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# ADJUSTMENT DYNAMICS AND RATIONAL PLAY IN GAMES*

by

Jeroen M. Swinkels

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# Adjustment Dynamics and Rational Play in Games*

JEROEN M. SWINKELS

*MEDS, Kellogg Graduate School of Management, Northwestern University*
and
*Department of Economics, Stanford University*
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When a given strategic situation arises repeatedly, the possibility arises that equilibrium predictions can be justified by a dynamic adjustment process. We examine *myopic adjustment dynamics*, a class that includes replicator dynamics from evolutionary game theory, simple models of imitation, models of experimentation and adjustment, and some simple learning dynamics. We present a series of theorems showing conditions under which behavior that is asymptotically stable under some such dynamic is strategically stable (Kohlberg and Mertens [1986]). This behavior is thus *as if* the agents in the economy satisfied the extremely stringent assumptions that game theory traditionally makes about rationality and beliefs.

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## I. INTRODUCTION

There is growing skepticism that sophisticated strategic behavior - satisfying, for example, sequential equilibrium or forward induction - is the natural end product of introspection by economic agents. Why, and under what circumstances, should we then believe in equilibrium and equilibrium refinements?

Many strategic situations of interest arise repeatedly. In some cases, fixed players will repeatedly find themselves in the same strategic situation, as, for example, competing firms. In other cases, a given strategic situation arises repeatedly amongst sets of anonymous players drawn at random from a large population. The interaction of drivers on the road seems a good example. Other cases fall between. While a lawyer preparing for a trial may never have faced quite the same legal situation before, an extensive record of similar past trials is available. Individuals preparing to negotiate the purchase of a new car have friends, various consumer publications, and their experience in other bargaining situations to guide their behavior. In all these situations, one might think of a process in which behavior adjusts over time based on the experience of participants.

In this paper, we try to understand when these processes will have implications for *as–if–rational* play. That is, when does stability under an adjustment process imply behavior that is *as if* the agents in the economy satisfied the stringent assumptions that game theory traditionally makes about rationality and congruence of beliefs?

Models of adjusting play have been extensively studied in both the learning and evolutionary game theory literatures. Some of these models are explicitly dynamic. Others, while at least partly based on intuitions about dynamic adjustment, are formulated in a static way.

The literature on static formulations had its genesis in the application of evolutionary ideas to game theory by Maynard Smith [1974,1982] and Maynard Smith and Price [1973]. They argued that many interactions in the natural world could be interpreted as strategic situations, and that mutation and natural selection would tend to push organisms toward optimal play. For some economic questions - in explaining altruistic behavior or tastes for example - a literal interpretation of these ideas from evolutionary biology may make sense. However, much more generally, both 'mutation' and 'natural selection' have close analogues in economic environments. In many situations there will be a general movement over time to strategies that perform well in their environment, whether by imitation, by the growth or bankruptcy of firms following superior or inferior strategies, or by a learning process.

An evolutionarily stable strategy profile is one that has the property that members of any small group of entrants to a population playing a strategy different from the status quo fare worse against the

post entry population than do individuals using the original strategy. As such, it attempts to capture in a static way the notion of stability of behavior in a population when a small mutation is followed by natural selection.[1]

The static approach implies a remarkable amount of as–if–rationality: In particular, van Damme [1991] shows that an evolutionarily stable strategy is proper (Myerson [1978]). Van Damme [1984] also shows that a proper equilibrium is sequential (Kreps and Wilson [1982]) in associated extensive forms. Thus, an evolutionarily stable strategy profile corresponds to a sequential equilibrium in associated extensive forms.

Swinkels ([1991a],[1991b]) extends this result, working with a much weaker static notion and deriving stronger implications. In that work, the entrants against which the status quo is tested are restricted to those that are best responses to the post entry environment. Solutions are allowed to be set valued. Such a set is called *equilibrium evolutionarily stable (EES)*. EES sets are robust to the iterative removal of weakly dominated strategies, satisfy the never a weak best response property (Kohlberg and Mertens [1986]), and depend only on the reduced normal form. Under some additional conditions (which are always satisfied for EES sets with a single element, and are generically true for EES sets for two person extensive form games) an EES set contains a stable component in the sense of both Kohlberg and Mertens [1986] and Hillas [1990].[2] For generic extensive form games, EES sets will correspond to a single outcome, with different elements reflecting different out of equilibrium behaviors. Because Hillas stable sets contain a proper element, this outcome is sequential.[3]

Of course, our intuitions about evolution and learning are largely about dynamics. And, indeed, these ideas have been intensively explored in explicitly dynamic frameworks (see for example Taylor and Jonker [1978], Friedman [1991], Foster and Young [1990] and Kandori, Mailath and Rob [1992]). In these models, there is typically a large population (or populations) from which sets of players, one for each player position, are randomly and repeatedly drawn to play the game. Players change their behavior over time based on their experiences, and perhaps some random factor. The state variable is typically the proportion of the population playing each pure strategy, i.e., the *population strategy profile*. This evolves in either a deterministic or stochastic fashion.

Particularly relevant for this paper is the replicator dynamic. For this dynamic, the proportionate rate of growth of the proportion of the population playing each pure strategy is linear in the difference between the payoffs to that pure strategy and the current average payoff within the population. The replicator dynamic arises naturally in biological games, where one interprets payoffs as the number of offspring. It also has some intuitive appeal as a model of imitation in an economic environment.

Friedman [1991] introduces *weak compatible dynamics*, which can be thought of as generalized replicator dynamics. For these dynamics, changes in behavior at any instant can be interpreted as reflecting attempts by myopic players to improve their strategy choices.

Given the intuition on which evolutionary stability is based, one might hope that evolutionarily stable strategies would correspond to asymptotically stable points under replicator dynamics or its generalizations. Evolutionary stability would then capture the idea that small (one time) mutations are driven from the population. However, as Taylor and Jonker [1978] show, evolutionary stability is sufficient, *but not necessary*, for asymptotic stability under replicator dynamics. Friedman shows that evolutionary stability is neither necessary nor sufficient for asymptotic stability under weak compatible dynamics. The relationship between the set valued notions discussed above and dynamics is equally murky.

This casts doubt on the import of the as–if–rationality results mentioned above, and bring us to our major question: Is as–if–rational behavior truly an implication of stability under this sort of adjustment process, or is it an artifact of the (perhaps over strong) static conditions?

In this paper, we examine *myopic adjustment dynamics*. Myopic adjustment dynamics generalize weak compatible dynamics, but retain the property that at each instant the direction of movement in each population's strategy is (at least weakly) payoff increasing given the current behavior of the opposing populations. While dynamics of this sort arise most naturally in large population, random matching models, it is important to note that the analysis of this paper depends only on the properties of the dynamic system, independent of the system's derivation. There is also reason to believe that the spirit of analysis introduced by this paper will be useful in understanding learning or evolutionary dynamics not included in the current analysis.

We begin with deterministic myopic adjustment dynamics that have as their state space the set of population strategy profiles and with the property that Nash equilibria are rest points (this set includes weak compatible dynamics). We show that if a strategy profile in asymptotically stable under some such dynamic, then it is hyperstable (Kohlberg and Mertens [1986]). Thus the above mentioned implications of the static notion of evolutionary stability are also implications of asymptotic stability under a myopic adjustment dynamic.

Unfortunately, for generic extensive forms, any Nash equilibrium that does not reach every information set is precluded both by evolutionary stability and by asymptotic stability under this sort of dynamic. The difficulty is that in such situations, any such Nash equilibrium will be a part of a non-trivial component of Nash equilibria, corresponding to different out of equilibrium behaviors. The dynamics we have discussed so far all have the property that they stop on each Nash equilibrium. Thus, no single element of the component of Nash equilibria can be asymptotically stable. The static notion of evolutionary stability similarly requires isolation in the set of Nash equilibria. Even for dynamics which need not stop on every Nash equilibrium (some of which we will discuss later), it will often be the case that the dynamic system does not select among various different out of equilibrium behaviors. Thus, strategies satisfying one or the other of these conditions will fail to exist precisely when concepts such as sequential equilibrium have power.

Motivated by this, we consider asymptotic stability for *sets* of strategy profiles.[4] Set valued notions are hard to interpret in standard rationality based game theory - what does it mean to say that rational players play a set? In dynamic environments, a set valued solution makes perfect sense: we predict that play, once in such a set, will remain in the set, without making any particular prediction about which element of the set which will be used at any instant in time.

As for sets satisfying rationality based solution concepts, sets which are asymptotically stable are most attractive when they correspond to a single outcome in an extensive form. We show that if a particular outcome in a generic two person extensive form game is asymptotically stable under such a dynamic (so that different limiting behaviors differ only at out of equilibrium information sets) then that outcome is hyperstable, and thus sequential and robust to forward induction.

More generally, for games with any finite number of players, if a set of strategies is asymptotically stable under some such dynamic, then it will contain a hyperstable subset if an additional topological condition is satisfied. If this set of strategies corresponds to a single outcome in the extensive form, then this outcome is sequential and robust to forward induction.

Thus, there are conditions under which adjustment dynamics can lead to play with very strong as–if-rationality properties. This is a remarkable and surprising result: strategic stability is motivated by very deep considerations involving idealized rational individuals. Why should this have anything to do with play that is dynamically stable under this extremely simplistic and myopic sort of adjustment rule? There are two keys to the connection. First, while the desiderata put forward by Kohlberg and Mertens are based on notions of rationality, the actual implementation of strategic stability involves robustness of sets of equilibria to perturbations in the underlying game. A set of Nash equilibria is strategically stable if it is structurally stable in the sense that close by games have Nash equilibria close to this set. But, asymptotic stability is itself a structurally stable property: many key implications of asymptotic stability survive small changes in the dynamic. The analysis hinges on relating these changes in the dynamic to the perturbations used in Kohlberg and Mertens' analysis. Starting from a myopic adjustment dynamic and an asymptotically stable set $\Theta$, and given a small perturbation to the game, we create a new dynamic which retains enough of the structure of the original dynamic to guarantee rest points near $\Theta$, but which has as rest points only Nash equilibria of the perturbed game. Thus $\Theta$ has the structural stability required by Kohlberg and Mertens.

These results *do not* provide a justification for the uncritical application of rationality based game theoretic concepts. In particular, for many games and many dynamics, convergence satisfying the conditions of our analysis does not occur. Rather, the results suggest that there are conditions under which some of these concepts are appropriate.

That these conditions are not innocuous is not necessarily a weakness of this type of analysis. In many games, the predictions of rationality based game theory are paradoxical, or in strong opposition to observed play. A theory which predicted as-if rationality in all games would thus be of questionable validity. It seems possible that understanding the effects of learning or evolution in game situations may provide a basis for a theory of the type of game, *and the type of setting*, in which various equilibrium notions should or should not apply.

After proving the main result, we turn to various extensions. We begin by dropping the requirement that every Nash equilibrium is a rest point of the system. We have two results for this type of dynamic. First, the results continue to hold for KM stability (but not for hyperstability) if the derivative field of the dynamic is Lipschitz continuous. Second, if the dynamic fails to stop on some Nash equilibria solely because it eliminates particular weakly dominated strategies, then the results hold for KM stability without the extra continuity condition.

Next, we turn to dynamics in which the direction of movement can depend on more than just the current population strategy profile. The analysis generalizes almost immediately when the state space is the cross product of the space of population strategy profiles with a compact convex subset of a Banach space. A key restriction to this analysis is that the added dimensions are allowed to affect which myopically improving direction is chosen, but not myopic improvement itself.

The condition of compactness in the last paragraph rules out time as a dimension of the state space. If a time varying dynamic has the property that time affects speed of movement, but not direction, or if the system admits a Lyapunov function, then the results go through.

An important instance of a time varying dynamic is provided by fictitious play models, and more generally, by models in which players respond not to actual play by their opponents, but rather to some perception of play that is formed from past play by the opponents. In some instances, our analysis can be made to apply to perceived play even though actual play meets few of the conditions of our analysis.

Next, we consider the extent to which the results can be recast in a discrete time framework. If next period's population strategy profile is continuous in this period's, then we can proceed without too much difficulty. The topological condition does need to be considerably strengthened, and the continuity condition is especially strong for a discrete time model.

In some situations, a set valued dynamic seems appropriate. We discuss these briefly.

