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A PROOF OF THE EXISTENCE OF SPECULATIVE EQUILIBRIA[*]

by

Matthew O. Jackson[**]

August 1992

[*]I benefitted from the comments of participants at the Hitotsubashi University International Symposium on Resource Allocations and Capital Accumulation in Market Economies, as well as seminars at Indiana University and the University of Western Ontario. I am very grateful to Jim Peck for many helpful conversations.

[**]Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208-2001, USA.

A Proof of the Existence of Speculative Equilibria*

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Abstract: The existence of speculative equilibria is proven in a simple overlapping generations, infinite horizon economy. In equilibrium, all agents bid for assets according to increasing functions of private information which is uncorrelated with the fundamental value (dividend stream) of the asset. This is a unique best response to the strategies of the other agents, which implies that speculative information is valuable.

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1. Introduction

A simple overlapping generations economy is examined, in which individuals can buy a unit of an asset when they are young, and then sell it when they are old. The asset pays a dividend each period to the current holder of the asset. Individuals have symmetric information about dividends and also observe private signals which are uncorrelated with the dividend stream. The private signals are, however, correlated within and across generations.

In the same model, Jackson and Peck (1991) demonstrated the existence of equilibria in which agents bid based on private signals, but only for a finite number of periods; with the remaining agents ignoring their signals. In these equilibria, the behavior of each generation depends on their anticipations of the behavior of the next generation. Since the remaining agents are indifferent between bidding and not bidding, these equilibria are somewhat tenuous. The fact that such equilibria can be constructed so that agents bid based on speculative information for an arbitrary number of periods, would lead one to conjecture that equilibria exist where all agents of every generation bid based on speculative information. This note confirms that conjecture. It is shown that there exist speculative equilibria in which all agents of every generation bid based on their private and non-fundamental information. These equilibria are strict Nash equilibria, so that every agent's unique best response is to bid according to speculative information. The implication is that speculative information is valuable and agents will expend resources developing such information.

The rationality of trading based on non-fundamental information is not new. This idea was discussed by Keynes (1936), among others, and has recently been explored in some detail by the sunspots literature.[1] The model discussed here extends the previous literature, since it decentralizes the information process. Rather than requiring that all agents observe the same "sunspot", agents have their own perceptions of market psychology.[2] This speculative information is valuable to the agents, since it is correlated with the actions of the other agents. As a result, equilibrium prices are increasing over time, even when agents are risk neutral and there is no fundamental value to the asset. The equilibrium prices are also

[1] An incomplete list includes Shell (1977), Azariadias (1981), Cass and Shell (1983), Peck (1988), Spear (1989), Peck and Shell (1991).

[2] More detailed discussions of the relation of this model to the existing literature, as well as illustrating examples, are given in Jackson and Peck (1991) and Jackson (1992).

more volatile than the underlying dividends. Thus the equilibria offer some insight into the equity premium and excess volatility puzzles.

2. The Economy

The following overlapping generations economy is borrowed from Jackson and Peck (1991). A new generation of agents is born at each time $t \in \{0, 1, 2, 3, \ldots\}$. Each generation consists of a finite number n of individuals who live for two periods. There is a single consumption good which agents consume in the second period of their life. Each individual born at time t is endowed with e_t units of the consumption good. Individuals' have von Neumann–Morgenstern preferences for consumption, which are represented by an increasing and concave utility function U_t.[3]

