

# School Matching

Tayfun Sönmez

Based on mostly:

Balinski & Sönmez (*JET* 1999), Abdulkadiroğlu & Sönmez (*AER* 2003),  
Pathak & Sönmez (*AER* 2008), and Pathak & Sönmez (*AER* in press)

# College Admissions, Student Placement & School Choice

- **College Admissions** (Gale & Shapley *AMM* 1962)
  - Models (many-to-one) two-sided matching markets.
  - Both schools and students are (potentially strategic) agents
- **Student Placement** (Balinski & Sönmez *JET* 1999)
  - Models centralized university admissions.
  - Students are (potentially) strategic agents
  - School seats are goods to be consumed
  - Priority at schools determined by exam scores
  - Under an adequate “fairness” axiom, model isomorphic to stable college admissions.

# College Admissions, Student Placement & School Choice

- **School Choice** (Abdulkadiroğlu & Sönmez *AER* 2003)
  - Models centralized assignment of public school seats to K-12 students.
  - Students are (potentially) strategic agents
  - School seats are goods to be consumed
  - Priorities at schools are exogenous
  - A version of the model isomorphic to stable college admissions.

# Student Placement: The Model

A **student placement problem** consists of

$I$	$=$	$\{i_1, \dots, i_n\}$	a set of students
$C$	$=$	$\{c_1, \dots, c_m\}$	a set of colleges
$R$	$=$	$(R_{i_1}, \dots, R_{i_n})$	a list of student preferences
$q$	$=$	$(q_1, \dots, q_m)$	a vector of college capacities
$T$	$=$	$\{t_1, \dots, t_k\}$	a set of skill categories
$f$	$=$	$(f^{i_1}, \dots, f^{i_n})$	a list of test scores
$t$	$:$	$C \longrightarrow T$	a function from $C$ to $T$

# Student Placement: The Model

Here

- $q_c$  is the capacity of college  $c$ ,
- $R_i$  is the preference of student  $i$  over colleges and the no college option,
- $f^i = (f_{t_1}^i, \dots, f_{t_k}^i)$  is a vector which gives the test score of student  $i$  in each category, and
- $t$  is a function which maps each college to a category.

# Associated College Admissions Problem

For each student placement problem we can construct an **associated college admissions problem** by assigning each college  $c$  a preference relation  $R_c$  based on the ranking in its category  $t(c)$ .

# Matching & Tentative Student Placement

- A **matching** is a function  $\mu : I \rightarrow C \cup \{c_0\}$  such that no college is assigned to more students than its capacity.

$\mu(i) = c_0$ : Student  $i$  is unmatched.

- A **tentative student placement** is a mapping  $\mu : I \rightrightarrows C \cup \{c_0\}$  such that no college is assigned to more students than its capacity.

**Remark:** Tentative student placement allows a student to be assigned more than one college.

- A (direct) **mechanism** is a function which assigns a matching for each student placement problem.

# Elimination of Justified Envy

- A matching  $\mu$  **eliminates justified envy** if whenever a student  $i$  prefers another student  $\tilde{i}$ 's assignment  $\mu(\tilde{i})$  to his own, he ranks worse than  $\tilde{i}$  in the category the college  $\mu(\tilde{i})$  is.
  - Closely related to **stability**: Isomorphism with stable college admissions.
  - Critical in the context of Turkish college admissions.
- A mechanism **eliminates justified envy** if it always selects a matching that eliminates justified envy.



## Simple Case: One Skill Category

- Given a priority ranking, the induced **simple serial dictatorship** assigns the first student his top choice, the next student his top choice among remaining seats, etc.

**Proposition:** If there is only one category (and hence only one ranking) then there is only one mechanism that is *Pareto efficient* which *eliminates justified envy*: The *simple serial dictatorship* induced by this ranking.

## Elimination of Justified Envy & Stability

- A matching is **individually rational** if no student prefers the no college option to his assignment.
- A matching is **non-wasteful** if no student prefers a college with one or more empty slots to his assignment.

**Lemma:** A matching is *individually rational, non-wasteful* and it *eliminates justified envy* if and only if it is *stable* for its associated college admissions problem.

# Turkish Mechanism: Multi-Category Serial Dictatorship

## Step 1:

- For each category  $t$ : Consider the ranking induced by the test scores in this category and assign the relevant seats to students with the induced simple serial dictatorship.
- Assign the no college option to any unmatched student.
- This in general leads to a tentative student placement.
- For each student  $i$  construct  $R_i^1$  from  $R_i$  as follows:
  - If the student is not assigned more than one college then  $R_i^1 = R_i$ .
  - If the student is assigned more than one college then obtain  $R_i^1$  by moving the no college option  $c_0$  right after the best of these assignments and otherwise keeping the ranking of the colleges the same.

Let  $R^1$  be the list of adjusted preferences.

# Turkish Mechanism: Multi-Category Serial Dictatorship

**Step  $k$ :** Construct  $R^k$  from  $R^{k-1}$  as it is described in Step 1.

The procedure terminates at the step in which no student is assigned more than one college. The **multi-category serial dictatorship (MSD)** selects this matching.