Section II covers basic definitions and strategic stability. Section III discusses dynamics. Section IV establishes the basic relationship between asymptotic stability under myopic adjustment dynamics and strategic stability. Sections V to X discuss the extensions. Section XI concludes. Most proofs are relegated to the appendix.

## II. Preliminaries

### Basic Definitions

A game $(S,\pi)$ consists of players $i \in N \equiv \{1,\ldots,n\}$, finite pure strategy sets $S_i$ with $S \equiv \Pi_{i\in N} S_i$, and payoff functions $\pi=(\pi_1,\ldots,\pi_n)$. The space of mixed strategies is $\Phi=\Pi_{i\in N}\Delta(S_i)$. The vector of weights given by the mixed strategy profile $\sigma=(\sigma_1,\ldots,\sigma_n)\in\Phi$, to $s=(s_1,\ldots,s_n)\in S$ is $\sigma(s)=(\sigma_1(s_1),\ldots,\sigma_n(s_n))$. $\pi$ is extended to $\Phi$ by the expected utility calculation. The set of Nash equilibria of $(S,\pi)$ is $N(S,\pi)$. $BR_i(\sigma)\subseteq\Phi_i$ is the set of player $i$'s best responses to $\sigma$.

$D(\mu,\nu)$ is the Euclidean distance between $\mu,\nu\in\mathbf{R}^m$. For $X\subseteq\mathbf{R}^m$, and $\mu\in\mathbf{R}^m$, $D(\mu,X)\equiv\inf_{\nu\in X}D(\mu,\nu)$. For $X\subseteq\mathbf{R}^m$, and $\varepsilon>0$, define $B_\varepsilon(X)=\{\gamma|D(\gamma,X)\leq\varepsilon\}$. Note that $B_\varepsilon(X)$ is closed. $Y$ is a neighborhood of $X$ if there is an open set containing $X$ but contained in $Y$. Thus, $B_\varepsilon(X)$ is a neighborhood of $X$. Int$(X)$ is the interior of $X$. Cl$(X)$ is its closure. For $x$ and $y$ functions on $S_i$, define $x\cdot y$ as $\sum_{s_i\in S_i} x(s_i)y(s_i)$. If $x,y$ are functions on $S$, $x\cdot y$ is similarly defined as $\sum_{i\in N}\sum_{s_i\in S_i} x(s_i)y(s_i)$. $\mathbf{R}_+$ denotes the nonnegative real numbers. When we refer to a subset of $\Phi$ as open or closed, we shall mean relative to $\Phi$, unless specifically stated otherwise.

### Strategic Stability

The idea of stability (Kohlberg and Mertens [1986]) is to examine the robustness of a set of equilibria to perturbations in the underlying game. A class $p$ of perturbed games and a metric $m$ is established. A set $\Theta\subseteq N(S,\pi)$ is $(m,p)$–stable if it is a minimal closed set such that every game in $p$ that is close to $(S,\pi)$ under $m$ has a Nash equilibrium close to $\Theta$ (in Euclidean distance).

For KM stability, a perturbation is generated by a completely mixed strategy profile $\gamma \in \Phi$, and a vector $\delta \in [0,1]^n$. The payoff to each pure strategy profile $s$ in the perturbed game is the payoff in the original game when each player plays $(1-\delta_i)s_i + \delta_i \gamma_i$. The distance from the perturbed game to the original game is $\max_{i \in N} \delta_i$.

For hyperstability, a perturbed game is obtained from the original game by first adding a finite number of redundant pure strategies, and then perturbing the payoffs to the pure strategies in the new game by a small amount. Every hyperstable set contains a KM stable subset. A major reason for being interested in results about hyperstability rather than just KM stability is that hyperstable sets contain a proper element. Thus, if a hyperstable set corresponds to a single outcome in an underlying extensive form, then this outcome will be sequential.

## III. Dynamics

We begin our exposition with deterministic dynamics that have state space $\Phi$. The standard interpretation of $\sigma \in \Phi$ will be that according to whatever matching technology is being used, and given the behavior of individuals, the total probability of drawing an n-tuple who (after any individual randomizations) play $s \in S$ is $\sigma(s)$. We refer to $\sigma$ as the *population strategy profile*.

For our purposes, it is convenient to summarize such a dynamic by a map $F: \Phi \times \mathbf{R}_+ \to \Phi$, where for $\sigma \in \Phi$, and $t \in \mathbf{R}_+$, if the population strategy profile is $\sigma$ at some time $t' \geq 0$, then it will be $F(\sigma,t)$ at time $t'+t$. For $i \in N$, and $s_i \in S_i$, $F(\sigma,t)(s_i)$ is the weight given to $s_i$ by $F(\sigma,t)$.

Definition 1: A dynamic $F$ is *admissible* if

(1.1) $F$ is continuous, and

(1.2) $F$ is right differentiable with respect to time. That is, for all $\sigma \in \Phi$,

$$f(\sigma) \equiv \lim_{t \downarrow 0} \frac{F(\sigma,t) - \sigma}{t}.$$

is a well defined real vector.

In many cases, the right derivative $f$ will be the primitive. If $f$ is Lipschitz continuous, then it will have a unique and continuous solution $F$. Note that (1.2) implies that $F(.,0)$ is the identity map.

For $\sigma \in \Phi$, $i \in N$, and $s_i \in S_i$, $f(\sigma)(s_i)$ is the time rate of change of the proportion of $s_i$ in the population strategy for player position $i$. By $f_i(\sigma)$, we will mean the restriction of $f(\sigma)$ to $S_i$.

We will formally analyze only the case in which the populations associated with each position of the game evolve independently. The analysis can be extended, along the lines of Swinkels [1991a,b] to cases in which several positions are symmetric and filled by players from a single population. This corresponds to restricting the state space to a subspace of $\Phi$ in which equality restrictions hold for some dimensions. Our results hold in these cases if the definition of strategic stability is correspondingly weakened to consider only perturbations satisfying the same equality restrictions.

REPLICATOR DYNAMICS AND MYOPIC ADJUSTMENT DYNAMICS

The replicator dynamic for a game $(S,\pi)$ is given by

$$f(\gamma)(s_i)=\gamma_i(s_i)\left[\pi_i(\gamma\setminus s_i)-\pi_i(\gamma)\right]$$

for $i\in N$, $s_i\in S_i$, and $\gamma\in\Phi$.[5]

Thus, among the non-extinct strategies, strategies that are currently doing well are growing relative to those that are not. Only the broad qualitative features of the replicator dynamic are needed for our results:

DEFINITION 2: An admissible dynamic $F$ is a *myopic adjustment dynamic* if $\forall\sigma\in\Phi$,

(2.1) $f_i(\sigma)\cdot\pi_i(\sigma_i\setminus.)\geq 0$, for all $i\in N$, and

(2.2) if $\sigma$ is Nash then $f(\sigma)=0$.

This is a mild relaxation of weak compatibility.[6] Condition (2.1) states that at any moment, the direction of movement for each player population is such that *holding the strategies of the other player positions constant*, payoffs are increasing. If the inequality in (2.1) is strict whenever $\sigma_i\notin BR_i(\sigma)$, then $F$ is a *strict myopic adjustment dynamic*. It is easily verified that the replicator dynamic is a myopic adjustment dynamic.

## IV. MYOPIC ADJUSTMENT DYNAMICS AND STRATEGIC STABILITY

Taylor and Jonker [1978] show that for a strategy profile $\sigma$ to be evolutionarily stable is sufficient but not necessary for asymptotic stability of $\sigma$ under replicator dynamics.[7] Friedman shows that asymptotic stability under a weak compatible dynamic need be neither necessary nor sufficient for evolutionary stability. Since myopic adjustment dynamics generalize weak compatible dynamics, this calls into question the relationship between asymptotic stability under myopic adjustment dynamics and as-if-rational play.

However,

THEOREM 1: *If a strategy profile $\sigma$ is asymptotically stable under a myopic adjustment dynamic, then $\{\sigma\}$ is hyperstable.*

This is a special case of Theorem 3 below. We have fully recovered the implications for rational play derived as implications of evolutionary stability and its point valued generalizations. Unfortunately, the weakness of those results reappears as well: If a given strategy profile in an extensive form leaves any information set unreached, then for generic payoffs either the strategy profile is a not a Nash equilibrium or it is part of a nontrivial component of Nash equilibria supporting the same outcome. An asymptotically stable element must be isolated in the set of rest points of the dynamic, and so, by (2.2), isolated in the set of Nash equilibria. Thus no such strategy profile could be asymptotically stable under this type of dynamic. The conditions of Theorem 1 fail for precisely the type of strategy profile where the result would be most interesting.

We thus consider a set valued notion of asymptotic stability.

DEFINITION 3: A set $Y \subseteq \Phi$ is *asymptotically stable* under the dynamic $F$ if it is closed and there is a neighborhood $Z$ of $Y$ such that

(3.1) for every neighborhood $W$ of $Y$ with $W \subseteq Z$, there is a neighborhood $V$ of $Y$ with $F(V,t) \subseteq W$ for all $t \geq 0$, and

(3.2) for each $\gamma \in Z$, $\lim_{t \to \infty} D(F(\gamma,t),Y)=0$.

If Y has a single element, then Definition 3 exactly corresponds to the standard notion of asymptotic stability. Note that $\lim_{t \to \infty} F(\gamma,t)$ is not required to exist. This allows for convergence to, for example, limit cycles. Because $Y$ is closed (and therefore compact since $\Phi$ is compact), for any neighborhood $X$ of $Y$ there is $\varepsilon > 0$ with $B_\varepsilon(Y) \subseteq X$. Minimality is not used in our analysis, and so is not imposed. Nonetheless, note that if $Y^k \to Y$ is a nested sequence of asymptotically stable sets with basin of attraction $Z$, then $Y$ is also asymptotically stable with basin of attraction $Z$. By Zorn's lemma, there is a minimal asymptotically stable set with basin of attraction $Z$.

Introducing the set valued notion helps matters considerably. Say that an outcome (distribution over terminal nodes) in an extensive form game is asymptotically stable under a dynamic $F$ if there is a set of strategy profiles generating this outcome that is asymptotically stable.

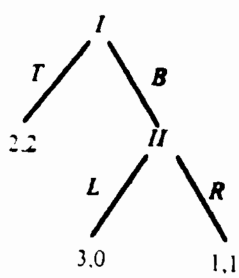
THEOREM 2: *For a generic 2 person extensive form game, if an outcome is asymptotically stable under a strict myopic adjustment dynamic then it is hyperstable.*

The proof of this hinges on three facts. First, if a set of strategy profiles corresponding to a particular outcome is asymptotically stable under a strict myopic adjustment dynamic, then that set must exactly correspond to the set of Nash equilibria supporting that outcome. Second, for generic two person games, the set of Nash equilibria supporting any particular outcome is convex (Swinkels [1991a, Lemma 8]). Third, for convex asymptotically stable sets, the topological condition in Theorem 3 is trivially satisfied.

To see the first claim, consider any $\sigma$ that is not Nash. Then, as $F$ is strict, there is some player $i$ who is moving in a strictly payoff increasing direction from $\sigma$. This must involve a change in the outcome. Thus, only Nash equilibria supporting the outcome can be included if the outcome is to be asymptotically stable. By (2.2), all such equilibria are included.[8]

The second facts fails for games with more than two players. An important open question is whether there is an interesting characterization of extensive form games for which sets of Nash equilibria supporting a particular outcome are sufficiently regular to guarantee the necessary topological condition of Theorem 3.

To see why some strengthening of (2.1) was needed, consider the generic extensive 2 person game and associated normal form illustrated in Fig. 1.




| I \ II | L | R |
|---|---|---|
| T | 2,2 | 2,2 |
| B | 3,0 | 1,1 |

FIGURE 1: A generic extensive form game $\Gamma_1$ and its normal form $G_1$.

Fig. 2 illustrates a copy of $\Phi$ for this game, and displays the gradient field for a particular dynamic.

