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ON CORES IN ECONOMIES WITH PUBLIC GOODS

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The behavior of the core of a private-goods economy as the number of agents is increased and the relationship between the core and the market equilibrium allocations in such economies have been carefully studied and are now well-understood. In this paper we investigate the corresponding issues in the presence of public goods.

Section I contains basic definitions and notation.

In Section II we consider the impact of increasing the number of agents on the core of an economy with public goods. As is well-known (Debreu and Scarf [3]), replicating a private-goods economy never causes the core to expand and, when the number of private goods is at least two, generally causes the core to shrink. We demonstrate that with public goods the situation is much different: if we replicate an economy with one private good and at least one public good, the core never shrinks and may well expand! Moreover, again in sharp contrast to the results obtained without public goods, the continuous representation of the core of a finite economy with public goods is a (generally proper) subset of the core of the continuous representation of that economy. (These terms are defined below. Note that the results depend critically on the existence of only one private good.)

These results reinforce the conclusion to be drawn from the examples given by Muench [7] and Milleron [6] that a generalization of the Debreu-Scarf [3] or Aumann [1] theorems which would link the core and Lindahl equilibria cannot be achieved if one adopts the definition of the core with public goods developed by Foley [5]. However, this natural generalization of the core concept developed for private-goods economies has been criticized frequently (see, e.g., Rosenthal [11]). In Section III we present an alternative concept under which coalitions are allowed certain powers to tax their complements to help produce public goods. This formulation appears to be promising for modeling various situations of resource allocation within the framework of a constitution, but again no interesting equivalence theorem is possible.

I. Definitions

Since we will want to consider both finite and continuum economies, we present our definitions in a general measure-theoretic context and specialize to one case or the other as desired.

An economy is an ordered quadruple

$$E = [(A, \mathcal{A}, \mu), \omega, \succsim, Z]$$

where $(A, \mathcal{A}, \mu)$ is a probability space; ω is an integrable function from A into $\mathbb{R}^n_+$ (the nonnegative orthant of $\mathbb{R}^n$); $\succsim$ is a mapping from A into the set of complete, continuous pre-orders on $\mathbb{R}^{n+m}_+$; and $Z \subseteq \mathbb{R}^{n+m}$.

Following the usual interpretation, $\mathbb{R}^{n+m}$ is the commodity space with n private goods and m public goods; A is the set of economic agents; $\mathcal{A}$ is the set of allowable coalitions; μ describes the relative sizes of the coalitions; $\left(\omega(a), 0\right)$ is the initial endowment density of agent a; $\succsim(a)$ is the preferences of a; and Z is the production set. For simplicity, we assume that the consumption set for each agent is $\mathbb{R}^{n+m}_+$.

Throughout, we assume $n \geqq 1$ and $n+m \geqq 2$. If $m \geqq 1$, E is a public-goods economy. If $m = 0$, E is a private-goods economy. An economy $[(A, \mathcal{A}, \mu), \omega, \succ, Z]$ is <u>finite</u> if A is a finite set, $\mathcal{A}$ is the power set of A and μ is the simple measure $\mu(S) = \#S/\#A$, where $\#S$ denotes the cardinality of S. (When E is finite, functions on A will be described by appropriate finite-dimensional vectors.) An economy is <u>atomless</u> if $(A, \mathcal{A}, \mu)$ is an atomless measure space (i.e., if $T \in \mathcal{A}$ and $\mu(T) > 0$, then there exists $S \in \mathcal{A}$, $S \subset T$ such that $\mu(T) > \mu(S) > 0$).

A <u>feasible allocation</u> for E is an ordered pair (x, y), where x is an integrable function from A into $\mathbb{R}_+^n$ and y is a point in $\mathbb{R}_+^m$, satisfying $\left(\int_A (x - \omega)d\mu, y\right) \in Z$. An <u>equal-treatment allocation</u> is a feasible allocation (x, y) with the property that $x(a) \sim(a)\ x(a')$ whenever a and a' are of the same type (i.e., have the same preferences and endowments). A set of allocations has the <u>equal-treatment property</u> if it consists of equal-treatment allocations only.

A feasible allocation (x, y) is <u>blocked</u> by the coalition $S \in \mathcal{A}$ if

(i) $\mu(S) > 0$

(ii) $\exists$ a feasible allocation (x', y') such that

$$\left(x'(a), y'\right) \succ(a) \left(x(a), y\right) \text{ a.e. in } S \text{ and } \left(\int_S (x' - \omega)d\mu, y'\right) \in Z.$$

The <u>core</u> is the set of feasible allocations which are not blocked. A <u>price system</u> is an ordered pair (p, π) where $p \neq 0$ is a point of $\mathbb{R}_+^n$ and π is an integrable function from A into $\mathbb{R}_+^m$. A <u>Lindahl equilibrium</u> is a quadruple $(\overline{p}, \overline{\pi}, \overline{x}, \overline{y})$ where $(\overline{p}, \overline{\pi})$ is a price system and $(\overline{x}, \overline{y})$ is a feasible allocation satisfying:

(i) for every $(z, y) \in Z$, $\left(\overline{p}, \int_A \overline{\pi}\right) \cdot (z, y) \leq \left(\overline{p}, \int_A \overline{\pi}\right)\left(\int_A (\overline{x} - \omega)d\mu, \overline{y}\right)$,

and

(ii) $(x, y) \succ(a) \left(\overline{x}(a), \overline{y}\right) \Rightarrow \overline{p} \cdot x + \overline{\pi}(a) \cdot y > \overline{p} \cdot \omega(a) = \overline{p} \cdot \overline{\mathbf{x}}(a) + \overline{\pi}(a) \cdot \overline{y}$.