There are k ($n > k > 0$) indivisible, infinitely lived assets which pay dividends (of the consumption good) each period. The dividend paid to the holder of an asset at time t is denoted d_t. Assets are initially held by agents born at time $t = 0$. The first opportunity for trade occurs at time $t = 1$. Markets are incomplete. The only market which exists is for the trade of the infinitely lived asset. At each time t, agents in the second period of their life (those born at time $t - 1$) may sell their assets to agents in the first period of their life (those born at time t). An agent of generation t who does not purchase an asset has final consumption

$$e_t(1 + r),$$

where $r \geq 0$ is a rate of return on stored consumption goods. An agent of generation t who purchases an asset at time t and sells it at time $t + 1$ has final consumption

$$(e_t - p_t)(1 + r) + d_{t+1} + p_{t+1}.$$

Period t begins with the birth of new agents and the death of old agents. Next, current holders of the infinitely lived assets receive their dividends d_t. After dividends have been paid, the sale of assets takes place. The agents of generation $t - 1$ who hold assets can sell them to agents of generation t. Period t ends with agents of generation $t - 1$ consuming, and the agents of generation t storing their goods and assets.

[3] Since symmetric equilibria are analyzed, it is important that agents of the same generation have the same utility function.

Trade

Assets are sold through a Vickrey auction. Young agents simultaneously submit bids $0 \leq b_t^i \leq e_t$. Each of the k highest bidders obtain one of the k assets. [Ties are broken according some fixed rule. The particular rule is unimportant since ties will occur with zero probability.] The winning bidders pay the same price, which is the value of the $k+1$–th highest bid. Each of the old agents who held a certificate receives the price.

As discussed in Jackson and Peck (1991) [see also Milgrom (1981)], the choice of a Vickrey auction makes it easy to analyze price formation and the value of speculative information. This follows from the competitive features of the Vickrey auction. The k winning bidders pay a price equal to the $k+1$–th highest bid, and thus the winning bidders cannot affect the price without losing the asset. The agent who had the $k+1$–th highest bid has no incentive to manipulate the price since he or she did not purchase an asset.

Information

Agents of each generation have symmetric information about the dividends. Agents see the history of prices up to the present. In addition, each agent i receives a real–valued private signal s_t^i. These private signals are random variables defined on a common probability space (Ω, F, μ). The signals are independent of the information about dividends.

The following notation will be useful: s_t^* denotes the $k+1$–th highest signal at time t, y_t^i denotes the k–th highest signal among the set $\{s_t^j\}_{j \neq i}$, and $h_t = (s_1^*, s_2^*, \ldots, s_{t-1}^*)$ is the history of s^*'s. Let $\mu(s_{t+1}^* \mid s_t^i = s,\ y_t^i = y,\ h_t = h)$ be a version of the conditional probability of s_t^*, given agent i of generation t's signal, the order statistic of other agents' signals, and the history of s^*. Since this is only one version of the conditional probability, all statements which follow should be read as holding almost always, even though this notation is suppressed.

The following conditions assure the existence of speculative equilibria.

(C1) For all t, i, and h, $\mu(s_{t+1}^* \mid s_t^i = s,\ y_t^i = y,\ h_t = h)$ first order stochastic dominates $\mu(s_{t+1}^* \mid s_t^i = \widetilde{s},\ y_t^i = \widetilde{y},\ h_t = \widetilde{h})$ whenever $s \geq \widetilde{s}$, $y \geq \widetilde{y}$, and $h \geq \widetilde{h}$. This dominance is strict when $s > \widetilde{s}$.

(C2) $\mu(s^*_{t+1} \mid s^i_t = s,\ y^i_t = y,\ h_t = h) = \mu(s^*_{t+1} \mid s^j_t = s,\ y^j_t = y,\ h_t = h)$ for all i, j, s, y, and h.

(C3) $\mu(y^i_t = s \mid s^i_t = s,\ h_t = h) = 0$ for all t, i, s, and h.

(C4) $D_t = E_t[\sum_{\tau=t+1}^{\infty} d_\tau/(1+r)^{\tau-t}]$ is well defined for all t. Endowments grow so that $e_{t+1} \geq (1+r)e_t$ for all t. And, $e_1 > D_1$.