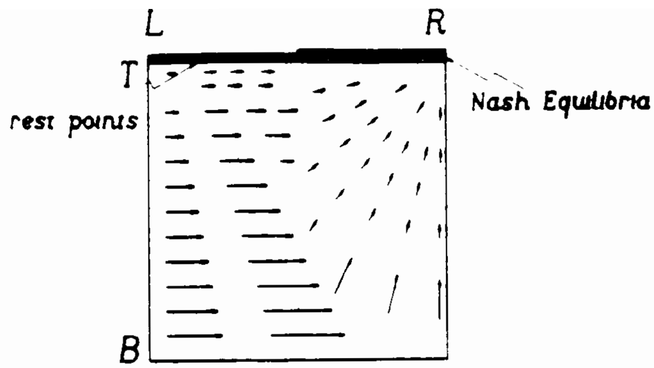



FIGURE 2: A myopic adjustment dynamic for $G_1$ under which $T\times\Phi_2$ is asymptotically stable.

This is a myopic adjustment dynamic under which the set $T\times\Phi_2$ is asymptotically stable. It is not strict because it includes elements of $T\times\{\sigma_2 \mid \sigma_2(R)<1/2\}$ as rest points.

THE MAIN THEOREM

We turn to a formal statement and proof of our main result.

THEOREM 3: *Let $(S,\pi)$ be a game, let $\Theta\subseteq\Phi$ be asymptotically stable under a myopic adjustment dynamic $F$, and assume there is a neighborhood $U$ of $\Theta$ contained in the basin of attraction of $\Theta$ which is homeomorphic to $\Phi$. Then, $\Theta$ contains a hyperstable subset.*

REMARK 1: In the proof that immediately follows, we work with any definition of stability in which perturbations can be interpreted as perturbations to payoffs to pure strategy profiles. This includes KM stability. In the next section, we show how this can be extended to games in which redundant pure strategies are added, covering hyperstability.[9]

REMARK 2: The homeomorphism assumption on $U$ and the assumption that $F$ is a continuous function are for the sake of Brouwer's fixed point theorem. By virtue of the Eilenberg–Montgomery fixed point theorem this could be weakened to $U$ being an acyclic absolute normal retract and $F$ an upper hemicontinous acyclic correspondence (see Border [1985], p. 73). It seems unlikely that the relaxation of the condition on $U$ is of practical significance. Extending the results to set valued dynamics would be valuable. However, as we discuss in Section X, acyclicity is very strong in this context.

REMARK 3: The existence of an appropriate $U$ is of course trivial if $\Theta$ is convex, establishing Theorems 1 and 2. A better understanding of when asymptotically stable sets admitting such a $U$ arise would be very desirable.

A key to the analysis that follows is to define, for any game of the form $(S,\rho)$, where $\rho$ may or may not equal $\pi$, a dynamic that has as its rest points precisely $N(S,\rho)$.

DEFINITION 4: The *canonical dynamic* for $(S,\rho)$ has gradient field given by

$$c_\rho(\sigma)(s_i)=\max[\rho_i(\sigma\backslash s_i)-\rho_i(\sigma),0]-\sigma_i(s_i)\sum_{t_i\in S_i}\max[\rho_i(\sigma\backslash t_i)-\rho_i(\sigma),0]$$

for $s_i\in S_i$, $i\in N$, and $\sigma\in\Phi$.

Note that $c_\rho$ is Lipschitz and so has a unique and continuous solution $C_\rho$. Also $\sum_{s_i\in S_i} c_\rho(\sigma)(s_i)=0$, and whenever $\sigma_i(s_i)=0$, $c_\rho(\sigma)(s_i)\geq 0$. Thus, $C_\rho$ maps $\Phi$ to $\Phi$.

The first term of $c_\rho$ increases weight on strategies that are performing better than the average. The second term reduces weight on all strategies proportionately so as to keep the system within $\Phi$. Thus, $c_\rho(\sigma)\cdot\rho(\sigma\backslash.)\geq 0$, with equality if and only if $\sigma\in N(S,\rho)$.

PROOF OF THEOREM: We must show that as $\rho\rightarrow\pi$, $(S,\rho)$ has Nash equilibria converging to $\Theta$. We do this by showing that for $\rho$ close enough to $\pi$ we can splice $C_\rho$ with $F$ in such a way that (1) the spliced map inherits enough of the structure of $F$ to guarantee a fixed point on $U$, and (2) any such fixed point is near $\Theta$ and also a fixed point of $C_\rho$.

Begin by choosing $V$, a closed neighborhood of $\Theta$ with $V\subseteq\text{Int}(U)$ and such that $F(V,t)\subseteq U$ for all $t\geq 0$. Such a $V$ exists by (3.1).

Next, we choose $T\geq 0$ such that $F(U,t)\subseteq U$ for all $t\geq T$. To see that such a $T$ exists, for each $\gamma\in U$, define $T(\gamma)=\inf\{t\geq 0 \mid F(\gamma,t)\in V\}$. If $T(.)$ is bounded on $U$ we will be finished since $F(V,t)\subseteq U$ $\forall t$. Assume $T(.)$ is not bounded on $U$. Since $U$ is compact, there is $\{\gamma^k\}_{k\in\mathbf{N}}\rightarrow\gamma$, with $\gamma^k,\gamma\in U$ and such that $\lim_{k\rightarrow\infty}T(\gamma^k)=\infty$. Since $V$ is a neighborhood of $\Theta$, there is $T'$ such that $F(\gamma,T')\in\text{Int}(V)$. But by the continuity of $F$, $F(\gamma^k,T')\in V$ for $k$ sufficiently large, contradicting that $\lim_{k\rightarrow\infty}T(\gamma^k)=\infty$.

We construct a particular partition of $U$. Choose $\varepsilon>0$ such that $B_{3\varepsilon}(\Theta)\subseteq\text{Int}(V)$. We will be interested in the following 5 regions:

| | |
|---|---|
| $A$ | $B_\varepsilon(\Theta)$ |
| $B$ | $\{\gamma \mid \varepsilon\leq D(\Theta,\gamma)\leq 2\varepsilon\}$ |
| $C$ | $\{\gamma \mid 2\varepsilon\leq D(\Theta,\gamma)\}\leq 3\varepsilon\}$ |
| $D$ | $\text{Cl}(V\backslash B_{3\varepsilon}(\Theta))$ |

$E \quad Cl(U \setminus V)$.

The various regions are displayed in Fig. 3.

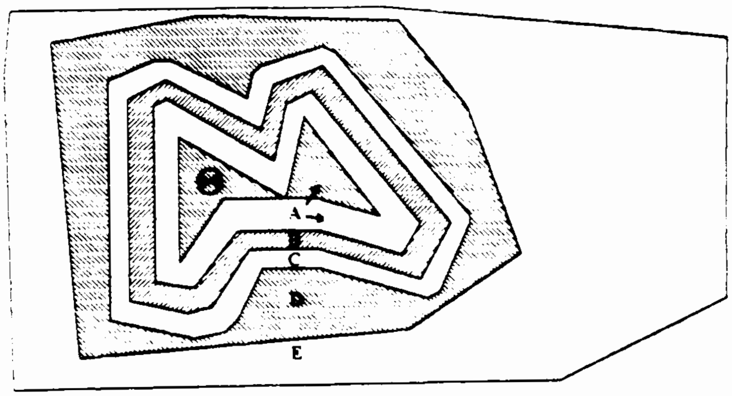



FIGURE 3: The regions used in constructing the composite map. $U$ is the entire set. $V$ is $A \cup B \cup C \cup D$.

$A$ and $C$ are disjoint closed sets. Thus by Urysohn's Lemma, there is $\alpha: U \to [0,1]$ with $\alpha(C)=1$, $\alpha(A)=0$, and $\alpha$ continuous. Similarly, $C$ and $E$ are disjoint and closed, so there is $\beta: U \to [0,1]$ with $\beta(E)=1$, $\beta(C)=0$, and $\beta$ continuous.

We now define the spliced map. For given $\rho$, consider $G_\rho: U \times \mathbf{R}_+ \to \Phi$ given by:

$$G_\rho(\sigma,t) = \begin{cases} C_\rho(\sigma,t) & \text{on } A \\ \alpha(\sigma)F(\sigma,t)+[1-\alpha(\sigma)]C_\rho(\sigma,t) & \text{on } B \\ F(\sigma,t) & \text{on } C \\ F(\sigma,\beta(\sigma)T+[1-\beta(\sigma)]t) & \text{on } D \\ F(\sigma,T) & \text{on } E \end{cases}$$

Note that the splice on $B$ takes place in the range, while the splice on $D$ takes place in the domain.

From the continuity of $C_\rho$, $\alpha$, $F$, and $\beta$, and from the agreement of the appropriate functions on $A \cap B$, $B \cap C$, $C \cap D$, and $D \cap E$, $G_\rho$ is continuous. Also, $B_\varepsilon(A \cup B) \subseteq U$, and $G_\rho(.,0)$ is the identity map. Thus, $G_\rho(A \cup B,t) \subseteq U$ for $t$ sufficiently small. Since $C \cup D \subseteq V$, $G_\rho(C \cup D,t) \subseteq U$ for all $t$. Finally, $G_\rho(E,t) \subseteq U$ by choice of $T$. Thus, for small $t>0$, $G_\rho(.,t)$ is a continuous map from $U$ to $U$. Since $U$ is homeomorphic to $\Phi$, $G_\rho(.,t)$ has a fixed point $\sigma^t$ for each $t$ sufficiently small. If $\gamma \in C \cup D \cup E$ is a fixed point of $G_\rho(.,t)$, then there is $t'>0$ such that $F(\gamma,t')=\gamma$. This is impossible as $C \cup D \cup E \subseteq U \setminus \Theta$ and $U$ is in the basin of attraction of $\Theta$ under $F$. Thus, $\sigma^t \in A \cup B$.

Let $\sigma_\rho$ be an accumulation point of $\{\sigma^t\}_{t \downarrow 0}$. We show that $\sigma_\rho$ is a rest point of $G_\rho$, i.e., that $G_\rho(\sigma^\rho,t')=\sigma^\rho \ \forall t' \geq 0$. Fix $t'>0$. For any $t \in \mathbf{R}_+$, define $r(t)=\min\{t^* \mid t^* \geq 0,\ t^*=t'-kt \text{ for } k \in \mathbf{N}\}$. By definition of $\sigma^t$, $G_\rho(\sigma^t,t')=G_\rho(\sigma^t,r(t))$. Since $\lim_{t \downarrow 0} r(t)=0$, and since $G_\rho$ is continuous, we have

$$G_\rho(\sigma,t') = \lim_{t\downarrow 0} G_\rho(\sigma',t') = \lim_{t\downarrow 0} G_\rho(\sigma',r(t)) = G_\rho(\sigma,0) = \sigma .$$

Since $t'>0$ was arbitrary, we are done.

Let $\sigma$ be a cluster point of $\{\sigma^\rho\}_{\rho\to\pi}$. Then, $\sigma$ is a rest point of $G_\pi$ (as $\rho\to\pi$, $\|c_\rho - c_\pi\|\to 0$. Thus, as $c_\rho$ is Lipschitz continuous, $C_\rho\to C_\pi$ pointwise (c.f Coddington and Levinson, 1955, p. 8, Theorem 2.1, or equation (†) in the appendix). But, then $G_\rho\to G_\pi$, and so $\sigma$ is a rest point of $G_\pi$). Also, if $\alpha(\sigma)=1$, then $\sigma\in B$, and $G_\pi(\sigma,.) = F(\sigma,.)$. Since $F$ has no rest points on $B$, $\alpha(\sigma)<1$. So, consider,

$$g_\pi(\sigma) \equiv \lim_{t\downarrow 0}\frac{G_\pi(\sigma,t)-\sigma}{t}$$

$$= \alpha(\sigma)f(\sigma)+[1-\alpha(\sigma)]c_\pi(\sigma)$$

Since $\sigma$ is a rest point of $G$, $g_\pi(\sigma)=0$ and thus $g_\pi(\sigma)\cdot\pi(\sigma\backslash.)=0$. Using (2.1) and $\alpha(\sigma)<1$, $c_\pi(\sigma)\cdot\pi(\sigma\backslash.)=0$ and so $\sigma\in N(S,\pi)$. By (2.2), $N(S,\pi)\cap A\cup B\subseteq\Theta$ and thus, $\sigma\in\Theta$. So, as $\rho\to\pi$, $\sigma^\rho\to\Theta$. Since $G=C_\rho$ on A, $\sigma^\rho\in N(S,\rho)$ for $\rho$ close to $\pi$, and so we are done.