If $(\bar{p}, \bar{\pi}, \bar{x}, \bar{y})$ is a Lindahl equilibrium, then $(\bar{x}, \bar{y})$ is a Lindahl allocation.

The nonemptiness of the set of Lindahl allocations is established in [9]. A stronger version of this theorem and a proof of the existence of equal-treatment Lindahl allocations are presented in [2]. Since under the assumptions of these theorems the Lindahl allocations belong to the core, these results give conditions for the nonemptiness of the core.

We shall be interested in the following assumptions on E.

A.1 (weak monotonicity for private goods)

For all $(x', y') \in \mathbb{R}_{+}^{n+m}$, $x \geqq x' \Rightarrow (x, y') \succsim(a)\ (x', y')$ for almost every $a \in A$.

A.2 (no free production)

$Z \cap \mathbb{R}_{+}^{n+m} \subset \{0\}$.

A.3 (strong monotonicity for private goods)

For all $(x', y') \in \mathbb{R}_{+}^{n+m}$, $x \geqq x' \Rightarrow (x, y') \succ(a)\ (x', y')$.

II. Expanding Cores

In this section we investigate the effect of increasing the number of agents on the core with public goods. We will limit our analysis to the case of a single private good $(n = 1)$.

We will first want to compare finite public-goods economies with differing numbers of agents, and, in particular, replica economies. To this end, given a finite economy $E = [(A, \mathcal{A}, \mu), \omega, \succsim, Z]$ with agents $A = \{a_1, \ldots, a_K\}$, we define the replica of E as the finite economy $E' = [(A', \mathcal{A}', \mu'), \omega', \succsim', Z']$ where

1) $A' = \{a_1', \ldots, a_{2K}'\}$;

2) $\omega'(a_{2k-1}') = \omega'(a_{2k}') = \omega(a_k)$, and

$\succsim'(a_{2k-1}') = \succsim'(a_{2k}') = \succsim(a_k)$, $k = 1, \ldots, K$; and

3) $Z' = Z$.

Thus, for each agent in E there are two agents in E' with the same preferences as the given agent, the same endowment as the given agent and a measure which is half the measure of the given agent. Note that the total endowment of resources of economy E' is the same as that of economy E. We will refer to the agents a_k in E and the agents a'_{2k-1} and a'_{2k} in E' as corresponding agents in the two economies. Note that in this setup, a coalition S in E with s members can provide each of its members a_k with (x_k, y) if and only if the coalition S' in E' made up of the 2s agents corresponding to the members of S can provide each of its members a'_{2k-1} and a'_{2k} with (x_k, y). To simplify notation, we will for the remainder of this section use primes to denote entities associated with the replica economy, and will index A and A' with the first K and the first 2K integers, respectively.

Theorem 1. Let E be a finite public-goods economy with $n = 1$ in which A.1 and A.2 hold and let $(\bar{x}, \bar{y}) = (\bar{x}_1, \ldots, \bar{x}_K, \bar{y})$ be a feasible allocation in E. Then the replicated allocation $(\bar{x}', \bar{y}') = (\bar{x}'_1, \ldots, \bar{x}'_{2K}, \bar{y}')$ defined by $\bar{x}'_{2k-1} = \bar{x}'_{2k} = \bar{x}_k$, $\bar{y}' = \bar{y}$ belongs to the core of E' if and only if $(\bar{x}, \bar{y})$ belongs to the core of E.

Proof. The "only if" part is trivial.

Clearly, if $(\bar{x}_1, \ldots, \bar{x}_K, \bar{y})$ is feasible in E, then $(\bar{x}'_1, \ldots, \bar{x}'_{2K}, \bar{y}')$ is feasible in E'. Suppose that $(\bar{x}', \bar{y}')$ is blocked in E'. Then there exists a coalition S' in E' which blocks $(\bar{x}', \bar{y}')$ via the allocation $(x', y') = (x'_1, \ldots, x'_{2K}, y')$: suppose initially that a'_{2k-1} belongs to S' if and only if a'_{2k} belongs to S'. But then the allocation (x, y) in E defined by $x_k = \frac{1}{2}\left(x'_{2k-1} + x'_{2k}\right)$, $y = y'$ is feasible for the coalition S made up of the s agents in E corresponding to the 2s agents in S', while $(x_k, y) \succ(a_k) (\bar{x}_k, \bar{y})$ for all $a_k \in S$, so that $(\bar{x}, \bar{y})$ is blocked in E.

