned}
\delta_{11} &= \delta'_{11} \\
\delta_{12} &= 0 \\
\delta_{21} &= \delta'_{21} \\
\delta_{22} &= \delta'_{22} + \delta'_{21}
\end{aligned}.$$

Then, for

$$\Delta = \begin{pmatrix} \delta_{11} & 0 \\ \delta_{21} & \delta_{22} \end{pmatrix}$$

so defined, the analysis of Example I, with d22 replaced by d'22-d'21, applies. Country 1 may contain people whose innate abilities are the equal of those in Country 2, but lack of education and of training and experience with an advanced technology leads to the skill component of the parameter a(1) being much smaller than a(2). Similarly, resources are scarce in Country 1, so that r(1) is small and hence so are e(1) and e*(1). Therefore, both d(11) and d(12) are small compared to Country 2, and so is w(1), i.e., what Country 1 can learn from 2 is small and what Country 1 can develop on its own is small. If this situation persists, the analysis of a more general version of equation (28) suggests that Country 1 will fall further and further behind relative to Country 2.

The fact that knowledge available in Country 2 is a public good and in principle also available to Country 1 is inconsequential if the acquisition of that knowledge requires ability, skill, effort and resources that Country 1 doesn't have or can't afford.

To change this outcome, Country 1 can adopt policies that increase the parameters d(11), d(12) or w(1). For example, if it is cheaper and easier to imitate than to discover, Country 1 could try to increase d(12) and w(1). If

Country 2 chooses a policy of growth by imitation, making $d'_{22} = 0$, and devoting its available persons and resources to learning from country 1, even then its knowledge and attainable production set will increase exponentially. It will eventually (in the limit as time goes to infinity) have its rate of growth catch up with that of Country 1.

This seems to have been the course followed by Japan during its early period of industrialization.[81] It seems to be the case that even today the Japanese research community is more heavily concentrated in R&D than in basic research.

Because of the nature of human capital, investment in Country 1's human capital is not necessarily attractive to Country 2 investors, even if the social return to investment in human capital in Country 1 is higher than it is in Country 2. Moreover, it is not clear that the marginal social return to human capital is higher in Country 1 than in Country 2, or even if it were, whether the fraction of it that could be captured by an investor is therefore higher in Country 1 than in Country 2.

It seems that it would not be difficult to augment the present model with economic and social institutions in which the economies of initially advanced and initially backward countries grow further apart under some conditions, and under other conditions initially backward countries catch up with and even surpass the initially advanced ones.

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[81] Mokyr (1991) points out that the British in the period of the Industrial Revolution borrowed freely from European knowledge, while other European societies were closed to outside knowledge for various reasons. He attributes part of the difference in the growth of British technology and of the economy as compared to Europe to this fact.

Section 7 Learning Curves

Learning curves are examples of an economic effect of the growth of knowledge. They appear in manufacturing. The two important properties of learning curves are:

i) that the number of *direct labor hours* required to produce a unit of product decreases as the *cumulative number produced* increases, and;

ii) that the *rate of reduction of direct labor hours* declines with cumulative output.

According to Argote and Epple,[82] the form of the learning curve conventionally assumed is

$$(7.1) \qquad \gamma = ax^{-b}$$

where a is the number of direct labor hours required to produce the first unit; x is the cumulative number of units produced; and b is a parameter measuring the rate at which labor hours are reduced as cumulative output increases. Cumulative output, x, is interpreted as a proxy for knowledge acquired through production.[83] Thus, learning curves ought to be derivable from the model of the growth of knowledge presented in Section 5. Such a derivation follows.

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[82] Argote, L. and Dennis Epple, [1990]. They give extensive references to the literature on Learning Curves.
[83] See, ibid.,

It is useful to have an example in mind, say, the assembly of an aircraft being built by some organization A. The aircraft to be assembled has already been designed, and the processes by which its parts are manufactured, and by which the assembly of them is to be carried out are also specified. The volumes of blueprints, or computer generated graphics, and other descriptions of the steps to be taken in assembling the aircraft exist. In short, a complete recipe, r, for the assembly of the aircraft is known and is to be carried out by those doing the assembly. Moreover, a facility has been constructed and equipped with all tools and machinery needed to execute the recipe, and the required labor force is in place.

Let z denote this fixed combination of inputs, and suppose that the rate of output, denoted y(t) at time t, is determined by a linear relation

$$(7.2) \qquad y(t) = n\, z,$$

where n is a positive scalar. This relation states that if the given recipe were carried out precisely as designed, the number of aircraft assembled per unit time would be the constant given by (7.2).

However, the labor force starts out with the knowledge of the prescribed recipe as embodied in the documents that specify it. Thus, the initial knowledge of the N people who make up the organization A is collectively

$$(7.3) \qquad \bigcup_{i\in A} K(i,0) = K(A,0) = \{r(0)\}$$

but, as a result of the experience of producing, they find out things about the recipe that they didn't know at the beginning. This knowledge allows them to improve the way the recipes are carried out—to produce from the fixed z a higher rate of output per unit time. Thus, n is not a constant, but depends on K(A, t). The recipe actually being used to assemble aircraft at t is r(t) rather than r(0) = r, r(t) being the modified version of r based on the knowledge acquired in the experience of production.

There are at least two ways of specifying the group's knowledge, corresponding to different assumptions about the way learning-by-doing occurs.