(C1) states that signals are correlated across generations and that higher signals are "good news" about future signals. (C2) states that agents have comparable information. Although they may obtain different realizations of signals, they are equivalent in their precision. This assumption is made for technical convenience. It permits us to concentrate on finding an equilibrium which is symmetric among agents of a given generation. (C3) that the probability of two agents observing exactly the same signal is zero. This assumption is also made for technical convenience, since the analysis of an auction is substantially more complicated when ties are possible. (C4) assures that agents can afford to bid the fundamental value of the assets.

3. The Existence of Speculative Equilibria

PROPOSITION. *If (C1)–(C4) are satisfied and agents of each generation observe the history of past prices, then there exists an equilibrium of the infinite horizon economy in which all agents bid according to an increasing function of their signals. Furthermore, if the support of an agent's signal is connected, then the equilibrium bidding strategy is the unique best response to the strategies of the other agents.*

The last statement implies that the equilibrium is strict. If other agents are bidding according to their private signals, then an agent's only choice is to do the same. An agent loses out by ignoring private information: even though this information is unrelated to the fundamental structure of the economy, it is correlated with the equilibrium actions of the other agents. The equilibrium will also be immune to refinements of the Nash equilibrium concept. Since the equilibrium is strict, it is also perfect, proper, etc.

The existence result in Jackson and Peck (1991) relied on showing that if agents of some generation t bid according to increasing functions of their signals, then it is best for agents of generation $t-1$ to do so as well. With such reasoning, an equilibrium can be established by working backwards from any T. However, one is not sure to be able to find bids which depend on signals for times past T which support the particular bidding functions at T. Here a different method of proof is taken which does not rely on on a change in behavior at any time. Bidding functions are identified for time $t+1$ which are functions of signals and which support bidding functions at time t, but which have the same functional form. With similar functional forms at each time, it is possible to string together an equilibrium for the infinite horizon.

Although the bidding functions of each generation have the same functional form, the equilibria not "stationary". This follows from the fact that agents will need to pay attention to the information revealed by the history of prices when they form their bids. These histories are of different lengths for different generations. Thus the equilibria involve agents going through similar, but not indentical types of calculations.[4]

PROOF: Let P_t denote the history of prices $\{p_1, p_2, \ldots, p_t\}$. Consider the bids defined below.

$$b_1(s_1^i) = a_1 s_1^i - f_1 \tag{1}$$

$$b_t(s_t^i, P_{t-1}) = a_t s_t^i - f_t(P_{t-1}), \tag{2}$$

where each scalar a_t is positive. Before defining the function f_t which appears in (2) above (f_1 is a scalar), it is helpful to define the following functions. If agents bid according to (1) and (2), then the price setting signal ($k+1$-th highest signal) of any time t can be inferred from the history of prices through time t. This is represented by $s_t(P_t)$ where

$$s_t(P_t) = [p_t + f_t(P_{t-1})]/a_t.$$

The history of price setting signals up to time t can be inferred from the price history

$$h_t(P_{t-1}) = \{s_1(P_1), s_2(P_2), \ldots, s_{t-1}(P_{t-1})\}.$$

The function $f_t(P_{t-1})$ [for $t > 1$] can now be defined to be the value of f which solves

$$E_{t-1}[U_{t-1}((e_{t-1} - p_{t-1})(1+r) + d_t + a_t s_t^* - f) \mid h_{t-1} = h_{t-1}(P_{t-2}),$$

$$s_{t-1}^i = y_{t-1}^i = s_{t-1}(P_{t-1})] = U_{t-1}(e_{t-1}(1+r)). \tag{3}$$

Notice that f_t is well defined, even though it appears in the definition of $s_t(P_t)$ above. This follows since the definition of f_t depends only on $s_{t-1}(P_{t-1})$ and $h_{t-1}(P_{t-2})$. To be

[4] There is also the technical point of assuring that bids are always feasible. Thus depending on the signal structure, parameters may have to be chosen to scale bids to be between 0 and an agent's endowment. For instance, if the size of signals grows faster than endowments it is not possible to bid the same constant times the signal each period.

careful about the order definition, first f_1 is chosen, then $s_1(P_1)$ is defined, then $f_2(P_1)$, then $s_2(P_2)$, then $f_3(P_2)$, etc.