Thus, if we can show that whenever $(\bar{x}', \bar{y}')$ is blocked in E' by a coalition T' then it can also be blocked by a coalition S' of the type assumed in the previous paragraph, we will have our result. Suppose then that $(\bar{x}', \bar{y}')$ is blocked by T' via the allocation (x', y'), and that $a'_{2k-1} \in T'$, $a'_{2k} \notin T'$. We can suppose that $x'_{2k-1} \leqq \omega'_{2k-1}$, or else we could remove this agent from T' and still have a blocking coalition. $\left(\sum_{i \in S'} (x'_i - \omega'_i) > 0\right.$ is impossible from A.2.$\left.\right)$ Consider then the coalition $\tilde{T}' = T' \cup \{a'_{2k}\}$ and an allocation $(\tilde{x}', \tilde{y}')$ where $\tilde{y}' = y'$, $\tilde{x}'_j = x'_j$ for $a'_j \in T'$ and $\tilde{x}'_{2k} = \omega'_{2k}$. This allocation is clearly feasible for $\tilde{T}'$, so to check that $\tilde{T}'$ blocks $(\bar{x}', \bar{y}')$ we need only show that $(\omega'_{2k}, y') \succ(a'_{2k}) (\bar{x}'_{2k}, \bar{y}')$. To see this, **note:** 1) that $(\omega'_{2k}, y') \succsim(a'_{2k}) (x'_{2k-1}, y')$ by assumption and the facts that $\succsim(a'_{2k}) = \succsim(a'_{2k-1})$ and $\omega'_{2k} = \omega'_{2k-1}$, 2) that $(x'_{2k-1}, y') \succ(a'_{2k}) (\bar{x}'_{2k-1}, \bar{y}')$ by the assumption that T' blocks $(\bar{x}', \bar{y}')$ and by $\succsim(a'_{2k}) = \succsim(a'_{2k-1})$, **and** 3) that $(\bar{x}'_{2k-1}, \bar{y}') = (\bar{x}'_{2k}, \bar{y}')$. Q.E.D. ||

The second sort of comparison we wish to make is between a finite economy and a corresponding economy with a continuum of agents. Given a finite economy $E = [(A, \mathcal{A}, \mu)\ \omega, \succsim, Z]$ where $A = \{a_1, \ldots, a_K\}$, we define a <u>continuous</u> <u>representation</u> of E to be an atomless economy E" = $[(A'', \mathcal{A}'', \mu''), \omega'', \succsim'', Z'']$ in which

1) A" is the union of disjoint measurable sets $A''_1, \ldots, A''_K$, with $\mu''(A''_k) = 1/K$, $k = 1, \ldots, K$;
2) $\omega''(a) = \omega(a_k)$ and $\succsim''(a) = \succsim(a_k)$ for all $a \in A''_k$; and
3) $Z'' = Z$.

If $(x, y) = (x_1, \ldots, x_K, y)$ is a feasible allocation in E, define the <u>continuous</u> <u>representation</u> of (x, y) as the feasible allocation (x^c, y) in E" defined by $x^c(a) = x_k$ for all a in A''_k, $k = 1, \ldots, K$. For private-goods economies, it is clear that the core of E'' is contained in the continuous representation of the core of E. Again the situation is reversed in public-goods economies.

Theorem 2. Let E be a finite public-goods economy with $n = 1$ in which assumptions A.1 and A.2 are met. Let E'' be a continuous representation of E and let $(\overline{x}^c, \overline{y})$ be the continuous representation of a feasible allocation $(\overline{x}, \overline{y})$ in E. Then $(\overline{x}^c, \overline{y})$ belongs to the core of E'' if and only if $(\overline{x}, \overline{y})$ belongs to the core of E.

Proof. Again the "only if" part is trivial.

Suppose $(\overline{x}^c, \overline{y})$ is blocked by a coalition S'' in A'' with (x'', y''). As in Theorem 1 we can assume that $x''(a) \leqq \omega''(a)$ for all a in S''. Let $S''_k = S'' \cap A''_k$, $k = 1, \ldots, K$, and let I be the set of indices such that $\mu''(S''_k) > 0$. The idea is to use the agents a_k, $k \in I$, to block $(\overline{x}, \overline{y})$ in E. Let $S = \{a_k \mid k \in I\}$.

Note first that, with A.1, since $\overline{x}^c$ is constant on A''_k, we may assume that x'' is a constant x''_k on S''_k. Consider any feasible allocation (x, y) in E where $x_k = x''_k$, $k \in I$, and $y = y''$. Then $(x_k, y) \succ(a_k) (\overline{x}_k, \overline{y})$, $k \in I$. Thus, we need only check whether $\left(\int_S (x - \omega) d\mu, \ y\right) \in Z$. But $\frac{1}{K} = \mu(a_k) \geqq \mu''(S''_k)$, while $(x_k - \omega_k) \leqq 0$. Thus, $\int_S (x - \omega) d\mu = \sum_{k \in I} \mu(a_k)(x_k - \omega_k) \leqq \sum_{k \in I} \mu''(S''_k)(x''_k - \omega''_k) = \int_{S''} (x'' - \omega'') d\mu''$. If $\int_{S''} (x'' - \omega'') d\mu'' = \int_S (x - \omega) d\mu$, we are done. If not, distribute the excess among the members of S in any arbitrary fashion. The resultant allocation (x^*, y) is then preferred by each agent in S (by A.1) and satisfies $\left(\int_S (x^* - \omega) d\mu, \ y\right) \in Z$. Q.E.D. ||

These two results indicate that increasing the number of agents does not cause the core to shrink in public-goods economies with a single private good. In fact, one may well expect the core actually to expand. Since per-capita production costs are a decreasing function of the size of the coalition producing them, small coalitions are relatively weak. This in turn means that the equal-treatment property enjoyed by core allocations in replicas of private-goods economies does not obtain. Thus, as the number of agents increases, opportunities for discriminating between identical agents are increased, and the core expands. It is possible to establish sufficient conditions for the core to expand under replication. However, we will limit ourselves here simply to providing an example illustrating the possibility that the inclusion in both Theorems 1 and 2 may be strict.