**Example:**  $I = \{i_1, i_2, i_3, i_4, i_5\}$ ,  $C = \{c_1, c_2, c_3\}$ ,  $q = (2, 1, 1)$ ,  $T = \{t_1, t_2\}$ ,  
 $t(c_1) = t_1$ ,  $t(c_2) = t(c_3) = t_2$ ,

$$\begin{array}{ll}
 R_{i_1} : c_2 - c_1 - c_0 & f^{i_1} = (9, 9) \\
 R_{i_2} : c_1 - c_2 - c_3 - c_0 & f^{i_2} = (8, 6) \\
 R_{i_3} : c_1 - c_3 - c_2 - c_0 & f^{i_3} = (7, 7) \\
 R_{i_4} : c_1 - c_2 - c_0 & f^{i_4} = (6, 8) \\
 R_{i_5} : c_2 - c_3 - c_1 - c_0 & f^{i_5} = (5, 5)
 \end{array}$$

Note that these scores induce the following rankings in categories  $t_1, t_2$ :

$$\begin{array}{l}
 t_1 : i_1 \ i_2 \ i_3 \ i_4 \ i_5 \\
 t_2 : i_1 \ i_4 \ i_3 \ i_2 \ i_5
 \end{array}$$

Step 1:

$$t_1 : \begin{array}{ccccc} i_1 & i_2 & i_3 & i_4 & i_5 \\ c_1 & c_1 & & & \end{array} \quad t_2 : \begin{array}{ccccc} i_1 & i_4 & i_3 & i_2 & i_5 \\ c_2 & - & c_3 & & \end{array}$$

Step 1 yields the following tentative student placement:

$$\nu^1 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ c_1, c_2 & c_1 & c_3 & c_0 & c_0 \end{pmatrix}$$

Having assigned at least one slot, preferences of students  $i_1, i_2, i_3$  are truncated:

$$\begin{aligned} R_{i_1}^1 &: c_2 - c_0 \\ R_{i_2}^1 &: c_1 - c_0 \\ R_{i_3}^1 &: c_1 - c_3 - c_0 \end{aligned}$$

For other students:  $R_{i_4}^1 = R_{i_4}$ , and  $R_{i_5}^1 = R_{i_5}$ .

**Step 2:** In Step 2 we first find the serial dictatorship outcomes for  $R^1$ .

$$t_1 : \begin{array}{ccccc} i_1 & i_2 & i_3 & i_4 & i_5 \\ - & c_1 & c_1 & & \end{array} \quad t_2 : \begin{array}{ccccc} i_1 & i_4 & i_3 & i_2 & i_5 \\ c_2 & - & c_3 & & \end{array}$$

Step 2 yields the following tentative student placement:

$$\nu^2 = \left( \begin{array}{ccccc} i_1 & i_2 & i_3 & i_4 & i_5 \\ c_2 & c_1 & c_1, c_3 & c_0 & c_0 \end{array} \right)$$

Having assigned two slots, preferences of student  $i_3$  is truncated:

$$R_{i_3}^2 : c_1 - c_0$$

For other students:  $R_{i_1}^2 = R_{i_1}^1$ ,  $R_{i_2}^2 = R_{i_2}^1$ ,  $R_{i_4}^2 = R_{i_4}^1$ , and  $R_{i_5}^2 = R_{i_5}^1$ .

**Step 3:** In Step 3 we first find the serial dictatorship outcomes for  $R^2$ .

$$t_1 : \begin{array}{ccccc} i_1 & i_2 & i_3 & i_4 & i_5 \\ - & c_1 & c_1 & & \end{array} \quad t_2 : \begin{array}{ccccc} i_1 & i_4 & i_3 & i_2 & i_5 \\ c_2 & - & - & - & c_3 \end{array}$$

Step 3 yields the following tentative student placement (which is also a matching):

$$\nu^3 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ c_2 & c_1 & c_1 & c_0 & c_3 \end{pmatrix}$$

Since no student is assigned more than slot in  $\nu^3$ , the algorithm terminates and  $\varphi^{msd}(R, f, q) = \nu^3$ .



## Mechanisms via Associated College Admissions Problem

- **Gale-Shapley college optimal stable mechanism (COSM):** The mechanism which selects the college optimal stable matching of the associated college admissions problem for each student placement problem.
- **Gale-Shapley student optimal stable mechanism (SOSM):** The mechanism which selects the student optimal stable matching of the associated college admissions problem for each student placement problem.

## Student-Proposing Deferred Acceptance Algorithm (Gale & Shapley AMM 1962)

**Step 1:** Each student proposes to her first choice. Each school tentatively assigns its seats to its proposers one at a time following their priority order. Any remaining proposers are rejected.

In general, at

**Step  $k$ :** Each student who was rejected in the previous step proposes to her next choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time following their priority order. Any remaining proposers are rejected.

# An Equivalence

**Theorem:** MSD = COSM.

**Remark:** This result is reminiscent of the classical Roth (*JPE* 1982) result that shows the equivalence of the NRMP mechanism and Gale-Shapley hospital optimal stable mechanism.

## Pareto Efficiency

**Example:**  $I = \{i_1, i_2\}$   $C = \{c_1, c_2\}$   $q = (1, 1)$   $T = \{t_1, t_2\}$   
 $t(c_1) = t_1, t(c_2) = t_2$

$$R_{i_1} : c_1 - c_2 - c_0 \quad f^{s_1} = (6, 8)$$

$$R_{i_2} : c_2 - c_1 - c_0 \quad f^{s_2} = (8, 6)$$

The algorithm terminates in one step resulting in the following Pareto inefficient matching:

$$\varphi^{msd}(R, f, q) = \begin{pmatrix} i_1 & i_2 \\ c_2 & c_1 \end{pmatrix}$$

**Theorem (Gale & Shapley AMM 1962):** SOSM Pareto dominates any other mechanism that eliminates justified envy.

## Strategy-Proofness

Example continued: Recall that

$$\varphi^{msd}(R, f, q) = \begin{pmatrix} i_1 & i_2 \\ c_2 & c_1 \end{pmatrix}$$

Now suppose  $i_1$  announces a fake preference relation  $\tilde{R}_{i_1}$  where only  $c_1$  is acceptable. In this case

$$\varphi^{msd}(\tilde{R}_{i_1}, R_{i_2}, f, q) = \begin{pmatrix} i_1 & i_2 \\ c_1 & c_2 \end{pmatrix}$$

and hence student  $s_1$  successfully manipulates MSD.