First it is shown that bids defined according to (1) and (2) form a (strict) equilibrium. The proof is then completed by showing that the factors f_1 and $\{a_1, a_2, \ldots\}$ can be chosen so that $0 \leq b_t(s_t^i, P_{t-1}) \leq e_t$ for all t and possible realizations of s_t^i and P_{t-1}.

Consider an agent i of generation $t-1$ who bids according to (2) when all other agents of generation $t-1$ and t also bid according to (2). It is shown below that even an agent who could condition on the value of y_{t-1}^i would not wish to change bids. This implies that an agent who could not condition on these values would not wish to change bids.

If $s_{t-1}^i = s < y = y_{t-1}^i$, then the agent will not get an asset, and so the agent's utility is $U_{t-1}(e_{t-1}(1+r))$. Any strategy which results in a different outcome must end up purchasing an asset. Such an alternative strategy will purchase the asset at a price of $p'_{t-1} = b(y, P_{t-2})$. Conditional on $y_{t-1}^i = y$ the agent's expected utility is

$$E_{t-1}[U_{t-1}((e_{t-1} - p'_{t-1})(1+r) + d_t + a_t s_t^* - f_t(P'_{t-1})) \mid h_{t-1} = h_{t-1}(P_{t-2}),$$

$$s_{t-1}^i = s,\ y_{t-1}^i = y], \tag{4}$$

where $P'_{t-1} = \{P_{t-2}, p'_{t-1}\}$. Since s is less than y, it follows from (C1) that this expected utility is less than

$$E_{t-1}[U_{t-1}((e_{t-1} - p'_{t-1})(1+r) + d_t + a_t s_t^* - f_t(P'_{t-1})) \mid h_{t-1} = h_{t-1}(P_{t-2}),$$

$$s_{t-1}^i = y_{t-1}^i = y]. \tag{5}$$

By the definition of f_t in (3), it follows that (5) is equal to $U_{t-1}(e_{t-1}(1+r))$. Thus (4) is less than $U_{t-1}(e_{t-1}(1+r))$ for any realization of s which is less than y. This means that any alternative strategy which wins an asset in such a situation results in an expected utility less than $U_{t-1}(e_{t-1}(1+r))$, the utility associated with (2). Thus the strategy (2) results in the highest possible utility level when $s_t^i = s < y = y_t^i$.

If $s_{t-1}^i = s > y = y_{t-1}^i$ then by following the bidding strategy (2) the agent will get an asset. Conditional on $y_t^i = y$, the agent's expected utility is

$$E_{t-1}[U_{t-1}((e_{t-1} - p_{t-1})(1+r) + d_t + a_t s_t^* - f_t(P_{t-1})) \mid h_{t-1} = h_{t-1}(P_{t-2}),$$

$$s_{t-1}^i = s,\ y_{t-1}^i = y]. \tag{6}$$

Since $s > y$, it follows from (C1) that (6) is greater than

$$E_{t-1}[U_{t-1}((e_{t-1} - p_{t-1})(1+r) + d_t + a_t s_t^* - f_t(P_{t-1})) \mid h_{t-1} = h_{t-1}(P_{t-2}),$$

$$s_{t-1}^i = y_{t-1}^i = y]. \tag{7}$$

From (3) it follows that (7) is equal to $U_{t-1}(e_{t-1}(1+r))$. This means that conditional on having won the asset, the agent's expected utility is greater than $U_{t-1}(e_{t-1}(1+r))$. Thus the strategy (2) does better than any alternative which does not obtain an asset in this situation.

By (C3) the case $s_{t-1}^i = s = y_{t-1}^i$ occurs with zero probability, and does not matter in evaluating (2) as a strategy.