The example is based on that provided by Muench [7]. He considers an atomless economy $E = [(A, \mathcal{A}, \mu), \omega, \succsim, Z]$ with one private and one public good in which the measure space $(A, \mathcal{A}, \mu)$ is the unit interval with Lebesgue measure, $\omega(a) \equiv 1$, $\succsim$ is constant on $[0, 1]$ and is described by the utility function $u(x, y) = x - e^{-y}$ and $Z = \{(z, y) \mid y \geqq 0,\ 2z + y \leqq 0\}$. In this economy, all core allocations involve $y = -\log \frac{1}{2}$. Thus, the core can be described solely in terms of the distribution of the private good or of utility. To do this, Muench adopts a Lorenz curve construction. Such a curve is a nondecreasing, continuous convex function from $[0, 1]$ onto $[0, 1]$ which gives the fraction of total utility possessed by the fraction of consumers with the lowest utility at a given allocation. Muench shows that the Lindahl equilibrium is the unique allocation with $x(a) \equiv 1 + \frac{1}{2} \log \frac{1}{2}$, $y = -\log \frac{1}{2}$, and that the core corresponds to all

Lorenz curves lying between $f(p) = p$, the curve corresponding to the Lindahl allocation, and

$$g(p) = \begin{cases} 0 \ , & p \leqq \frac{1}{2} \\ \dfrac{p - \frac{1}{2} + \frac{1}{2} \log \frac{1}{2p}}{\frac{1}{2} + \frac{1}{2} \log \frac{1}{2}} \ , & p > \frac{1}{2} \ . \end{cases}$$

Consider now a sequence of finite economies $E^N = [(A^N, \mathcal{A}^N, \mu^N), \omega, \succsim, Z]$, where ω, $\succsim$ and Z are as in Muench's example and A^N contains 2^N elements. Thus E^{N+1} is the replica of E^N. It is a simple matter to check that both the core and the continuous representation of the core of E^1 correspond to those Lorenz curves lying between the curves f and g which are piecewise linear on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$. The correspondence for E^2 is with those Lorenz curves in the same region that are piecewise linear on $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$ and $[\frac{3}{4}, 1]$, etc. Thus, at each N, the core of E^N corresponds to a strict subset of the core of E^{N+1}, while the core of E is a strict superset of the continuous representation of the core of each E^N. This is illustrated in figure 1, where the heavy lines are f and g, the lighter solid lines represent core allocations in E^1 (which, of course, also correspond to core allocations in E^2 and E) and the broken line represents a core allocation of E^2 which is not one in E^1.

It may be worth noting that our results are not dependent on representing finite economies by probability spaces. However, if this approach is not taken, one faces the problem that as the number of agents is increased the output of the public goods can grow without bound while the per-capita cost of producing it goes to zero. If, to handle this, one makes preferences and/or endowments dependent on the number of agents in the economy and correspondingly alters the allocations which one compares between economies (as in Milleron [6]), our results can be reproduced.

