Introduction to Games in Logic
Semantic Game

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Introduction

- Logic is the study of the language of mathematics (structures and their relations) and of mathematics through its language.
- The study of this language (objectual language) also uses a language (metalanguage) and standard mathematical reasoning itself.
- Games are all over mathematical logic since probably Henkin that used games to give meanning to sentences in infinitary logic (1950s).
- Hintikka made the case for using games as an alternative to the compositional approach to semantics.

Introduction

- There are three important types of games related to logic:
  1. Semantic games: chracterize satisfaction.
  2. Model existence game: charcterize model existence or consistency.
  3. Separation game or Ehrenfeucht - Fraisse game. Characterizes elementary equivalence.

Introduction

- Once the connection between logic and games has been made, logical principles such as:
  1. Bivalence (every sentence has exactly one truth value, either it is true or false - a semantic concept)
  2. Law of excluded middle (for every sentence ϕ, $(\phi \vee \neg\phi)$ is true) can be seen as results of the theory of games.
- For example bivalence is an inmediate consequence of he Zermelo or Gale-Steward theorem.

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Games Mathematicians Play

- A hint to the relation of games and logic.

Example (Game of Nim)

There are six stacked boxes. Two players. Each player may retrive one or two boxes per turn. The player that retrives the last box wins.

Consider the following structure $\mathfrak{N} = (\{1,2\}, W^{\mathfrak{N}})$ were $W^{\mathfrak{N}}$:

$$W^{\mathfrak{N}} = \{(a_0, b_0, \ldots, a_2, b_2) : \exists n \leq 2, \sum_{i=0}^{n}(a_i + b_i) = 6, a_i, b_i \in \{1,2\}\}. \tag{1}$$

Notice $W^{\mathfrak{N}}$ is defined using symbols not present in the language.

Example (Game of Nim)

Now consider the following first orden formula:

$$\phi = \forall x_0 \exists y_0 \forall x_1 \exists y_1 W(x_0, y_0, x_1, y_1). \tag{2}$$

Nim is a model of ϕ if and only if player II ($\exists$) has a winning strategy.

- The continuity and uniform continuity of a real valued function on the real line can also be intepreted as an extensive form games of perfect information where quantifiers define players.

Example (Continuity)

$f : \mathbb{R} \to \mathbb{R}$ is continuous if the following sentence is true:

$$\forall x \forall \epsilon \exists \delta (|x - y| < \delta \to |f(x) - f(y)| < \epsilon) \tag{3}$$

Example (Continuity)

It is uniformly continuous if the following sentence is true:

$$\forall \epsilon \exists \delta \forall x \forall y (|x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon) \tag{4}$$

Game: Player I choose an element of $\mathbb{R}$ whenever the quantifier $\forall$ appears and player II choose an element of $\mathbb{R}$ whenever $\exists$ appears.

Example (Banach - Mazur)

In 1928 the Polish mathematician Mazur invented the following game. Let I_0 be a closed interval of the real line, $A \subset I_0$ and $B = I_0/A$.There are two players A and B. The game (A, B) is played in the following way.

1. A plays first and chooses a closed interval $I_1 \subset I_0$.
2. Then B chooses a closed interval $I_2 \subset I_1$.
3. Repeat 1 and 2 to obtain sequences: $(I_{2n-1})_{n=1,\dots}$ of actions for A and $(I_{2n})_{n=1,\dots}$ sequences of actions for B.

If $\cap_{n\geq 0} I_n$ has an element in common with A then A wins. Otherwise B wins. Mazur proved that if A is category I then the game is determined in favor of B. Banach proved the converse.

Example (Nondecidable determined games)

Some games can be determined but it is not known in favor of whom they are determined.

- Chess is good example.
- Consider the following arithmetic class of games (see Jones (1982)). Let $P(x_1, ..., x_L)$ be a polynomial in L variables $(x_i)_{i=1,..,L}$.
  There are two players. Player I starts choosing a nonnegative integer x_1. Player II chooses next a nonnegative integer x_2, etc. The last player wins if $P(x_1, ..., x_L) = 0$ otherwise it loses. Arithmetic games are determined.