It has been shown that even if the agent were able to condition on the value of y_t^i, the agent could not do better than the strategy suggested by (2). In fact any strategy which does not result in the same outcome as (2) for every s_t^i and y_t^i, results in a lower utility in some cases. Thus (2) is a best reply for the agent. If the support of s_t^i is connected, then any alternative strategy would result in a different outcome for some realization of s_t^i and y_t^i, and so (2) is a strict best response.

To complete the proof it must be shown that the factors f_1 and $\{a_1, a_2, \ldots\}$ can be chosen so that $0 \leq b_t(s_t^i, P_{t-1}) \leq e_t$ for all t and possible realizations of s_t^i and P_{t-1}.

Without loss of generality, assume that the support of every s_t^i is a subset of (0,1). [5] Let $\delta = e_1 - D_1$. By (C4), $\delta > 0$. Let $f_1 = D_1 + \frac{\delta}{4}$ and let $a_t = \delta/2^{t+1}$. This assures that for all s_1^i and P_0

$$D_1 + \frac{\delta}{4} \leq b_1(s_1^i, P_0) \leq D_1 + \frac{\delta}{2} \tag{8}$$

Next it is shown that, for all s_t^i and P_{t-1}, if

$$D_t + (1+r)^{t-1}\frac{\delta}{2^{t+1}} \leq b_t(s_t^i, P_{t-1}) \leq (1+r)^{t-1}(D_1 + \frac{\delta}{2}) + \sum_{\tau=2}^{t}(1+r)^{t-\tau}\frac{\delta}{2^{\tau+1}} \tag{9}$$

then

$$D_{t+1} + (1+r)^t\frac{\delta}{2^{t+2}} \leq b_{t+1}(s_{t+1}^i, P_t) \leq (1+r)^t(D_1 + \frac{\delta}{2}) + \sum_{\tau=2}^{t+1}(1+r)^{t+1-\tau}\frac{\delta}{2^{\tau+1}}. \tag{10}$$

By Jensen's inequality and the definition of f_t from (3), it follows that for all P_t

$$-p_{t-1}(1+r) \leq f_{t+1}(P_t) \leq a_{t+1} + E_t(d_{t+1}) - p_{t-1}(1+r).$$

Then from (2), it follows that

$$(1+r)p_t - a_{t+1} - E_t(d_{t+1}) \leq b_{t+1} \leq (1+r)p_t + a_{t+1}. \tag{11}$$

Substituting in from (9) for p_t and substituting for a_{t+1}, (11) becomes

$$\begin{aligned}(1+r)(D_t + (1+r)^{t-1}\frac{\delta}{2^{t+1}}) - \frac{\delta}{2^{t+2}} - E_t(d_{t+1}) \leq b_{t+1} \\ \leq (1+r)[(1+r)^{t-1}(D_1 + \frac{\delta}{2}) + \sum_{\tau=2}^{t}(1+r)^{t-\tau}\frac{\delta}{2^{\tau+1}}] + \frac{\delta}{2^{t+2}}.\end{aligned} \tag{12}$$

Noting that $(1+r)D_t - E_t(d_{t+1}) = D_{t+1}$, (12) simplifies to (10). By induction, since (8) holds and since (9) implies (10), it follows that that (9) holds for all t. Notice that the left hand side of (9) is at least 0, and the right hand side of (9) is no larger than $(1+r)^{t-1}(D_1+\delta)$. Since $D_1 + \delta = e_1$, the right hand side of (9) is no larger than $(1+r)^{t-1}e_1$, which by (C4) is no more than e_t. Thus it has been shown that $0 \leq b_t(s_t^i, P_{t-1}) \leq e_t$ for all t, s_t^i, and P_{t-1}. ∎

[5] This assumption is without loss of generality since the real line is homeomorphic to (0,1). To work without this assumption, simply replace the s_t^i on the right hand side of equations (1) and (2) by $1/2 + s_t^i/2\sqrt{1+(s_t^i)^2}$.

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