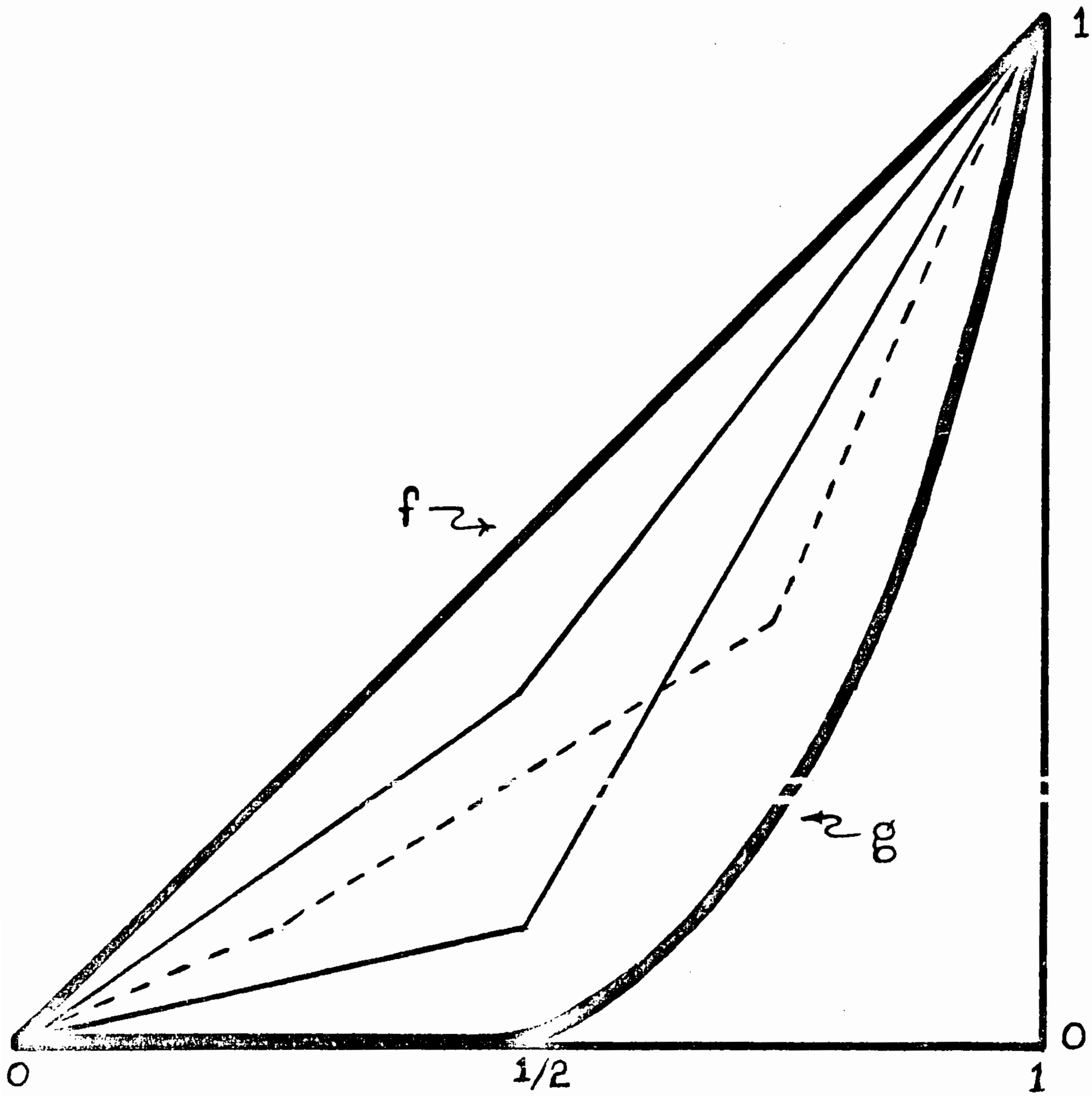

Figure 1

On the other hand, it should be clear that our results are heavily dependent on there being only one private good. If there are multiple private goods, we would expect the effect of increasing the number of agents to be ambiguous: there would be some tendency from the private-goods side for the core to contract, while the effect of the public goods would be to expand the core. The net effect would seem to be very difficult to specify in general.

III. Fiscal Laws and φ-Cores

We now wish to consider the second issue of concern: the effect of public goods on the relationship between the core and the allocations arising from a price system. Muench's example showed that under the standard definition of the core (which requires a blocking coalition to produce all of the public goods it wants from its own resources) the core and Lindahl equilibria need not coincide in atomless economies, while Milleron exhibited a sequence of core allocations in replica economies which did not converge to a Lindahl allocation. The results of Section II indicate that these examples cannot be considered exceptional.

However, the usual definition of the core with public goods is not the only possible extension of that developed in private-goods economies, and, indeed, it is not clear that it is even the most useful definition for capturing an intuitive notion of social stability (Rosenthal [11]). Thus, one might conjecture that some other meaningful definition of blocking power might yield more positive results.

Such a definition would have to make blocking easier for small coalitions. Three particular ways of doing this come to mind. Two of these have been investigated already in the literature. Rosenthal [11] and Richter [8] studied the impact of recognizing that it may be advantageous for a complementary coalition to contribute to the provision of a public good, while Ellickson [4] and Roberts [10] considered the core with semi-public goods, where the cost of producing the public good is a nondecreasing function of the size of the coalition. However, these approaches, although interesting, have not been successful for our purpose. In this section we examine the prospects for reducing the size of the core by granting certain coalitions the power to tax for purposes of producing public goods. Though we do not provide a final answer to the above question, our results seem to indicate that further work in this direction is not likely to prove fruitful from an economic point of view.

A <u>fiscal law</u> is a correspondence φ which associates to each economy $\mathcal{E}$, each coalition $S \in \mathcal{Q}$, and each proposed production activity $(z, y) \in Z$, the set of forced contributions $\varphi(\mathcal{E}, S, z, y) \subseteq \mathbb{R}^n_+$ which S can legally extract from its complement for use in the production of public goods y.

In order to φ-block, a coalition must be able to implement a program which it prefers and which requires not more input from its complementary coalition than is specified by φ.

Let φ be a fiscal law. The feasible allocation (x, y) is <u>φ-blocked</u> by the coalition $S \in \mathcal{Q}$ if

(i) $\mu(S) > 0$; and

(ii) $\exists$ a feasible allocation (x', y') and $h \in \varphi\left(\mathcal{E}, S, \int_A (x' - \omega)d\mu, y'\right)$ such that $\left(x'(a), y'\right) \succ_{(a)} \left(x(a), y\right)$ a.e. in S, $\left(\int_A (x' - \omega)d\mu, y'\right) \in Z$, and $h = \int_{A\setminus S} (x' - \omega)d\mu$.

