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Conditioning and Aggregation of Preferences

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The author is at the Department of Finance, J.L. Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60208-2006, tel. (708) 467-2328, e-mail: c-skiadas@nwu. Some of the material of an earlier version of this paper is now incorporated in Skiadas (1994a). I am especially indebted to Darrell Duffie and Larry Epstein for their advice and encouragement during my student years, and beyond. I am grateful for discussions with Don Brown, Eddie Dekel, Pierre-Yves Geoffard, Birgit Grodal, Faruk Gul, Peter Klibanoff, David Kreps, Marco LiCalzi, Erzo Luttmer, Mark Machina, David Schmeidler, Karl Vind, and Martin Weber. Of course, I alone am responsible for this paper's shortcomings.

Abstract

This paper develops a general framework for modeling choice under uncertainty that extends subjective expected utility to include non-separabilities, state-dependence, and/or acts that do not admit a natural state-contingent structure. The theory is also consistent with the fact that decision makers may not be able to list or analyze all possible consequences of an act. Given a minimal monotonicity assumption on conditional preferences, it is shown that ordinal conditional utilities of acts can be chosen so that the conditional utility of any act with respect to any information, $\mathcal{G}$, can be computed from the conditional utility of the act given any information finer than $\mathcal{G}$, using an "aggregator" map that depends only on $\mathcal{G}$. General conditions are provided, under which aggregators take the form of conditional expectations. Various applications are discussed by further specifying the structure of acts and conditional preferences, including disappointment aversion, and the subjective value of information. The formal framework also serves as a basis for thinking about issues of bounded rationality and Knightian uncertainty.

1. Introduction

This paper presents a general framework for modeling choice under uncertainty that overcomes some of the limitations of Savage's (1954) approach, but also includes the latter as a special case.

Savage defined preferences over acts that are mappings from states of nature to consequences. These preferences are separable,[1] in the sense that the ranking of two acts given an event (set of states) does not depend on the consequences of these acts at states outside the event. The relaxation of separability has been discussed in connection to systematic empirical violations of the expected utility hypothesis, notably by Machina (1989). Numerous "non-expected" utility theories implicitly or explicitly violate the assumption of separability. (Relevant surveys include Fishburn (1988), and Karni and Schmeidler (1990).) Savage's theory also assumes that preferences are state-independent, that is, the ranking of consequences given a state does not depend on the state. The limitations of state independence, and models of state-dependent preferences are surveyed by Karni (1987), and Karni and Schmeidler (1990), (see also Karni (1993a,b)). Another important limitation of Savage's framework arises from the fact that humans have only limited ability to imagine, let alone analyze, all possibilities arising in a situation. It can even be argued that an exhaustive description of all possible states and consequences is inherently impossible since further analysis and information can always be used to refine or fundamentally alter current perceptions.

In defense of Savage's theory, there is a class of arguments whose central position is that apparent limitations of the theory disappear once consequences are interpreted broadly enough. (See, for example, Raiffa (1968) and Hammond (1989).) A problem with this type of arguments is their incorporation of subjective consequences such as disappointment, regret, ambiguity, frame of mind, and so on. A well known difficulty with subjective consequences in the Savage setting is that they may give rise to impossible acts. (An example is an act whose consequence in the state

[1] The term "separable" is of course used in the literature in various contexts and with various meanings. Our use of the term, outlined here, will be given a precise definition later on. In particular, separability is to be regarded distinct from additivity.

of excellent weather is disappointment at the weather.) More fundamentally, one would like all subjective aspects of choice to be captured by the properties of preferences, rather than the nature of acts or events.[2] This paper develops a formulation of conditional preferences and their aggregation that allows for non-separability and state-dependence, without the use of subjective consequences. Moreover, the theory is consistent with the notion that a decision maker may have limited ability to perceive states of nature and their consequences.

The basic theory starts with a set of primitives that includes states, events, acts, conditional preferences, but no consequences. States and events are used pretty much in the standard sense, although the decision maker need not be able to condition with respect to every event. On the other hand, acts are given no structure whatsoever, they are assumed to be merely labels of possible courses of action. Given an event F that is perceived by the decision maker, there is a complete and transitive preference order $\succeq^F$ over the space of acts X. Given acts x and y, the statement $x \succeq^F y$ has the interpretation that the decision maker can visualize the situation of event F occurring, and he imagines that, given that situation, x would have been a preferred course of action than y. With conditional preferences taken as given, the paper discusses how conditional preferences with respect to events are aggregated to give the conditional preferences with respect to coarser events, through ordinal utility representations.

A special type of act is a state-contingent act, defined as an act that is a mapping of the form $x : \Omega \to C$, where Ω is a state-space and C is some consequence space. Given the assumption of state-contingent acts, we call preferences separable if for every event F, $x = y$ on F implies that $x \sim^F y$, where $\sim^F$ is the indifference relation associated with $\succeq^F$. In this paper we do not assume that preferences are necessarily separable. That means that Savage's construction of conditional preferences does not apply in general, and $x \succeq^F y$ cannot be reduced to a statement

[2] This methodological restriction is consistent with the development of subjective probability. Theories of subjective expected utility such as Savage's were motivated by the fact that probabilities are usually subjective in nature, and therefore should not be part of the description of an act. Instead, subjective probability assessments should be revealed by preferences over unambiguous acts. The same approach seems appropriate for any subjective aspect of acts, including consequences and events.

about the relative ranking of the restrictions of x and y on F. Moreover, the paper's main results also apply in settings in which acts do not have a state-contingent structure at all (and hence separability is not even defined).

Applications and extensions of the formal theory are discussed in Section 5. One set of applications involves state-contingent acts without the assumption of preference separability. An example of this type, considered in Section 5.1, is a theory of disappointment aversion that extends the utility theories of Dekel (1986) and Gul (1991). Another set of applications involves acts that do not have a state-contingent structure at all. Representative of this type of application is a theory of the subjective value of information, to be found in Section 5.2. An important reason for eliminating consequences from the set of primitives is the subjective and ambiguous nature of the consequences of acts given various events in most realistic decision situations. Conditional preferences indirectly represent an agent's imperfect perception of consequences. This "bounded rationality" aspect of conditional preferences is further discussed in Section 5.3, and is related to Knightian uncertainty, as exhibited in Ellsberg's (1961) well known examples, in Section 5.4. Other extensions include dynamic formulations and subjective probability under non-separable preferences, studied in Skiadas (1994a,b). While the paper focuses entirely on personal choice, it will be clear that the formal setting admits a reinterpretation in terms of social choice (where "states" become indices of agents).

The central results of the paper are best introduced in a finite setting. Suppose that an agent perceives all the events in a finite algebra,[3] $\mathcal{F}$, of subsets of some state-space Ω. Conditional preferences with respect to events in $\mathcal{F}$ are assumed to be coherent, meaning that for any disjoint events F and G, $x \succeq^F y$ and $x \succeq^G y$ implies $x \succeq^{F \cup G} y$. (A strict version of this will also be assumed, but is ignored here for simplicity.) For example, suppose that the agent imagines the possibilities of good weather and bad weather, and decides that under either scenario x would be a preferred course of action than y. Then x should be preferred to y unconditionally. In the case of state-contingent acts, coherence is one of the ingredients of Savage's "sure-thing principle," but it is emphatically more general, because separability

[3] An algebra is a set of sets that is closed with respect to the Boolean operations of union, intersection, and complementation.

is not assumed. Given any algebra of events $\mathcal{G} \subseteq \mathcal{F}$, it will be convenient to define a state-contingent preference order $\succeq^{\mathcal{G}}$, representing the agent's conditional preferences given information $\mathcal{G}$. This is accomplished by letting $\succeq^{\mathcal{G}}_{\omega}$ be equal to $\succeq^{G}$, where G is the smallest event in $\mathcal{G}$ that contains the state ω. If the space of acts X is also finite, $\succeq^{\mathcal{G}}$ has a state-contingent utility representation $U^{\mathcal{G}} : \Omega \times X \rightarrow I\!R$, in the sense that $x \succeq^{\mathcal{G}}_{\omega} y$ if and only if $U^{\mathcal{G}}(\omega, x) \geq U^{\mathcal{G}}(\omega, y)$.

One of the main results of this paper is that conditional utilities can be chosen so that, for every subalgebra $\mathcal{G}$ of $\mathcal{F}$, there is a function $A[\cdot \mid \mathcal{G}]$, mapping $\mathcal{F}$-measurable random variables to $\mathcal{G}$-measurable ones, such that $U^{\mathcal{G}}(x) = A[U^{\mathcal{H}}(x) \mid \mathcal{G}]$ whenever $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{F}$. Moreover, the maps $A[\cdot \mid \mathcal{G}]$, called "aggregators," are separable, in the sense that the value of $A[V \mid \mathcal{G}]$ on some event G in $\mathcal{G}$ does not depend on the values that V takes outside G. This is true, even with non-separable conditional preferences over state-contingent acts. Like conditional expectations, aggregators have the property: $\mathcal{G} \subseteq \mathcal{H}$ implies that $A[V \mid \mathcal{G}] = A[A[V \mid \mathcal{H}] \mid \mathcal{G}]$. While there are various functional forms that aggregators can assume, another main result of the paper provides additional conditions, under which aggregation is additive (that is, $A[V \mid \mathcal{G}] = E[V \mid \mathcal{G}]$ for some expectation operator E). Additive aggregation reduces to state-dependent expected utility only under the additional assumption of state-contingent acts and separable preferences.

Building on the basic ideas just outlined, the paper develops a theory that includes an infinite number of acts and events. In addition to the many well known reasons for considering the infinite case, the assumption of a finite number of events imposes an arbitrary "rationality bound" on the decision maker, in a sense explained in Section 5.2. Parts of the theory that are straightforward in the finite case, present considerable complications in the infinite case. For example, the construction of conditional preferences with respect to algebras relies on a measure-theoretic result developed in Appendix I. The theorem of additive aggregation with an infinite number of events uses the martingale convergence theorem. The result seems to be new, even in the case of separable preferences.

The rest of the paper is organized in four sections and two appendices. Section 2 defines coherent conditional preferences with respect to events and with respect to σ-algebras, and studies their interrelationship. Section 3 investigates the existence of conditional utility representations that are related through aggregator maps. Section 4 discussed the case of additive aggregation, while Section 5 is on applications. The appendices provide some auxiliary results and proofs. On a first reading, the reader is encouraged to assume throughout that there is only a finite number of events, and that the only null event is the empty set. (Null events are needed only for the formulation of conditional preferences with respect to infinite σ-algebras.)

2. Conditional Preferences

In this section we define coherent conditional preferences with respect to σ-algebras and with respect to events. The main result is that the two notions are essentially equivalent, in the sense that each uniquely determines the other under a natural notion of consistency.

Throughout the paper, we take as primitive the quadruple $(\Omega, \mathcal{F}, \mathcal{N}, X)$, with the following definitions and interpretations. The set Ω is the *state space*, and its elements are called *states*. The set $\mathcal{F}$ is a σ-algebra[4] of subsets of Ω, called *events*. The decision maker is assumed to be aware of the events in some sub-σ-algebra of $\mathcal{F}$. The set X consists of *acts* that are not assumed to have any special structure; they are merely labels of possible courses of action. Finally, $\mathcal{N}$ is a subset of $\mathcal{F}$, and consists of the *null* events. Null events represent "insignificant" scenarios, in that they do not affect decisions. We assume that (a) any event that is a subset of a null event is null, (b) the countable[5] disjoint union of null events is null,[6] and (c) Ω is not null. We use the term *information class* to describe a set of complete σ-algebras of events, where an algebra is *complete* if it contains all the null events.

The decision maker expresses conditional preferences over X, relative to the perceived events. To describe these preferences, several preliminary definitions are required. Let $B(X)$ be the set of all binary relations[7] on X. A member of $B(X)$ is

[4] A *σ-algebra* on Ω is a set of subsets of Ω that contains the empty set and is closed under complementation and countable disjoint unions.

[5] In this paper, countable will always mean countable or finite.

[6] That is, if $\{ F_n \}$ is a sequence of null events with $F_i \cap F_j = \emptyset$ for $i \neq j$, then $\bigcup_n F_n$ is also null. A set of events $\mathcal{N}$ satisfying (a) and (b) is known as a *σ-ideal.*

[7] A *binary relation* R on X is a subset of X^2. We write $x \, R \, y$ to denote $(x, y) \in R$.

a *preference order* if it is complete and transitive.[8] A *state-contingent relation* is a function $R : \Omega \to B(X)$ such that, for all acts x and y, $\{ x\, R\, y \}(= \{ \omega : x\, R_\omega\, y \})$ is an event. We identify state-contingent relations that differ only on null events. A state-contingent relation R is *complete* if, for any acts x and y, there exists a null event N (depending on x and y) such that $\{ x\, R\, y \} \cup \{ y\, R\, x \} \cup N = \Omega$. A state-contingent relation R is *transitive* if, for any acts x, y, and z, there exists a null event N (depending on x, y, and z) such that $\{ x\, R\, y \} \cap \{ y\, R\, z \} \subseteq \{ x\, R\, z \} \cup N$. A *state-contingent preference order* $\succeq$ is a complete and transitive state-contingent relation. (Notice that this is not the same as saying that $\succeq_\omega$ is a preference order for all ω outside a null event.) The associated *state-contingent strict preference order* $\succ : \Omega \to B(X)$, and *state-contingent indifference relation* $\sim : \Omega \to B(X)$, are defined by $(x \succ_\omega y) \Leftrightarrow (\text{not } y \succeq_\omega x)$, and $(x \sim_\omega y) \Leftrightarrow (x \succeq_\omega y \text{ and } y \succeq_\omega x)$, respectively.

In order to simplify the exposition, we adopt the following conventions throughout the paper. Suppose q represents any state-contingent statement of the form $x\, R\, y$, where x and y are acts or random variables, and R is any (state-contingent) relation, including ordinary equalities and inequalities. The statement "q" should always be interpreted as "$q(\omega)$ for all ω outside a null event." We write $\{ q \} = \{ \omega : q(\omega) \}$, while "$q$ on F" means that the event $F \backslash \{ q \}$ is null.

Given any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, a state-contingent preference order $\succeq$ is $\mathcal{G}$-*measurable* if $\{ x \succeq y \} \in \mathcal{G}$ for all acts x and y. A conditional preference with respect to $\mathcal{G}$ could then be simply defined as a $\mathcal{G}$-measurable state-contingent preference order. We wish to extend this definition, however, by requiring that conditioning with respect to some information does not contradict conditioning with respect to coarser information. This is the requirement of coherence, formalized in the following definition. The notation $\mathcal{G} \cap F$, where $\mathcal{G}$ is a σ-algebra and F is an event, denotes the set $\{ G \cap F : G \in \mathcal{G} \}$.

DEFINITION 1 (ALGEBRA-CPF). *An algebra-conditional preference family, or algebra-CPF for short, is a set of the form* $\{ \succeq^{\mathcal{G}} : \mathcal{G} \in \Phi \}$ *such that*

(a) Φ *is an information class.*

[8] Relation R is *complete* if for any $x, y \in X$, $x\, R\, y$ or $y\, R\, x$, and *transitive* if $x\, R\, y$ and $y\, R\, z$ implies $x\, R\, z$.