- A mechanism is **strategy-proof** if truthtelling is a dominant strategy in its associated preference revelation game.
- Theorem (Dubins & Freedman AMM 1981, Roth MOR 1982): SOSM is *strategy-proof*.
- Theorem (Alcalde & Barberà ET 1994): SOSM is the only mechanism that *eliminates justified envy*, and is *individually rational*, *non-wasteful*, and *strategy-proof*.

## Respecting Improvements

Example further continued: Recall that

$$\varphi^{msd}(R, f, q) = \begin{pmatrix} i_1 & i_2 \\ c_2 & c_1 \end{pmatrix}$$

Now suppose student  $s_1$  scores worse in both tests and his new test scores are  $\tilde{f}^{i_1} = (5, 5)$ .

In this case

$$\varphi^{msd}(R, \tilde{f}^{i_1}, f^{i_2}, q) = \begin{pmatrix} i_1 & i_2 \\ c_1 & c_2 \end{pmatrix}$$

and student  $i_1$  is rewarded by getting his top choice as a result of a worse performance!

- A mechanism **respects improvements** if a student never receives a worse assignment as a result of an increase in one or more of his test scores.
- **Theorem:** SOSM respects improvements.
- **Theorem:** SOSM is the only mechanism that is *individually rational*, *non-wasteful* and that *eliminates justified envy* and *respects improvements*.



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**Bottomline:** In an environment where elimination of justified envy cannot be sacrificed (eg. when priorities obtained through exams as in Turkey), SOSM is the unambiguous winner!

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**However there is one bit of bad news...**

## Efficiency Cost of Elimination of Justified Envy

**Example:** There are three students  $i_1, i_2, i_3$  and three schools  $c_1, c_2, c_3$ , each of which has only one seat. Preferences and school priorities are as follows:

$$R_{i_1} : c_2 - c_1 - c_3$$

$$\pi_{c_1} : i_1 - i_3 - i_2$$

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$$R_{i_3} : c_1 - c_2 - c_3$$

$$\pi_{c_3} : i_2 - i_1 - i_3$$

Only  $\mu$  eliminates justified envy but it is Pareto dominated by  $\nu$ :

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

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While SOSM Pareto dominates any mechanism that eliminates justified envy, SOSM itself is not Pareto efficient!

## School Choice vs. Student Placement

- School Choice model (Abdulkadiroğlu & Sönmez 2003) builds on the Student Placement model (Balinski & Sönmez 2003) and pays particular attention to mechanisms used in U.S. school districts and their shortcomings.
- While elimination of justified envy is **imposed** throughout B&S (1999) making SOSM the unique winning mechanism, it is only **desired** under A&S (2003).

As such, the two mechanisms SOSM and TTC emerge as plausible mechanisms for school choice.

# School Choice

- **School choice problem:**
  - There are a number of students, each of whom should be assigned a seat at one of a number of schools.
  - Each school  $s$  has a maximum capacity  $q_s$  but there is no shortage of the total seats.
  - Each student  $i$  has strict preferences  $R_i$  over all schools and each school  $s$  has a strict priority ordering  $\pi_s$  of all students.
- **Priorities: Exogenous**

# Differences with College Admissions & Student Placement

- Differences with College Admissions:

- Students are (possibly strategic) agents; school seats are objects to be consumed.
- Elimination of justified envy is *plausible* but *not a must*.

If imposed, then school choice is *isomorphic* to stable college admissions, although the desirable axioms are still different since only one side of the market are agents.

- Differences with Student Placement:

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- Differences with Student Placement:

- Priorities are exogenous.
- Elimination of justified envy is *plausible* but *not a must*.

Thus, perhaps, there are plausible mechanisms that are Pareto efficient.



## Boston Mechanism

The most widely used mechanism by far is the mechanism used by Boston Public Schools (BPS) in the period 1988-2005:

- 1 For each school a priority ordering is exogenously determined.  
In case of BPS, priority of student  $i$  at a given school  $s$  depends on
  - whether student  $i$  lives in the **walk-zone** of school  $s$ , ,
  - whether student  $i$  has a **sibling** already attending school  $s$ , and
  - a lottery number to break ties.
- 2 Each student submits a preference ranking of the schools.
- 3 The final phase is the student assignment based on preferences and priorities:

# Boston Mechanism

**Round 1:** In the first round only the first choices of the students are considered. For each school  $s$ , consider the students who have listed  $s$  as first choice and assign seats of school  $s$  to them one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her first choice.

**Round  $k$ :** Consider the remaining students. In Round  $k$  only the  $k^{\text{th}}$  choices of these students are considered. For each school with still available seats, consider the students who have listed it as their  $k^{\text{th}}$  choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her  $k^{\text{th}}$  choice.

## Very Easy to Manipulate

- **Major difficulty:** The Boston mechanism is not **strategy-proof**.
- Even if a student has very high priority at school  $s$ , she loses her priority to students who have top ranked school  $s$  unless she lists it as her top choice!
- Hence the Boston mechanism gives parents strong incentives to overrank schools where they have high priority.