The <u>φ-core</u> is the set of all feasible allocations which are not φ-blocked. The usual definition of core corresponds to the case in which φ is identically zero ($\varphi \equiv \{0\}$).

Fiscal laws are defined in a sufficiently general fashion so that many political forms and mechanisms for providing public goods may be treated as special cases. It should be clear that under political forms such as majority rule, if all majorities are given broad enough powers by the fiscal law, the φ-core will generally be empty. Thus, the only classes of fiscal laws which are interesting for φ-core analysis are those in which either only very special coalitions have the power to force contributions or in which the power to force contributions is somehow curtailed.

To motivate this discussion somewhat more, consider Muench's example. In that example there is only one type of agent. Define the fiscal law $\bar{\varphi}(E, S, z, y) = \left\{z' \in \mathbb{R}^n_+ : z' \leq \frac{\mu(A \setminus S)z}{\mu(A)}\right\}$. In this case a coalition may tax its complement at the same rate at which it contributes itself. It is easy to see that for Muench's example the $\bar{\varphi}$-core coincides with the unique Lindahl allocation.

For economies with more than one type of agent, however, $\bar{\varphi}$ provides certain coalitions the power to tax away more than the total endowments of their complements. Generalizing the $\bar{\varphi}$ idea to economies with a finite number K of types, where A_k denotes the set of agents of type k ($k = 1, \ldots, K$), define

$$\bar{\bar{\varphi}}(E, S, z, y) = \left\{z' \in \mathbb{R}^n_+ : z' \leq \Big(1 - \alpha(S)\Big)z\right\}$$

where

$$\alpha(S) = \max_{k \in \{1, \ldots, K\}} \frac{\mu(S \cap A_k)}{\mu(A_k)} .$$

Note that $\bar{\bar{\varphi}} = \bar{\varphi}$ when $K = 1$ and that $\bar{\bar{\varphi}}$ embodies the notion of taxation at the same rate if we restrict taxation to that subset of the complement of S which has the same profile as S. Unfortunately, it is easy to show the following.

Theorem 3. Suppose $n = 1$. Under assumption A.3, if A is atomless and if $\mu(A_1) = \mu(A_2) = \ldots \mu(A_K)$, the $\bar{\bar{\varphi}}$-core equals the set of equal-treatment $\{0\}$-core allocations.

Proof. Suppose (x, y) is an equal-treatment allocation which is $\bar{\bar{\varphi}}$-blocked by the coalition S with (x', y'). Form a coalition $\bar{S}$ with $\mu(\bar{S} \cap A_k) = \left(1 - \alpha(S)\right)\mu(A_k) + \mu(S \cap A_k)$ for $k = 1, \ldots, K$. Allocate $\left(\frac{1}{\mu(S \cap A_k)} \int_{S \cap A_k} x'(a)d\mu\,,\ y'\right)$ to each agent in $\bar{S} \cap A_k$ and $\left(\omega(a),\ y'\right)$ to each agent in $A \setminus \bar{S}$. Clearly $\bar{S}$ $\{0\}$-blocks (x, y) with this allocation, since (x, y) has the equal-treatment property. Thus, every equal-treatment $\{0\}$-core allocation is in the $\bar{\bar{\varphi}}$-core.

Conversely, every $\bar{\bar{\varphi}}$-core allocation is in the $\{0\}$-core. If a $\bar{\bar{\varphi}}$-core allocation (x, y) did not have the equal-treatment property, there would be a coalition S with $\mu(S \cap A_1) = \ldots = \mu(S \cap A_K) > 0$, $\frac{1}{\mu(S \cap A_k)} \int_{S \cap A_k} x d\mu \le \frac{1}{\mu(A_k)} \int_{A_k} x d\mu$ for $k = 1, \ldots, K$ with strict inequality holding for at least one k. Through the fiscal law $\bar{\bar{\varphi}}$, S can achieve $\left(\frac{1}{\mu(A_k)} \int_{A_k} x d\mu\,,\ y\right)$ for each of its agents of type k, $k = 1, \ldots, K$. By then redistributing private goods within S, each agent in S can be made strictly better off than at (x, y) by assumption A.3. $\|$

Since there may exist Lindahl allocations which do not have the equal-treatment property, the $\bar{\bar{\varphi}}$-core clearly does not generally coincide with the set of Lindahl allocations in atomless economies. Nor does it generally coincide even with the set of equal-treatment Lindahl allocations. To see this, artificially split the set of agents in the Muench economy into two types (or slightly alter the preferences of half the agents in a region of the consumption set which is far from the origin). Now the set of equal-treatment core allocations is strictly larger than the set of equal-treatment Lindahl allocations in this economy.

Even if there is some fiscal law φ for which the φ-core and the set of Lindahl equilibria are closely related, this is of little economic interest unless φ can be implemented at small social cost. In particular, any fiscal law which could not be applied unless the preferences of economic agents were known publicly would seem impractical, both because of the massive amounts of data involved and because of the difficulties of inducing agents to reveal their preferences correctly. We are therefore interested in the class of fiscal laws which do not depend on the preferences of agents in the economy. We shall now establish that if $\hat{\varphi}$ is a fiscal law which is independent of the preferences of all agents, then even for "nice" atomless economies, the $\hat{\varphi}$-core does not generally coincide with either the set of Lindahl allocations or the set of equal-treatment Lindahl allocations.

