# Dynamic Mechanism Design:

## Revenue Equivalence, Profit Maximization, and Information Disclosure

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# Motivation

- Mechanism Design: auctions, taxation, etc...
- Standard model: one-time information, one-time decisions
- Many real-world settings
  - Information arrives over time (serially correlated)
  - Sequence of decisions
  - Non-time-separable technology/preferences

# Examples

- Sequential procurement auctions
  - bidders acquire information, invest, learn by doing...
  - intertemporal capacity constraints
- New "experience goods"
  - valuation dynamics driven by consumption ("experimentation")
  - price discrimination by menu of price paths
- Advance sales (e.g., flight tickets)
  - buyers receive information, make investments over time
  - price discrimination on early info. by menu of price-refund options

# State of the Literature

- Efficient dynamic mechanisms:
  - Athey-Segal,Bergemann-Valimaki ...
- Special cases of profit-maximization: typically one agent, Markov process
  - Baron-Besanko: two-period monopoly regulation
  - Courty-Li: two-period advance ticket sales
  - Eso-Szentes: two-period, one decision
  - Battaglini: infinite horizon with 2 types in each period
- Hanging questions:
  - Necessary + sufficient conditions for incentive compatibility with many agents, many periods, non-Markov processes, continuous types
  - Properties of profit-maximizing mechanisms
  - Important technical assumptions

# What's Different about Dynamic Mechanisms?

- How to derive transfers, payoffs from nonmonetary allocations ("revenue equivalence")?
- $\hookrightarrow$ Must control for multi-period contingent deviations

# Payoff Non-equivalence with Discrete Future Types

- What assumptions on type-process are needed?

## Example

Payoff: $\theta_2 x_2 - p_1 - p_2$

- $2^{nd}$ period consumption: $x_2 \in \{0, 1\}$, no consumption in $1^{st}$ period
- Types: $\theta_2 \in \{H, L\}$ and $\theta_1 = \Pr\{\theta_2 = H\} \in [0, 1]$
- Mechanism:
- 1st period: nothing
- 2nd period: post price $q$, with $L \leq q \leq H$
- Allocation $x_2(H) = 1$, $x_2(L) = 0$ for any $\theta_1$, regardless of $q$!
- Equilibrium payoff: $V(\theta_1) = \theta_1(H - q)$

- Revenue Equivalence at $t = 1$ fails because of disconnected type space at $t = 2$ (despite connected type-space at $t = 1$)

# Payoff Non-equivalence with Discontinuous Transitions

## Example (continued)

- Payoff: $\theta_2 x_2 - p_1 - p_2$
- Types: $\theta_1, \theta_2 \in [0,1]$ with

$$f_2(\theta_2|\theta_1) = \begin{cases} 1 & \text{if } \theta_1 < \frac{1}{2} \\ 2\theta_2 & \text{if } \theta_1 \geq \frac{1}{2} \end{cases}$$

- Mechanism:
  - 1st period: advance contract with posted price $q$ with $q \in (\frac{1}{2}, \frac{2}{3})$
  - 2nd period: execute contract
- Allocation $x_2(\theta_1) = 1$ iff $\theta_1 \geq \frac{1}{2}$ regardless of $\theta_2$, regardless of $q$!
- Eq. payoff: $V(\theta_1) = 0$ if $\theta_1 < \frac{1}{2}$, and $V(\theta_1) = \frac{2}{3} - q$ if $\theta_1 \geq \frac{1}{2}$
- E.g., if $V(0) = 0$, then $V(1) \in \left[0, \frac{1}{6}\right]$

- Revenue Equivalence at $t = 1$ fails because of discontinuous transitions

# Results of this Paper

- Incentive compatibility $\Rightarrow$ Formula expressing agents' eq. payoffs
  - Summarizes "first-order" multi-period IC (cf. Mirrlees)
  - Technical "smoothness" conditions for this to hold
- Sufficient conditions for "global" incentive compatibility
- In quasilinear multi-agent environments, with statistically independent types across agents:
  - Revenue Equivalence Theorem
  - Principal's expected profits = expected "dynamic virtual surplus"
  - Profit-maximizing mechanisms
  - Dynamics of distortions
- Applications: sequential auctions, mechanisms for selling new goods, etc.