REMARK 4: Note that (2.1) was only needed on a set playing the role of $B$. This observation could be a first step toward results about dynamics that are only approximately myopic in the sense that players whose play is very close to optimal may move in non-improving directions. Since $B$ is a closed set containing no Nash equilibria, there is a strictly positive lower bound on $B$ for the amount by which some player is short of an optimum. If (2.1) holds when $\sigma_i$ is suboptimal by this amount, then the analysis goes through.

REMARK 5: Note that the continuity of $F$ was only necessary on $U$. This is important, because in many interesting examples a given dynamic will be discontinuous only on the boundaries of the basin of attraction. Consider the coordination game and associated dynamic shown in Fig. 4.

| | | *II* | |
|---|---|---|---|
| | | *L* | *R* |
| *I* | *L* | 1,1 | 0,0 |
| | *R* | 0,0 | 3,3 |

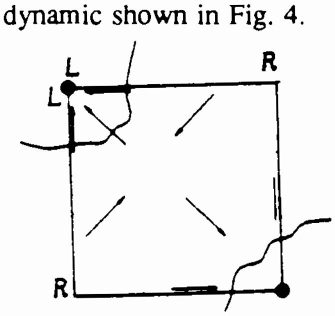



FIGURE 4: A coordination game $G_2$ and an associated dynamic.

As time passes, a constant flow of players change their action to a best response to the current population (as long as players not playing a best response exist). Let $\iota$ denote the set of strategy profiles at which one or the other player is indifferent. I.e., $\iota \equiv \{\sigma \mid \sigma_i(L)=2/3$ for $i=1$ or $2\}$. The specification of $f$ on $\iota$ is immaterial to our discussion. While $f$ is discontinuous both on the boundaries of $\Phi$, and on $\iota$, the associated $F$ is discontinuous only on $\iota$. But, for either asymptotically stable point, $U$ can be taken to exclude $\iota$. Since only the continuity of $F$ is required (as opposed to that of $f$), and that only on a neighborhood of $\Theta$, the framework extends to this example.

HYPERSTABILITY

Consider adding a redundant strategy to $(S,\pi)$. That is for some $i$, augment $S_i$ with a pure strategy $r_i$, where $r_i$ is equivalent to some mixture $\gamma_i \in \Phi_i$. Let $S'=S_{-i}\times S_i \cup r$, let $\pi'$ be the appropriately augmented payoff function, and let $\Phi'$ be the space of mixed strategy profiles for $S'$. For $\sigma \in \Phi'$, let $E(\sigma)$ be the strategy profile in $\Phi$ which is equivalent to $\sigma$. That is, define $E:\Phi' \to \Phi$ by $E(\sigma)=\sigma \backslash \phi_i$ where

$$\phi_i(s_i)=\sigma_i(s_i)+\sigma_i(r_i)\gamma_i(s_i) \text{ for } s_i \in S_i.$$

Let $\Theta \in \Phi$ be asymptotically stable under a myopic adjustment dynamic $F$ for $(S,\pi)$. To extend Theorem 3 to hyperstability, we will show that $E^{-1}(\Theta)$ satisfies all the conditions of Theorem 3, and so is strategically stable relative to payoff perturbations in $(S',\pi')$. Since the argument can be repeated a finite number of times, this implies that $\Theta$ is hyperstable in $(S,\pi)$.

So, for $\sigma \in \Phi'$, define

$$F'(\sigma,t) = \underset{\eta \in E^{-1}(F(E(\sigma),t))}{\operatorname{argmin}} D(\eta,\sigma).$$

That is, for any $\sigma$ in $\Phi'$, first project back into $\Phi$, then translate the projection by $F$, and finally return to $\Phi'$ by taking the point in $\Phi'$ that is closest to $\sigma$ subject to being equivalent to the translated projection. Fig. 5 may help.

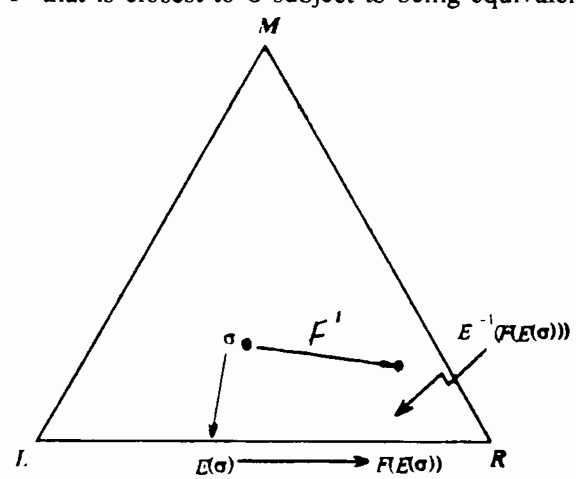



FIGURE 5: The derivation of $F'$. $\Phi'$ is the full simplex. $\Phi$ is the simplex with pure elements $L$ and $R$. In this case, $M$ is equivalent to $2/3L+1/3R$. Dashed lines are $E$-equivalence classes of $\Phi'$.

Then, $E(F'(\sigma,t))=F(E(\sigma),t)$, so that $F'$ operates on $E$-equivalence classes of $\Phi'$ in the same way as $F$ operates on $\Phi$. From this, we conclude that $E^{-1}(\Theta)$ is asymptotically stable under $F'$. Continuity of $F'$ is obvious as $E^{-1}(F(E(\sigma),t)$ is a continuous correspondence in $\sigma$ and $t$, while $D(.,.)$ is strictly concave. Seeing that $f'$ is well defined is a little involved. However, note that

$$\lim_{t\downarrow 0}\left(\frac{F'(\sigma,t)-\sigma}{t}\right)\pi'(\sigma\backslash.)$$

is well defined and equal to $f(E(\sigma))\cdot\pi(E(\sigma)\backslash.)\geq 0$ even if $f'$ is not well defined. As the only use of (1.2) and (2.1) in the proof of Theorem 3 is to ensure that this product is well defined and non-negative, we can avoid the direct proof. Also note that if $\sigma$ is a Nash equilibrium of $(S',\pi')$, then $E(\sigma)$ is a Nash equilibrium of $(S,\pi)$. Because $F$ satisfies (2.2), $F(E(\sigma),t)=E(\sigma)$ for all $t\geq 0$, from which we conclude that $F'(\sigma,t) = \sigma$ for all $t\geq 0$, so that (2.2) is satisfied by $F'$. Finally, $E^{-1}(U)$ is homeomorphic to $\Phi'$ (see the appendix). Taken together, this implies that Theorem 3 carries through for $E^{-1}(\Theta)$ and $F'$.

## V. Dynamics That Can Pass Through Nash Equilibria

In general, a set $\Theta$ that is asymptotically stable under an admissible dynamic satisfying (2.1) but not (2.2) need not contain a hyperstable subset. Consider $G_1$ (Fig. 1) along with the dynamic of Fig. 6.

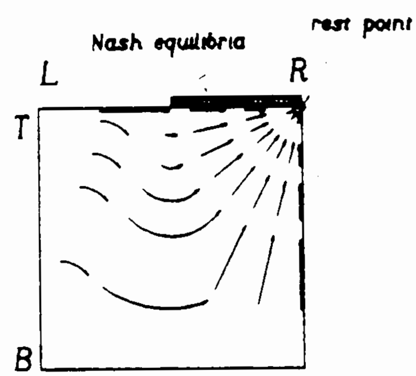



FIGURE 6: A dynamic for $G_1$ under which $(T,R)$ is asymptotically stable.

While $(T,R)$ is asymptotically stable under this dynamic, and the dynamic satisfies (2.1), $G_1$ has as its unique hyperstable set $\{\sigma \mid \sigma_1(T)=1,\ \sigma_2(L)\geq 1/2\}$. It is easy to see where the proof of Theorem 3 fails: For small perturbations in which $\rho_2(T,L)>\rho_2(T,R)$ the canonical dynamic must travel from right to left on $T\times\Phi_2$, while the dynamic of Fig. 6 travels from left to right. These will cancel each other somewhere on $T\times\Phi_2$ near $(T,R)$, while the only Nash equilibria of the perturbed game will have $\sigma_2(L) \approx 1/2$.

The strategy profile $(T,R)$ is KM stable. This holds in general if $f$ is Lipschitz continuous.

THEOREM 4: *Let* $(S,\pi)$ *be a game, let* $\Theta\subseteq\Phi$ *be asymptotically stable under an admissible dynamic* $F$ *satisfying* (2.1) *and such that the associated* $f$ *is Lipschitz continuous, and assume there is a neighborhood* $U$ *of* $\Theta$ *contained in the basin of attraction of* $\Theta$ *which is homeomorphic to* $\Phi$. *Then,* $\Theta$ *contains a KM stable subset.*

See the appendix for a proof. We also have:

THEOREM 5: *If a particular outcome* $\zeta$ *in a generic two person game is asymptotically stable under a dynamic that satisfies* (2.1) *strictly whenever* $\sigma_i \notin BR_i(\sigma)$, *and if the basin of attraction is large enough to include a neighborhood of the set of Nash equilibria supporting* $\zeta$, *then* $\zeta$ *is KM stable.*

As for Theorem 2, the idea is to appeal to the convexity of sets of Nash equilibria corresponding to a particular outcome in generic two person extensive form games. Since the dynamic is strict, the outcome must be Nash, and the asymptotically stable set must be a subset of the Nash equilibria. Thus, if the basin of attraction includes a neighborhood of the set of Nash equilibria supporting the outcome, then an $\varepsilon$-ball around this set, for $\varepsilon>0$ sufficiently small, will be the necessary $U$.

The dynamic for $G_1$ illustrated in Fig. 7 illustrates the need for the continuity of $f$.

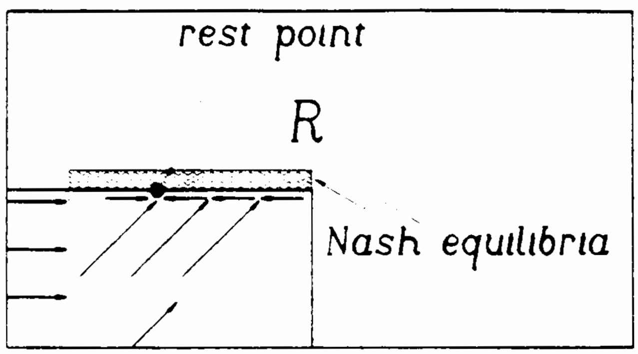



FIGURE 7: A dynamic for $G_1$ illustrating the need for continuity of $f$ in Theorem 4. (2.1) is satisfied, but $\zeta$ is an asymptotically stable point that is not KM stable. Note that $f$ is discontinuous as $\sigma(T)\to 1$.

$F$ is continuous in initial conditions and time and satisfies (2.1). Since $\zeta$ involves a weakly dominated strategy, it is not KM stable. The construction used in proving Theorem 4 fails for this example: For any perturbed dynamic of the sort used in that proof, $(T,R)$ becomes the unique asymptotically stable set.

WEAK DOMINANCE

The dynamic of Fig. 6 violated (2.2) only in that it eliminated a weakly dominated strategy. For dynamics of this sort, we can reclaim the KM stability implication without the extra condition of Lipschitz continuity on $f$.

THEOREM 6: *Let* (S,$\pi$) *be a game. For each* $i\in N$, *let* $R_i\in S_i$ *be a set of weakly dominated strategies, and assume F is such that* $f(\sigma)(r_i)\leq 0$ *for all* $\sigma\in\Phi$, $r_i\in R_i$. *Assume that f satisfies* (2.1), *and satisfies* (2.2) *for any Nash equilibrium not involving* $\cup_{i\in N}R_i$. *If* $\Theta$ *is asymptotically stable under F, and there is U a neighborhood of* $\Theta$ *contained in the basin of attraction of* $\Theta$ *homeomorphic to* $\Phi$, *then* $\Theta$ *contains a subset that is KM stable.*

The proof is in the appendix.

## VI. RICHER STATE SPACES

The analysis generalizes almost immediately to a state space of the form $\Phi\times\Psi$, where $\Psi$ is a compact convex subset of a Banach space. Note that compactness of $\Psi$ rules out time as a dimension of the state space. We will discuss time varying dynamics in the next section.

A *generalized state space dynamic* is a map $F:\Phi\times\Psi\times\mathbb{R}_+\rightarrow\Phi\times\Psi$. Define $P_\Phi:\Phi\times\Psi\rightarrow\Phi$ as the projection map onto $\Phi$.