Example

Consider the game defined by the polynomial $P(x_1, \ldots, x_5) = x_1^2 + x_2^2 + 2x_1x_2 - x_3x_5 - 2x_3 - 2x_5 - 3$. This can be rewritten as $P(x_1, \ldots, x_5) = (x_3 + 2)(x_5 + 2)$. Player *II* has a winning strategy if and only if there are infinite prime numbers of the form $n^2 + 1$, an unsolved problem in 1982.

Example (Nondecidable determined games)

Some games can be determined but it is not known in favor of whom they are determined.
There are games which are determined but the winning strategy is not effectively computable.
Rabin (1957) has given an example based on simple sets (in fact, finite extensive form games of perfect information solvabe by bakcward induction).
Jones (1981, 1982) has shown how to reinterpret this games in terms of arithmetic games.

Contenido

Propositional Logic

- The language of propositional logic has the following symbols.
  1. Atomic propositions: $\{p_0, p_1, \ldots\}$
  2. Logical symbols: $\neg, \wedge, \vee, (,)$
- Valid formulas are finite strings of symbols of the language build recursively using logical symbols $(\widetilde{P})$.

Propositional Logic

- A valuation funcion is any funtion $v : P \to \{0, 1\}$.
- Valuations play the same role as models in predicative or first order logic.
- A formula $\phi \in \widetilde{P}$ is true under valuation v if $v(\phi) = 1$. We denote this by $v \models \phi$.
- Given a set of formulas Γ we say $\Gamma \models \phi$ if and only for all valuation v such that $v(\psi) = 1$ for all formula $\phi \in \Gamma$ then $v(\psi) = 1$.

Propositional Logic

Theorem (Monotonicity)

If Γ, Γ' are sets of formulas such that $\Gamma \subseteq \Gamma'$ and $\Gamma \models \phi$ then $\Gamma' \models \phi$.

- Propositional logic is monotonic.
- A formula ϕ is in negation normal form (NNF) if all negation symbols that appear in ϕ appear in front of atomic propositions.
- Every formula is logically equivalent to a formula in negation normal form.

Propositional Logic: Semantic game

- Semantic game: The semantic game for propositional logic is a special case of the semantic game of predicative logic for the case of sentences with no quantifiers.

Propositional Logic: A Non-monotone version

- Consider the following alternative definition of the (satisfaction) the relation $\models$. We say $\Gamma \models_{CK} \phi$:
  1. If ϕ is a propostional symbol and $\phi \in \Gamma$.
  2. If $\phi = \psi_1 \wedge \psi_2$ then $\Gamma \models_{CH} \psi_1$ and $\Gamma \models_{CH} \psi_2$.
  3. If $\phi = \psi_1 \vee \psi_2$ then $\Gamma \models_{CH} \psi_1$ or $\Gamma \models_{CH} \psi_2$.
  4. If $\phi = \neg\psi$ then it is not the case that $\Gamma \models_{CH} \psi$.
- Notice that with this definition, $\models_{CH}$ is not monotone (take $q \models_{CH} \neg p$ but $q, p \not\models_{CH} \neg p$).

Example

Consider the following set of formulas $\Gamma = \{p \vee q, \neg q\}$. Then $\Gamma \models p$ but $\Gamma \not\models_{CH} p$. It follows that $\models_{CH}$ is a strictly stronger relation than $\models$.

Contenido

Predicative Logic

Definition (Alphabet)

The alphabet of first order language consists of:

1. A set of logical symbols A: countable variables, $V = \{v_0, v_1, ...\}$, connectives $(\wedge, \vee)$ and quantifiers symbols $(\forall, \exists)$, an equality symbol and parentheses.
2. A set of non logical symbols S (a signature): a countable set of constants, a set of n-ary functions and n-ary relation symbols.

- Finite strings of symbols:
  1. Terms.
  2. Atomic formulas.
  3. Formulas.

Semantics

Definition (Structure)

An S-structure $\mathfrak{M}$ is a a pair, $\mathfrak{M} = (M, \sigma)$ where M is a set called the universe and σ is a function that interprets in M all elements of the symbol set S. We denote this interpretations in M by $c^{\mathfrak{M}}, f^{\mathfrak{M}}, R^{\mathfrak{M}}$, etc.