## So Easy to Manipulate, it is All Over the News!

Consider the following quotation from St.Petersburg Times:

*Make a realistic, informed selection on the school you list as your first choice. It's the cleanest shot you will get at a school, but if you aim too high you might miss.*

*Here's why: If the random computer selection rejects your first choice, your chances of getting your second choice school are greatly diminished. That's because you then fall in line behind everyone who wanted your second choice school as their first choice. You can fall even farther back in line as you get bumped down to your third, fourth and fifth choices.*

## Evidence from Education Literature

Glenn (*PI* 1991) states

*As an example of how school selections change, analysis of first-place preferences in Boston for sixth-grade enrollment in 1989 (the first year of controlled choice in Boston) and 1990 shows that the number of relatively popular schools doubled in only the second year of controlled choice. The strong lead of few schools was reduced as others "tried harder."*

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Highly **optimistic** scenario!

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Highly **optimistic** scenario!

More plausible scenario: **Learning**

## Evidence from 2004-2005 BPS School Guide

*For a better choice of your “first choice” school . . . consider choosing less popular schools.*



## Lack of Elimination of Justified Envy

- Boston mechanism does not eliminate justified envy: Priorities are lost unless school ranked as top choice.
- B&S (1999): If *elimination of justified envy* is to be maintained, then SOSM is the big winner!
- **Caution:** Recall the efficiency cost of elimination of justified envy.

# Equilibria of the Boston Mechanism

**Theorem (Ergin & Sönmez JPubE 2006):** The set of Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism is equal to the set of stable matchings of the associated college admissions game under the true preferences.

## Efficiency of the Boston Mechanism

**Corollary:** The dominant-strategy equilibrium outcome of the SOSM either Pareto dominates or equal to the Nash equilibrium outcomes of the Boston mechanism.

**Remark 1:** The preference revelation game induced by the Boston mechanism is a huge “coordination game” with lots of uncertainty. So it is unrealistic to expect a Nash equilibrium in practice.

That said, it is perhaps even less natural to expect an off-equilibrium outcome that is “better” than its best equilibrium.

**Remark 2:** Ergin & Sönmez (*JPubE* 2006) show that the above result no longer holds under incomplete information.

## Efficiency of the Boston Mechanism

- Abdulkadiroğlu, Che & Yasuda (*AER* 2011) further show that, if
  - ① all students have identical ordinal preferences but different cardinal preferences, and
  - ② all students have the same claims for each school (and thus priorities at each school is constructed with a uniform lottery)

then any student weakly prefers any symmetric Bayesian Nash equilibria of the Boston mechanism to the dominant strategy outcome of the SOSM.

- Observe that under the above assumptions not only the resulting model is quite different than school choice, but also SOSM is identical to pure random allocation.

## Interim Summary for the Boston Mechanism

- Highly vulnerable to manipulation.
- Does not respect priorities (in the sense of elimination of justified envy).
- Efficiency comparison with SOSM is less clear, but only because the analysis of the preference game induced by the Boston mechanism relies on strong assumptions.

# Doing Away with Elimination of Justified Envy: TTC

We can adopt the **Top Trading Cycles** mechanism (TTC) to school choice:

## Step 1:

- Assign a *counter* for each school which keeps track of how many seats are still available at the school. Initially set the counters equal to the capacities of the schools.
- Each student “points to” her favorite school. Each school points to the student who has the highest priority.
- There is at least one cycle. Every student in a cycle is assigned a seat at the school she points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed. Counters of all other schools stay put.

**Step k:** Repeat Step 1 for the remaining “economy.”

# Efficiency & Strategy-Proofness of TTC

- TTC simply **trades priorities** of students among themselves starting with the students with highest priorities.
- TTC inherits the plausible properties of Gale's TTC:

**Theorem:** The TTC mechanism is *Pareto efficient* and *strategy-proof*.

## Adoption of SOSM in NYC

- Shortly after A&S (2003) was published in June 2003, NYC and Boston both adopted the SOSM. However the two reforms evolved in very different ways, and they are summarized in
  - Abdulkadiroğlu, Pathak & Roth (*AER P&P* 2005) for NYC, and
  - Abdulkadiroğlu, Pathak, Roth & Sönmez (*AER P&P* 2005) for Boston.
- **May 2003:** NYCDOE Director of Strategic Planning contacted Alvin Roth for advice on the design of a new high school matching mechanism after the **collapse** of their mechanism.
  - Unlike most other school districts, NYCDOE did not have a direct mechanism prior to 2003.
  - Their mechanism gave students incentives to manipulate their preferences (reminiscent of those under the Boston mechanism), and it gave schools the ability to manipulate their priority ranking as well as to conceal capacity.
  - NYCDOE failed to assign roughly 30 percent of students via its mechanism in its final run, a very visible failure that required abandoning it in haste.



## Adoption of SOSM in NYC

- **October 2003:** NYCDOE adopted SOSM for high school admissions. Strategy-profness of the SOSM made it particularly attractive.

*“For more than a generation, parents and students have been unhappy with the admissions process to New York City high schools. The new process is a vast improvement, as it provides greater choice, **equity** and efficiency. For example, for the first time, students will be able to list preferences as true preferences, limiting the need to game the system.*

*This means that students will be able to rank schools without the risk that naming a competitive school as their first choice will adversely affect their ability to get into a school they rank lower.”*

*Peter Kerr, Director of Communications, NYCDOE*

## Adoption of SOSM in Boston

- Unlike in NYCDOE, BPS was quite satisfied with its mechanism.
- **September 2003:** The *Boston Globe* published an article on A&S (2003), describing the flaws of the Boston mechanism, and advocating the adoption of SOSM.
- **October 2003:**
  - Following a series of e-mail exchanges, Valerie Edwards, then Strategic Planning Manager at BPS, invited Sönmez to Boston to present the case against the Boston mechanism.