First, suppose that each worker learns equally from himself and the others. This is expressed by the condition that, for each i and j in A,

(7.4) $$d_{ij} = d_i.$$

In that case the differential equation system (10) is,

(7.5) $$k\begin{pmatrix} \dot{k}(1,t) \\ \vdots \\ \dot{k}(N,t) \end{pmatrix} = \begin{pmatrix} \delta_1 & \dots & \delta_1 \\ \vdots & \vdots & \vdots \\ \delta_N & \dots & \delta_N \end{pmatrix} \begin{pmatrix} k(1,t) \\ \vdots \\ k(N,t) \end{pmatrix},$$

and so

(7.6) $$\frac{\partial \sum_{i=1}^{N_1} k(i,t)}{\partial t} = \sum_{i=1}^{N} \dot{k}(i,t) = \sum_{j=1}^{N} \delta_j \sum_{i=1}^{N} k(i,t) = N\bar{\delta} \sum_{i=1}^{N} k(i,t),$$

where,

$$\sum_{j=1}^{N} \delta_j = N\bar{\delta}.$$

This is equivalent to the differential equation

(7.6') $$\dot{k}(t) = N\bar{\delta}\,\bar{k}(t)$$

Now,

$$\nu(t) = b\sum_{i=1}^{N} k(i,t),$$

where b is the coefficient b_{11} from equation (2.1), and k*(i,t) = k(i,t), (because the only knowledge involved here is of the recipe(s) actually being used). It follows from (7.6') that

(7.7) $$\nu(t) = b\bar{k}(t) = bce^{N\bar{\delta}t}.$$

Let y(t) be the number of aircraft assembled per unit time, say, per day. Then $\frac{1}{y(t)}$ is the number of days it takes to assemble one aircraft, and some fixed fraction of $\frac{N}{y(t)}$ is the number of hours of direct labor per aircraft. Then the number of hours of direct labor per aircraft at time t is given by

(7.8) $$\gamma = \frac{c}{y(t)}.$$

For simplicity assume that

b = 1.

Then

(7.9) $$\nu(\bar{k}(t)) = \bar{k}(t),$$

and

c =1.

Let x(t) denote the accumulated output at time t. Then,

(7.10) $$\begin{aligned} x(T) &= \int_0^T y(t)dt = \int_0^T Ce^{N\bar{\delta}t}dt \\ &= \frac{1}{N\bar{\delta}}\left(e^{N\bar{\delta}T} - 1\right). \end{aligned}$$

Solving equation (7.10) for T in terms of x yields

$$T = (1/N\bar{\delta}) \ln (x(T) + 1).$$

Writing t = T, substituting from (7.10) into (7.2) and using (7.9), gives the rate of production y as a function of cumulative production x. Namely,

$$y = z \exp (\ln (N\bar{\delta} x + 1).$$

Hence, equation (7.8) gives the direct labor per aircraft as a function of cumulative output.

(7.11) $$\gamma = \frac{1}{y} = \frac{z}{N\bar{\delta}x + 1}.$$

The function defined by (7.11) is a learning curve in the sense that it has the two essential properties i) and ii) above—direct labor hours decline with cumulative output, and the rate of decline diminishes with cumulative output.

However, 7.11 the learning curve (7.11) is not the one conventionally assumed, according to Argote and Epple. That curve can be obtained by the second way of characterizing the group's knowledge at t. This is done as follows.

It follows from (12) of Section 5 or (2.1) of Section 6, that

$$\bar{k}^{*}(t) = \begin{pmatrix} \left(c_1 e^{\lambda_1 t} s_{11} + c_2 e^{\lambda_2 t} s_{21} + \cdots + c_N e^{\lambda_N t} s_{N1}\right) \\ + \\ + \\ \left(c_1 e^{\lambda_1 t} s_{1N} + c_2 e^{\lambda_2 t} s_{2N} + \cdots + c_N e^{\lambda_N t} s_{NN}\right) \end{pmatrix}$$

Or, under the assumptions made,

(7.12) $$\bar{k}(t) = N \sum_{j=1}^{N} c_j e^{\lambda_j t} s_j .$$

Equation (7.12) represents the function $\bar{k}(t)$ as a superposition of exponentials. Let

$$\mu(t) = \ln\left(N \sum_{j=1}^{N} c_j e^{\lambda_j t} s_j \right).$$

Then,

$$Ce^{\mu(t)} = N\sum_{j=1}^{N} c_j e^{\lambda_j t} s_j .$$

Consider the following differential equation for $\overline{k}(t)$.

$$\frac{\partial\overline{k}(t)}{\partial t} = \beta(t)\overline{k}(t), \tag{7.13}$$

which is an equation with a variable coefficient that is satisfied by $\overline{k}(t)$ for b = m. Broaden the class of differential equations to include (7.13). The function

$$\beta(t) = d \ln t,$$

yields an equation in this class. Then, the following is among the solutions of (7.13), (7.14), but not all solutions can be guaranteed to be increasing.

$$\overline{k}(t) = ct^d .$$

Therefore,

$$y = zct^d ,$$

Furthermore,

$$x(T) = zc\int_0^T t^d dt = \frac{C}{d + 1} T^{d+1} .$$

Solving for T in terms of x, we get

$$T = \left(\frac{x(d+1)}{C} \right)^{\frac{1}{d+1}}.$$

Then, substituting in the equation for y gives

$$y(x) = C\left(\frac{x(d+1)}{C} \right)^{\frac{d}{d+1}}.$$

It follows that

$$(7.15) \qquad \gamma(x) = \frac{1}{y(x)} = ax^{-b},$$

where

$$b = \frac{d}{d+1},$$

and

$$a = \frac{1}{C}\left(\frac{d+1}{C} \right)^{\frac{-d}{d+1}}.$$

Equation (7.15) is the commonly used form of the learning curve.

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