DEFINITION 5: A generalized state space dynamic $F$ is *admissible* if

(5.1) $F$ is continuous, and

(5.2) $P_\Phi(F)$ is right differentiable with respect to time. That is,

$$f(\sigma,\psi)\equiv\lim_{t\downarrow 0}\frac{P_\Phi(F(\sigma,\psi,t))-\sigma}{t}$$

is well defined for all $(\sigma,\psi)\in\Phi\times\Psi$.

DEFINITION 6: An admissible generalized state space dynamic $F$ is a *myopic adjustment dynamic* if for all $(\sigma,\psi)\in\Phi\times\Psi$,

(6.1) $f_i(\sigma,\psi)\cdot\pi_i(\sigma_i\backslash.)\geq 0$, for all $i\in N$, and

(6.2) if $\sigma$ is Nash then $f(\sigma,\psi)=0$.

Thus, the other dimensions of the state space can determine which myopically improving direction is chosen at each point, but they are not permitted to affect myopic improvement.

We have the following analogue to Theorem 1:

THEOREM 7: *If* $(\sigma,\psi)\in\Phi\times\Psi$ *is asymptotically stable under a generalized state space myopic adjustment dynamic, then* $\{\sigma\}$ *is hyperstable.*

This is a special case of Theorem 8. We cannot generalize Theorem 2 to this framework, i.e., it need not be the case that for a generic 2 person extensive form game, an outcome is asymptotically stable under a strict generalized state space myopic adjustment dynamic only if it is strategically stable. While $P_\Phi(\Theta)$ is exactly equal the set of Nash equilibria supporting the outcome, and so is convex, $\Theta$ itself could be more oddly shaped. Particular dynamics might preclude this pathology. Theorem 3 goes through with little change:

THEOREM 8: *Let* $(S,\pi)$ *be a game, and let* $\Psi$ *be a compact convex subset of a Banach space. Let* $\Theta\subseteq\Phi\times\Psi$ *be asymptotically stable under a generalized state space myopic adjustment dynamic* $F$, *and assume there is a neighborhood* $U$ *of* $\Theta$ *contained in the basin of attraction of* $\Theta$ *which is homeomorphic to* $\Phi\times\Psi$. *Then,* $P_\Phi(\Theta)$ *contains a hyperstable subset.*

The proof is much like that of Theorem 3, except that the appropriate fixed point theorem is Schauder's rather than Brouwer's. See the appendix for details.

## VII. TIME VARYING DYNAMICS

Consider dynamics of the form $F:\Phi\times\mathbf{R}_+\times\mathbf{R}_+\to\Phi$, with the interpretation that if the system is at $\sigma\in\Phi$ at time $t$, then at time $t+t'$, it is at $F(\sigma,t,t')$. As before, assume that $F$ is continuous, that

$$f(\sigma,t)\equiv\lim_{t'\downarrow 0}\frac{F(\sigma,t,t+t')-\sigma}{t'}$$

is well defined for $(\sigma,t)\in\Phi\times\mathbf{R}_+$, and that for all $(\sigma,t)\in\Phi\times\mathbf{R}_+$, $f(\sigma,t)$ satisfies

(1) $f_i(\sigma,t)\cdot\pi_i(\sigma_i\backslash.)\geq 0$, and,

(2) If $\sigma\in(S,\pi)$ then $f(\sigma,t)=0$.

Consider mimicking the proof of Theorem 3. One constructs the same partition as before, and constructs the spliced map $G(\sigma,t)$ as

$$G(\sigma,t) = \begin{cases} C_\rho(\sigma,t) & \text{on } A \\ \alpha(\sigma)F(\sigma,0,t)+[1-\alpha(\sigma)]C_\rho(\sigma,t) & \text{on } B \\ F(\sigma,0,t) & \text{on } C \\ F(\sigma,0,\beta(\sigma)T+[1-\beta(\sigma)]t) & \text{on } D \\ F(\sigma,0,T) & \text{on } E \end{cases}$$

As before, for small $t$, $G$ is a continuous mapping from $U$ to $U$ and so has fixed points. Now, however, one cannot conclude that a fixed point of $G(.,t)$ on $C\cup D\cup E$ is a cycle of $F$. In particular, since the dynamic is time varying, it is quite consistent that $F(\sigma,0,t')=\sigma$ for some $t'>0$ and $\sigma\in U\backslash\Theta$, but that nonetheless $\lim_{t\to\infty}D(F(\sigma,0,t),\Theta)=0$. Thus, we cannot conclude that the map $G$ has a fixed point on $A\cup B$. Except for this step, the proof goes through as before.

To recapture Theorem 3, we need to rule out $F(\sigma,0,t')=\sigma$ for $t'>0$ and $\sigma\in U\backslash\Theta$. For at least one special case, this is easily done. This is when (a) time affects the speed of movement at any point, but not the direction of movement, and (b) as time goes by, the dynamic is not slowed down 'too fast'. For such dynamics one can find $\hat{F}$ a time invariant dynamic and $k:\mathbf{R}_+\to\mathbf{R}_+$ a strictly increasing bijection, such that $F(\sigma,t,t') = \hat{F}(\sigma,k(t')-k(t))$. Thus, if $\sigma$ recurs once, then it recurs forever, albeit possibly at longer and longer intervals. That is, if ever $F(\sigma,0,t)=\sigma$ for $t>0$ and $\sigma\in\Phi$, then for every $T>0$ there is $t'>T$ such that $F(\sigma,0,t')=\sigma$. For $\sigma\in U\backslash\Theta$, this contradicts asymptotic stability, and so cannot happen. The definition of 'too fast' is the standard one in updating procedures; for the speed of the system to fall as $1/t$ is not too fast, while for the speed to fall as $1/t^2$ is (if the dynamic slows as $1/t^2$ then $k$ will not be onto). Such dynamics can arise when movement of the population strategy is caused by new entrants whose entry rate (as a proportion of the existing population) is not constant, but who follow a fixed entry rule as a function of the current population strategy profile. Similarly, for various 'summary statistic' dynamics (see the next section), time enters only in how 'responsive' the dynamic is to new observations.

Another situation in which one can conclude that such cycles cannot occur is when the system admits a Lyapunov function.[10] Whether there are more general conditions implying acyclicity of $F(.)$ on $U\backslash\Theta$ is an open question. We summarize the positive part of this discussion with:

THEOREM 9: *Let* $(S,\pi)$ *be a game. Let* $F:\Phi\times\mathbf{R}_+\times\mathbf{R}_+\to\Phi$ *be a time varying dynamic satisfying* (1) *and* (2). *Assume* $\Theta\subseteq\Phi$ *is asymptotically stable under* $F$, *and that there is a neighborhood* $U$ *of* $\Theta$ *contained in the basin of attraction of* $\Theta$ *which is homeomorphic to* $\Phi$. *Assume in addition that* $F(\sigma,0,t)\neq\sigma$ *for* $\sigma\in U\backslash\Theta$ *and* $t>0$. *Then,* $\Theta$ *contains a hyperstable subset.*

A formal proof is omitted. It is interesting that the compatibility condition is only needed on $f$ at time 0 (note that the definition of $G$ involves only $F(\sigma,0,t)$). This is evidence of the strength of the acyclicity condition: For an arbitrary convex component of Nash equilibria, one could imagine choosing $f$ at time 0 consistent with definition (1) and (2), while modifying the dynamic as time passes to give asymptotic stability. The acyclicity assumption is what rules out this construction.

This general construction could be extended to other non-compact extensions of the state space: Consider dynamics of the form $F:\Phi\times\Omega\times\mathbb{R}_+\rightarrow\Phi\times\Omega$, where $\Omega$ is not compact. By analogy to the previous section, one could work with the projection $P_\Phi F$ of the dynamic onto $\Phi$ and ask for asymptotic stability of $\Theta\subseteq\Phi$ relative to $P_\Phi F$. The proof of Theorem 3 then goes through directly in terms of this projection, *if* one can somehow ensure that $P_\Phi F(\sigma,\omega,t)=\sigma$ never holds for $\sigma\in U\backslash\Theta$ and $t>0$.

## VIII. Summary Statistic Dynamics

Until now, we have interpreted $\Phi$ as representing actual play in the population at each instant in time. In some models, it is fruitful to interpret $\Phi$ as capturing perceived play. Consider models with the following characteristics:

(1) At each instant in time there is a prediction for the play of each population.

(2) At each instant, those players who are called upon to play choose a best response to this prediction. When there is more than one optimal response, players use some rule that is a function only of their current prediction to select over best responses.

(3) The equation of motion of the prediction held about population $i$ is of the form

$$\frac{\partial\sigma^c(t)}{\partial t}=v\times\left(\sigma(t)-\sigma^c(t)\right),$$

where $\sigma^c(t)$ is conjectured play at time $t$, $\sigma(t)$ is actual play at time $t$, and $v$ is a positive scalar.

Actual play is almost certainly discontinuous, in both time and initial conditions, and may or may not be myopically improving relative to current actual play. Perceived play is considerably better behaved. Population $i$ plays best responses to the perception of population $-i$'s play. Population $-i$'s perception of population $i$'s play moves in the direction of $i$'s actual play. Thus, $-i$'s perception of *i's* play moves in a direction that is myopically improving relative to *i's* perception of $-i$'s play, i.e., perceived play satisfies (2.1). These perceptions are continuous in time, and may well in particular examples be continuous in initial conditions, at least on a region of some asymptotically stable set. In such examples, we can conclude that if perceived play is asymptotically stable (and satisfies the other conditions of

Theorem 3 or 4), then the asymptotically stable set of perceived plays will contain a strategically stable subset. Finally, note that if actual play converges to some convex region, then perceived play will as well.

A useful extension is to allow v to be a function of $t$. For example, setting $v(t) = 1/t$ generates a continuous time version of a fictitious play model (take $t \in [1,\infty)$ to avoid definitional difficulties). Over time speed is affected, but not direction. In particular, for this example, let $\hat{F}(\sigma,t)=F(\sigma,\ln(t))$, where $F$ is the original dynamic. Then, $\hat{F}$ is a time invariant dynamic that inherits myopic adjustment, asymptotic stability and continuity from $F$. Depending on whether any given example satisfies the other assumptions, dynamic stability of fictitious play may well imply strategic stability.

## IX. Discrete Time Dynamics

We will consider simple discrete time dynamics with state space $\Phi$. Such a dynamic can be represented by a map $F:\Phi\rightarrow\Phi$, with the interpretation that $\sigma\in\Phi$ is carried to $F^t(\sigma)$ in $t$ periods. Initially, we will assume $F$ is continuous.

DEFINITION 7: A discrete time dynamic $F$ is a *myopic adjustment dynamic* if for all $\sigma\in\Phi$,

(7.1) $\pi_i(\sigma\backslash F_i(\sigma))\geq\pi_i(\sigma)$ for all $i\in N$, and

(7.2) if $\sigma\in N(S,\pi)$ then $F(\sigma)=\sigma$

This is a direct translation of Definition 2. In particular, (7.1) states that, holding the actions of the other player positions fixed, population $i$ moves in a payoff increasing direction.

We have the following parallel to Theorem 3. Note that the condition on $U$ has been considerably strengthened. We will discuss this immediately following the proof.

THEOREM 10: *Let $(S,\pi)$ be a game, let $\Theta\subseteq\Phi$ be asymptotically stable under a discrete time myopic adjustment dynamic $F$, and assume there is $U$ a compact convex neighborhood of $\Theta$ such that $U$ is in the basin of attraction of $\Theta$ under $F$, and $U$ is forward invariant under $F$. Then, $\Theta$ contains a hyperstable subset.*

The proof (appendix) largely parallels that of Theorem 3. The major task is in constructing an analog to the canonical dynamic for the discrete time environment.