(*b*) *For every* $\mathcal{G}$ *in* Φ, $\succeq^{\mathcal{G}}$ *is a* $\mathcal{G}$*-measurable state-contingent preference order.*

(*c*) *(Coherence) For all* $\mathcal{G}, \mathcal{H}$ *in* Φ, x, y *in* X, *and non-null* F *in* $\mathcal{H} \cap \mathcal{G}$ *such that* $\mathcal{H} \cap F \subseteq \mathcal{G} \cap F$,

$$(x \succeq^{\mathcal{G}} y \text{ on } F) \Rightarrow (x \succeq^{\mathcal{H}} y \text{ on } F)$$
$$(x \succ^{\mathcal{G}} y \text{ on } F) \Rightarrow (x \succ^{\mathcal{H}} y \text{ on } F).$$

Alternatively, given any event F, we can consider a preference order $\succeq^F$ (in $B(X)$) representing an agent's conditional preference given F.

DEFINITION 2 (EVENT-CPF). *An* ***event-conditional preference family****, or event-CPF for short, is a set,* $\mathcal{E} = \{ \succeq^G : G \in \mathcal{G} \}$, *of preference orders on* X, *such that*

(*a*) $\mathcal{G}$ *is a complete sub-*σ*-algebra of* $\mathcal{F}$.

(*b*) *(Coherence). For every countable collection of pairwise disjoint non-null events* $G_1, G_2, \ldots$ *in* $\mathcal{G}$ *whose union is* G, *and for all* x, y *in* X,

$$(x \succeq^{G_n} y \text{ for all } n) \Rightarrow (x \succeq^{G} y)$$
$$(x \succ^{G_n} y \text{ for all } n) \Rightarrow (x \succ^{G} y).$$

(*c*) *(Irrelevancy of Null Events). For all non-null events* F *and* G *such that*[9] $F \triangle G$ *is null,* $\succeq^F = \succeq^G$.

Throughout the paper, we identify event-CPFs that differ only on null events.

The following is a natural notion of consistency between an algebra-CPF and an event-CPF.

DEFINITION 3 (CONSISTENT CPFS). *The algebra-CPF* $\mathcal{A} = \{ \succeq^{\mathcal{H}} : \mathcal{H} \in \Phi \}$ *and the event-CPF* $\mathcal{E} = \{ \succeq^G : G \in \mathcal{G} \}$ *are* ***consistent*** *if*

(*a*) $\mathcal{G} \in \Phi$, *and every element of* Φ *is a sub-*σ*-algebra of* $\mathcal{G}$.

(*b*) *For all* $\mathcal{H} \in \Phi$ *and* $H \in \mathcal{H}$,

$$(x \succeq^{\mathcal{H}} y \text{ on } H) \Leftrightarrow (x \succeq^{F} y \text{ for all non-null } F \in \mathcal{H} \cap H).$$

If $\mathcal{A}$ and $\mathcal{E}$, as given in Definition 3, are consistent, it is not hard to show that the strict preference version of (b) also holds: For all $\mathcal{H} \in \Phi$ and $H \in \mathcal{H}$,

$$(x \succ^{\mathcal{H}} y \text{ on } H) \Leftrightarrow (x \succ^{F} y \text{ for all non-null } F \in \mathcal{H} \cap H).$$

[9] We define $F \triangle G = (F \backslash G) \cup (F \backslash G)$, the symmetric difference of F and G.

EXAMPLE 1 (ADDITIVE CPFS). Suppose P is a probability[10] on $\mathcal{F}$ such that $\mathcal{N} = \{ F : P(F) = 0 \}$, and $U : \Omega \times X \to I\!R$ is a state-contingent utility such that $U(\cdot, x)$ is measurable and $\int_\Omega |U(x)| dP < \infty$, for every x. An event-CPF, $\mathcal{E} = \{ \succeq^F : F \in \mathcal{F} \}$, is then defined by letting $x \succeq^F y$ if and only if $\int_F U(x)\, dP \geq \int_F U(y)\, dP$, for every non-null event F. Given any information class Φ, it is easy to check that the only algebra-CPF, $\mathcal{A} = \{ \succeq^{\mathcal{G}} : \mathcal{G} \in \Phi \}$, consistent with $\mathcal{E}$ is defined by

$$\{ x \succeq^{\mathcal{G}} y \} = \{ E[\, U(x) \mid \mathcal{G} \,] \geq E[\, U(y) \mid \mathcal{G} \,] \}, \quad \mathcal{G} \in \Phi,$$

where the conditional expectations are taken with respect to P. Conversely, $\mathcal{A}$ uniquely determines $\mathcal{E}$, provided that Φ contains all complete σ-algebras generated by single events. This completes the example.

Generalizing the above example, we now show that, subject to weak conditions, there is a one-to-one correspondence between algebra and event-CPFs, defined by consistency. As indicated in the Introduction, the case of a finite number of events is straightforward; consistency can be viewed as a way of defining an algebra-CPF in terms of an event-CPF. With $\mathcal{G}$ being an infinite σ-algebra, however, it is not at all clear what value should be assigned to $\succeq^{\mathcal{G}}_\omega$. It is here that the notion of null events becomes crucial.

THEOREM 1. *Suppose that $\mathcal{N} = \{ F \in \mathcal{F} : P(F) = 0 \}$ for some probability P, and that $\mathcal{E} = \{ \succeq^G : G \in \mathcal{G} \}$ is an event-CPF. Given any information class Φ of sub-σ-algebras of $\mathcal{G}$, there exists a unique algebra-CPF of the form $\{ \succeq^{\mathcal{H}} : \mathcal{H} \in \Phi \}$ that is consistent with $\mathcal{E}$.*

PROOF: The proof is based on a measure-theoretic result developed in Appendix I. Let Φ be as stated, and consider any $\mathcal{H} \in \Phi$, and $x, y \in X$. Call a set $H \in \mathcal{H}$ *light* if it is non-null and $x \succeq^H y$, and *dark* if it is non-null and $y \succ^H x$. With this definition, every set in $\mathcal{H}$ is exactly one of null, light, or dark. By coherence, a countable disjoint union of light sets is light, and a countable disjoint union of dark sets is dark. Also, any two non-null sets H_1 and H_2 in $\mathcal{H}$ such that $P(H_1 \triangle H_2) = 0$ are either both light or both dark. A set $H \in \mathcal{H}$ is defined to be white (respectively,

[10] Throughout the paper, probability measures are assumed to be countably additive.

black) if every set $F \in \mathcal{H} \cap H$ is light (respectively, dark). Theorem 8 (applied to $(\Omega, \mathcal{H}, P^{\mathcal{H}})$, where $P^{\mathcal{H}}$ is the restriction of P to $\mathcal{H}$) says that there exist unique (up to null events) disjoint sets B and W in $\mathcal{H}$ such that $B \cup W = \Omega$, B is black or empty, and W is white or empty. For each $\omega \in \Omega$, let $x \succeq_{\omega}^{\mathcal{H}} y$ if and only if $\omega \in B$. It is now straightforward to confirm that this construction gives a unique algebra-CPF $\{ \succeq^{\mathcal{H}} : \mathcal{H} \in \Phi \}$ that is consistent with $\mathcal{E}$. ■

To state a partial converse of Theorem 1, we define a σ-algebra of events to be ***countably generated*** if it is the smallest complete σ-algebra that contains some countable set of events.

THEOREM 2. *Suppose that $\mathcal{A} = \{ \succeq^{\mathcal{G}} : \mathcal{G} \in \Phi \}$ is an algebra-CPF, and for every $\mathcal{G} \in \Phi$, Φ contains every countably generated sub-σ-algebra of $\mathcal{G}$. Then there exists a unique event-CPF of the form $\{ \succeq^{G} : G \in \mathcal{G} \}$ consistent with $\mathcal{A}$.*

PROOF: Given any event $H \in \mathcal{G}$, let $\mathcal{H} = \sigma(\{ H \} \cup \mathcal{N})$ be the σ-algebra generated by H and the null events. Define, for every $x, y \in X$, $x \succeq^{H} y$ if and only if $x \succeq^{\mathcal{H}} y$ on H. It is then tedious but straightforward to confirm all the properties that show that $\{ \succeq^{G} : G \in \mathcal{G} \}$ is an event-CPF consistent with $\mathcal{A}$. Uniqueness (up to null events) is immediate, since the above construction is dictated by consistency. ■

We will use the term ***conditional preference family***, or CPF for short, to mean either an algebra-conditional preference family, or an event-conditional preference family. The precise meaning will always be clear from the context.

3. Conditional Utilities and Aggregators

Having introduced conditional preference families, we now discuss their utility representation. We show that coherence of conditional preferences translates into an aggregation property of ordinal conditional utilities.

We begin with a natural definition of the utility representation of a CPF. A ***state-contingent utility*** is any function of the form $U : \Omega \times X \to I\!R$ such that $U(\cdot, x)$ is $\mathcal{F}$-measurable for every x.

DEFINITION 4. *A utility representation of a CPF, $\{ \succeq^{\mathcal{G}} : \mathcal{G} \in \Phi \}$, is a set, $\mathcal{U} = \{ U^{\mathcal{G}} : \mathcal{G} \in \Phi \}$, of state-contingent utilities such that*

(a) *Given any* $\mathcal{G} \in \Phi$, $U^{\mathcal{G}}(\cdot, x)$ *is* $\mathcal{G}$*-measurable for every* x.

(b) *For all* $\mathcal{G} \in \Phi$, $U^{\mathcal{G}}$ *represents* $\succeq^{\mathcal{G}}$ *in the sense that for all* $G \in \mathcal{G}$ *and* $x, y \in X$,

$$(x \succeq^{\mathcal{G}} y \text{ on } G) \Leftrightarrow (U^{\mathcal{G}}(x) \geq U^{\mathcal{G}}(y) \text{ on } G).$$

Given standard results on the utility representation of preferences, it is not surprising that only weak conditions guarantee that a CPF has a utility representation. What is interesting, however, is that almost equally weak conditions give rise to utilities representations that satisfy

$$\mathcal{H} \subseteq \mathcal{G} \quad \text{implies} \quad U^{\mathcal{H}}(x) = A[\, U^{\mathcal{G}}(x) \mid \mathcal{H}\,] \text{ for all } x \in X, \tag{1}$$

for appropriate mappings $A[\,\cdot \mid \mathcal{G}\,]$ that we call aggregators. Furthermore, aggregators have the properties listed in the following definition. We let $M^{\mathcal{G}}$ represent the set of all $\mathcal{G}$-measurable random variables.

DEFINITION 5 (AGGREGATOR FAMILY). *An aggregator family is a set,* $\{\, A[\,\cdot \mid \mathcal{G}\,] : \mathcal{G} \in \Phi \,\}$, *of maps of the form* $A[\,\cdot \mid \mathcal{G}\,] : D^{\mathcal{G}} \to D^{\mathcal{G}} \cap M^{\mathcal{G}}$, *where* Φ *is an information class,* $D^{\mathcal{G}} \subseteq M^{\mathcal{F}}$, *and the following conditions are satisfied for all* $\mathcal{G}, \mathcal{H} \in \Phi$ *and* $W, V \in D^{\mathcal{G}}$:

(a) *(Separability)* $V = W$ *on* $G \in \mathcal{G}$ *implies* $A[\, V \mid \mathcal{G}\,] = A[\, W \mid \mathcal{G}\,]$ *on* G.

(b) *(Projection Property) (i)* $V \in D^{\mathcal{G}} \cap M^{\mathcal{G}}$ *implies* $A[\, V \mid \mathcal{G}\,] = V$; *(ii)* $\mathcal{H} \subseteq \mathcal{G}$ *implies* $D^{\mathcal{G}} \subseteq D^{\mathcal{H}}$ *and* $A[\, A[\, W \mid \mathcal{G}\,] \mid \mathcal{H}\,] = A[\, W \mid \mathcal{H}\,]$.

(c) *(Coherence) If* $H \in \mathcal{H} \subseteq \mathcal{G}$, *then*

$$(A[\, W \mid \mathcal{G}\,] \geq A[\, V \mid \mathcal{G}\,] \text{ on } H) \Rightarrow (A[\, W \mid \mathcal{H}\,] \geq A[\, V \mid \mathcal{H}\,] \text{ on } H),$$

$$(A[\, W \mid \mathcal{G}\,] > A[\, V \mid \mathcal{G}\,] \text{ on } H) \Rightarrow (A[\, W \mid \mathcal{H}\,] > A[\, V \mid \mathcal{H}\,] \text{ on } H).$$

A utility representation $\mathcal{U} = \{\, U^{\mathcal{G}} : \mathcal{G} \in \Phi \,\}$ *of a CPF is said to admit aggregation if* (1) *is satisfied for some aggregator family.*

It is straightforward to show that if $\{\, A[\,\cdot \mid \mathcal{G}\,] : \mathcal{G} \in \Phi \,\}$ is an aggregator family and U is a state-contingent utility, then $\{\, A[\, U \mid \mathcal{G}\,] : \mathcal{G} \in \Phi \,\}$ is the utility representation of a CPF. We now show essentially the converse of this statement, and we close the section with examples of aggregator families.

The following result generalizes a standard utility representation theorem (see, for example, Fishburn (1970), Theorem 3.1), to obtain utility representations that admit aggregation. Given a CPF $\mathcal{A} = \{ \succeq^{\mathcal{G}} : \mathcal{G} \in \Phi \}$, we say that the set $Z \subseteq X$ is *$\mathcal{A}$-dense* if for any acts x, y, any $\mathcal{G}$ in Φ, and any non-null G in $\mathcal{G}$, $x \succ^{\mathcal{G}} y$ on G implies that there exists a z in Z such that $\{ x \succeq^{\mathcal{G}} z \succeq^{\mathcal{G}} y \}$ has a non-null intersection with G.

THEOREM 3. *Every CPF, $\mathcal{A}$, for which there exists a countable $\mathcal{A}$-dense subset of X has a utility representation that admits aggregation.*

COROLLARY 1. *If X is countable, then every CPF has a utility representation that admits aggregation.*

Theorem 3 can be further specialized if X is endowed with a topology and preferences are continuous. Recall that a preference order $\succeq$ is *continuous* if all sets of the form $\{ x : x \succeq y \}$ and $\{ y : x \succeq y \}$ are closed. A CPF $\{ \succeq^{\mathcal{G}} : \mathcal{G} \in \Phi \}$ is *continuous* if, for every $\mathcal{G}$, $\succeq^{\mathcal{G}}_{\omega}$ is continuous for every ω outside a null event. The following generalizes a standard result of Debreu (1952).

COROLLARY 2. *Suppose that X is a connected, separable topological space. Then every continuous algebra-CPF has a utility representation that admits aggregation.*

A partial converse of Theorem 3 is

THEOREM 4. *Suppose that the CPF $\mathcal{A} = \{ \succeq^{\mathcal{G}} : \mathcal{G} \in \Phi \}$ has a utility representation, Φ is countable,*[11] *and under some probability measure the null events are exactly the events of zero probability. Then there exists a countable $\mathcal{A}$-dense subset of X.*

We close with examples of aggregator families. A constructive characterization of the functional form of all aggregator families is an open problem.