To see this, we shall consider three "nice" atomless economies and show that no preference-independent $\hat{\varphi}$ can work for all three.

In each of the economies, A is the unit interval, μ is Lebesque measure, $n = m = 1$, $\omega(a) = 1$ all $a \in [0, 1]$, and $Z = \{(z, y) \in \mathbb{R}^2 : 2z + y \leq 0,\ y \geq 0\}$. The economies differ only in the preferences of the agents, defined by the following utility functions.

Economy 1: $u_a^1(x, y) = x - e^{-y}$ $\qquad a \in [0, 1]$

Economy 2: $$u_a^2(x, y) = \begin{cases} \dfrac{x}{1+\alpha} - e^{-y} & a \in [0, \frac{1}{2}] \\ \dfrac{x}{1-\alpha} - e^{-y} & a \in (\frac{1}{2}, 1] \text{ where } 0 < \alpha < 1 \end{cases}$$

Economy 3: $u_a^3(x, y) = x + \frac{y}{3}$ $\qquad a \in [0, 1]$.

Note that economy 1 is Muench's example and that all three economies satisfy the assumptions typically made in the public-goods literature.

It is a straightforward matter to compute the unique Lindahl allocations of each of these economies, all of which have the equal-treatment property. The information is summarized as follows.

Economy 1:

$$\bar{x}_a = 1 + \frac{1}{2}\log\frac{1}{2} \qquad a \in [0, 1]$$

$$\bar{y} = -\log\frac{1}{2}$$

$$u_a^1(\bar{x}_a, \bar{y}) = \frac{1}{2} + \frac{1}{2}\log\frac{1}{2} \qquad a \in [0, 1].$$

Economy 2:

$$\bar{x}_a = \begin{cases} 1 + \frac{1}{2}(1+\alpha)\log\frac{1}{2} & a \in [0, \frac{1}{2}] \\ 1 + \frac{1}{2}(1-\alpha)\log\frac{1}{2} & a \in (\frac{1}{2}, 1] \end{cases}$$

$$\bar{y} = -\log\frac{1}{2}$$

$$u_a^2(\bar{x}_a, \bar{y}) = \begin{cases} \dfrac{1-\alpha}{2(1+\alpha)} + \dfrac{1}{2}\log\dfrac{1}{2} & a \in [0, \frac{1}{2}] \\ \dfrac{1+\alpha}{2(1-\alpha)} + \dfrac{1}{2}\log\dfrac{1}{2} & a \in (\frac{1}{2}, 1]. \end{cases}$$

Economy 3:

$$\bar{x}_a = 1 \qquad a \in [0, 1]$$

$$\bar{y} = 0$$

$$u_a^3(\bar{x}_a, \bar{y}) = 1 \qquad a \in [0, 1].$$

From Muench [7], we know that for small $\varepsilon > 0$, the allocation $(\tilde{x}, \tilde{y})$ is in the $\{0\}$-core of economy 1, where

$$\tilde{x}_a = \begin{cases} 1+\frac{1}{2}\log\frac{1}{2}-\varepsilon & a \in [0, \frac{1}{2}] \\ 1+\frac{1}{2}\log\frac{1}{2}+\varepsilon & a \in (\frac{1}{2}, 1] \end{cases} .$$

$$\tilde{y} = -\log\frac{1}{2} .$$

If our claim is false, then there exists $\hat{\varphi}$ independent of preferences such that $(\tilde{x}, \tilde{y})$ is $\hat{\varphi}$-blocked in economy 1. We shall demonstrate that this hypothesis leads to the $\hat{\varphi}$-blocking of a Lindahl allocation in one of the other economies, a contradiction.

Let S be the $\hat{\varphi}$-blocking coalition, $S_1 = S \cap [0, \frac{1}{2}]$, and $S_2 = S \cap (\frac{1}{2}, 1]$. For any coalition B, let h(B) be the contribution by B to public-goods production at the $\hat{\varphi}$-blocking allocation.

Consider

(1) $\mu(S_1) - h(S_1) - \mu(S_1)e^{-2h(A)} > \mu(S_1)(\frac{1}{2}+\frac{1}{2}\log\frac{1}{2}) - \mu(S_1)\varepsilon$ if $\mu(S_1) > 0$,

(2) $\mu(S_2) - h(S_2) - \mu(S_2)e^{-2h(A)} > \mu(S_2)(\frac{1}{2}+\frac{1}{2}\log\frac{1}{2}) + \mu(S_2)\varepsilon$ if $\mu(S_2) > 0$, and

(3) $\mu(S) - h(S_1) - h(S_2) - \mu(S)e^{-2h(A)} \leq \mu(S)(\frac{1}{2}+\frac{1}{2}\log\frac{1}{2})$.