Together with Abdulkadiroğlu, Pathak and Roth, he presented to BPS the case against the Boston mechanism, and proposed two strategy-proof alternatives.
  - While skeptical prior to meeting, BPS staff was convinced strategizing was likely occurring, to the detriment of students and families.
  - They invited the team to carry out an empirical study of the Boston mechanism to support the results in A&S (2003) and the working paper versions of Chen & Sönmez (*JET* 2006) and Ergin & Sönmez (*JPubE* 2006) presented in the meeting.

## Adoption of SOSM in Boston

- **September 2004:** In their report to BPS, Student Assignment Task Force recommended the adoption of **TTC**.
- **July 2005:** BPS gave up the Boston mechanism and adopted **SOSM**.

**Lesson Learned in the Process:** Over emphasizing certain features of a mechanism via its name, in this case **trade**, can scare off the policy makers!

Policy makers at BPS were mostly worried about the trade of sibling priorities.

## Sincere and Strategic Players in the Boston Mechanism

- Superintendent Payzant's May 2005 Report to School Committee:  
"A strategy-proof algorithm **levels the playing field** by diminishing the harm done to parents who do not strategize or do not strategize well."
- Also recall how NYCDOE, Director of Communications linked strategy-proofness to equity when NYC adopted SOSM in 2003.
- **The Goal of Pathak & Sönmez (AER 2008)**: Formal investigation of the above perspective.
- P&S (2008) analyze the Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism when part of the players are *sincere* and others are *strategic*.

## Modeling Choice

We assume that unsophisticated parents are **truthful**.

- Natural default behavior.
- Chen & Sönmez (*JET* 2006): Even in controlled experiments with relatively sophisticated subjects, about 20% of participants report their true preferences under the Boston mechanism.
- Hastings, Kane, Staiger (2005): In Charlotte, “we believe that the extent of strategic manipulation in the first year was limited and that parents were generally reporting their true preferences.”
- Since truth-telling is a dominant strategy under SOSM, this is the relevant case for comparative static analysis.

## Model with Sincere & Strategic Students

- Consider A&S (2003) school choice model with the following additional structure partitioning students into two groups.

$$\underbrace{I}_{\text{set of students}} = \underbrace{\mathcal{N}}_{\text{sincere students}} \cup \underbrace{\mathcal{M}}_{\text{strategic students}}$$

- For each sincere student  $i \in \mathcal{N}$ , restrict the strategy space to be a singleton, corresponding to truthful preference revelation.
- Focus on the Nash equilibria of the preference revelation game induced by the Boston mechanism.

# Augmented Priorities

- Given an economy  $(R, \pi)$  and a school  $s$ , partition the set of students  $I$  into  $|S| = \ell$  sets as follows
  - $I_1$ : Strategic students + Sincere students who rank  $s$  as first choice
  - $I_2$ : Sincere students who rank  $s$  as second choice
  - $I_3$ : Sincere students who rank  $s$  as third choice
  - $\vdots$
  - $I_\ell$ : Sincere students who rank  $s$  as last choice

- Given an economy  $(R, \pi)$  and a school  $s$ , uniquely construct an **augmented priority ordering**  $\tilde{\pi}_s$  as follows:
  - each student in  $I_1$  has higher priority than each student in  $I_2$ ,
  - each student in  $I_2$  has higher priority than each student in  $I_3$ ,
  - $\vdots$
  - each student in  $I_{\ell-1}$  has higher priority than each student in  $I_\ell$ , and
  - for any  $k \leq \ell$ , priority among students in  $I_k$  is based on  $\pi_s$ .
- $\tilde{\pi} = (\tilde{\pi}_s)_{s \in S}$ .
- $(R, \tilde{\pi}_s)$ : **The Augmented Economy**



# Characterization of Nash Equilibrium Outcomes

- **Proposition:** The set of Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism for the original economy  $(R, \pi)$  is equal to the set of stable matchings for the augmented economy  $(R, \tilde{\pi})$ .
- **Observations:**
  - Sincere students lose their priority to strategic students.
  - Set of Nash equilibrium outcomes inherits all the properties of the set of stable matchings for  $(R, \tilde{\pi})$ : Set of students who are single is the same in all equilibrium outcomes, set of occupied seats always the same, lattice structure, Pareto-dominant equilibrium allocation, etc.

# Equilibrium Assignments of Sincere Students

- **Proposition:** Fix an economy  $(R, \pi)$  and a sincere student  $i \in \mathcal{N}$ . Student  $i$  receives the same assignment at each Nash equilibrium outcome of the “Boston game.”
- Useful result for comparative statics.

## Comparing Mechanisms for Sincere Students

- Is a sincere student always better off under the SOSM?

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No.

- A sincere student can prefer the Boston mechanism since
  - she gains priority at her second choice school at the expense of sincere students who rank it third or lower,
  - she gains priority at her third choice school at the expense of sincere students who rank it fourth or lower, etc.

Loosely speaking, a sincere student may luck out (at the expense of another sincere student) under the Boston mechanism!