EXAMPLE 2 (ADDITIVE AGGREGATION). An aggregator family is *additive* if there exists an expectation operator E such that, for every $\mathcal{G} \in \Phi$ and $V \in \mathcal{D}^{\mathcal{G}}$, $E|V| < \infty$

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[11] Alternatively, a continuity-separability argument can be made. As discussed in Section 5.3, Φ can be regarded as a complete, separable metric space. The assumption of countability can then be replaced with a proper notion of continuity of the conditional preference $\succeq^{\mathcal{G}}$ with respect to $\mathcal{G}$. The details of this approach (that uses the result proved here) are left to the interested reader.

and $A[V \mid \mathcal{G}] = E[V \mid \mathcal{G}]$. The utility representation $\mathcal{U}$, as given in Definition 4, is ***additive*** if (1) holds for an additive aggregator family. The term "P-additive" is used to signify the underlying probability. Clearly, every probability P defines a P-additive aggregator for every Φ, by simply taking $D^{\mathcal{G}}$ to be the space of all P-integrable random variables. Additive aggregators in turn can be used to define additive CPFs (see Example 1). An axiomatic development of additive aggregation is presented in the following section.

EXAMPLE 3 (EXTREMAL AGGREGATION). Suppose that Ω is finite, $\mathcal{N} = \{\emptyset\}$, and Φ is an information class. An aggregator family is then defined by letting, for all $\mathcal{G} \in \Phi$, $D^{\mathcal{G}} = M^{\mathcal{F}}$ and $A[V \mid \mathcal{G}] = \max\{W \in M^{\mathcal{G}} : W \leq V\}$.

EXAMPLE 4 (ORDINAL EQUIVALENCE). Let $f : I\!R \to I\!R$ be a strictly increasing continuous function, and let $\{A[\cdot \mid \mathcal{G}] : \mathcal{G} \in \Phi\}$ be an aggregator family. A new aggregator family can be defined by letting $\tilde{A}[W \mid \mathcal{G}] = f^{-1}(A[f(W) \mid \mathcal{G}])$, for every $\mathcal{G} \in \Phi$ and W such that $f(W) \in D^{\mathcal{G}}$. Aggregator families related like that are ordinally equivalent, in the sense that they correspond to different utility representations of the same CPF.

EXAMPLE 5 (AGGREGATOR MIXING). Let $\{\Omega_1, \Omega_2, \ldots\}$ be a countable partition of Ω, and denote by π the σ-algebra it generates together with the null events. Let Φ be an information class with the property that $\pi \in \Phi$ and, for every $\mathcal{G} \in \Phi$, either $\mathcal{G} \subseteq \pi$ or $\pi \subseteq \mathcal{G}$. For every $n \in \{1, 2, \ldots\}$, suppose $\{A_n[\cdot \mid \mathcal{G}] : \mathcal{G} \in \Phi\}$ is an aggregator family. These aggregator families can be "mixed," using a third aggregator family $\{A_0[\cdot \mid \mathcal{G}] : \mathcal{G} \in \Phi\}$, by letting

$$A[V \mid \mathcal{G}] = A_0\Big[\sum_{n=1}^{\infty} A_n[V \mid \mathcal{G} \vee \pi] 1_{\Omega_n} \mid \mathcal{G}\Big], \quad \mathcal{G} \in \Phi,$$

where $\mathcal{G} \vee \pi$ is the σ-algebra generated by $\mathcal{G} \cup \pi$. It is not hard to confirm that $\{A[\cdot \mid \mathcal{G}] : \mathcal{G} \in \Phi\}$ is an aggregator family.

4. Additive Aggregation

This section presents sufficient conditions for additive aggregation, as defined in Examples 1 and 2. The result extends standard additive representation results for separable preferences. In the case of an infinite number of events, the theorem of additive aggregation presented is new even under preference separability.

We formulate the results on additive aggregation in terms of an event-CPF $\mathcal{E} = \{ \succeq^G : G \in \mathcal{G} \}$. The connection to additive algebra-CPFs and additive aggregators is given in Examples 1 and 2. The desired representation is given in the following definition.

DEFINITION 6. *The pair (U, P), where U is a $\mathcal{G}$-measurable state-contingent utility and P is a probability on $\mathcal{G}$, is an **additive representation** of the CPF $\{ \succeq^G : G \in \mathcal{G} \}$ if $\int_\Omega |U(x)| \, dP < \infty$ for every x, $G \in \mathcal{G} \cap \mathcal{N} \Leftrightarrow P(G) = 0$, and*

$$x \succeq^G y \Leftrightarrow \int_G U(x) \, dP \geq \int_G U(y) \, dP, \quad G \in \mathcal{G} \backslash \mathcal{N}, \quad x, y \in X.$$

*The representation (U, P) is **unique** if any other additive representation of the form $(\tilde{U}, P)$ satisfies $\tilde{U} = \alpha U + \beta$ for some $\alpha \in (0, \infty)$ and some (P-integrable) random variable β. The representation (U, P) is **continuous** if $\int_G U \, dP$ is continuous for every event G in $\mathcal{G}$.*[12]

Clearly, unique additive aggregation does not uniquely determine the underlying probability measure. By the Radon-Nikodym theorem, if (U, P) is an additive representation of $\mathcal{E}$, so is (V, Q), where Q is any measure equivalent to P, and $V(x) = U(x)(dP/dQ)$, $x \in X$. Moreover, (V, Q) inherits uniqueness and continuity from (U, P). In Skiadas (1994b) the setting is extended to give conditions under which there is a unique probability measure consistent with additive aggregation, and that measure has the interpretation of subjective probability.

That not every CPF has an additive representation follows from well known counterexamples dating back to Scott and Suppes (1953), formulated in the context of state-contingent acts over a finite number of states, and separable preferences. (Examples are also discussed in Fishburn (1970), and Krantz et. al. (1971). A counterexample to additivity in the setting of this paper that does not rely on

[12] Equivalently, for every net $\{ x_\gamma \}$ in X converging to $x \in X$, the net of random variables $\{ U(x_\gamma) \}$ converges to $U(x)$ weakly in $L^1(\Omega, \mathcal{F}, P)$. (See, for example, Dunford and Schwartz (1988) Chapter IV, Exercise 25.)

preference separability can be found in Skiadas (1992).) We now develop sufficient conditions for additivity, considering first the case of a finite $\mathcal{G}$, followed by the case of a countably generated $\mathcal{G}$.

To motivate our assumptions, it is helpful to consider the basic idea of our approach. Suppose that $\mathcal{G}$ is generated by a partition $\{G_1, \ldots, G_n\}$ of Ω. We wish to provide sufficient conditions, under which we can consistently define a preference order $\succeq$ on X^n by letting $(x_1, \ldots, x_n) \succeq (y_1, \ldots, y_n) \Leftrightarrow x \succeq^{\Omega} y$, where $x \sim^{G_i} x_i$ and $y \sim^{G_i} y_i$ for all i. Moreover, we will assume conditions under which Debreu's (1960) additive representation theorem, as generalized by Krantz, Luce, Suppes, and Tversky (1971), can be applied on the preference order $\succeq$. Among these assumptions will be that X is some connected topological space, and that all conditional preferences are continuous. Following Debreu, we call an event $G \in \mathcal{G}$ *essential* if it is non-null and $x \succ^{G} y$ for some $x, y \in X$.

The following assumption is one way of guaranteeing that the preference order $\succeq$ introduced above is well defined on X^n and continuous.

ASSUMPTION 1. *The following conditions hold:*

(a) (Solvability) Given any pairwise disjoint essential events $G_1, \ldots, G_n$ in $\mathcal{G}$, and any acts $x_1, \ldots, x_n$, there exists an act x such that $x \sim^{G_i} x_i$ for all i.

(b) X is Hausdorff and compact.[13]

With separable preferences over state-contingent acts, solvability is automatically satisfied, because one can construct x to yield the same consequences as x_i on G_i, for every i. In its general form, solvability can be thought of as the availability of acts that exactly compensate for not following certain courses of action under corresponding scenarios. Act x could, for example, be a state-contingent monetary payoff that, on the occurence of event G_i, would exactly compensate for not having followed act x_i, for every i. Of course, such an act may be an artifact, not naturally occuring in the initial formulation of the problem.

Compactness of X in Assumption 1 serves the purpose of showing continuity of

[13] For the relevant background on topological notions, the reader is referred to Kelley (1955) or Dugundji (1966).

$\succeq$. Since this is not a valid assumption in many contexts, we consider the following alternative condition.

ASSUMPTION 2. *(Continuous Solvability) Given any pairwise disjoint essential events $G_1, \ldots, G_n$ in $\mathcal{G}$, and any acts $x_1, \ldots, x_n$, there exists an act x with the following property: for every neighborhood N of x, there exist neighborhoods $N_1, \ldots, N_n$ of $x_1, \ldots, x_n$, respectively, such that $x'_i \in N_i$ for all i implies that there exists $x' \in N$ such that $x' \sim^{G_i} x'_i$ for all i.*

Assumption 2 has the same interpretation as solvability, with the additional intuition that sufficiently small perturbations of the acts x_i correspond to small perturbations of the "compensating" act x.

In case the solvability assumptions appear too strong, it is instructive to consider some alternative formulations. The following assumption implies solvability, and can replace continuous solvability in the additive representation theorem.

ASSUMPTION 3. *Given any pairwise disjoint essential events $G_1, \ldots, G_n$ in $\mathcal{G}$, and any $x \in X$, the set $A = \{ x' \in X : x' \sim^{G_i} x, \ i = 1, \ldots, n-1 \}$ is connected, and for every $y \in X$ there exist $\overline{z}, \underline{z} \in A$ such that $\overline{z} \succeq^{G_n} y \succeq^{G_n} \underline{z}$.*

Assumption 3 weakens the exact solvability requirement of Assumption 1, at the cost of assuming that the sets A are connected. This idea can be taken even further. It is shown in Lemma 6 (Appendix II) that Assumption 3 is implied by

ASSUMPTION 4. *Let $G_1, \ldots, G_n$ be any pairwise disjoint essential events in $\mathcal{G}$, and x any act. Every set of acts of the form $\{ y : y \, R_i \, x, \ i = 1, \ldots, n \}$, with $R_i \in \{ \succeq^{G_i}, \preceq^{G_i}, \sim^{G_i} \}$ for every i, is connected. Furthermore, if $R_i \in \{ \succeq^{G_i}, \preceq^{G_i} \}$ for all i, then for every $y \in X$ there exists a $z \in X$ such that $z \, R_i \, x$ for $i = 1, \ldots, n-1$, and $z \, R_n \, y$.*

Any of the above assumptions, together with connectedness of X, continuity of preferences, and a strict version of coherence, gives additive aggregation in the case that $\mathcal{G}$ is finite and contains enough essential events.

THEOREM 5. *Suppose that $\mathcal{G}$ is finite, and that the CPF $\mathcal{E} = \{ \succeq^G : \ G \in \mathcal{G} \}$ satisfies the following conditions:*

(a) $\mathcal{G}$ *contains at least three essential events.*

(b) X *is a connected topological space, and* $\succeq^G$ *is a continuous preference order on* X *for every non-null* G *in* $\mathcal{G}$.

(c) *For every pair of non-null disjoint events* F *and* G *in* $\mathcal{G}$, *and any acts* x *and* y, $x \succ^F y$ *and* $x \succeq^G y$ *implies* $x \succ^{F \cup G} y$.

(d) *Any one of Assumptions 1, 2, 3, or 4 holds.*

Then $\mathcal{E}$ *has a continuous and unique additive representation.*

Remark 1. The proof in Appendix II shows that Theorem 5 remains valid if $\mathcal{G}$ is not an algebra, provided that every element of $\mathcal{G}$ is a union of some of the events $G_1, \ldots, G_n$, and that $\Omega \in \mathcal{G}$.

We now provide a result that allows the extension of Theorem 5 to the case in which $\mathcal{G}$ is countably generated. Countably generated σ-algebras cover most situations in practice. For example, the Borel sets of any separable space (such as a Euclidean space) are countably generated, and it can be shown (see Billingsley (1986), Exercise 20.1) that $\mathcal{G}$ is countably generated if and only if it is generated by a random variable. The following result relies on the martingale convergence theorem. An alternative approach can be based on the methods of Vind (1990). [14] Given a sequence of events $\{ G_n \}$, the notation $G_n \uparrow G$ means that $m < n$ implies $G_m \subset G_n$ and $G = \bigcup_n G_n$. The notation $G_n \downarrow G$ means that $(\Omega \backslash G_n) \uparrow (\Omega \backslash G)$.

THEOREM 6. *Suppose that* $\mathcal{G}$ *is countably generated, and that the CPF* $\mathcal{E} = \{ \succeq^G : G \in \mathcal{G} \}$ *satisfies the following conditions, in addition to conditions (a) through (d) of Theorem 5:*

(e) *For any sequence* $\{ G_n \}$ *of events in* $\mathcal{G}$, *and any acts* x *and* y *such that* $x \succeq^{G_n} y$ *for all* n, $G_n \uparrow G$ *or* $G_n \downarrow G$ *implies that* $x \succeq^G y$.

(f) *There exist acts* $\underline{x}$ *and* $\overline{x}$ *such that, for every non-null event* G *in* $\mathcal{G}$ *and any act* x, $\overline{x} \succeq^G x \succeq^G \underline{x}$ *and* $\overline{x} \succ^G \underline{x}$.

[14] In the context of separable preferences, Vind (1990) takes a probability defining the null events as given, and assumes a sense of continuity of preferences relative to that probability. In Theorem 6 null events are taken as primitive, and a probability is constructed so that all null events are the events of probability zero.

(g) *For any sequence of events* $\{ G_n \}$ *in* $\mathcal{G}$ *such that* $G_n \downarrow \emptyset$, *there exists a sequence of acts* $\{ x_n \}$ *from which a subsequence* $\{ x_{n(k)} \}$ *can be extracted that converges to* $\underline{x}$, *but at the same time* $x_{n(k)} \sim^{G_{n(k)}} \overline{x}$ *for every* k.

Then $\mathcal{E}$ *has a continuous and unique additive representation.*

Remark 2. The proof of the theorem shows that the theorem could be stated in a more general form. Conditions (a) through (d) can be replaced by the weaker: Every finite subalgebra $\mathcal{H}$ of $\mathcal{G}$ is contained in a σ-algebra $\mathcal{H}' \subseteq \mathcal{G}$ such that $\{ \succeq^G : G \in \mathcal{H}' \}$ has a continuous and unique additive representation. Condition (g) need only be assumed for events $\{ G_n \}$ in some countable algebra generating $\mathcal{G}$. Finally, instead of coherence of $\mathcal{E}$, one need only assume: $x \succeq^F y$ and $x \succeq^G y$ implies $x \succeq^{F \cup G} y$, for all disjoint non-null events F and G. Under conditions (c) and (e), this condition implies coherence (Definition 2(b)). This completes the remark.

Of the new assumptions of Theorem 6, (e) has a clear meaning, and (f) is technically motivated, serving to provide appropriate bounds to utilities when limits are taken. Assumption (g) states that the worst act $\underline{x}$ is (topologically) close to acts that, conditionally on sufficiently "small" events, are as desirable as the best act $\overline{x}$. Section 5.1 provides an example with state-contingent acts and an explicit norm topology, in which all of the assumptions of Theorem 6 are satisfied.