(1) and (2) follow from the definition of $\hat{\varphi}$-blocking. (3) follows from the hypothesis that the Lindahl allocation in economy 1 is not $\hat{\varphi}$-blocked. From (1), (2), and (3), $\mu(S_1) > \mu(S_2)$. Let $\theta_1 = \frac{1}{\mu(S_1)} h(S_1)$,

$$\theta_2 = \begin{cases} \frac{1}{\mu(S_2)} h(S_2) & \text{if } \mu(S_2) > 0 \\ 0 & \text{if } \mu(S_2) = 0 \end{cases} \quad \text{and} \quad \theta = \frac{1}{\mu(S)} h(S). \quad \text{Then}$$

(1a) $1 - \theta_1 - e^{-2h(A)} > \frac{1}{2}+\frac{1}{2}\log\frac{1}{2} - \varepsilon$, (2a) $1-\theta_2-e^{-2h(A)} > \frac{1}{2}+\frac{1}{2}\log\frac{1}{2}+\varepsilon$ if $\mu(S_2) > 0$, and

(3a) $1-\theta-e^{-2h(A)} \le \frac{1}{2}+\frac{1}{2}\log\frac{1}{2}$.

Manipulating (1a) and (2a) further yields:

$$(4)\quad \frac{1}{1+\alpha}(1-\theta_1)-e^{-2h(A)} > \frac{1-\alpha}{2(1+\alpha)}+\frac{1}{2}\log\frac{1}{2}+\frac{\alpha}{1+\alpha}\left[\frac{1}{2}-\frac{1}{2}\log\frac{1}{2}-\frac{\varepsilon}{\alpha}-e^{-2h(A)}\right],$$

$$(5)\quad \frac{1}{1-\alpha}(1-\theta_2)-e^{-2h(A)} > \frac{1+\alpha}{2(1-\alpha)}+\frac{1}{2}\log\frac{1}{2}-\frac{\alpha}{1-\alpha}\left[\frac{1}{2}-\frac{1}{2}\log\frac{1}{2}-\frac{\varepsilon}{\alpha}-e^{-2h(A)}\right]$$

if $\mu(S_2) > 0$.

Case 1: $e^{-2h(A)}+\frac{\varepsilon}{\alpha} \le \frac{1}{2}-\frac{1}{2}\log\frac{1}{2}$.

Consider the same $\hat{\varphi}$-blocking coalition S in economy 2 with forced contributions as above (possible under the hypothesis that $\hat{\varphi}$ is independent of preferences) and a transfer of private goods in the amount $\mu(S_1)\alpha\left(\frac{1}{2}-\frac{1}{2}\log\frac{1}{2}-\frac{\varepsilon}{\alpha}-e^{-2h(A)}\right)$ from S_1 to S_2. (If $\mu(S_2) = 0$, no transfer is necessary.) From (4) the average agent in S_1 prefers this to the Lindahl allocation in economy 2. The average agent in S_2 then receives utility

$$\frac{1}{1-\alpha}(1-\theta_2)-e^{-2h(A)}+\frac{\mu(S_1)}{\mu(S_2)}\frac{\alpha}{1-\alpha}\left(\frac{1}{2}-\frac{1}{2}\log\frac{1}{2}-\frac{\varepsilon}{\alpha}-e^{-2h(A)}\right)$$

which he also prefers to the Lindahl allocation in economy 2 from (5). Since the average allocation for S_1 and S_2 may be distributed uniformly over S_1 and S_2, respectively, the Lindahl allocation in economy 2 is $\hat{\varphi}$-blocked.

Case 2: $e^{-2h(A)}+\frac{\varepsilon}{\alpha} > \frac{1}{2}-\frac{1}{2}\log\frac{1}{2}$.

From (1), (2), and the defining condition for case 2,

$$\frac{1}{2}-\frac{1}{2}\log\frac{1}{2}-\theta+\varepsilon > e^{-2h(A)} > \frac{1}{2}-\frac{1}{2}\log\frac{1}{2}-\frac{\varepsilon}{\alpha}\ .$$

Since ϵ can be taken arbitrarily small but positive, let $\epsilon^n \to 0$ and let S^n, h^n, and θ^n, respectively, be the corresponding blocking coalition, total contribution function and average contribution by the members of S^n. Then the above inequalities imply that $\lim\sup \theta^n = 0$. Since $\theta^n \geqq 0$, this in turn means $\lim \theta^n$ exists and equals zero. Thus, $\lim h^n(A)$ exists and equals $-\frac{1}{2} \log \left(\frac{1}{2} - \frac{1}{2} \log \frac{1}{2}\right) > 0$. Then, for large enough n, the coalition S^n can $\hat{\varphi}$-block the Lindahl allocation in economy 3, since it will be receiving approximately $-\log \left(\frac{1}{2} - \frac{1}{2} \log \frac{1}{2}\right)$ of public good while contributing an arbitrarily small amount of its private good. This establishes the result.

In conclusion, we feel that the results of Section 2 clear up considerably the nature of the phenomena observed in the examples of Milleron and Muench. The examples of Section 3 show that, for at least the class of blocking rules we have considered, core-like concepts cannot be expected to have the same relationship with the Lindahl equilibria in public-goods economies as the core has with the competitive equilibria in private-goods economies. On the basis of this we feel that the use of core-like notions is not likely to be fruitful if one is interested in further clarifying the role of prices in large economies with public goods and more generally with externalities of any sort.

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