# Comparing Mechanisms for Strategic Students

- **Proposition:** Fix an economy  $(R, \pi)$  and a strategic student  $i \in \mathcal{M}$ . The assignment of student  $i$  under the Pareto-dominant Nash equilibrium outcome of the Boston mechanism is at least as good as her assignment under the dominant strategy equilibrium outcome of the SOSM.
- **Remarks:**
  - Coordination at Pareto-dominant Nash equilibrium unlikely.
  - Strategic players could be worse off in other Nash equilibrium outcomes of the Boston mechanism.

Hence the case for the above result would have been stronger, if the Boston game had a unique equilibrium.

## Virtually Unique Stable Matching in the Case of Boston

- However computational experiments (with data from school years 2005-06 and 2006-07) suggest that while multiple equilibria is a theoretical possibility under the Boston game, it likely affects a very small fraction of students.

	Fraction of Sincere Students			
	20%	40%	60%	80%
2005-06				
Grade K2	0.14	0.08	0.04	0.01
Grade 6	0.38	0.20	0.07	0.01
2006-07				
Grade K2	0.03	0.01	0.00	0.00
Grade 6	0.24	0.14	0.05	0.01

## Resistance to Change by the WZPG

- June 8th, 2005: Community testimony from WZPG leader

*There are obviously issues with the current system. If you get a low lottery number and don't strategize or don't do it well, then you are penalized. But this can be easily fixed. When you go to register after you show you are a resident, you go to table B and the person looks at your choices and lets you know if you are choosing a risky strategy or how to re-order it.*

*Don't change the algorithm, but give us more resources so that parents can make an informed choice.*

- Leadership at WZPG must have agreed with the previous result that suggests that strategic families are better off under the Boston mechanism!



## Comparative Statics: Becoming Strategic

- What happens when a parent joins WZPG?
- **Proposition:** A sincere student weakly benefits from becoming strategic under the Pareto dominant Nash equilibrium of the Boston game whereas all veteran strategic students get weakly worse off.
- **Remark:** Therefore information sharing and coordination of preferences are among more likely roles of parents groups such as WZPG (rather than educating parents to strategize).

## School Admissions Reforms in the Last Decade

- While all admissions mechanism reforms we are aware of involved heated debates on “gaming of the system,” not all of them resulted in adoption of strategy-proof mechanisms.

Indeed NYC reform is one such reform since they adopted a “capped” version of SOSM with 12 choices.

- Sometimes practical concerns preclude the use of an uncapped version of SOSM.

**Example:** There are more than 500 high schools in NYC!

- So how does one compare two mechanisms in terms of incentive-compatibility when they are both manipulable?

This is the focus of Pathak & Sönmez (*AER* forthcoming), with an emphasis on recent reforms in Chicago and England.

# School Admissions Reforms in the Last Decade

- New mechanisms, **with** direct involvement of matching theorists:
  - ✓ 2003: New York City
  - ✓ 2005: Boston
- Mechanisms abandoned/illegal, **without** direct involvement of matching theorists:
  - ✓ 2007: England
  - ✓ 2009: Chicago
- Discussions about the vulnerability of mechanisms to manipulation played a key role in each of these reforms.

Chicago Sun-Times November 12, 2009  
**8th-graders' shot at elite high schools better**

*Poring over data about eighth-graders who applied to the city's elite college preps, Chicago Public Schools officials discovered an alarming pattern.*

*High-scoring kids were being rejected simply because of the order in which they listed their college prep preferences.*

*"I couldn't believe it," schools CEO Ron Huberman said. "It's terrible."*

*CPS officials said Wednesday they have decided to let any eighth-grader who applied to a college prep for fall 2010 admission re-rank their preferences to better conform with a new selection system.*

*Previously, some eighth-graders were listing the most competitive college preps as their top choice, forgoing their chances of getting into other schools that would have accepted them if they had ranked those schools higher, an official said.*

*Under the new policy, Huberman said, a computer will assign applicants to the highest-ranked school they qualify for on their list.*

*"It's the fairest way to do it." Huberman told Sun-Times.*

# Chicago Selective Enrollment High Schools

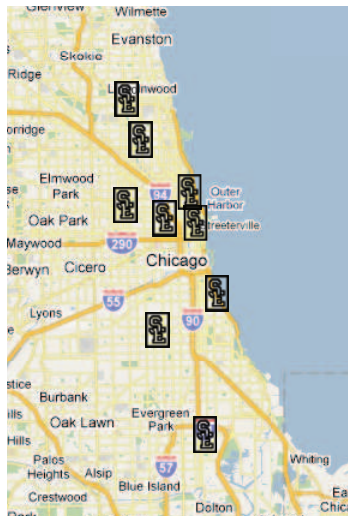
9 selective high schools

**Applicants:** Any current 8<sup>th</sup> grader in Chicago

**Composite Test Score:**  
Entrance exam + 7<sup>th</sup> grade scores

Up to Fall 2009, system worked as follows:

- Take entrance test
- Rank up to 4 schools



# Chicago Selective Enrollment Mechanism

**Round 1:** Only the first choices of the students are considered. For each school, assign seats of the to the students who have listed it as their first choice one at a time following their composite test score until either there are no seats left or there is no student left who has listed it as her first choice.

In general, for  $k = 2, \dots, 4$

**Round  $k$ :** Consider the remaining unplaced students. In Round  $k$  only the  $k^{\text{th}}$  choices of these students are considered. For each school with remaining seats, consider the students who have listed it as their  $k^{\text{th}}$  choice and assign the remaining seats to these students one at a time following their composite test score until either there are no seats left or there is no student left who has listed it as her  $k^{\text{th}}$  choice.