# Environment (as seen by one agent)

- In each period $t = 1, \ldots, T$
  - Agent privately observes $\theta_t \in \Theta_t \subset \mathbb{R}$
  - Decision $y_t \in Y_t$
- Histories:

$$
\begin{aligned}
y^t &= (y_1, \ldots, y_t) \in Y^t = \prod_{\tau=1}^{t} Y_\tau, \\
\theta^t &= (\theta_1, \ldots, \theta_t) \in \Theta^t = \prod_{\tau=1}^{t} \Theta_\tau
\end{aligned}
$$

  full histories: $y = y^T \in Y = Y^T$, $\theta = \theta^T \in \Theta = \Theta^T$
- Technology:

$$\tilde{\theta}_t \sim F_t(\cdot|\theta^{t-1}, y^{t-1})$$

  - allows learning-by-doing, information acquisition, etc.
- Agent's payoff: $U(\theta, y)$

# Mechanisms

- Revelation principle (Myerson 86) $\Rightarrow$ *direct* mechanisms:
- In each period $t$
  - Agent observes $\theta_t \in \Theta_t$
  - Agent submits report $m_t \in \Theta_t$
  - Mechanism draws $y_t \in Y_t$ from probability distribution $\Omega_t(\cdot|m^t, y^{t-1})$
    - Randomization allows e.g. dependence on other agents' messages
- *(Randomized direct) mechanism*:

$$\Omega = \left\langle \Omega_t : \Theta^t \times Y^{t-1} \to \Delta\left(Y_t\right)\right\rangle_{t=1}^{T}$$

- Agent's reporting *strategy*:

$$\sigma = \left\langle \sigma_t : \Theta^t \times \Theta^{t-1} \times Y^{t-1} \to \Theta_t \right\rangle_{t=1}^{T}$$

- *Truthful strategy:*

$$\sigma_t(\theta^t, m^{t-1}, y^{t-1}) \equiv \theta_t \quad \text{for all } t, \quad \text{all } (\theta^t, m^{t-1}, y^{t-1})$$

# Stochastic Process and Expected Payoffs

- Histories:

$$H = \{(\theta^s, m^t, y^u) : \quad s \geq t \geq u \geq t-1\}$$

- Technology $F$, mechanism $\Omega$, strategy $\sigma$, and history $h \in H \implies$ probability measure $\mu[\Omega, \sigma]|h$ on $\Theta \times \Theta \times Y$
  - $\mu[\Omega]|h$ if $\sigma$ is truthful
  - $\mu[\Omega, \sigma]$ if $h$ is null history

- $\mathbb{E}^{\mu[\Omega,\sigma]|h}[U(\tilde{\theta}, \tilde{y})] =$ resulting exp.payoff

- *Value function*:

$$V(h) = \sup_{\sigma} \mathbb{E}^{\mu[\Omega,\sigma]|h}[U(\tilde{\theta}, \tilde{y})]$$

# Incentive Compatibility

## Definition

Mechanism $\Omega$ is *incentive compatible at history $h$ (IC at $h$)* if

$$\mathbb{E}^{\mu[\Omega]|h}[U(\tilde{\theta}, \tilde{y})] = V(h)$$

- Focus on ex ante rationality:

## Definition

Mechanism $\Omega$ is *ex-ante incentive compatible (ex-ante IC)* if it is IC at $\varnothing$

- Ex-ante IC implies IC at truthful histories (i.e., on eq.path) with $\mu[\Omega]$-prob. 1

# First-Order IC in Static Model (Mirrlees, Myerson)

- Assume $T = 1$
- Mechanism $\Omega$ is IC at each $\theta$:

$$V(\theta) \equiv \sup_{m \in \Theta} \int_Y U(\theta, y) d\Omega(y|m) = \int_Y U(\theta, y) d\Omega(y|\theta)$$

- Envelope Theorem:

$$V'(\theta) = \int_Y \frac{\partial U(\theta, y)}{\partial \theta} d\Omega(y|\theta)$$

- Quasilinear setting:
  - $U(\theta, \underbrace{(x, p)}_{y}) = u(\theta, x) + p$
  - $\Rightarrow$ Revenue Equivalence, characterization of optimal mechanisms

# First-Order IC in Dynamic Model: Heuristic Derivation

- Mechanism $\Omega$ is IC at (truthful) history $h = (\theta^t, \theta^{t-1}, y^{t-1})$:

$$\begin{aligned} V(h) &= \mathbb{E}^{\mu[\Omega]|h}[U(\tilde{\theta}, \tilde{y})] \\ &= \int U(\theta, y) \prod_{\tau=t}^{T} \left[ d\Omega_\tau(y_\tau | m^\tau, y^{\tau-1}) dF_{\tau+1}\left(\theta_{\tau+1} | \theta^\tau, y^\tau\right)\right] \Bigg|_{m=\theta} \end{aligned}$$

- Differentiate wrt current type $\theta_t$:

  1. in $U(\theta, y) \Rightarrow \mathbb{E}^{\mu[\Omega]|h}\left[\partial U(\tilde{\theta}, \tilde{y}) / \partial \theta_t\right]$
  2. in $F_{\tau+1}(\theta_{\tau+1}|\theta^\tau, y^\tau) \Rightarrow$ integrate by parts, differ. within integral:

$$-\mathbb{E}^{\mu[\Omega]|h}\left[\int \frac{\partial V((\tilde{\theta}^\tau, \theta_{\tau+1}), \tilde{\theta}^\tau, \tilde{y}^\tau)}{\partial \theta_{\tau+1}} \frac{\partial F_{\tau+1}(\theta_{\tau+1}|\tilde{\theta}^\tau, \tilde{y}^\tau)}{\partial \theta_t} d\theta_{\tau+1}\right]$$

  3. Derivatives wrt report $m_t = \theta_t$: vanish by (appropriate version of ) Envelope Thm

# Technical Assumptions

- Don't want to impose "smoothness" on mechanism
- "Smooth" environment needed to iterate Envelope Thm backward
- Ensure one can differentiate totally and under expectations
    - Need new assumptions on kernels $F_t$

# Technical Assumptions

1. $\Theta_t = (\underline{\theta}_t, \overline{\theta}_t)$ with $-\infty \leq \underline{\theta}_t \leq \overline{\theta}_t \leq +\infty$
2. $\partial U(\theta, y)/\partial \theta_t$ exists and bounded uniformly in $(\theta, y)$
3. "Full Support": $F_t(\theta_t|\theta^{t-1}, y^{t-1})$ strictly increasing in $\theta_t$
4. $\int |\theta_t| \, dF_t(\theta_t|\theta^{t-1}, y^{t-1}) < +\infty$
5. For $\tau < t$, $\partial F_t(\theta_t|\theta^{t-1}, y^{t-1})/\partial \theta_\tau$ exists and bounded in abs. value by an integrable function $B_t(\theta_t)$
6. $F_t(\cdot|\theta^{t-1}, y^{t-1})$ continuous in $\theta^{t-1}$ in total variation metric
7. $F_t(\cdot|\theta^{t-1}, y^{t-1})$ abs. continuous, with density $f_t(\cdot|\theta^{t-1}, y^{t-1})$ (only to simplify formulas)

# Payoff via FOC: Formal Result

## Theorem

*Under Assumptions 1-7, if* $\Omega$ *is IC at* $h^{t-1} = (\theta^{t-1}, \theta^{t-1}, y^{t-1})$, *then* $V(\theta_t, h^{t-1})$ *is Lipschitz continuous in* $\theta_t$, *and for a.e.* $\theta_t$,

$$\frac{\partial V(\theta_t, h^{t-1})}{\partial \theta_t} = \mathbb{E}^{\mu[\Omega]|(\theta_t, h^{t-1})}\left[\sum_{\tau=t}^{T} J_t^{\tau}(\tilde{\theta}, \tilde{y})\frac{\partial U(\tilde{\theta}, \tilde{y})}{\partial \theta_\tau}\right] \qquad \text{(IC-FOC)}$$

*where*

$$\underbrace{J_t^{\tau}(\theta, y)}_{\text{"Total information index"}} = \sum_{K\in\mathbb{N},\ l\in\mathbb{N}^K: t=l_0<\ldots<l_K=\tau} \prod_{k=1}^{K} I_{l_{k-1}}^{l_k}(\theta, y)$$

$$\underbrace{I_t^{\tau}(\theta, y)}_{\text{"Direct information index"}} = -\frac{\partial F_\tau(\theta_\tau|\theta^{\tau-1}, y^{\tau-1})/\partial\theta_t}{f_\tau(\theta_\tau|\theta^{\tau-1}, y^{\tau-1})}$$

# Example: AR(k) Process

$$\theta_t = \sum_{l=1}^{k} \phi_l \theta_{t-l} + \varepsilon_t$$

- $\varepsilon_t \sim G_t$, independent across $t$; $\theta_t$ public for $t \leq 0$
- $F_\tau(\theta_\tau|\theta^{\tau-1}, y^{\tau-1}) = G_\tau\left(\theta_\tau - \sum_{l=1}^{k} \phi_l \theta_{\tau-l}\right)$
- $I_t^\tau(\theta, y) = -\frac{\partial F_\tau(\theta_\tau|\theta^{\tau-1}, y^{\tau-1})/\partial\theta_t}{f_\tau(\theta_\tau|\theta^{\tau-1}, y^{\tau-1})} = \phi_{\tau-t}$
- $J_t^\tau(\theta, y) = \sum_{K\in\mathbb{N},\ l\in\mathbb{N}^K : t=l_0<\ldots<l_K=\tau} \prod_{k=1}^{K} \phi_{l_k - l_{k-1}}$ "impulse response" constants
- AR(1):

$$I_t^\tau(\theta, y) = \begin{cases} \phi_1 & \text{if } \tau = t+1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad J_t^\tau(\theta, y) = (\phi_1)^{\tau-t}.$$