Finally, all of the above theory can be "localized." That is, we can assume that every act has a neighborhood on which $\mathcal{E}$ has an additive representation. The methodology of Chateauneuf and Wakker (1993) can then be applied to obtain a "global" additive representation, provided that all indifference sets are connected. The details of such an argument follow closely that of Chateauneuf and Wakker, and are left to the interested reader.

5. Applications and Extensions

In the last three sections we have defined conditional preferences with respect to events or σ-algebras, we have discussed their utility representation and a general form of aggregation, and finally provided additional conditions for additive aggregation. This section discusses applications and extensions of this theory.

A class of applications involves state-contingent acts, just as in Savage (1954), but non-separable preferences. Section 5.1 presents such an example, through a theory of disappointment aversion. Beyond non-separable preferences, the formal setting of this paper allows us to talk about preferences over acts that do not have a natural state-contingent structure. Section 5.2 presents such an example in the form of the subjective value of information. The reason for the lack of an explicit state-contingent structure of acts can be the fact that a decision maker has limited ability to contemplate possible resolutions of uncertainty and their consequences. This "bounded rationality" interpretation is further discussed in Section 5.3, and related to Knight's (1921) and Ellsberg's (1961) concept of uncertainty (or ambiguity) in Section 5.4.

Further extensions are presented in related papers. Applications to dynamic choice are considered in (Skiadas 1994a), where the methodology of this paper is used to axiomatize utilities with intertemporal dependencies. Subjective probability under state-dependence and non-separable preferences, based on the theory of additive aggregation, is considered in (Skiadas 1994b). Finally, Klibanoff and Skiadas (1994) develop a theory of the aggregation of conditional utilities using the multiple-prior approach of Gilboa and Schmeidler (1989).

5.1. Non-separable Preferences and Disappointment Aversion

In this subsection we consider the case in which acts have a state-contingent structure, just as in Savage's setting. We relax, however, the assumption of preference separability, by allowing the possibility that conditional preferences over consequences are affected by initial expectations, for example, because of disappointment or relief.

Throughout the subsection, we take as given a measurable space $(C, \mathcal{C})$, and we assume that X consists of all measurable functions of the form $x : \Omega \to C$. The interpretation of $x(\omega)$ is that of an objective consequence of act x at state ω. We take as primitive an event-CPF, $\mathcal{E} = \{ \succeq^F : F \in \mathcal{F} \}$, that is consistent with an algebra-CPF with utility representation $\mathcal{U} = \{ U^{\mathcal{G}} : \mathcal{G} \in \Phi \}$. We assume that Φ includes $\mathcal{F}$ and the trivial σ-algebra $\mathcal{T}$ (generated by the null events). For simplicity, we write $U = U^{\mathcal{F}}$ and $u = U^{\mathcal{T}}$. (The utility u represents $\succeq^{\Omega}$.) We also define $I = \{ u(x) : x \in X \}$. To avoid technicalities, we begin with the assumption that $\mathcal{F}$ is finite and $\mathcal{N} = \{\emptyset\}$. This assumption is relaxed in Theorem 7 below.

Preference separability is the requirement that the ranking of two acts given some event does not depend on the consequences of these acts outside the event. Formally,

$$(x = y \text{ on } F) \Rightarrow (x \sim^F y), \qquad x, y \in X, \quad F \in \mathcal{F}. \tag{2}$$

Given our assumption of a finite number of events, it is straightforward to show that (2) is equivalent to the existence of a function f, whose domain is a subset of $\Omega \times C$, such that $U(\omega, x) = f(\omega, x(\omega))$ for all $(\omega, x) \in \Omega \times X$. If $\mathcal{U}$ is additive, as in Example 2 and Theorem 5, we then have the state-dependent expected utility representation $u(x) = E(f(x))$.

A weaker condition than (2), allowing for non-separable choice, is

$$(x = y \text{ on } F \text{ and } x \sim^\Omega y) \Rightarrow (x \sim^F y), \quad x, y \in X, \quad F \in \mathcal{F}. \tag{3}$$

The idea behind (3) is that conditional preferences may be affected by the ex-ante ranking of acts (where there is no information available ex-ante). For example, the actual realization of an act may lead to disappointment (elation), because the ex-post ranking of an act is inferior (superior) relative to its ex-ante ranking. This is formalized by

$$(x = y \text{ on } F \text{ and } y \succeq^\Omega x) \Rightarrow (x \succeq^F y), \quad x, y \in X, \quad F \in \mathcal{F}. \tag{4}$$

Condition (4) is a statement of (weak) disappointment aversion: if the ex-ante assessment is that y is better than x, but x and y are identical on F, then given the occurrence of F, the agent is better off choosing x, rather than y, since the latter leads to disappointment. A strict version of (4), modeling strict disappointment aversion, is

$$(x = y \text{ on } F \text{ and } y \succ^\Omega x) \Rightarrow (x \succ^F y), \quad x, y \in X, \quad F \in \mathcal{F}. \tag{4'}$$

Given our assumption of a finite number of events, it is easy to show that condition (3) is equivalent to the existence of a function, f, whose domain is a subset of $\Omega \times C \times I$, such that $U(\omega, x) = f(\omega, x(\omega), u(x))$ for all $(\omega, x) \in \Omega \times X$.

Moreover, the function f can be chosen to be nonincreasing (respectively, strictly decreasing) in its last argument if and only if (4) (respectively, (4′)) is satisfied. If $\mathcal{U}$ is additive, as in Theorem 5, we obtain

$$u(x) = E[f(x, u(x))], \quad x \in X. \tag{5}$$

Under assumption (4) (or (4′)), equation (5) uniquely determines u, and therefore the entire CPF through the equation $U(x) = f(x, u(x))$. To see that, suppose that for a given x the equation $v = E[f(x, v)]$ is satisfied for $v = v_1$ and $v = v_2$, where $v_1 < v_2$. Then the monotonicity of f in its utility argument implies that $v_1 = E[f(x, v_1)] \geq E[f(x, v_2)] = v_2$, a contradiction.

The class of utilities just introduced extends the class of utilities axiomatized by Dekel (1986), who considered a von Neumann-Morgenstern setting of preferences over distributions, weakening the independence axiom to the betweeness axiom of Chew (1983). The interpretation of disappointment aversion was provided by Gul (1991) in a narrower class of preferences than that considered by Dekel. (Gul's intention was to explain some of the observed violations of expected utility, by providing a minimal extension of the von Neumann-Morgenstern framework that captures the notion of disappointment aversion.) Disappointment was also modeled in special settings by Bell (1985), and Loomes and Sugden (1986).

The remainder of this subsection presents a theory of disappointment aversion under additive aggregation and an infinite number of events. The infinity of events presents a number of technical difficulties that are overcome with a number of special assumptions, including

Assumption 5. *The following conditions hold:*

(a) There is a probability P such that $(\Omega, \mathcal{F}, P)$ is a non-atomic[15] *probability space, and $P(F) = 0 \Leftrightarrow F \in \mathcal{N}$. The σ-algebra $\mathcal{F}$ is countably generated.*

(b) $C = [0, 1]^d$, for some positive integer d, and $\mathcal{C}$ consists of the usual Borel sets.

(c) A sequence $\{x_n\}$ in X converges to x if and only if $\{x_n\}$ converges to x in

[15] For a discussion of non-atomic spaces see, for example, Dudley (1989), Section 3.5.

probability[16] *(under P).*

(d) *Let $\overline{x}(\omega) = (1, \ldots, 1)$ and $\underline{x}(\omega) = (0, \ldots, 0)$ for all $\omega \in \Omega$. Then $\overline{x} \succ^F \underline{x}$ and $\overline{x} \succeq^F x \succeq^F \underline{x}$ for all non-null $F \in \mathcal{F}$ and $x \in X$.*

(e) *For any acts x, y, and any non-null event F, $x \geq y$ on F and $y \succeq^\Omega x$ implies $x \succeq^F y$.*

We think of the elements of C as consumption bundles. The choice of the interval $[0, 1]$ in Assumption 5(b) is arbitrary; C could be the product of any finite number of compact intervals. Assumption 5(d) states that $\overline{x}$ and $\underline{x}$ represent the absolute best and absolute worst, respectively. Consequently, the utility of $(0, \ldots, 0)$ on some event cannot be decreased because of disappointment, and analogously the utility of $(1, \ldots, 1)$ on some event cannot be enhanced because of relief. In other words, disappointment or relief is allowed to modify the utility of non-extreme consequences only. Assumption 5(e) is a stronger version of condition (3). On the one hand, it expresses a weak form of disappointment aversion, in the sense that preferences can be separable (condition (2)) or disappointment averse (condition (4)). On the other hand, condition (e) expresses the idea that more consumption is (weakly) preferred to less consumption, provided there is no disappointment.

The representation theorem that follows specializes Theorem 6. Notice that in this context connectedness of X is implied by the fact that X is a convex set. The separable-preference case is treated more generally by Grodal and Mertens (1968).

THEOREM 7. *Suppose that, in addition to Assumption 5, conditions (a) through (d) of Theorem 5, and condition (e) of Theorem 6 hold (with $\mathcal{G} = \mathcal{F}$). Then $\mathcal{E}$ has a continuous and unique additive representation (U, P). Moreover, there exists a function $f : \Omega \times C \times I \to I\!R$ that is nondecreasing in its second argument and nonincreasing in its third argument, such that $U(x) = f(x, u(x))$ and $u(x)$ uniquely solves (5) for every $x \in X$. The function f is strictly decreasing in its last argument if and only if (4′) is satisfied, and it does not depend on its last argument if and only if (2) holds.*

The following is an example satisfying all the assumptions of Theorem 7.

[16] X can be considered as a complete metric space under the Ky Fan metric (see, for example, Dudley (1989), Section 9.2).

EXAMPLE 6. Suppose $C = [0,1]$, and the function $f : \Omega \times C \times [0,1] \to [0,1]$ has the following properties: (a) for every $(\omega, v) \in \Omega \times [0,1]$, $f(\omega, \cdot, v)$ is continuous and strictly increasing, with $f(\omega, 0, v) = 0$ and $f(\omega, 1, v) = 1$; and (b) for every $(\omega, c) \in \Omega \times C$, $f(\omega, c, \cdot)$ is nonincreasing and continuous. Let X and $(\Omega, \mathcal{F}, P)$ be as in Assumption 5. We claim that a utility u is uniquely defined by (5). Uniqueness is a consequence of (b), as argued earlier. To show existence, we observe that $E[f(x, v)] - v$ is a continuous function of v that is nonnegative for $v = 0$ and nonpositive for $v = 1$, and therefore vanishes somewhere on $[0,1]$. Given the function u, implicitly defined by (5), we define $\mathcal{E}$ by letting $U(x) = f(x, u(x))$. It is easy to check then that all the assumptions of Theorem 7 hold.

5.2. Subjective Value of Information

In the last subsection we considered an application involving non-separable preferences over state-contingent acts. This subsection is on the subjective value of information, an application in which acts do not admit a natural state-contingent representation in the sense of Savage at all.

One usually thinks of information as being valuable indirectly, because it presents planning advantages that result to higher utility of, say, state-contingent consumption. However, information is also "consumed" and valued directly. Information can have entertainment value, but can also be a source of distress, even when state-contingent consumption is not affected by that information. On the other hand, even when an economic agent is in principle only interested in the indirect value of information, it is usually impractical to consider and evaluate all possible optimal actions implied by that information. For example, it is unrealistic to assume that the subscriber to a news report contemplates all optimal plans that are contingent upon all possible resolutions of uncertainty provided by the report. Instead, the subscriber places a direct subjective value on the report, conditionally on some algebra representing the agent's current perception of possible resolutions of uncertainty.

Throughout this subsection, we assume that X is an information class. An "act" is then a σ-algebra, and represents information in the usual sense. While all of this paper's general results apply to this setting, we discuss in some detail the additive aggregation theory. For purposes of comparison, we begin with a simple example of the indirect value of information.

EXAMPLE 7. Suppose C is a consumption space, for example, the non-negative cone of a Euclidean space, and Z is a space of feasible consumption plans. Every element of Z is of the form $z : \Omega \to C$ (and is measurable relative to some σ-algebra of subsets of C). Given state-contingent utility $u : \Omega \times C \to I\!R$, we assume that, for any σ-algebra of events $\mathcal{G}$, the indirect conditional utility of information is well defined by $U^{\mathcal{G}}(x) = \max\{\, E[\, u(z) \mid \mathcal{G}\,] \,:\, z \in Z \text{ is } x\text{-measurable}\,\}$ for any $x \in X$ such that $x \supseteq \mathcal{G}$. For simplicity, we assume that the maximum is achieved and is finite.[17] It is easy to check then that $\mathcal{H} \subseteq \mathcal{G} \subseteq x$ implies $U^{\mathcal{H}}(x) = E[\, U^{\mathcal{G}}(x) \mid \mathcal{H}\,]$. In particular, we can define $V(x) = U^{x}(x)$, $x \in X$, and extend the definition of the conditional utility $U^{\mathcal{G}}$ to the whole of X, by letting

$$U^{\mathcal{G}}(x) = E[\, V(x) \mid \mathcal{G}\,], \quad x \in X. \tag{6}$$

This completes the example.

In contrast to the above example, the theory of additive aggregation leads to an additive utility representation $\{\, U^{\mathcal{G}} \,:\, \mathcal{G} \in \Phi\,\}$ of the agent's CPF over X. Defining the function $V : \Omega \times X \to I\!R$ by $V(x) = U^{x}(x)$, we obtain representation (6) once again, with the new interpretation of $V(x)$ as the subjective value of information x. Of course, the additive representation result requires some topological and solvability assumptions that we now consider in turn.

First, we need to endow X with a topology. A natural[18] choice is the pointwise-convergence topology introduced by Cotter (1986), under which X becomes a Polish

[17] For example, this can be derived from standard continuity-compactness assumptions. More generally, we could replace the maximum with a supremum, at the cost of technical difficulties in showing the additivity statement of this Example. These difficulties can be dealt with, however, using the techniques of Rockafellar (1975).

[18] Another alternative would be to use the topology of Boylan (1971), which is analogous to the Hausdorff topology on spaces of sets, and has been shown by Allen (1983) to possess many nice properties for the purposes of economic theory. Cotter's topology used here is weaker than Boylan's topology, and has most of the properties investigated by Allen.

space.[19] The topology is defined in terms of an underlying probability P, so that a net $\{x_\gamma\}$ in X converges to $x \in X$ if and only if, for every integrable random variable r, $\{E[r \mid x_\gamma]\}$ converges to $E[r \mid x]$ in probability.[20] Cotter showed that changing the underlying probability P to one defining the same null events results in an equivalent topology. Therefore, P can be chosen arbitrarily as long as $P(F) = 0 \Leftrightarrow F \in \mathcal{N}$. For the rest of this subsection, we assume that X is endowed with the Cotter topology.

Additive aggregation also requires that X be connected. The following is an example in which this is true.