Example

Some important examples are:

1. $S_{ar} = (+, \cdot, 0, 1)$ The language of fields.
2. $S_{ar}^{<}$ is the language of ordered fields (the language of fields plus an strict order relation).
3. One can use differente languages to represent similar structures. $S_G = (\circ)$ or $S_G = (\circ, 1)$ or even $S_G = (\circ, ^{-1}, 1)$ for the language of groups. But there are important differences: (1) Axioms will be different and may or not have existential quantifiers, (2) Substructures will be very diffeent.
4. $S_G = \langle E \rangle$, were E is a binary relation is the language of graphs.

Semantics

- The definition of what truth means in a particular intepretation is Tarski definition of satisfaction.

Definition (Satisfaction)

We say that a sentence φ is satisfied in $\mathfrak{M}$ if is satisfies Tarski conditions:

1. The satsifaction of atomic formulas and $(\phi \wedge \psi)$, $(\phi \vee \psi)$ and $\neg\phi$ is the natural one and corresponds to standard mathematical practice.
2. $\mathfrak{M} \models \forall x \phi$. if and only if for all $m \in M$, $\mathfrak{M} \models \phi[m]$.
3. For the existential quantifier a similar definition applies.

- Let ϕ be a first order formula in NNF, $\mathfrak{M}$ a structure.
- The semantic game $G(\mathfrak{M}, \phi)$ is a two player win-lose finite horizon perfect information game (N, H, P, u) where:
  1. $N = \{I, II\} = \{Abelard, Eloise\}$.
  2. $H = \bigcup\{H_\varphi : \varphi \in Subf(\phi)\}$ where is defined recursively as follow:
     - If $\varphi = \phi$, then $H_\varphi = \{(\phi)\}$.
     - If $\varphi = \varphi_1 \circ \varphi_2$, then $H_{\varphi_i} = \{(\check{h}, \varphi_i) : h \in H_\phi\}$.
     - If $\varphi = Qx\varphi$, then $H_\varphi = \{(\check{h}, (x, a)) : h \in H_\phi, a \in M\}$.

Semantic Game

- There is one unique initial node ϕ
- There are three types of actions (branches): either subformulas ψ of ϕ, or subformulas ψ and elements of the universe: (ψ, a).
- Leafes or terminal nodes are atomic formulas or negations of atomic formulas.
- Player I (nature, falsifier, Abelard) plays with $(\forall, \wedge)$.
- Player II (verifier, Eloise) plays with $(\exists, \vee)$.
- The verifier wins the game on a terminal history if the formula at the terminal node is satisfied by the current assignment. Otherwise the falsifier wins.

Semantic Game

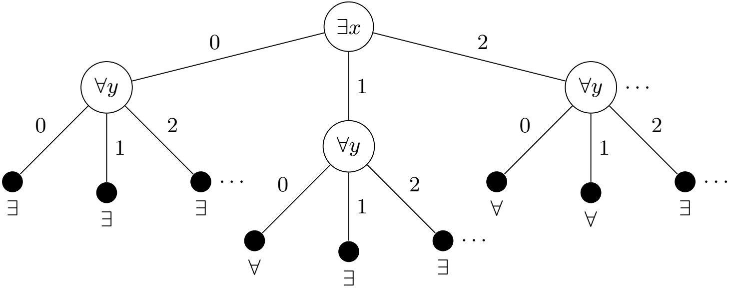

Figure 3.1 The semantic game for $\exists x \forall y (x \leq y)$ in $\mathbb{N}$

Semantic Game

Definition (Game Theoretic Semantics)

Let ϕ be a first order sentence. ϕ is true in structure $(\mathfrak{M}, s)$ if Eloise has a winning strategy in the associated semantic game.

Theorem (Equivalence between Compositional and GTS)

Under the axiom of choice both definitions are equivalent. In fact they are equivalent to Skolem semantics.