Consider attempting to further mimic the construction of Theorem 3 without the condition that

U is forward invariant. It is easily shown that there is $T$ such that $F^t(U)\subseteq U\ \forall t\geq T$. Could one then splice $F^T$ and $F$ together in some manner to create an aggregate map that is onto $U$? Since $F^t$ is only defined for positive integers, splicing continuously in the domain is impossible. The obvious alternative is to splice in the range. That is one might consider a splice of the form

$$\beta(\sigma)F^T(\sigma)+[1-\beta(\sigma)]F(\sigma),$$

replacing $F(\sigma,\beta(\sigma)T+[1-\beta(\sigma)]t)$ in the definition of $G$ in Theorem 3. The difficulty is that there does not seem to be any natural condition ruling out $\sigma=\beta(\sigma)F^T(\sigma)+[1-\beta(\sigma)]F(\sigma)$.[11] Unlike when we convexified in the range, we can no longer conclude that such a fixed point of the map would correspond to a cycle of $F$.

The assumption that $F$ is continuous is particularly bothersome in this context. A very natural discrete time model is one in which in each period one of a finite set of players switch pure strategies based on their experiences. As an example, consider the game of Fig. 8.

| | | *II* | |
|---|---|---|---|
| | | *L* | *R* |
| *I* | *L* | 0,0 | 4,2 |
| | *R* | 2,4 | 0,0 |

FIGURE 8: A game for which a particular discrete time dynamic (see text) is discontinuous, but nonetheless, Theorem 10 can still be made to apply.

Assume pairs of players are drawn from a single population, so that the state space is simply [0,1]. If less than 1/3 of the population plays $R$, $R$ is the best response. If more than 1/3 of the population plays $R$, $L$ is the best response. Consider a dynamic in which if $R$ is strictly better than $L$, one $L$ player per period switches from $L$ to $R$, and conversely if $L$ is strictly better than $R$. Under this dynamic, a small interval around $\sigma(R)\ =\ 1/3$ is asymptotically stable. However, $F$ will be discontinuous at $\sigma(R)=1/3$, with the system overshooting $\sigma(R)=1/3$ when it starts sufficiently close by on each side.

For this case, the theorem can still be made to apply: In particular, the only discontinuity is within the asymptotically stable set. Since the construction of Theorem 10 pastes the (continuous) discrete canonical dynamic onto a neighborhood of the asymptotically stable set, $G$ will be continuous despite the discontinuity in $F$.

More generally, however, it is difficult to see how to incorporate this sort of discontinuity into our theory. For example, if the example above is modified to allow separate populations for players $I$ and $II$, then the discontinuity will not be confined to the asymptotically stable set, and we are at an impasse.

## X. Set Valued Dynamics

The possibility that non–modelled factors might affect the direction of movement leads us to set valued maps $F$. I.e., if $\sigma$ is the position of the system at time 0, then $F(\sigma,t)$ is the set of possible positions at time $t$. Now, to have any hope of applying a fixed point theorem, we need $F(.,t)$ to be well behaved. For $t$ large, this seems unreasonable. Even if the set of directions in which a particle can move is (for example) convex at any instant, $F(\sigma,t)$ can be badly behaved for $t$ large. Consider the system of Fig. 9.

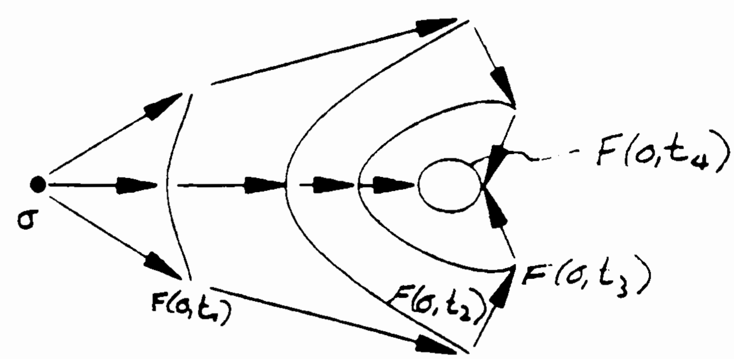



FIGURE 9: While $f(\sigma)$ is convex at $t$=0, and single valued thereafter along each path, $F(\sigma,t_4)$ is homeomorphic to a circle.

Starting from $\sigma$, the system can move in any direction in the convex fan indicated. At every other point, $f(\sigma)$ is single valued. Despite this, $F(\sigma,t_4)$ is homeomorphic to a circle.

It does seem more reasonable to ask that short term behavior is well behaved. Thus, we assume that there is $\underline{t}>0$ such that for $0<t<\underline{t}$, and $\sigma\in\Phi$, $F(\sigma,t)$ is acyclic (has the same homology groups as a singleton).

DEFINITION 8: A set $Y \subseteq \Phi$ is *asymptotically stable* under the dynamic $F$ if it is closed and there is a neighborhood $Z$ of $Y$ such that

(8.1) for every neighborhood $W$ of $Y$ with $W \subseteq Z$, there is a neighborhood $V$ of $Y$ with $F(V,t) \subseteq W$ for all $t \geq 0$, and

(8.2) for each $\gamma \in Z$, $\lim_{t \to \infty} D(F(\gamma,t),Y)=0$,

where $D(F(\gamma,t),Y) \equiv \sup_{\sigma \in F(\gamma,t)} D(\sigma,Y)$.

DEFINITION 9: For $F$ a set valued dynamic, $f$ is defined by

$$f(\sigma)=\{d \mid d=\lim_{t \downarrow 0} \frac{\sigma'-\sigma}{t}, \text{ where } \sigma' \in F(\sigma,t) \ \forall t\}.$$

We assume that $F(.,.)$ is upper-hemicontinuous, and that $F(\sigma,0)=\sigma \ \forall \ \sigma \in \Phi$.

DEFINITION 10: A set valued dynamic $F$ is a *myopic adjustment dynamic* if for all $\sigma \in \Phi$

(10.1) $d_i \cdot \pi_i(\sigma \backslash .) \geq 0 \ \forall \ d_i \in f_i(\sigma)$ and $\forall \ i \in N$, and

(10.2) if $\sigma \in N(S,\pi)$ then $\sigma \in F(\sigma,t) \ \forall t \geq 0$.

THEOREM 11: *Let $F$ be a set valued myopic adjustment dynamic satisfying the above assumptions, and let $\Theta$ be asymptotically stable under $F$. Assume there is $U$ a neighborhood of $\Theta$ in $\Phi$ which is homeomorphic to $\Phi$ and forward invariant under $F$. Then, $\Theta$ contains a hyperstable subset.*

See the appendix for the proof. The assumption that $U$ is forward invariant can be dropped if we are willing to assume $F(.,t)$ acyclic for $t$ large.

## XI. CONCLUSION

We have shown that there are conditions under which asymptotic stability of behavior under a dynamic adjustment process can imply behavior that is as if the members of the economy satisfied all the rationality and commonality of beliefs assumptions that underlie traditional game theory. The results in this paper, while hardly complete, do cover a wide class of situations.

It seems likely that the analysis could be adapted to dynamics that have other state spaces or do not satisfy the myopic improvement condition (2.1). Given a dynamic $F$ on some state space, the key is the construction of another dynamic for nearby games that (a) stops only on states that correspond in some way to Nash equilibria, and (b) does not cancel $F$ on some region playing the role of $B$. The combination of the canonical dynamic and the myopic adjustment condition is one way of doing this. There are surely others.

The existing treatment of discrete dynamics is unsatisfactory. It seems possible that a different framework might prove productive. Finally, a treatment of stochastic dynamics would be valuable.

## APPENDIX

PROOF THAT $E^{-1}(U)$ IS HOMEOMORPHIC TO $\Phi'$: (Nothing in this proof is fundamental to understanding the results of the paper.) First, assume that $X \subseteq \mathbf{R}^n$ has an interior, and that $h:X \to B^n$ is a homeomorphism. Let $x \in Int(X)$ (rel $\mathbf{R}^n$). We will show that $h(x) \in Int(B^n)$ (rel $\mathbf{R}^n$). We first argue that if $x \in Int(X)$ (rel $\mathbf{R}^n$) then, $X\backslash x$ is not contractible. We start from the fact that for any $x \in \mathbf{R}^n$, and any $\varepsilon>0$, $B_\varepsilon(x)\backslash x$ is not contractible. This is standard from algebraic topology. So, since $x \in Int(X)$, $\exists \varepsilon>0$ s.t. $B_\varepsilon(x) \subseteq X$. Assume $X\backslash x$ is contractible, i.e., assume there is $H:X\backslash x \times [0,1] \to X\backslash x$ s.t.

(1) $H(y,0) = y \ \forall \ y \in X\backslash x$, and

(2) there is $z \in X\backslash x$ such that $H(y,1) = z \ \forall \ y \in X\backslash x$

For $y$ in $B_\varepsilon(x)\backslash x$ and $t$ in $[0,1]$, define $\bar{H}(y,t)$ as the closest point in $B_\varepsilon(x)$ to $H(y,t)$. $\bar{H}$ is continuous since $H$ is continuous, $B_\varepsilon(x)$ is strictly convex, and distance is convex. If $H(y,t) \in B_\varepsilon(x)$, then $H(y,t) = \bar{H}(y,t) \neq x$, while if $H(y,t) \notin B_\varepsilon(x)$, then $\bar{H}(y,t) \in \partial B_\varepsilon(X) \neq x$. Thus, $\bar{H}$ is onto $B_\varepsilon(X)\backslash x$ and so $\bar{H}$ induces a contraction of $B_\varepsilon(X)\backslash x$, which is a contradiction.

So, assume the theorem is false. Then, for some $x \in Int(X)$, $h(x) \in \partial B^n$. Consider the restriction of $h$ to $X\backslash x$. This is a homeomorphism between $X\backslash x$ and $B^n\backslash h(X)$. But, $X\backslash x$ is not contractible, while $B^n\backslash h(x)$ is convex and so trivially contractible. As contractibility is a topological invariant, we have a contradiction.

So, choose $D$ and $h$, where $D$ is a strictly convex compact set and $h:U \to D$ is a homeomorphism (to see that we can choose $D$ strictly convex note that $\Phi$ is homeomorphic to $B_\varepsilon(\sigma)$ for any $\sigma \in \text{Int}(\Phi)$ and $\varepsilon>0$: for any given ray from $\sigma$, points which are a fraction $\lambda$ along the ray to the boundary of $D$ get mapped to points which are a fraction $\lambda$ of the way to the boundary of $B_\varepsilon(\sigma)$). Define $j:E^{-1}(U) \to D \times [0,1]$ by $j(\sigma)=(h(E(\sigma)),\sigma(r_i))$. Then, $j$ is a homeomorphism between $E^{-1}(U)$ and $K \equiv j(E^{-1}(U))$. Define $A:D \to [0,1]$ by $A(b)=\max\ \{a \mid (b,a) \in K\}$. Note that an equivalent definition of $A(b)$ is $\max\{a \mid \exists \sigma \in E^{-1}(h^{-1}(b))$ with $\sigma(r_i)=a\}$, from which it is clear that $A$ is continuous, and that $K=\{(b,a) \mid b \in D,\ a \in [0,A(b)]\}$. From above, boundary points of $D$ correspond to boundary points of $U$. By the equivalent definition of A, it is thus clear that $A(b)=0 \Rightarrow b$ on the boundary of $D$. Define $K^c$ as the convex hull of $K$. $K^c$ is compact. Define $A^c:D \to [0,1]$ by $A^c(b)=\max\ \{a \mid (b,a) \in K^c\}$. If $A(b)=0$, then since $b$ is on the boundary of $D$, and since $D$ is strictly convex, $b$ cannot be obtained as any nontrivial convex combination. Thus, $A^c(b)=0$ also. Since $A^c \geq A$, we have that $A^c(b)=0 \Leftrightarrow A(b) = 0$. Define $m{:}K \to K^c$ by

$$m(b,a) = \begin{cases} (b,(A^c(b)/A(b))a) & \text{for } A(b)>0 \\ (b,a) & \text{else.} \end{cases}$$

$m$ is clearly one to one and onto. It is trivially continuous at any $(b,a)$ where $A(b)>0$. So, let $\{(b^k,a^k)\}_{k\in N}$ be a sequence from $K$ with limit $(b,a)$, where $A(b)=0$. Then, $a=0$ must hold, and so $m(b,a) = (b,0)$. But, since $K^c$ is compact, and since $A^c(b) = 0$ also holds, $m(b^k,a^k) \to (b,0)$. Thus, $m$ is continuous. Continuity of $m^{-1}$ is entirely analogous. So, $K^c$ is a compact convex set, and $j \circ m$ a homeomorphism between $E^{-1}(h^{-1}(U)$ and $K^c$. Since $K^c$ is homeomorphic to $\Phi'$, we are done. ∎