EXAMPLE 8. Suppose that $\{B_t : t \in [0,1]\}$ is d-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$, and $\{\mathcal{F}_t : t \in [0,1]\}$ is the augmented filtration generated by B (see, for example, Karatzas and Shreve (1988), Section 2.7). We assume that B_0 is constant, and therefore $\mathcal{F}_0$ is trivial (contains events of probability one or zero). We also assume that $\mathcal{F}$ is generated by $\bigcup_{t=0}^{\infty} \mathcal{F}_t$. We define X to be the set of all complete sub-σ-algebras of $\mathcal{F}$. We will show that X is connected under the Cotter topology. It suffices to show that every $x \in X$ is path-connected to $\mathcal{F}_0$ (see, for example, Dugundji (1966), paragraphs 5.2 and 5.3). Given any $x \in X$, let $x_t = x \cap \mathcal{F}_t$. By the discussion of Section 2.7 of Karatzas and Shreve (1988), we have that $x_t = \bigcap_{u \geq t} x_u$ and x_t is generated by $\bigcup_{u \leq t} x_u$. This fact, together with Lévy's theorem (see, for example, Billingsley (1986), Theorems 35.5 and 35.7), implies that the mapping $t \mapsto x_t$ is continuous. Since $x_0 = \mathcal{F}_0$ and $x_1 = x$, we have shown that x is path-connected to $\mathcal{F}_0$, and therefore the whole of X is connected. This completes the example.

Another assumption that was employed to obtain additive aggregation is solvability. As always, to justify such an assumption, one may wish to extend the space of acts by considering the product space of the given act space and a space of state-contingent acts with, say, monetary consequences. A special case in which solvability comes for free, however, is when conditional preferences are separable.

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[19] That is, there exists a metric d on X such that (X, d) is a complete, separable metric space.

[20] Or, equivalently, in L_1, since the set $\{E[r \mid x_\gamma]\}$ is uniformly integrable.

Separability in this context is defined by

$$(F \in x \cap y \text{ and } x \cap F = y \cap F) \Rightarrow (x \sim^F y), \quad x, y \in X. \tag{7}$$

For example, the (implied) CPF of Example 7 is easily seen to be separable. One can easily show that (7) implies continuous solvability (Assumption 2).

A consequence of condition (7) under additive aggregation is that the value of a partition can be computed as the sum of the value of its elements, in the following sense. Given (7), we can consistently define $v(G) = V(\mathcal{G})$, where $\mathcal{G}$ is the complete σ-algebra generated by G. Let X^f consist of all the complete σ-algebras generated by finite partitions of Ω, and for every $x \in X^f$, let $\pi(x)$ be a finite partition of Ω that generates x. Given (7), representation (6) then gives

$$V(x) = \sum \{ v(F) : F \in \pi(x) \}, \quad x \in X^f.$$

Under continuous additive aggregation, this representation completely determines the underlying event-CPF, because $\int_F V \, dP$ is continuous for every event F, and Cotter has shown that X^f is dense in X. The above additive representation of V on X^f was also discussed by Gilboa and Lehrer (1991). Their result is cardinal in character, taking V as a primitive, and assuming an additivity-type condition on V directly. Gilboa and Lehrer also noted that V is monotonically increasing ($x \supseteq y \Rightarrow V(x) \geq V(y)$) if and only if for any disjoint (non-null) events F and G, $v(F \cup G) \leq v(F) + v(G)$. The indirect utility of information of Example 7 is easily seen to be monotonically increasing.

In Skiadas (1994a) the above theory is extended to a dynamic setting, generalizing the treatment of preferences for the timing of resolution of Kreps and Porteus (1978).

5.3. Bounded Rationality Interpretations

In this subsection we briefly consider some interpretations of a CPF that relate to the limitations of a decision maker in perceiving possible resolutions of uncertainty and associated consequences of acts. These limitatations have been an important motivation in considering preferences over acts that do not necessarily have a state-contingent structure.

Given a CPF $\mathcal{A} = \{ \succeq^{\mathcal{G}} : \mathcal{G} \in \Phi \}$, we define a σ-algebra of events $\mathcal{B}$ to be a *rationality bound* of $\mathcal{A}$ if every conditional preference $\succeq^{\mathcal{G}}$ in $\mathcal{A}$ is $\mathcal{B}$-measurable. The *rationality limit*, $\mathcal{R}$, of $\mathcal{A}$ is the least rationality bound of $\mathcal{A}$, that is, the intersection of all rationality bounds of $\mathcal{A}$. The rationality limit $\mathcal{R}$ of an agent's CPF can be interpreted as the agent's perception of the possible resolutions of uncertainty. Assuming for simplicity that $\mathcal{R} \in \Phi$, the conditional utility $U^{\mathcal{R}}(x)$ reflects the agent's subjective perception of the consequences of x given each of the events in $\mathcal{R}$. In the case that acts are $\mathcal{R}$-measurable state-contingent acts, $U^{\mathcal{R}}$ can be given more specific structure through the assumption of separability, or some weaker version such as disappointment aversion. More realistically, acts may have consequences that obtain in not well understood circumstances, or have consequences of subjective and ambiguous nature. We are therefore interested in the case that state-contingent acts are not $\mathcal{R}$-measurable, or more generally in cases in which acts do not have a state-contingent structure in terms of unambiguous consequences. Providing $U^{\mathcal{R}}$ with more structure in such cases is an important research challenge. In the remainder of this subsection we briefly review a sample of ideas in this direction, with the understanding that a lot more remains to be said. The discussion of ambiguous consequences is continued in the following subsection in the context of Knightian uncertainty.

A general approach to explaining a conditional utility relative to a rationality limit is to assume that it represents an aggregated version of a conditional utility relative to a finer perception of uncertainty. For example, one can think of $U^{\mathcal{R}}$ as being derived from a larger theory of rational behavior that incorporates learning. The agent is really capable of formulating a conditional utility with respect to an algebra finer than $\mathcal{R}$, say $U^{\mathcal{F}}$, but has learned to implement the decision rule corresponding to the utility $U^{\mathcal{R}} = A[\, U^{\mathcal{F}} \mid \mathcal{R}\,]$. The latter requires of the agent to be aware of only $\mathcal{R}$, thus saving on information processing and gathering costs, while incurring learning or memorization costs. (Of course the question of the structure of $U^{\mathcal{F}}$ remains.) In another interpretation, one can think of $U^{\mathcal{R}}$ on some event as being derived from other agents or some social environment. For example, an agent that does not understand the implications of act x on event G in $\mathcal{R}$ may seek the advice of an expert. The expert, being aware of an algebra of events $\mathcal{G}$ such that

$\mathcal{G} \cap G \supseteq \mathcal{R} \cap G$, can communicate to the agent the utility $U^{\mathcal{R}} = A[\, V \mid \mathcal{R}\,]$ for some state-contingent utility V. Through the utility V, the expert not only summarizes information, but necessarily passes on to the agent subjective value judgements. Example 5 can be interpreted as the aggregation of the opinions of many experts, with expert n providing the conditional utility $V1_{\Omega_n}$ and corresponding aggregator family. The result is a compound CPF with a finer rationality bound than the agent's.

From another perspective, the rationality limit of a CPF represents not an agent's true finest perception of possible events, but rather the limitation of the CPF as a model. While the agent perceives events not in $\mathcal{R}$, these are of a subjective or ill defined nature, and hence not included in the primitives of the model. An example in this direction can be based on the work of Kreps (1976, 1992) as follows. Suppose that the elements of X represent opportunity sets. As Kreps (1992) points out, it may be infeasible for an agent to pick an optimal consequence given an x in X and state of the world, because of the possibility of unforseen contingencies, or more generally bounded rationality as discussed above. This motivates the introduction of direct preferences over X. The general results of this paper all apply to the case in which X consists of subsets of some space.[21] Assuming for simplicity that the rationality limit $\mathcal{R}$ of an agent is generated by some finite partition $\{\, R_1, \ldots, R_n \,\}$ of Ω, Krep's theory of preferences for flexibility can be applied wholesale on each of the primitive preferences $\succeq^{R_i}$. This way the conditional utility $U^{\mathcal{R}}$ on R_i can be represented as the value of the maximization of some state-dependent utility over subjective "sub-states" of R_i, that were not included as events in $\mathcal{R}$ because their nature is unknown to the modeler. This is essentially an idea of Kreps (1992), except that we have avoided having to postulate preferences over state-contingent acts with X as a consequence space.

5.4. Knightian Uncertainty

The notion of uncertainty (or ambiguity, or vagueness) as distinct from risk, dates back to Knight (1921), and has received considerable attention since Ellsberg's

[21] To endow X with a topology, the Hausdorff topology can be adopted following Kreps (1976), or alternatively the approach of Berliant (1986) can be taken, who also showed connectedness (required by the additive aggregation result).

(1961) well known experiments. For a survey of this literature, the reader can consult Karni and Schmeidler (1990), and Camerer and Weber (1991). In this subsection, we relate the setting of this paper to Knightian uncertainty.

We call a CPF $\{\succeq^{\mathcal{G}} : \mathcal{G} \in \Phi\}$ *full* if Φ contains every complete σ-algebra of events. The following Ellsberg-type example shows that a full CPF that satisfies preference separability (condition (2), assuming state-contingent acts) is not consistent with the notion of "ambiguity aversion."

EXAMPLE 9. An experiment consists of tossing a perfectly fair coin, and at the same time blindly pulling a ball from an urn containing red and green balls in completely unknown proportions. The natural state-space is $\Omega = \{hr, hg, tr, tg\}$, with the obvious interpretation. Acts are assumed to be state-contingent monetary payoffs. An agent's conditional preferences over these acts are represented by the CPF $\{\succeq^F : F \subseteq \Omega\}$. Given any event F, we denote by (F) the act whose consequences are a payoff of \$100 at every state in F, and no payoff at every other state. We let $H = \{hr, hg\}$ be the event of heads, and analogously we define the events T, R, and G. Because of symmetry, we naturally assume that $(H) \sim^G (T)$ and $(R) \sim^T (G)$. Let now x be the act $(\{hr, tg\})$. Preference separability (condition (2)) and the above symmetry indifferences imply that $x \sim^R (H)$ and $x \sim^G (T) \sim^G (H)$. Similarly, $x \sim^H (R)$ and $x \sim^T (G) \sim^T (R)$. Coherence then gives $x \sim^\Omega (H) \sim^\Omega (R)$, precluding the possibility that $(H) \succ^\Omega (R)$ on the grounds that (H) involves a less "ambiguous" risk than (R) does. This completes the example.

This example suggests two approaches to ambiguity aversion. In the first one separability is relaxed, and the ambiguity of an act is reflected in the values of conditional preferences. In the second approach preferences are modeled through a non-full CPF, and ambiguity is modeled in the form of aggregation, while preserving separability. We now briefly consider the two approaches, and then discuss them from the point of view of bounded rationality. It should also be pointed out that there are theories of Knightian uncertainty (such as Bewley's (1986)) that do not conform to this categorization.

Approach 1. In this approach the ambiguity of an act is reflected in its conditional utilities, while separability is not valid. In the context of Example 9, the agent reasons that on the event that R occurs (H) would have been a preferred act than x, since the latter would have involved the bad feeling of having chosen an ambiguous act. Thus the separability condition $x \sim^R (H)$ is violated. Section 4.1 of Camerer and Weber (1992) surveys some specific existing attempts to model ambiguity aversion by adjusting the utility of a consequence. We outline here a general method for the case of state-contingent acts.

Suppose that an agent, endowed with no information, subjectively ranks acts according to their "ambiguity." We model that by a complete and transitive order $\succeq_a$ on X, where $x \succeq_a y$ denotes the subjective assessment that x is more ambiguous than y. The agent's conditional preferences are modeled through the CPF $\{ \succeq^F : F \in \mathcal{F} \}$. Assuming that acts are state-contingent, the assumption of ambiguity aversion can then be stated as

$$(x = y \text{ on } F \text{ and } y \succeq_a x) \Rightarrow x \succeq^F y, \quad F \in \mathcal{F} \backslash \mathcal{N}, \ x, y \in X.$$

Using arguments analogous to those of Section 5.1,[22] one obtains the representation $U^{\mathcal{F}}(\omega, x) = f(\omega, x(\omega), a(x))$, where $a : X \to I\!R$ is an "ambiguity index" that ordinally represents $\succeq_a$, and f is non-increasing in its third argument. Under additive aggregation, $U^{\mathcal{G}}(x) = E[\, f(x, a(x)) \mid \mathcal{G}\,]$. This is of course not a complete model of ambiguity aversion, since nothing has been said about the structure of $\succeq_a$. It does show, however, that aggregation in the sense of this paper is in principle compatible with ambiguity aversion.

Approach 2. In this approach ambiguity aversion is reflected in the form of aggregation, while allowing preference separability. To accomplish that, we start with a non-full CPF of the form $\{ \succeq^{\Omega}, \succeq^{\mathcal{F}} \}$, with utility representation $\{ u, U^{\mathcal{F}} \}$. Coherence in this context is equivalent to the existence of a monotone function A,

[22] This is a general technique for introducing preference non-separabilities. For example, disappointment aversion is obtained by setting $\succeq_a = \succeq^{\Omega}$. More generally, given complete and transitive orders $\succeq_i$, $i = 1, \ldots, n$, on X, the condition $(x = y$ on F and $y \succeq_i x$ for all $i) \Rightarrow (x \succeq^F y)$ leads to the representation $U^{\mathcal{F}}(\omega, x) = f(\omega, x(\omega), a_1(x), \ldots, a_n(x))$, where $a_i : X \to I\!R$ ordinally represents $\succeq_i$. For example, to model disappointment aversion and ambiguity aversion together, we let $n = 2$, $\succeq_1 = \succeq_{\Omega}$, and $\succeq_2 = \succeq_a$, where $\succeq_a$ is a subjective ranking of acts with respect to ambiguity.

mapping random variables to real numbers, such that $u(x) = A[U^{\mathcal{F}}(x)]$. The mapping A can be any of the aggregation rules that have been proposed in the literature in order to capture ambiguity aversion. Examples include a Choquet integral, as in Schmeidler (1989), or a minimum expected value relative to a set of priors, as in Gilboa and Schmeidler (1989). Since ambiguity aversion is captured in the form of the aggregator, rather than the utility, preference separability can be assumed, implying that $U^{\mathcal{F}}(\omega, x) = f(\omega, x(\omega))$ for some function f.[23] Suppose now that we wish to extend the CPF $\{ \succeq^{\Omega}, \succeq^{\mathcal{F}} \}$ to a full CPF, while preserving separability and ambiguity aversion. Example 9 shows that there can be a sub-σ algebra of $\mathcal{F}$, $\mathcal{G}$, such that $x \sim^{\mathcal{G}} y$ does not imply $x \sim^{\Omega} y$, contradicting coherence. For such a $\mathcal{G}$, there does not exist an aggregator map $A^{\mathcal{G}}$ for which we can write $u(x) = A^{\mathcal{G}}[U^{\mathcal{G}}(x)]$ for all x in X.

While it is not our purpose here to discuss the various advantages and disadvantages of the two approaches, we conclude with a comment from the point of view of bounded rationality. We argue that the second approach is limited in its ability to incorporate ambiguous consequences. The following example illustrates the point.