- New Chicago mechanism ( $SD^4$ )
  - ✓ Rank up to 4 schools
  - ✓ Students ordered by composite score
  - ✓ The first student obtains her top choice, the second student obtains her top choice among remaining, and so on.
- Somewhat surprising midstream change, given that both mechanisms are manipulable...

# Framework

**Players:**  $i = 1, \dots, N$

**Allocations:**  $A$

**Preferences:**  $R_i$  over  $A$ , strict version  $P_i$

**Problem:**  $R = (R_1, \dots, R_N)$

**Direct Mechanism:**  $\psi$  map from preference profile to outcome

- Mechanism  $\psi$  is **manipulable** by player  $i$  at problem  $R$  if there exists a type  $R'_i$  such that

$$\psi(R'_i, R_{-i}) P_i \psi(R).$$

- Problem  $R$  is **vulnerable** under mechanism  $\psi$  if  $\psi$  is manipulable by some player at  $R$ .



# Comparing Mechanisms

- Mechanism  $\psi$  is **at least as manipulable as** mechanism  $\varphi$  if for any problem where mechanism  $\varphi$  is manipulable, mechanism  $\psi$  is also manipulable.
- Mechanism  $\psi$  is **more manipulable than** mechanism  $\varphi$  if
  - ✓  $\psi$  is at least as manipulable as  $\varphi$ , and
  - ✓ there is at least one problem where  $\psi$  is manipulable though  $\varphi$  is not.

**Equivalent Definition:** If truth-telling is a Nash equilibrium of the game induced by mechanism  $\psi$ , it is also Nash equilibrium of the game induced by mechanism  $\varphi$  (even though the converse does not hold).

# Admissions Reform in Chicago

**Proposition:** Suppose there are at least  $k$  schools and let  $k > 1$ . The old Chicago mechanism ( $\text{CHI}^k$ ) is more manipulable than the truncated serial dictatorship Chicago adopted ( $\text{SD}^k$ ) in Fall 2009.

- Outrage expressed in quotes from Chicago Sun-Times:

*"I couldn't believe it," schools CEO Ron Huberman said. "It's terrible."*

suggests that the old mechanism was quite undesirable.

- To make this precise, we need to consider a class of mechanisms:
  - stable mechanisms?
  - not satisfied by many school choice mechanisms, including Chicago's old one

- A matching is **strongly unstable** if a student who ranks school  $s$  as his first choice loses a seat to a student who has a lower composite score.
- A **weakly stable** matching is one that is not strongly unstable. A weakly stable mechanism always produces a weakly stable matching
  - ✓ Old Chicago mechanism is weakly stable
  - ✓ New mechanism is weakly stable
  - ✓ Potentially a very large class

**Theorem:** The old Chicago mechanism ( $\text{CHI}^k$ ) is at least as manipulable as any weakly stable mechanism.

## Chicago in 2010-2011

- Last two results suggest that the new mechanism in Chicago is an improvement in terms of discouraging manipulation.
- However, the choice of a “truncated” mechanism is still sub-optimal both in terms of efficiency and incentive compatibility.
- Possible to have an efficient and non-manipulable mechanism by considering all choices.
- In 2010-11 school year, Chicago decided to increase the number of submitted choices to 6, but the resulting mechanism is still manipulable.

## Constrained School Choice

- We consider this issue more generally by returning to an environment where students are not necessarily ordered in the same way at each school.
- Vulnerability of school choice mechanisms to manipulation played a role in NYC's adaptation of a version of the SOSM in NYC, where students can rank up to 12 choices.

NYC DOE press release on change: “to reduce the amount of gaming families had to undertake to navigate a system with a shortage of good schools” (New York Times, 2003)

- Based on the strategy-proofness of the SOSM, the following advice was given to students:

You must now rank your **12** choices according to your true preferences.

## Constrained School Choice

Next result formalizes the idea that the greater the number of choices students can make, the less vulnerable this mechanism is to manipulation:

**Proposition:** Let  $\ell > k > 0$  and suppose there are at least  $\ell$  schools. The SOSM where students can rank  $k$  schools is more manipulable than the SOSM where students can rank  $\ell$  schools.

**Corollary:** The 2009 Chicago mechanism ( $SD^4$ ) is more manipulable than the newly adopted 2010 Chicago mechanism ( $SD^6$ ).

## Admissions Reform throughout England

- Aside from Boston (which used the Boston mechanism until 2005), variants of this mechanism have been used in several U.S. school districts including: Cambridge MA, Charlotte-Mecklenburg NC, Denver CO, Miami-Dade FL, Minneapolis MN, Providence RI, and Tampa-St. Petersburg FL.
- U.S. is not the only country where versions of the Boston mechanism are used to assign students to public schools.
- A large number of English Local Authorities had been using what they referred to as “first preference first” systems until 2007.

## Admissions Reform throughout England

- Formally, a **first preference first (FPF)** mechanism is a hybrid between the SOSM and the Boston mechanism: Under this mechanism, a school selects to be either a **first preference first school** or an **equal preference school**, and the outcome is determined by the student-proposing deferred acceptance algorithm, where
  - 1) the base priorities for each student are used for each equal preference school, whereas
  - 2) the base priorities of students are adjusted so that
    - any student who ranks school  $s$  as his first choice has higher priority than any student who ranks school  $s$  as his second choice,
    - any student who ranks school  $s$  as his second choice has higher priority than any student who ranks school  $s$  as his third choice, etc.for each first preference first school.
- The Boston mechanism is a special case of this mechanism when all schools are first preference first schools and the SOSM is a special case when all schools are equal preference schools.