PROOF OF THEOREM 4: Let $\rho$ be an arbitrary KM perturbed payoff function. Then, there is $\delta=(\delta_1,\ldots,\delta_n)$, with $\delta_i \in (0,1)$ $\forall i \in N$ and $\gamma=(\gamma_1,\ldots,\gamma_n)$ with $\gamma_i \in \mathrm{Int}(\Delta_i)$ $\forall i \in N$, such that for each $\sigma \in \Phi$,

$$\rho(\sigma)=\pi((1-\delta)\sigma+\delta\gamma),$$

where $(1-\delta)\sigma+\delta\gamma$ is a convenient shorthand for

$$((1-\delta_1)\sigma_1+\delta_1\gamma_1,\ldots,(1-\delta_n)\sigma_n+\delta_n\gamma_n).$$

For $\sigma \in \Phi$, define $f_\rho(\sigma)$ by

$$f_{\rho i}(\sigma) = f_i((1-\delta)\sigma+\delta\gamma\backslash\sigma_i),\ i \in N. \tag{*}$$

Then, $f_\rho$ inherits Lipschitz continuity from $f$. If $\sigma_i(s_i)=0$ then $((1-\delta)\sigma+\delta\gamma\backslash\sigma_i)_i(s_i)=0$, and so

$$f_{\rho i}(\sigma)(s_i) = f_i((1-\delta)\sigma+\delta\gamma\backslash\sigma_i)(s_i) \geq 0$$

since $F$ is onto $\Phi$. Further, $\sum_{s_i \in S_i} f_{\rho i}(\sigma)(s_i)=0$. Thus $f_\rho$ has a unique and continuous solution $F_\rho{:}\Phi\times\mathbb{R}_+ \to \Phi$. Finally,

$$\begin{aligned} f_{\rho i}(\sigma)\cdot\rho_i(\sigma) &= f_i((1-\delta)\sigma+\delta\gamma\backslash\sigma_i)\cdot\pi_i((1-\delta)\sigma+\delta\gamma\backslash.) \\ &= f_i((1-\delta)\sigma+\delta\gamma\backslash\sigma_i)\cdot\pi_i(((1-\delta)\sigma+\delta\gamma\backslash\sigma_i)\backslash.) \\ &\geq 0 \end{aligned}$$

since $f$ satisfies (2.1). So, $f_\rho$ satisfies (2.1) relative to $\rho$.

Let $\rho \to \pi$. From (*) and continuity of $f$, $f_\rho \to f$ pointwise. Since $\Phi$ is compact, and each $f_\rho$ and $f$ is continuous, $\|f_\rho - f\| \to 0$.

Chose $V$ a neighborhood of $\Theta$ such that $F(V,t) \subseteq U$ $\forall t \geq 0$. Choose any $\lambda>0$ such that $B_\lambda(\Theta) \subseteq \mathrm{Int}(V)$. We will show that sufficiently close by KM perturbed games $(S,\rho)$ have Nash equilibria in $B_{2\lambda}(\Theta)$. Now, for each $\rho$, $F_\rho$ is an approximate solution to $F$. Thus, (see, for example, Coddington and Levinson [1955, page 8, Theorem 2.1].) for all $t \geq 0$, and for all $\sigma \in \Phi$,

$$|F(\sigma,t)-F_\rho(\sigma,t)| \leq \frac{\|f-f_\rho\|}{b}\left[e^{bt}-1\right],$$

where $b$ is the Lipschitz coefficient for $f$. Choose $T>0$ such that $F(U,t)\subseteq B_{\lambda/2}(\Theta)$ $\forall t\geq T$. Then in particular, if $f_\rho$ is sufficiently close to $f$, then $\forall$ $t\leq 2T$, and $\forall$ $\gamma\in U$,

$$\|F(\gamma,t)-F_\rho(\gamma,t)\|\leq\lambda/2.$$

Choose any such $\rho$. Consider any $t\geq T$ and $\gamma\in U$. Now, $F(\gamma,T)\in B_{\lambda/2}(\Theta)$ and $\|F(\gamma,T)-F_\rho(\gamma,T)\|\leq\lambda/2$ by construction. Thus, $F_\rho(\gamma,T)\in B_\lambda(\Theta)$. Since $B_\lambda(\Theta)\subseteq V$, we can apply the same argument repeatedly to conclude $F_\rho(\gamma,kT)\in B_\lambda(\Theta)$, where $k$ is the integer such that $2T>t\text{-}Kt\geq T$. Now, since $t\text{-}Kt\geq T$, $F[F_\rho(\gamma,kT),t\text{-}kT]\in B_{\lambda/2}(\Theta)$, and since $t\text{-}kT<2T$,

$$\|F_\rho[F_\rho(\gamma,kT),t\text{-}kT]\text{-}F[F_\rho(\gamma,kT),t\text{-}kT]\|\leq\lambda/2.$$

Finally, note that $F_\rho[F_\rho(\gamma,kT),t\text{-}kT]=F_\rho(\gamma,t)$. Thus, $F_\rho(U,t)\subseteq V$ $\forall t\geq T$. Also, $F_\rho$ has no rest points on $U\backslash B_\lambda(\Theta)$.

Choose any such $\rho$. Partition $U$ and define $G_\rho$ as in the proof of Theorem 3, with $B_\lambda(\Theta)$ playing the role of $\Theta$ and $F_\rho$ the role of $F$, and with $T$ taken from the above construction. By choosing $\varepsilon$ small enough in this construction, this can be done such that $A\cup B\subseteq B_{2\lambda}(\Theta)$. By the same analysis as in Theorem 3, there is $\sigma\in A\cup B$, $\sigma$ a rest point of $G_\rho$. Then $g_\rho(\sigma)=0$ and so

$$(1-\alpha(\sigma))c_\rho(\sigma)\cdot\rho(\sigma\backslash.)+\alpha(\sigma)f_\rho(\sigma)\cdot\rho(\sigma\backslash.) = g_\rho(\sigma)\cdot\rho(\sigma\backslash.) = 0.$$

The second term of the LHS is non-negative since $f_\rho$ satisfies (2.1) relative to $\rho$. Thus, the first term of the LHS must be 0. As in Theorem 3, $\alpha(\sigma)<1$ must hold, since $\alpha(\sigma)=0$ would imply a rest point of $F_\rho$ on $B\subseteq U\backslash B_\lambda(\Theta)$. Thus $c_\rho(\sigma)\cdot\rho(\sigma\backslash.) = 0$ and so $\sigma\in N(S,\rho)$. Since $A\cup B\subseteq B_{2\lambda}(\Theta)$, we are done. ∎

PROOF OF THEOREM 6: Let $(S,\pi_z)$ be obtained from $(S,\pi)$ be subtracting a positive constant $z$ from $\pi_i(\sigma\backslash r_i)$ for $i\in N$, $\sigma\in\Phi$, and $r_i\in R_i$. In $(S,\pi_z)$, elements of $R_i$ are strictly dominated, and so there are no Nash equilibria of this game that put positive weight on $\cup_{i\in N}R_i$. Thus, $F$ satisfies (2.2) for $(S,\pi_z)$. As $f(\sigma)(r_i)\leq 0$ for $r_i\in R_i$, $F$ also satisfies (2.1). Thus by Theorem 3, $\Theta$ contains a hyperstable subset for $(S,\pi_z)$. Consider any small KM perturbation of the original game. Note that a weakly dominated strategy remains weakly dominated in KM-perturbations. Subtract $z$ as before to create a small perturbation of $(S,\pi_z)$ that thus has a Nash equilibrium $\gamma$ near $\Theta$. $\gamma$ does not use elements of $\cup_{i\in N}R_i$. Add back $z$. Since the elements of $R_i$ remain weakly dominated, $\gamma$ remains a Nash equilibrium. ∎

PROOF OF THEOREM 8: Define $Q_\rho:\Phi\times\Psi\times\mathbb{R}_+\to\Phi\times\Psi$ by $Q_\rho(\sigma,\psi,t)=(C_\rho(\sigma,t),\psi)$.

Choose $\lambda>0$. We will show that for $\rho$ sufficiently close to $\pi$, $N(S,\rho)\cap B_\lambda(P_\Phi(\Theta))\neq\varnothing$. Choose $V\subseteq\Phi\times\Psi$ partition $U$, and choose $\alpha$ and $\beta$ as before. By compactness of $\Psi$, we can choose $\varepsilon$ such that $B_{3\varepsilon}(\Theta)\subseteq\text{Int}(V)\cap B_\lambda(\Theta)$. Define $G_\rho$ as before, substituting $Q_\rho$ for $C_\rho$. As before, for small $t$, $G(.,t)$. is a continuous map from $U$ to $U$ and so has a fixed point $\sigma'$ for each $t$ sufficiently small. Since there is no assumption that $\Psi$ is finite dimensional, the relevant fixed point theorem is Schauder's (see for example Deimling, [1985, p. 60]) rather than Brouwer's.[12] As before, a fixed point of $G(.,t)$ on $C \cup D \cup E$ is a positive length cycle of $F$, contradicting asymptotic stability, and so there is a fixed point $\sigma'$ of $G(.,t)$ in $A \cup B$ for $t$ sufficiently small and so (again using compactness of $\Psi$) a rest point $\sigma$ of $G$.

It remains to show $\sigma \notin B$. So, for any $\gamma \in B$ consider

$$g(\gamma) \equiv \lim_{t \downarrow 0} \frac{P_\Phi(G(\gamma,t)-\gamma)}{t}$$

$$= \alpha(\gamma)f(\gamma)+[1-\alpha(\gamma)]c_\rho(\gamma).$$

This expression is identical to the corresponding expression in Theorem 3. Thus, $\sigma \in A$. But, then by definition of $Q_\rho$, $P_\Phi(\sigma) \in N(S,\rho)$. Finally, since $\sigma \in A \subseteq B_\lambda(\Theta)$, $P_\Phi(\sigma) \in B_\lambda(P_\Phi(\Theta))$. ∎

PROOF OF THEOREM 10: For $\sigma \in \Phi$ define $\lambda(\sigma)$ by $\lambda(\sigma) = \max\{\lambda \mid 0 \le \lambda \le 1,\ \sigma + \lambda c_\rho(\sigma) \in \Phi\}$. For $\Lambda \ge 0$, the *discrete canonical dynamic with scaling factor* $\Lambda$ for $(S,\rho)$ is given by

$$J_\rho(\sigma) = \sigma + \left(\frac{\lambda(\sigma)}{\Lambda}\right) c_\rho(\sigma).$$

The role of $\Lambda$ will be clear shortly. By construction, $J_\rho(\sigma)$ maps $\Phi$ to $\Phi$. Continuity of $J_\rho$ follows from the continuity of $\lambda$ and $c_\rho$. Since $c_\rho$ defines a continuous dynamic remaining within the simplex, $\lambda(\sigma)>0$ $\forall\sigma$. Thus

$$\rho_i(\sigma \backslash J_{\rho i}(\sigma)) - \rho_i(\sigma) = \left(\frac{\lambda(\sigma)}{\Lambda}\right) c_{\rho i}(\sigma)\, \mathrm{p}(\sigma \backslash .) \ge 0$$

with equality if and only if $\sigma \in N(S,\rho)$. As before, consider an arbitrary neighborhood $M$ of $\Theta$. Let $\varepsilon>0$ be such that $B_{3\varepsilon}(\Theta) \subseteq M \cap U$. Partition $U$ as follows

| | |
|---|---|
| $A$ | $B_\varepsilon(\Theta)$ |
| $B$ | $\{\gamma \mid \varepsilon \le D(\Theta,\gamma) \le 2\varepsilon\}$ |
| $C$ | $U \cap \{\gamma \mid 2\varepsilon \le D(\Theta,\gamma)\}$ |

Chose $\alpha: U \to [0,1]$ with $\alpha(C)=1$, $\alpha(A)=0$, and $\alpha$ continuous. Then, for any given $\Lambda$, consider the following spliced map:

$$G(\sigma) = \begin{cases} J_\rho(\sigma) & \text{on } A \\ (\sigma)F(\sigma)+[1-\alpha(\sigma)]J_\rho(\sigma) & \text{on } B \\ F(\sigma) & \text{on } C \end{cases}$$

Continuity of $G$ is again clear from the continuity of $J_\rho$, $\alpha$, $F$, and $\beta$, and from the agreement of the appropriate functions on $A \cap B$, $B \cap C$, $C \cap D$, and $D \cap E$.