EXAMPLE 10. Consider two different contracts, C_1 and C_2. An agent that enters contract C_i is entitled to monetary payoffs contingent on well defined resolutions of uncertainty. The agent knows that there are perfect enforcement mechanisms that guarantee that he will receive the appropriate payoff in every contingency. The rules of the contracts, however, are very complicated and written in a language that the agent does not comprehend. To make the case extreme, we assume that the agent is completely unaware of the contracts' contingencies. Having received advice from trusted experts, however, the agent has formed a subjective assessment of their value. He finds them to be of equal value. (We assume there is no market on these contracts.)

Consider now the setting of Example 9, and let $\mathcal{R}$ be the algebra of all subsets of $\Omega = \{ hr, hg, tr, tg \}$. If desired, $\mathcal{R}$ can be viewed as a coarse algebra of events in

[23] Klibanoff and Skiadas (1994) show that separability is not a necessary ingredient of this construction. For example, in the presence of disappointment aversion, we could have the representation $U^{\mathcal{F}}(\omega, x) = f(\omega, x(\omega), u(x))$, just as in Section 5.1.

a much larger state-space that includes all the contingencies relevant to contracts C_1 and C_2. Following Approach 2, the agent's preferences are represented by the CPF $\{\succeq^{\Omega}, \succeq^{\mathcal{R}}\}$, and separability is assumed. For example, the CPF could be defined by utilities $\{u, U^{\mathcal{G}}\}$, where $U^{\mathcal{G}}(\omega, x) = f(\omega, x(\omega))$ for some function f, and $u(x) = A[f(x)]$ for some aggregator A. For any event F, let $(C_i, C_j; F)$ denote the act in which the agent enters contract C_i on the event F and contract C_j otherwise. The act that involves entering contract C_i for sure is denoted C_i. Consider the acts $x = (C_1, C_2; H)$ and $y = (C_1, C_2; R)$. We assume that $C_1 \sim^{\mathcal{R}} C_2$, since the contracts are viewed of equal value by the agent, and the coin and urn are known to the agent not to be part of any of the contracts. By the separability assumption then, $x \sim^{\mathcal{R}} y \sim^{\mathcal{R}} C_1$, and by coherence, $x \sim^{\Omega} y \sim^{\Omega} C_1$. This completes the example.

In contrast to the conclusion of the example, one would expect that an agent averse to ambiguity may not be indifferent between x and C_1, and x and y. Act x may be viewed as "hedging" the uncertain (and unmodeled) nature of the two contracts by mixing them through a well understood randomizing device. (Gilboa and Schmeidler (1989) use this idea to define ambiguity aversion). On the other hand, y may be viewed as more ambiguous than x, since it replaces the coin toss with a more ambiguous randomization process. This problem can be avoided by modeling uncertainty in more detail than that allowed by $\mathcal{R}$. Such a solution would contradict, however, the interpretation of $\mathcal{R}$ as a rationality limit in the sense of the last subsection. Given a finer description of uncertainty one would still be able to formulate an example analogous to Example 10, as long as the model does not perfectly capture reality. Approach 1 is therefore limited in its ability to capture ambiguity aversion in the presence of ambiguous consequences. On the other hand, Approach 2 does not seem to suffer from this problem. In the context of Example 10, the conclusion $x \sim^{\mathcal{R}} y \sim^{\mathcal{R}} C_1$ is not necessarily valid if separability is not assumed.

APPENDIX I: On Black and White Sets

This appendix develops a measure theoretic result used in the proofs of Theorems 1, 4, and 7. The result is a generalization of the Hahn decomposition theorem.

Given is a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is a σ-algebra of subsets of

Ω, and P is a countably additive probability on $\mathcal{F}$. The elements of $\mathcal{F}$ are called *events*. A subset of an event F that is also an event is a *subevent* of F. An event N is *null* if $P(N) = 0$. Events that are not null are designated either *light* or *dark*, but not both. These are primitive notions and are required to satisfy the following conditions:

(a) A countable disjoint union of light (respectively, dark) events is light (respectively, dark).

(b) Two non-null sets whose symmetric difference is null are either both light or both dark.

An event F is defined to be *white* (respectively, *black*) if it is non-null and every non-null subevent of F is light (respectively, dark). We can now state the main result.

THEOREM 8. *Every non-null event F is black, or it is white, or it is the disjoint union of a white subevent of F and a black subevent of F. This decomposition of F is unique up to null events.*

EXAMPLE 11. Let μ be a finite signed measure on $(\Omega, \mathcal{F})$. Let P be defined by letting $P(F) = |\mu|(F)/|\mu|(\Omega)$, where $|\mu|$ represents the absolute variation of μ. Let an event F be dark (respectively, light) if $P(F) > 0$ and $\mu(F) \geq 0$ (respectively, < 0). Then the above theorem reduces to the Hahn decomposition theorem for finite signed measures. (See Halmos (1974), Theorem 29A, whose exposition inspired the proof that follows.)

LEMMA 1. *Every non-null event that is not black (respectively, not white) has a white (respectively, black) subevent.*

PROOF: Suppose F is a non-null event that is not black. Let L_1 be a light subevent of F. If L_1 is white we are done. Otherwise, let n_1 be the smallest integer for which there exists a dark subevent $D_1 \subseteq L_1$, with $P(D_1) \geq 1/n_1$. Then $L_2 \equiv L_1 - D_1$ is light, (since the disjoint union of two dark events is dark). Repeating the process, we either arrive at a white event L_n for some integer n, or produce a disjoint sequence $\{ D_k : k = 1, 2, \ldots \}$ of dark events and a sequence $\{ n_k : k = 1, 2, \ldots \}$ of integers. In the latter case let $D \equiv \bigcup_{k=1}^{\infty} D_k$ which is also dark, being the countable disjoint union of dark events. We claim that $L_1 - D$ is white. It cannot be a null event for then L_1 would have to be dark. On the other hand, suppose it contained a dark subevent G. Since $P(D_k) \to 0$, $n_k \to \infty$, as $k \to \infty$, and therefore $P(G) > 1/n_k$ for all large enough k. But this contradicts the choice of the integers n_k. ■

PROOF OF THEOREM 8: If the given event F is white there is nothing to show. If not, the Lemma shows that it has a black subevent. Let $\alpha \equiv \sup\{\, P(B) : B \in \mathcal{F} \cap F$, and B is black $\}$. Let $B_1, B_2, \dots$ be a sequence of black subevents of F such that $P(B_n) \to \alpha$ as $n \to \infty$. Then $B \equiv \bigcup_{n=1}^{\infty} B_n$ is black, and $P(B) = \alpha$. To see the first claim, suppose B is not black, and therefore (by the Lemma) it contains a white subevent W. But then, for at least one n, $W \cap B_n$ is a non-null subevent of W, and therefore also a light subevent of B_n, contradicting the fact that B_n is black. The second claim follows since $\alpha \geq P(B) \geq P(B_n) \to \alpha$. If $F - B$ is null we are done. If not, then it must be white. For if $F - B$ contained a black subevent B', then $P(B \cup B') > \alpha$, a contradiction. We have shown the first part of the theorem. Uniqueness is immediate. ■

Appendix II: Proofs

Proof of Theorem 3

Suppose that $Z = \{\, z_1, z_2, \dots \}$ is an $\mathcal{A}$-order dense subset of X. We fix any $\mathcal{G} \in \Phi$, and we define, for every $(\omega, x) \in \Omega \times X$,

$$d_n^{\mathcal{G}}(\omega, x) = \begin{cases} 0, & \text{if } z_n \succ_\omega^{\mathcal{G}} x; \\ 1, & \text{if } x \sim_\omega^{\mathcal{G}} z_n; \\ 2, & \text{if } x \succ_\omega^{\mathcal{G}} z_n. \end{cases}$$

Let $d_n^{\mathcal{G}}(x)$ denote the random variable $d_n^{\mathcal{G}}(\cdot, x)$. In the case that Z has a last element z_N, we let $d_n^{\mathcal{G}}(x) = 0$ for $n > N$. A candidate conditional utility is now defined by the formula

$$U^{\mathcal{G}}(x) = \sum_{n=1}^{\infty} \frac{d_n^{\mathcal{G}}(x)}{3^{2n-1}}.$$

Notice that there is a one-to-one correspondence between values of $U^{\mathcal{G}}(x)$ and sequences $\{\, d_n^{\mathcal{G}}(x) : n = 1, 2, \dots \}$. We will show that $\mathcal{U} = \{\, U^{\mathcal{G}} : \mathcal{G} \in \Phi \,\}$ is a utility representation of $\mathcal{A}$ that admits aggregation.

We first show that $\mathcal{U}$ is a utility representation of $\mathcal{A}$. Clearly, $U^{\mathcal{G}}(x)$ is $\mathcal{G}$-measurable for each $x \in X$. We now show that, for every $G \in \mathcal{G}$, $x \succeq^{\mathcal{G}} y$ on G if and only if $U^{\mathcal{G}}(x) \geq U^{\mathcal{G}}(y)$ on G. For any $G \in \mathcal{G}$, suppose $x \succeq^{\mathcal{G}} y$ on G. Then $d_n^{\mathcal{G}}(x) \geq d_n^{\mathcal{G}}(y)$ on G, for all n, and therefore $U^{\mathcal{G}}(x) \geq U^{\mathcal{G}}(y)$ on G. Conversely, suppose that $U^{\mathcal{G}}(x) \geq U^{\mathcal{G}}(y)$ on $G \in \mathcal{G}$, but $y \succ^{\mathcal{G}} x$ on some $G' \in \mathcal{G} \cap G$. Since Z is $\mathcal{A}$-dense, there exists integer m and some non-null $G'' \in \mathcal{G} \cap G'$, such that $y \succeq^{\mathcal{G}} z_m \succeq^{\mathcal{G}} x$ on G'', while of course $y \succ^{\mathcal{G}} x$ on G''. It follows that $d_m^{\mathcal{G}}(y) > d_m^{\mathcal{G}}(x)$ on G'', while $d_n^{\mathcal{G}}(y) \geq d_n^{\mathcal{G}}(x)$ on G'' for all n. Therefore, $U^{\mathcal{G}}(y) > U^{\mathcal{G}}(x)$ on G'', contradicting our original assumption. This completes the proof that $\mathcal{U}$ is a utility representation of $\mathcal{A}$.

It remains to show that $\mathcal{U}$ admits aggregation. For this purpose, we will use the following result.

LEMMA 2. *Suppose that $x, y \in X$; $\mathcal{G}, \mathcal{H}, \mathcal{I} \in \Phi$; and $I \in \mathcal{I} \subseteq \mathcal{G} \cap \mathcal{H}$. If $U^{\mathcal{G}}(x) = U^{\mathcal{H}}(y)$ on I, then $U^{\mathcal{I}}(x) = U^{\mathcal{I}}(y)$ on I.*

PROOF: We assume that $\mathcal{I} = \mathcal{G} \cap \mathcal{H}$. The more general statement follows immediately from this special case by coherence. Suppose $U^{\mathcal{G}}(x) = U^{\mathcal{H}}(y)$ on I. Then $d_n^{\mathcal{G}}(x) = d_n^{\mathcal{H}}(y)$ on I, for every n. It follows that, for each n, there exist disjoint (and possibly null) events in $\mathcal{I}$, I_1^n, I_2^n, and I_3^n whose union is I, and such that $d_n^{\mathcal{G}}(x) = d_n^{\mathcal{H}}(y) = i$ on I_i^n, for $i \in \{1, 2, 3\}$. We now claim that $x \sim^{\mathcal{I}} y$ on I. To see that, suppose $\{x \succ^{\mathcal{I}} y\} \cap I$ is non-null. Then, since Z is $\succeq^{\mathcal{I}}$-order dense, there exists integer n and event $F \in \mathcal{I} \cap I$ such that $x \succeq^{\mathcal{I}} z_n \succeq^{\mathcal{I}} y$ on F, while of course $x \succ^{\mathcal{I}} y$ on F. But it is not hard to check, using coherence, that this leads to a contradiction on each of the events $F \cap I_i^n$, $i \in \{0, 1, 2\}$. ■

To construct the aggregators, we let

$$D^{\mathcal{G}} = \{ U^{\mathcal{H}}(x) : \mathcal{H} \supseteq \mathcal{G},\ \mathcal{H} \in \Phi;\ x, y \in X \}.$$

We define the mappings $A[\cdot \mid \mathcal{G}] : D^{\mathcal{G}} \to D^{\mathcal{G}} \cap M^{\mathcal{G}}$ through (1) of Section 3. That this is a consistent definition is an immediate consequence of Lemma 2, as is the separability of the aggregators. The remaining properties in Definition 5 are straightforward implications of (1) and coherence. This completes the proof of the Theorem 3. ■

Proof of Corollary 1

The set X is $\mathcal{A}$-order dense in itself. ■

Proof of Corollary 2

Let $Z = \{ z_1, z_2, \dots \}$ be a countable (topologically) dense subset of X. We claim that Z is also $\mathcal{A}$-order dense. To see that, let $\mathcal{G} \in \Phi$, and suppose $x \succ^{\mathcal{G}} y$ on a non-null event $G \in \mathcal{G}$. Let $G_n = \{ x \succeq^{\mathcal{G}} z_n \succeq^{\mathcal{G}} y \} \cap G$. By continuity of $\succeq^{\mathcal{G}}$, there exists a null event N, such that $\succeq^{\mathcal{G}}_{\omega}$ is continuous for every ω in $G \backslash N$. Fix such an ω, and define the open sets $X_1 = \{ z : x \succ^{\mathcal{G}}_{\omega} z \}$ and $X_2 = \{ z : z \succ^{\mathcal{G}}_{\omega} y \}$. Since $X = X_1 \cup X_2$ is connected, X_1 and X_2 have a non-empty open intersection. It follows that there exists an act z in Z, such that $x \succeq^{\mathcal{G}}_{\omega} z \succeq^{\mathcal{G}}_{\omega} y$. Therefore, $G \subseteq (\bigcup_n G_n) \cup N$, and at least one of the $G_n \subseteq G$ is non-null. This completes the proof of Corollary 2. ■

Proof of Theorem 4

Since Φ is assumed to be countable, it suffices to prove the special case in which Φ is a singleton. The general case follows by taking the union of all the countable order-dense subsets corresponding to each of the elements of Φ. For the rest of the proof, we assume that $\Phi = \{ \mathcal{F} \}$. For any $\mathcal{G} \subseteq \mathcal{F}$, the case of $\Phi = \{ \mathcal{G} \}$ is identical to the one given here, after restricting the available events and P to $\mathcal{G}$. To simplify notation, we drop all the superscripts $\mathcal{F}$. Thus U is a $\mathcal{F}$-measurable state-contingent utility representing the state-contingent preference $\succeq = \succeq^{\mathcal{F}}$. The following is a generalization of the argument reported in Fishburn (1970), Theorem 3.1, using the result of Appendix I.

Let $\mathcal{K}$ denote the set of bounded open intervals with rational endpoints.

LEMMA 3. *For each $I \in \mathcal{K}$, there exists a sequence $\{ z^I_1, z^I_2, \dots \}$ of elements of X, and a sequence $\{ F^I_0, F^I_1, \dots \}$ of events, with the following properties:*

(a) The sets $F^I_0, F^I_1, \dots$ form a partition of Ω.