## Ban of FPF Mechanism in 2007

- 2003 School Admissions Code in England requires all local authorities to coordinate public school admissions.
- While a majority of local authorities adopted truncated versions of the SOSM after (or in anticipation) of the 2003 code, more than 60 local authorities adopted the FPF mechanism (including several that adopted the Boston mechanism).
- The FPF mechanism was banned throughout England with the 2007 School Admissions Code along with other mechanisms that use **“unfair oversubscription criteria.”**

**Section 2.13:** *In setting oversubscription criteria the admission authorities for all maintained schools must not:*

*... give priority to children according to the order of other schools named as preferences by their parents, including 'first preference first' arrangements.*

## Ban of FPF mechanism in 2007

- Rationale given by Department for Education and Skills:  
*'first preference first' criterion made the system unnecessarily complex to parents*
- Education Secretary Alan Johnson remarked that the FPF system “forces many parents to play an ‘admissions game’ with their children’s future.”
- Great deal of public discussion throughout England.

## Ban of FPF Mechanism in 2007

Some Local Education Authorities abandoned the FPF mechanism prior to 2007 ban:

- ✓ Pan London Admissions Authority adopted the truncated SOSM with 6 choices in 2005
  - designed to “make the admissions system fairer” and “create a simpler system for parents”*
- ✓ Newcastle: 2003, from Boston-3 to SOSM-3; later changed to SOSM-4 (currently, 97 schools citywide).
- ✓ Brighton and Hove (Boston-3 to SOSM-3 out of 9): change will “hopefully eliminate the need for tactical preferences.”
- Changes involve moving between two truncated mechanisms, which is a striking parallel with Chicago
- Interesting that participants themselves (not matching theorists) re-organized these markets.

The following result shows that school choice reforms in England in the period 2003-2007 are consistent with the objective of **eliminating the need for tactical preferences**.

**Proposition:** Suppose there are more than  $k$  schools where  $k > 1$ . The FPF mechanism when participants can only rank  $k$  schools is more manipulable than the SOSM where students can rank  $k$  schools.

**Corollary:** Suppose there are more than  $k$  schools where  $k > 1$ . The Boston mechanism when participants can only rank  $k$  schools is more manipulable than the SOSM where students can rank  $k$  schools.

## Recent School Admissions Reforms

Allocation System	Year	From	To	Manipulable (More or Less?)	Allocation System	Year	From	To	Manipulable (More or Less?)
Boston Public Schools (K, 6, 9)	2005	Boston	GS	Less	Merton	2006	FPF <sup>5</sup>	GS <sup>2</sup>	Less
Chicago Selective High Schools	2009	Boston <sup>1</sup>	SD <sup>1</sup>	Less	Newcastle	2005	Boston <sup>1</sup>	GS <sup>2</sup>	Less
	2010	SD <sup>4</sup>	SD <sup>5</sup>	Less		2010	GS <sup>3</sup>	GS <sup>4</sup>	Less
Ghana - Secondary Schools	2007	GS <sup>3</sup>	GS <sup>3</sup>	Less	North Lincolnshire	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
	2008	GS <sup>4</sup>	GS <sup>5</sup>	Less	North Somerset	2007*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
Seattle Public Schools	1999	Boston	GS	Less	North Tyneside	2007*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
	2009	GS	Boston	More	Oldham	2007*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
England					Peterborough	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Bath and North East Somerset	2007*	FPF <sup>2</sup>	GS <sup>3</sup>	Less	Plymouth	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Bedford and Bedfordshire	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Poole	2007*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
Blackburn with Darwen	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Portsmouth	2007*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
Blackpool	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Richmond	2005	FPF <sup>5</sup>	GS <sup>5</sup>	Less
Bolton	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Sefton primary	2007	Boston <sup>1</sup>	GS <sup>3</sup>	Less
Bradford	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Sefton secondary	2007	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Brighton and Hove	2007	Boston <sup>1</sup>	GS <sup>3</sup>	Less	Slough	2006*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
Calderdale	2006	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Somerset	2007*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
Cornwall	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	South Gloucestershire	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Cumbria	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	South Tyneside	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Darlington	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Southampton	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Derby	2005*	FPF <sup>4</sup>	GS <sup>3</sup>	Less	Stockton	2007*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
Devon	2006*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Stoke-on-Trent	2007*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
Durham	2007	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Suffolk	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Ealing	2006*	FPF <sup>6</sup>	GS <sup>5</sup>	Less	Sunderland	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
East Sussex	2007	Boston <sup>1</sup>	GS <sup>3</sup>	Less	Surrey	2007	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Gateshead	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less		2010	GS <sup>2</sup>	GS <sup>5</sup>	Less
Halton	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Sutton	2006	FPF <sup>5</sup>	GS <sup>5</sup>	Less
Hampshire	2007	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Swindon	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Hartlepool	2007	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Tameside	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Isle of Wright	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Telford and Wrekin	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Kent	2007	Boston <sup>1</sup>	GS <sup>3</sup>	Less	Torbay	2007*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
Kingston upon Thames	2007*	FPF <sup>3</sup>	GS <sup>4</sup>	Less	Warrington	2007*	FPF <sup>3</sup>	GS <sup>2</sup>	Less
Knowsley	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Warwickshire	2007*	FPF <sup>7</sup>	GS <sup>2</sup>	Less
Lancashire	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Wilgan	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less
Lincolnshire	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Wales				
Luton	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less	Wrexham County Borough	2011	FPF <sup>3</sup>	GS <sup>2</sup>	Less
Manchester	2007*	FPF <sup>3</sup>	GS <sup>3</sup>	Less					