Now, $c_\rho$ is bounded, and therefore

$$D(\sigma, J_\rho(.)) = D(0, \lambda(\sigma) c_\rho(\sigma)/\Lambda)$$

is also bounded for any fixed $\Lambda$. Choosing $\Lambda$ large enough, we can conclude that $J_\rho(A \cup B) \subseteq U$. From the convexity of $U$, we can conclude $G(A \cup B) \subseteq U$ and so $G(U) \subseteq U$.

We can now argue as before that $G(.)$ has a rest point in $A \cup B$, and that for $\rho$ close enough to $\pi$, this rest point must actually be in $A$, and so a Nash equilibrium of $(S, \rho)$. ■

PROOF OF THEOREM 11: Chose $\varepsilon>0$ such that $B_{3\varepsilon}(\Theta) \subseteq \text{Int}(U)$. Define $A$, $B$, and $C$ as in the proof of Theorem 10, and choose $\alpha$ as before. For given $\rho$, consider $G_\rho: U \times [0, \underline{t}) \to \Phi$ given by:

$$G_\rho(\sigma, t) = \begin{cases} C_\rho(\sigma, t) & \text{on } A \\ \alpha(\sigma) F(\sigma, t) + [1 - \alpha(\sigma)] C_\rho(\sigma, t) & \text{on } B \\ F(\sigma, t) & \text{on } C \end{cases}$$

$G_\rho(., t)$ is upper hemicontinuous and has acyclic values. $G_\rho(A \cup B, t) \subseteq U$ for $t$ sufficiently small because $G_\rho$ is upper hemicontinuous, $B_\varepsilon(A \cup B) \subseteq U$, and $G_\rho(\sigma, 0) = \sigma \ \forall \sigma$. Since $U$ is forward invariant, $G_\rho(., t): U \to U$ and so, by the Eilenberg-Montgomery fixed point theorem (Border, [1985, p. 73]) has a fixed point $\sigma^t$ for each $t$ sufficiently small.

A fixed point of $G_\rho(., t)$ on $C$ is impossible as $C \subseteq U \backslash \Theta$ and $U$ is in the basin of attraction of $\Theta$ under $F$. Thus, $\sigma^t \in A \cup B$ for $t$ sufficiently small. Taking a subsequence if needed, let $\sigma_\rho = \lim_{t \downarrow 0} \sigma^t$.

We show that $\sigma_\rho$ is a rest point of $G_\rho$, i.e., that $\sigma_\rho \in G_\rho(\sigma_\rho, t') \ \forall t' \geq 0$. For any $t \in [0, \underline{t})$ and $t'>0$ note

$$G_\rho(\sigma^t, r(t)) \subseteq G_\rho(\sigma^t, t'), \tag{†}$$

where $r(t) = \min\{t^* \mid t^* \geq 0,\ t^* = t' - kt \text{ for } k \in \mathbf{N}\}$. Let $t \downarrow 0$ and take a sequence from the LHS. By upper hemicontinuity, since $(\sigma^t, r(t)) \to (\sigma^\rho, 0)$ this converges to an element of $G(\sigma^\rho, 0)$, i.e., to $\sigma^\rho$. But, then by (†) and upper hemicontinuity of $G$, $\sigma^\rho \in G(\sigma^\rho, t')$.

Let $\sigma$ be a cluster point of $\{\sigma^\rho\}_{\rho \to \pi}$. Then, $\sigma$ is a rest point of $G_\pi$. Also, if $\alpha(\sigma)=1$, then $\sigma \in B$, and $G_\pi(\sigma, .) = F(\sigma, .)$. Since $F$ has no rest points on $B$, $\alpha(\sigma)<1$. So, consider,

$$g_\pi(\sigma) = \alpha(\sigma) f(\sigma) + [1 - \alpha(\sigma)] c_\pi(\sigma)$$

Since $\sigma$ is a rest point of $G$, $0 \in g_\pi(\sigma)$. But, then for some $d \in f(\sigma)$,

$$\alpha(\sigma) d + [1 - \alpha(\sigma)] c_\pi(\sigma) = 0$$

Using (9.1) and $\alpha(\sigma)<1$, this implies $c_\pi(\sigma)=0$, and so $\sigma \in N(S, \pi)$. By (9.2), $N(S, \pi) \cap A \cup B \subseteq \Theta$. Thus, $\sigma \in \Theta$. As before, we are done. ■

## REFERENCES


BORDER, K. (1985), *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, Cambridge, UK.

CODDINGTON, E. AND LEVINSON, N. (1955), *Theory of Ordinary Differential Equations*, McGraw-Hill, New York.

DEIMLING, K. (1985), *Nonlinear Functional Analysis*, Springer–Verlag: Berlin

FOSTER, D. AND P. YOUNG (1990), "Stochastic Evolutionary Game Dynamics," *Theoretical Population Biology*, 38, 219–232.

FRIEDMAN, D. (1991), "Evolutionary Games in Economics," *Econometrica*, 59, 637–666.

HILLAS, J., (1990), "On the Definition of the Strategic Stability of Equilibria," *Econometrica*, 58, 1365–1390.

KANDORI M., MAILATH, G., AND R. ROB (1992), "Learning, Mutation, and Long Run Equilibria in Games," CARESS working paper #91–01R, University of Pennsylvania.

KREPS, D. AND R. WILSON (1982), "Sequential Equilibria," *Econometrica*, 50, 863–894.

KOHLBERG, E. AND J.F. MERTENS (1986), "On the Strategic Stability of Equilibria," *Econometrica*, 54, 1003–1037.

MAYNARD SMITH, J. (1974), "The Theory of Games and the Evolution of Animal Conflicts," *J. Theoretical Biology*, 47, 209–221.

MAYNARD SMITH, J. (1982), *Evolution and the Theory of Games*, Cambridge University Press, Cambridge, UK.

MAYNARD SMITH, J. AND G. PRICE (1973), "The Logic of Animal Conflict," *Nature*, 246, 15–18.

MYERSON, R. (1978), "Refinement of the Nash Equilibrium Concept," *Int. J. Game Theory*, 7, 73–80.

SWINKELS, J. (1991a), "Evolutionary Stability with Equilibrium Entrants," Research Report #9, Stanford Institute for Theoretical Economics. Forthcoming *J. Econ. Theory*.

SWINKELS, J. (1991b), "Stability and Evolutionary Stability," Research Report #10, Stanford Institute for Theoretical Economics. Forthcoming *J. Econ. Theory*.

TAYLOR, P., AND L. JONKER (1978), "Evolutionarily Stable Strategies and Game Dynamics," *Mathematical Biosciences*, 40, 145–156.

VAN DAMME, E. (1984), "A Relation Between Perfect Equilibria in Extensive form Games and Proper Equilibria in Normal Form Games," *Int. J. Game Theory*, 13, 1–13.

VAN DAMME, E. (1991), *Stability and Perfection of Nash Equilibria*, 2nd Ed., Springer–Verlag: Berlin.

## ENDNOTES

1. Aside from connections to stability under any particular dynamic process, evolutionary stability (and its offspring) can also be interpreted as strong forms of the familiar 'no profitable entry' condition. This may help explain the considerable appeal of evolutionary stability to economists despite its unsatisfactory dynamic foundations.

2. Stability is an overworked term in economics in general, and this paper in specific. To avoid terminological confusion, stability in the sense of Kohlberg and Mertens [1986] henceforth will be referred to as *KM stability*. The general area pioneered by Kohlberg and Mertens will be referred to as *strategic stability*.

3. A proviso: EES comes in two flavors: in one, which applies only to symmetric games, players for the various player positions are modelled as being drawn from a single population. In the other, players for different player positions are drawn from independent populations. Sets which satisfying the former version will (under the additional conditions) contain a subset which satisfies a symmetric version of strategic stability. Sets which satisfy the later condition will (under the additional conditions) contain a subset satisfying the original definition of stability. In either case, the set will contain a proper element.

4. This was also the motivation for the set valued notions of evolutionary stability introduced in Swinkels [1991a].

5. An alternative specification has the right hand side of the previous expression divided through by $\pi_i(\gamma)$ (where one imposes the condition that $\pi_i(\gamma)>0$ for all $\gamma$). For symmetric games in which players are drawn from a single population, this changes the speed but not the direction of the dynamic at each point. The solution curves are thus invariant. When populations corresponding to different player positions evolve independently, the difference between the two specifications clearly matters. Our analysis covers either case.

6. Under weak compatibility, $f_i(\sigma)=0$ must hold any time all non-extinct strategies for position $i$ are performing equally well. Further, whenever all non-extinct strategies for position $i$ are not performing equally well, the inequality in (2.1) is required to be strict.

7. They also show that in the 'regular' case, the converse holds. While regularity is generically satisfied for normal form games, the set of normal form games generated from any given non-trivial extensive form is itself zero measure. Thus, one cannot conclude that regularity is satisfied by 'most' of the equilibria of games we find interesting. See also the discussion following Theorem 1.

8. This consideration of the extensive form also tells us that, while formally quite simple, the relaxation of weak compatibility implicit in Definition 2 is quite important. Consider any outcome in the extensive form which does not reach every information set. If the outcome can be supported by a Nash equilibrium, then any strategy profile yielding this outcome has the property that all strategies for any player position present in positive measure for any strategy profile consistent with this equilibrium must perform equally well. Weak compatible dynamics must be at rest in those circumstances. Thus, for an outcome to be asymptotically stable under a weak compatible dynamic, the corresponding set of strategy profiles must include all strategy profiles generating this outcome. By the fact that (2.2) must be strict whenever not all strategies present in positive measure for a player position are performing equally well, and by piecewise differentiability of the gradient field (also assumed by Friedman), if this set contains any non-Nash equilibrium outcomes, then the set cannot be asymptotically stable. Thus, except in the most trivial cases, an outcome which does not reach every information set could never be asymptotically stable under a weak compatible dynamic.

9. Whether the results can be extended to Hillas stable sets remains open.

10. Generalizing the concept of a Lyapunov function for asymptotically stable sets is straightforward. For this state space, say that $\Lambda$ is a Lyapunov function for the set $\Theta\subseteq\Phi$ relative to the dynamic system $F$ if

(1) the set of minimizers of $\Lambda$ is precisely $\Theta$, and

(2) $\dot{\Lambda}(\sigma)\equiv\left.\frac{\partial\Lambda(F(\sigma,t',t)}{\partial t}\right|_{t=t'} < 0$ for all $\{\sigma,t'\}\in U\backslash\Theta\times\mathbb{R}^{+}$,

where the derivative is interpreted as a right hand derivative if needed. Then, $\Theta$ is asymptotically stable under $F$. Condition (2) implies that $F$ has no recurrent points in $U\backslash\Theta$.

11. More correctly, there does not seem to be any natural condition ruling out $\sigma=\beta(\sigma)F^T(\sigma)+[1-\beta(\sigma)]F(\sigma)$ which is not stronger than the condition assumed. In particular, if the dynamic system has an associated Lyapunov function $\Lambda$ with the property that $\Lambda((1-\alpha)\sigma+\alpha(\gamma))\leq\min(\Lambda(\sigma),\Lambda(\gamma))$, then clearly $\sigma=\beta(\sigma)F^T(\sigma)+[1-\beta(\sigma)]F(\sigma)$ could never arise, since $\Lambda(F^T(\sigma))<\Lambda(\sigma)$, $\Lambda(F(\sigma))<\Lambda(\sigma)$ and so $\Lambda(\beta(\sigma)F^T(\sigma)+[1-\beta(\sigma)]F(\sigma)) < \Lambda(\sigma)$. However, in such a circumstance, any set of the form $\{\sigma \mid \Lambda(\sigma)\leq b\}$, $b>0$, is a forward invariant neighborhood of $\Theta$ which is (trivially) homeomorphic to a compact convex set.

12. Schauders theorem only requires $\Psi$ to be a closed and bounded subset of a Banach space. We use the (stronger) compactness condition elsewhere in the proof. Schauders theorem also requires a compact (rather than merely continuous) map. Since $\Psi$ is a compact space, continuous maps on $\Phi\times\Psi$ are compact maps.