(b) $U(z^I_k) \in I$ on F^I_k; $k = 1, 2, \dots$

(c) For every $x \in X$, $P\left(\{ U(x) \in I \} \cap F^I_0\right) = 0$.

PROOF: We use the terminology of Appendix I, with a non-null event F being light if the statement of the lemma is true with F in place of Ω and with F^I_0 null. Noticing that every non-null subevent of a light event is light, it is easy to check that the assumptions of Theorem 8 are satisfied, and Ω has a black and white decomposition. Letting F_0^I be the black part of Ω the lemma follows. ■

Let $Z_1 = \{ z_k^I : I \in \mathcal{K}; k = 1,2,\dots \}$. We will require that Z contains Z_1, but this will not be enough. We now define what will be the remaining elements of Z. Let D consist of all the triples $(x,y;F)$, where $(x,y) \in X^2$ and F is a non-null event, such that $x \succ y$ on F and, for all $z \in X$, $P(\{ x \succ z \succ y \} \cap F) = 0$. The following lemma extracts a countable subset of X^2 that will be used to complete the definition of Z.

LEMMA 4. *There exists a countable set $C = \{ (x_k, y_k) : k = 1,2,\dots \} \subseteq X^2$ with the property that, for each $(x,y;F) \in D$, there exists a k such that $P(\{ x_k \sim x \succ y \sim y_k \} \cap F) > 0$.*

PROOF: We assume $D \neq \emptyset$, for otherwise the result is trivial. Given $(x,y) \in X^2$, we can apply Theorem 8 with a non-null event F being dark if $(x,y;F) \in D$, to obtain the "black and white" decomposition: $\Omega = B(x,y) \cup W(x,y)$. Consider now the finite measure space $(\Omega \times I\!R, \mathcal{F} \times \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ-algebra on $I\!R$, and $\mu = P \times l$, where l is any finite measure on $I\!R$ equivalent to Lebesgue measure (e.g. $l(dr) = (1/2)\exp(-|r|)dr$). For each $x, y \in X$ we define the set $S(x,y) = \{ (\omega, r) : \omega \in B(x,y), U(x,\omega) > r > U(y,\omega) \} \subseteq \Omega \times I\!R$. (It is not hard to verify that $S(x,y) \in \mathcal{F} \times \mathcal{B}$.) By definition of the sets $B(x,y)$, the sets $S(x,y)$ are disjoint up to μ-null sets. It follows that only at most a countable number of them can be of positive μ measure, say $S(x_1,y_1), S(x_2,y_2), \dots$ Finally, let $C = \{ (x_1,y_1), (x_2,y_2), \dots \}$ to complete the proof. ■

Let[24] $Z_2 = \{ z : (z,x) \in C \text{ for some } x \in X \}$, and define $Z = Z_1 \cup Z_2$. Clearly, Z is countable. We now show that Z is $\mathcal{A}$-order dense. Suppose $x \succ y$ on a non-null event F. If $(x,y;F) \in D$, then Lemma 4 gives a $z \in Z_2$, such that $P(\{ x \sim z \succ y \} \cap F) > 0$, implying $P(\{ x \succeq z \succeq y \} \cap F) > 0$. Otherwise, there is a $z \in X$ such that $P(\{ U(x) > U(z) > U(y) \} \cap F) > 0$. It follows that there are rationals α and β such that $P(\{ U(x) \geq \alpha > U(z) > \beta \geq U(y) \} \cap F) > 0$. But then Lemma 3 shows

24 Alternatively, we could let $Z_2 = \{ z : (x,z) \in C \text{ for some } x \in X \}$.

that there exists a k such that $P(\{ U(x) \geq \alpha > U(z_k^{(\alpha,\beta)}) > \beta \geq U(y) \} \cap F) > 0$. Since $z_k^{(\alpha,\beta)} \in Z$, we have proved that Z is $\mathcal{A}$-order dense. This completes the proof of Theorem 4. ■

Proof of Theorem 5

We first show two lemmas that relate Assumptions 1, 3, and 4.

LEMMA 5. *Assumption 3 implies solvability (Assumption 1(a)).*

PROOF: Suppose that Assumption 3 holds. We use induction on the number of events $G_1, \ldots, G_n$. For $n = 1$, solvability holds trivially. Suppose now that, given the disjoint essential events $G_1, \ldots, G_n$ and acts $x_1, \ldots, x_n$, there exists an $x \in X$ such that $x \sim^{G_i} x_i$ for $i = 1, \ldots, n-1$. Let A be the set of all such acts x, and define the sets $A_1 = A \cap \{ z : z \succeq^{G_n} x_n \}$ and $A_2 = A \cap \{ z : z \preceq^{G_n} x_n \}$. Both A_1 and A_2 are closed in A, and non-empty by assumption. Since A is connected, there exists $x \in A_1 \cap A_2$ that necessarily satisfies $x \sim^{G_i} x_i$ for all i. ■

LEMMA 6. *Assumption 4 implies Assumption 3.*

PROOF: Suppose that Assumption 4 holds, and $G_1, \ldots, G_n$ and x are as in the statement of either assumption. We form the following inductive hypothesis for $m < n$.

(H_m) : If $R_i \in \{ \succeq^{G_i}, \preceq^{G_i} \}$ for $i = m+1, \ldots, n$, then for every $y \in X$, there exists $z \in X$ such that $z \sim^{G_i} x$ for $i = 1, \ldots, m$, and $z \, R_i \, x$ for $i = m+1, \ldots, n$.

Notice that (H_0) is assumed, while (H_{n-1}), together with the connectedness assumption, gives Assumption 3. It remains to show that $(H_{m-1}) \Rightarrow (H_m)$. Let A_1 (respectively, A_2) be the set of all $z \in X$ such that $z \sim^{G_i} x$ for $i = 1, \ldots, m-1$, $z \succeq^{G_m} x$ (respectively, $z \preceq^{G_m} x$), and $z \, R_i \, z$ for $i = m+1, \ldots, n$. If (H_{m-1}) holds, then A_1 and A_2 are non-empty. Moreover, their union is connected, and each is closed relative to their union. It follows that $A_1 \cap A_2$ is non-empty, which implies (H_m). ■

We now proceed with the proof of the theorem. Since $\mathcal{G}$ is assumed finite, there is a partition $\Pi = \{ G_1, \ldots, G_n \}$, of Ω, with each G_i non-null, such that $\mathcal{G}$ is generated by Π and the null events of $\mathcal{G}$. For simplicity, we write $\succeq^i$ instead of $\succeq^{G_i}$, and analogously for strict preferences and indifferences.

We define a relation $\succeq$ on X^n, by letting

$$(x_1, \ldots, x_n) \succeq (y_1, \ldots, y_n) \quad \Leftrightarrow \quad x \succeq^{\Omega} y,$$

where x and y are any acts such that $x \sim^i x_i$ and $y \sim^i y_i$, for all i. Solvability and coherence guarantee the soundness of the definition. Because of Lemmas 5 and 6, solvability holds under any of Assumptions 1-4.

LEMMA 7. *Under any one of Assumptions 1-4, $\succeq$ satisfies the assumptions of Theorem 6.13 in Krantz, Luce, Suppes, and Tversky (1971).*

PROOF: Under Assumptions 1 or 2, we will show that $\succeq$ is continuous in the product topology of X^n, and therefore Theorem 6.14 in Krantz et. al. (1971) applies. The proof of the latter shows that the assumptions of Theorem 6.13 are then satisfied. Under Assumption 3, we will confirm the conditions of Theorem 6.13 directly. This will also cover the case of Assumption 4, because of Lemma 6. We now consider these three cases in turn.

CASE 1. Suppose that Assumption 1 holds, and that, for every i, $\{ x_i^{\gamma} \}$ is a net converging to x_i, with $(x_1^{\gamma}, \ldots, x_n^{\gamma}) \succeq (y_1 \ldots, y_n)$ for all γ. Then there exist corresponding net $\{ x^{\gamma} \}$ and $y \in X$, such that $x^{\gamma} \sim^i x_i^{\gamma}$ and $y \sim^i y_i$ for all i, and $x^{\gamma} \succeq^{\Omega} y$ for all γ. Since X is assumed compact, there is a subnet $\{ x^{\gamma(\alpha)} \}$ that converges to some $x \in X$. Since X is assumed Hausdorff, the subnet $\{ x_i^{\gamma(\alpha)} \}$ converges to x_i for every i. By continuity of each of the preferences $\succeq^i$, we then have that $x \sim^i x_i$ for all i. Continuity of $\succeq^{\Omega}$ implies that $x \succeq^{\Omega} y$. Therefore, $(x_1, \ldots, x_n) \succeq (y_1 \ldots, y_n)$. Since the same argument also applies with all orders reversed, we have shown that $\succeq$ is continuous in the product topology of X^n. It is straightforward to check that the remaining of the assumptions of Theorem 6.14 in Krantz et. al. (1971) apply.

CASE 2. Suppose that Assumption 2 holds, and that $(x_1, \ldots, x_n) \succ (y_1, \ldots, y_n)$, with $x \sim^i x_i$ and $y \sim^i y_i$ for all i, and $x \succ^{\Omega} y$. Then there exists a neighborhood N of x such that $x' \in N$ implies $x' \succ y$. Assumption 2 allows the construction of a neighborhood of $(x_1, \ldots, x_n)$ in X^n (with the product topology), all elements of which are strictly preferred to $(y_1, \ldots, y_n)$. The same argument applies with all orders reversed, showing that $\succeq$ is continuous in the product topology. The proof is then completed as in Case 1.

CASE 3. Suppose that Assumption 3 holds. In this case, we verify directly each of the assumptions of Theorem 6.13 of Krantz et. al., following an argument that parallels the proof of Theorem 6.14. The only non-obvious properties to be shown are restricted solvability, and that every bounded standard sequence is finite.

We first show restricted solvability. Suppose that

$$(x_1, \ldots, x_{n-1}, \overline{x}_n) \succeq (y_1, \ldots, y_n) \succeq (x_1, x_2, \ldots, x_{n-1}, \underline{x}_n).$$

We are to show that there exists $x_n \in X$ such that $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$. Let A be the set of all $x \in X$ such that $x \sim^i x_i$ for $i = 1, \ldots, n-1$. Also let A_1 (resp. A_2) consist of all x in A such that $x \succeq^\Omega y$ (resp. $x \preceq^\Omega y$), where $y \sim^i y_i$ for all i. Our assumptions imply that A is connected, and each of A_1 and A_2 are non-empty and closed. Therefore, $A_1 \cap A_2$ is non-empty. It is easy to confirm that any $x_n \in A_1 \cap A_2$ has the desired property. Since the numbering of the components is arbitrary, this shows restricted solvability.

Finally, we show that every strictly bounded standard sequence is finite. Let $\{ x^k : k = 1, 2, \ldots \}$ be an infinite standard sequence on component $j \in \{ 1, \ldots, n \}$. We are to show that there is no $\overline{x} \in X$ such that $\overline{x} \succeq^j x^k$ for all k. For each k, we define the open set $B^k = \{ x \in X : x^k \succ^j x \}$. We have to show that $B = \bigcup_{k=1}^{\infty} B^k$ has an empty complement. Since B is open and X is connected, it suffices to show that $\Omega \backslash B$ is open. The proof is then completed exactly as the proof of Theorem 6.14 of Krantz et. al.

This completes the proof of Lemma 7. ■

From Lemma 7 it follows that there exist functions $u_i : X \to I\!\!R$, $i \in \{ 1, \ldots, n \}$, such that for all $x_i, y_i \in X$,

$$(x_1, \ldots, x_n) \succeq (y_1, \ldots, y_n) \quad \Leftrightarrow \quad \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i). \tag{8}$$

Any other such functions, $\tilde{u}_i : X \to I\!\!R$, satisfy

$$\tilde{u}_i = \alpha u_i + \beta_i, \quad i \in \{ 1, \ldots, n \},$$

for some $\alpha > 0$ and $\beta_i \in I\!\!R$. Moreover, each u_i is continuous.

Given any probability P on $\mathcal{G}$ such that $P(G) = 0$ if and only if G is null, we define a state-contingent utility U, by letting

$$U(x, \omega) = \frac{u_i(x)}{P(G_i)}, \quad \omega \in G_i, \quad i \in \{1, \ldots, n\}, \quad x \in X,$$

and, say, $U(x, \omega) = 0$ for all ω in the remaining null set. We show that (U, P) is a continuous and unique additive representation of $\mathcal{E}$. Consider any non-null event G in $\mathcal{G}$, and let $I = \{ i : G_i \backslash G \in \mathcal{N} \}$. Given any $x, y, z \in X$, by solvability, there exist acts x' and y' such that $x' \sim^i x$ and $y' \sim^i y$ for all $i \in I$, while $x' \sim^i z \sim^i y'$ for all $i \notin I$. By coherence, and (8) it follows that

$$\begin{aligned} x \succeq^F y \quad &\Leftrightarrow \quad x' \succeq y' \\ &\Leftrightarrow \quad \sum_{i \in I} u_i(x) + \sum_{i \notin I} u_i(z) \geq \sum_{i \in I} u_i(y) + \sum_{i \notin I} u_i(z) \\ &\Leftrightarrow \quad \int_G U(x)\, dP \geq \int_G U(x)\, dP. \end{aligned}$$

Continuity and uniqueness, follow from the corresponding results about the u_i's stated above. ■

Proof of Theorem 6

Since $\mathcal{G}$ is countably generated, there exists an increasing sequence of finite algebras $\{ \mathcal{G}_n : n = 1, 2, \ldots \}$ such that $\mathcal{G} = \sigma(\bigcup_n \mathcal{G}_n)$. We assume that $\mathcal{G}_1$ contains at least three essential events. By Theorem 5, $\mathcal{E}_n = \{ \succeq^G : G \in \mathcal{G}_n \}$ has a continuous and unique additive representation (U_n, P_n), for every n. We also assume, without loss in generality, that $G \in \mathcal{N} \Rightarrow x \sim^G y$ for all $x, y \in X$. (It is easy to check that redefining $\mathcal{E}$ to satisfy this condition does not change the assumptions of the theorem, and that the conclusion of the theorem does not depend on the definition of $\mathcal{E}$ at null events.)

Since U_n is state-dependent, the measure P_n can be chosen arbitrarily, as long as $P_n(G) = 0 \Leftrightarrow G \in \mathcal{G}_n \cap \mathcal{N}$. We can then inductively define the sequence of measures $\{ P_n \}$ to be *compatible*, meaning that P_n is the restriction of P_{n+1} to $\mathcal{G}_n$. Given these probabilities, the state-contingent utilities are uniquely determined by the condition: $U_n(\underline{x}) = 0$ and $\int U_n(\overline{x})\, dP_n = 1$. We adopt this condition, thus making the sequence $\{(U_n, P_n)\}$ *compatible*, in the sense that $\{P_n\}$ is compatible, and U_n is the expectation of U_{n+1} conditional on $\mathcal{G}_n$, under measure P_{n+1}. We let $u = \int U_n \, dP_n$ be the corresponding unconditional utility (representing $\succeq^\Omega$). The function u is continuous, and does not depend on the choice of the index n used in its definition. Also $u(\overline{x}) = 1$ and $u(\underline{x}) = 0$.