Contenido

Logic of Imperfect Information

- Consider the following sentences:
  1. $\forall x \forall y \exists z R(x, y, z)$
  2. $\forall x \forall y (\exists z / \{y\}) R(x, y, z)$
- The first sentence is a standard first order formula. The second sentence is an IF sentence. The expression $\exists z / \{y\}$ means that z is independent of y even if z is in the scope of a quantifier of y.

Logic of Imperfect Information

- Logic of Imperfect Information is also called: Independence Friendly Logic.
- IF logic has more expressive power.
- Propositional IF logic is equivalent to Kleene's strong three valued logic.
- Predicative IF logic is logically equivalent to existential second order logic.
- Perfect recall fragment of IF logic is equivalent to first order logic.
- IF logic does not have a complete proof system.
- IF logic is a compact logic.

Example (Uniform Continuity)

$f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if the following first order sentence is true:

$$\forall \epsilon \exists \delta \forall x \forall y (|x - y| < \delta \to |f(x) - f(y)| < \epsilon) \tag{5}$$

or equivalently in IF logic:

$$\forall x \forall \epsilon (\exists \delta / \{x\}) \forall y (|x - y| < \delta \to |f(x) - f(y)| < \epsilon) \tag{6}$$

- Several properties are lost when passing from first order to IF logic. For example, there is no guarantee that every IF sentence is either true or false. Consider the following leaidng example:

$$\forall x(\exists y/\{x\})x = y \tag{7}$$

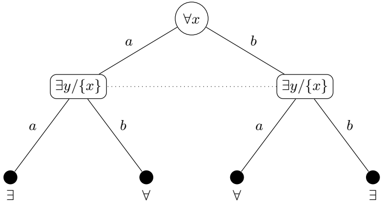

Figure 4.1 The semantic game for $\forall x(\exists y/\{x\})x = y$ in the structure $\mathbb{M} = \{a, b\}$

- Notice the difference between the previous IF sentence with the following first order sentences:

$$\forall x(\exists y)x = y \tag{8}$$

and

$$\exists y(\forall x)x = y \tag{9}$$

In first order logic the first one is clearly true in any structure. The second one is false in any structure with at least two elements.

- In IF logic the value of sentence may depend even in variables not present in the formula. In first order logic: $\forall x \exists y R(x, y)$ is equivalent to $\forall x \exists z \exists y R(x, y)$.
- Now consider the following example of IF logic:

$$\forall x \exists z (\exists y / \{x\}) x = y \tag{10}$$

  The informational restriction can be circunvented by storing in z the value of x.
- Therefore, $(\exists y / \{x\}) x = y$ has a different meanning in the previous two formulas depending z which does not appear in $(\exists y / \{x\}) x = y$.
- What this example suggests is the meaning of sentences is context specific.

Logic of Imperfect Information: Syntax

- Terms are defined in the same way as in first order logic.
- Atomic formulas are defined in the same way.
- IF formulas are formulas closed under connectives and quantifiers but for simplicity negation symbols can only appear in front of atomic formulas.
- IF formulas are closed under quantifiers conditioned to slash sets: If ϕ is an IF formula then $Qx/W\phi$ is an IF formula were W is a finite set of variables.

Game Theoretic Semantics

Definition (Game Theoretic Semantics)

Given a sentence ϕ, we say:

$$\mathfrak{M} \models^{+} \phi \textit{ iff Eloise has a winnign strategy in} G(\mathfrak{M}, \phi) \tag{11}$$

$$\mathcal{M} \models^{-} \phi \textit{ iff Abelard has a winnign strategy in} G(\mathfrak{M}, \phi) \tag{12}$$

- The first relation characterizes truth, the second false sentences.
- Notice that not being true does not mean it is false.

- Consider the following formula:

$$\exists w \forall x (\exists y / w)(\exists z / w, x)(z = x \wedge y \neq w) \tag{13}$$

- It asserts that there is an inyection that is not the entire universe (Dedekind definition of infinite). The Skolem form asserts there are two function f, g such that:

$$\exists w \forall x (g(f(x)) = x \wedge f(x) \neq w) \tag{14}$$

  Therefore f is inyective but not the entire universe.
- There is no formula logically equivalent to this one in first order logic.
- The formula does not have perfect recall (Eloise forgets her choice of w).