Since $\overline{x} \succ^G \underline{x}$ for every non-null event G, $U_n(\overline{x})$ is strictly positive outside a null event N_n in $\mathcal{G}_n$. Let Z_n be equal to $U_n(\overline{x})$ outside N_n, and equal to 1 on N_n. We define a new additive representation (V_n, Q_n) of $\mathcal{E}_n$, by letting $dQ_n/dP_n = Z_n$ and $V_n = U_n/Z_n$. It is easy to check that $\{(V_n, Q_n)\}$ is also compatible. Moreover, for every n, we have $V_n(\overline{x}) = 1$ and $V_n(\underline{x}) = 0$, while $u = \int V_n \, dQ_n$.

Let Q^0 be the unique finitely additive measure on the algebra $\mathcal{G}^0 = \bigcup_n \mathcal{G}_n$ with the property that, for every n, Q_n is the restriction of Q^0 to $\mathcal{G}_n$. Condition (g) implies that Q^0 is also countably additive. Suppose $\{G_n\}$ is a sequence of events in $\mathcal{G}^0$ such that $G_n \downarrow \emptyset$. To show countable additivity of Q^0, it suffices to show that $Q^0(G_n) \downarrow 0$. Clearly, it suffices to prove that for a subsequence of $\{G_n\}$. To simplify notation, we assume then that the necessary subsequence of condition (g) has already been taken, and that there exists a sequence $\{x_n\}$ of acts such that $x_n \to \underline{x}$, while $x_n \sim^{G_n} \overline{x}$ for all k. Using the fact that $V_n(\overline{x}) = 1$ and $V_n(x_n) \geq 0$, we have

$$u(x_n) = \int_\Omega V_n(x_n) \, dQ_n \leq \int_{G_n} V_n(x_n) \, dQ_n = \int_{G_n} V_n(\overline{x}) \, dQ_n = Q_n(G_n) = Q^0(G_n).$$

Therefore, $0 \leq Q^0(G_n) \leq u(x_n) \to u(\underline{x}) = 0$, as $n \to \infty$, proving countable additivity of Q^0.

By Caratheodory's extension theorem (see Halmos (1950), Theorem 13.A), Q^0 extends to a unique probability Q on $\mathcal{G}$. Compatibility of $\{(V_n, Q_n)\}$ implies that $\{(V_n(x), \mathcal{G}_n) : n = 1, 2, \ldots\}$ is a bounded (taking values between zero and one) Q-martingale, for every act x. By the martingale convergence theorem (see Billingsley (1986) or Dudley (1989)), for every act x, $\{V_n(x)\}$ converges almost surely to a random variable $V(x)$, valued in $[0, 1]$, and $V_n(x) = E^Q[V(x) \,|\, \mathcal{G}_n]$, where E^Q denotes expectation under measure Q.

We claim that for all $G \in \mathcal{G}$,

$$x \succeq^G y \Leftrightarrow \int_G V(x)\, dQ \geq \int_G V(y)\, dQ, \quad x, y \in X. \tag{9}$$

This follows by a standard monotone class argument (see Halmos (1950), Theorem 6B). By condition (e), the class $\mathcal{M}$ of all $G \in \mathcal{G}$ for which (9) holds is a monotone class and contains the algebra $\mathcal{G}^0$. By the monotone class theorem, $\mathcal{M} = \mathcal{G}$, and (9) holds for all $G \in \mathcal{G}$. The probability Q represents the null events in $\mathcal{G}$, because by (9) and the assumption that $x \sim^G y$ for all null G in $\mathcal{G}$, $Q(G) = 0 \Leftrightarrow \overline{x} \sim^G \underline{x} \Leftrightarrow G \in \mathcal{N} \cap \mathcal{G}$. We have therefore shown that (V, Q) is an additive representation of $\mathcal{E}$.

To show that (V, Q) is a unique additive representation, suppose that $(\tilde{V}, Q)$ is also an additive representation of $\mathcal{E}$. Since (V_n, Q_n) is a unique additive representation of $\mathcal{E}_n$, for every n, there exists a positive constant α_n and a random variable β_n such that $E^Q[\, \tilde{V} \mid \mathcal{G}_n\,] = \alpha_n E^Q[\, V \mid \mathcal{G}_n\,] + \beta_n$. Using the equalities $V(\overline{x}) = 1$ and $V(\underline{x}) = 0$, we obtain $\alpha_n = E^Q(\tilde{V}(\overline{x}) - \tilde{V}(\underline{x}))$ and $\beta_n = E^Q[\, \tilde{V}(\underline{x}) \mid \mathcal{G}_n\,]$. Letting $n \to \infty$, and using the martingale convergence theorem, we obtain that $\tilde{V} = \alpha V + \beta$, where $\alpha = \alpha_n$ and $\beta = \tilde{V}(\underline{x})$.

Next we show that (V, Q) is continuous. Given any event G in $\mathcal{G}$, let $\mathcal{H} \subseteq \mathcal{G}$ be any finite algebra containing G. Then, by Theorem 5 and this proof, the restriction of $\mathcal{E}$ on $\mathcal{H}$ has a continuous and unique additive representation $(V^{\mathcal{H}}, Q^{\mathcal{H}})$, where $V^{\mathcal{H}}(x) = E^Q[\, V(x) \mid \mathcal{H}\,]$ for all $x \in X$, and $Q^{\mathcal{H}}$ is the restriction of Q on $\mathcal{H}$. Therefore $\int_G V\, dQ = \int_G V^{\mathcal{H}}\, dQ^{\mathcal{H}}$ is continuous.

Finally, V and Q are uniquely determined by the requirement that (V, Q) is a unique additive representation of $\mathcal{E}$, and $V(\overline{x}) = 1 = 1 - V(\underline{x})$. To see that, suppose that (U, P) also satisfies these conditions. Then $(U(dP/dQ), Q)$ is also an additive representation of $\mathcal{E}$, and therefore $V = \alpha U(dP/dQ) + \beta$, for $\alpha > 0$ and random variable β. Evaluating at $\underline{x}$ and $\overline{x}$, one easily obtains that $V = U$ and $(dP/dQ) = 1$.
■

Proof of Theorem 7

Theorem 6 (with $\mathcal{G} = \mathcal{F}$) implies the existence of a continuous and unique additive representation (V, Q) of $\mathcal{E}$ such that $V(\overline{x}) = 1$ and $V(\underline{x}) = 0$. (To show that condition (g) holds, let x_n be equal to $\overline{x}$ on G_n and equal to $\underline{x}$ on G_n^c.) An additive representation (U, P) of $\mathcal{E}$ is then obtained by letting $U = (dQ/dP)V$. In fact, it is easy to check (using Theorem 30B in Halmos (1974)) that all the assumptions and conclusions of the theorem remain valid if P is replaced by the equivalent measure Q. Without loss of generality, we therefore assume that $U = V$ and $P = Q$. In particular, $U(\overline{x}) = 1$, $U(\underline{x}) = 0$, and $I = [0, 1]$.

Let X^0 consist of the acts that are of the form $x = \sum_{n=1}^{\infty} c_n 1_{F_n}$, where $c_n \in C$ for all n, and $\{ F_1, F_2, \dots \} \subseteq \mathcal{F} \backslash \mathcal{N}$ is a countable partition of Ω. A standard exercise shows that X^0 is dense in X. Our plan is to first show the result on the space X^0, and then use continuity of (U, P) and Assumption 5(e) to extend it to the whole of X. We begin with a preliminary lemma.

LEMMA 8. *Given any $z \in X$ (respectively, X^0) and $v \in (0, 1)$, there exists a countable partition of Ω, $\{ F_1^{z,v}, F_2^{z,v}, \dots \}$, and a sequence $\{ x_1^{z,v}, x_2^{z,v}, \dots \} \subseteq X$ (respectively, X^0), such that $u(x_n^{z,v}) = v$ and $x_n^{z,v} = z$ on $F_n^{z,v}$ for every n.*

PROOF: We adopt the terminology and result of Appendix I. Given the pair (z, v), let a non-null event F be light if the statement of the lemma is true with F in place of Ω, and dark otherwise. It is easy to check that countable disjoint unions of light (respectively, dark) events are light (respectively, dark). Theorem 8 then gives a "black and white" decomposition of Ω. We will show that no event is black, and therefore Ω is white, which proves the lemma.

Let F be any non-null event. We will show that there exists act $x \in X$ (respectively, in X^0) such that $u(x) = v$ and $\{ x = z \} \cap F$ is non-null, implying that F is not black. Since the probability space $(\Omega, \mathcal{F}, P)$ is assumed to be non-atomic, we can construct a decreasing sequence $\{ F_n \}$ of non-null subevents of F with a null intersection (see, for example, Exercise 2.17 of Billingsley (1986)). Given such a sequence, define $\overline{x}_n = z 1_{F_n} + \overline{x} 1_{F_n^c}$ and $\underline{x}_n = z 1_{F_n} + \underline{x} 1_{F_n^c}$. Since $\{ \overline{x}_n \}$ ($\{ \underline{x}_n \}$) converges to $\overline{x}$ ($\underline{x}$), and u is continuous with $u(\overline{x}) = 1$ ($u(\underline{x}) = 0$), it follows that for some n, $u(\overline{x}_n) > v > u(\underline{x}_n)$. Fixing this n, and for every $\alpha \in (0, 1)$, let $x^\alpha = \alpha \overline{x}_n + (1 - \alpha) \underline{x}_n$. Then $u(x^\alpha)$ varies continuously with α, and achieves values both above and below v. Therefore, for some α, $x = x^\alpha$ has the required properties. Notice that if z is in X^0, so is x^α. ■

We define the function $f : \Omega \times C \times I \to \mathbb{R}$ as follows. First, we let $f(\omega, c, 1) = 1$

and $f(\omega, c, 0) = 0$ for all $(\omega, c) \in \Omega \times C$. For any given $(c, v) \in C \times (0, 1)$, we choose events $F_n^{c,v}$ and acts $x_n^{c,v}$, $n = 1, 2, \ldots$, as in Lemma 8, where we have abused notation in denoting by c the act identically equal to c. We then define $f(\omega, c, v) = U(\omega, x_n^{c,v})$ for all $\omega \in F_n^{c,v}$. With f so defined, we claim that $U(x) = f(x, u(x))$ for all $x \in X^0$. To show that, suppose that $x \in X^0$, $x = c \in C$ on some non-null event F, and $u(x) = v$. Then $F = \bigcup_n F_n$, where $F_n = F_n^{c,v} \cap F$ and the $F_n^{c,v}$ are the events used in the definition of f. For any given n, we have that $x = x_n^{c,v} = c$ on F_n, while $u(x) = u(x_n^{c,v}) = v$. By Assumption 5(e) and the definition of f, it follows that $U(x) = U(x_n^{c,v}) = f(c, v) = f(x, u(x))$ on F_n. Since this holds for every n, we have shown that $U(x) = f(x, u(x))$ on F. Since $x \in X^0$, this argument need only be applied on countably many such events F.

We have proved that $x \in X^0$ implies $U(x) = f(x, u(x))$. From this fact and Assumption 5(e), it follows that f is nondecreasing in its second (consumption) argument, and nonincreasing in its third (utility) argument. It is also immediate that f is strictly increasing in its utility argument if (4) holds, and not dependent on the utility argument if (2) holds. Finally, we use a continuity argument to show that $U(x) = f(x, u(x))$ for all $x \in X$.

Let x be any element of X. We define the events $\Omega_1 = \{ U(x) = f(x, u(x)) \}$, $\Omega_2 = \{ U(x) < f(x, u(x)) \}$, and $\Omega_3 = \{ U(x) > f(x, u(x)) \}$. We are to show that Ω_2 and Ω_3 are null. Suppose that Ω_2 is non-null. As in the proof of Lemma 8, the non-atomicity of $(\Omega, \mathcal{F}, P)$ implies the existence a non-null event $F \subseteq \Omega_2$ such that $u(\overline{x} 1_F + \underline{x} 1_{F^c}) \leq u(x)$. For such an event F, we will show that $\int_F U(x)\, dP \geq \int_F f(x, u(x))\, dP$, a contradiction.

Since x is measurable and bounded, there exists a decreasing sequence $\{ x_n \}$ in X^0 such that $x_n \downarrow x$ almost surely as $n \to \infty$.

LEMMA 9. *There exists a sequence $\{ \tilde{x}_n \}$ in X^0, converging to an act $\tilde{x}$, such that, for all n, $\tilde{x}_n = x_n$ on F and $u(\tilde{x}_n) = u(x)$. Consequently, $\tilde{x} = x$ on F and $u(\tilde{x}) = u(x)$.*

PROOF: For every $\alpha \in [0, 1]$, we define $x^\alpha = \alpha \overline{x} + (1 - \alpha) \underline{x}$ and $x_n^\alpha = x_n 1_F + x^\alpha 1_{F^c}$. By Assumption 5(e), u is nondecreasing. Therefore, $u(x_n^1) \geq u(x)$, since $x_n^1 \geq x$, and $u(x_n^0) \leq u(\overline{x} 1_F + \underline{x} 1_{F^c}) \leq u(x)$, since $x_n^0 \leq \overline{x} 1_F + \underline{x} 1_{F^c}$ and the last inequality holds by the choice of F. Since $u(x_n^\alpha)$ varies continuously in α, we conclude that the set $A_n = \{\, \alpha : u(x_n^\alpha) = u(x) \,\}$ is non-empty and closed, for every n. Let $\alpha_n = \max A_n$ and $\tilde{x}_n = x_n^{\alpha_n}$. We now argue that the sequence $\{\, \alpha_n \,\}$ is nondecreasing. Since $\{\, x_n \,\}$ is decreasing, $x_{n+1}^{\alpha_n} \leq \tilde{x}_n$, and therefore $u(x_{n+1}^{\alpha_n}) \leq u(\tilde{x}_n) = u(x)$. On the other hand, $u(x_{n+1}^1) \geq u(x)$, and therefore α_{n+1} is to be found in the interval $[\alpha_n, 1]$. This shows that $\{\, \alpha_n \,\}$ is nondecreasing and therefore has a limit $\tilde{\alpha}$. Letting $\tilde{x} = x 1_F + x^{\tilde{\alpha}} 1_{F^c}$, the result follows. ■

Lemma 9 and the monotonicity of f in consumption imply that

$$\int_F f(x, u(x))\, dP \leq \int_F f(x_n, u(x))\, dP = \int_F f(\tilde{x}_n, u(\tilde{x}_n))\, dP = \int_F U(\tilde{x}_n)\, dP.$$

(The last equality holds because $\tilde{x}_n \in X^0$.) Letting $n \to \infty$, we obtain

$$\int_F f(x, u(x))\, dP \leq \int_F U(\tilde{x})\, dP = \int U(x)\, dP,$$

where we have used the continuity of (U, P), and Assumption 5(e). This inequality contradicts the fact that F is a non-null subevent of Ω_2, and shows that Ω_2 must be null. A symmetric argument shows that Ω_3 is null, and the proof of Theorem 7 is complete. (The uniqueness part was argued in Section 5.1.) ■

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