# Alternative Approach: Independent-shock Representation

- $\theta_t = z(\varepsilon^t; y^{t-1})$ where $\varepsilon_t \sim G_t$, support in $\mathbb{R}$, independent across $t$
- E.g. AR(k): $\theta_t = \sum_{l=1}^{k} \phi_l \theta_{t-l} + \varepsilon_t$
- Two representations are equivalent: Given mechanism $\Omega$ for $F$, there exists $\hat{\Omega}$ for $(G, z)$ that induces same distribution on $\Theta \times Y$ as $\Omega$. And vice versa.
- Alternative route: have agent report $(\varepsilon_t)_{t=1}^{T} \longrightarrow$ mechanism $\hat{\Omega}$
- Redefine utility in terms of $\varepsilon$: $\hat{U}(\varepsilon, y) \equiv U(z(\varepsilon; y), y)$
- With serially independent shocks, IC-FOC formula simplifies to

$$\frac{\partial \hat{V}\left(\varepsilon_t, h^{t-1}\right)}{\partial \varepsilon_t} = \mathbb{E}^{\mu[\hat{\Omega}]|(\varepsilon_t, h^{t-1})} \left[ \frac{\partial \hat{U}(\tilde{\varepsilon}, \tilde{y})}{\partial \varepsilon_t} \right]$$

  where $h^{t-1} = (\varepsilon^{t-1}, \varepsilon^{t-1}, y^{t-1})$
- Simpler proof: sufficient to consider period-$t$ deviations

# Independent Shocks: Results

**Theorem**

*Any $F$ admits "canonical" independent-shock representation in which for all $t$, $\tilde{\varepsilon}_t \sim \mathcal{U}(0,1)$.*

- Proof by induction on $t$ using "prob. integral transform thm":

$$z_t(\varepsilon^t; y^{t-1}) = F_t^{-1}(\varepsilon_t | z^{t-1}(\varepsilon^{t-1}; y^{t-2}), y^{t-1})$$

- Given model specified in terms of $F$, two routes to payoff equivalence:
  1. Work with $F$ and impose Assumptions 1-7 from above
  2. Convert $F$ into independent shocks $(G, z)$ and identify assumptions on $F, U$ that ensure $\hat{U}$ is "smooth"

- Turns out that assumptions required for 1 and 2 are not nested:
  - 1 rules out "shifting atoms" (e.g., fully persistent types)
  - 2 rules out "growing atoms" but allows for shifting atoms

# Independent Shocks: Assumptions for IC-FOC

- New conditions:
  (a) $U(\cdot, y)$ equi-Lipschitz and *continuously* differentiable in $\theta$
  (b) $F_t^{-1}(\varepsilon|\cdot, y^{t-1})$ equi-Lipschitz and *continuously* diff in $\theta^{t-1}$
  (c) $F_t^{-1}(\cdot|\theta^{t-1}, y^{t-1})$ equi-Lipschitz and *continuously* diff. in $\varepsilon$.

**Theorem**

*Suppose $(U, F)$ satisfies assumptions (1)-(2) + (a)-(c). Then $\hat{U}(\varepsilon, y)$ is equi-Lipschitz continuous and differentiable in $\varepsilon$. It follows that if $\hat{\Omega}$ is IC at history $h^{t-1} = (\varepsilon^{t-1}, \varepsilon^{t-1}, y^{t-1})$, then*

$$\frac{\partial \hat{V}\left(\varepsilon_t, h^{t-1}\right)}{\partial \varepsilon_t} = \mathbb{E}^{\mu[\hat{\Omega}]|(\varepsilon_t, h^{t-1})}\left[\frac{\partial \hat{U}(\tilde{\varepsilon}, \tilde{y})}{\partial \varepsilon_t}\right] \quad a.e.$$

# Quasilinear Settings with multiple agents

- Agents $i = 1, ..., N$
- $(x_t, p_t)$, where $p_t \in \mathbb{R}^N$, $x_t = (x_{1t}, ..., x_{Nt}) \in X_t \subset \prod X_{it}$
- $U_i(\theta, (x, p)) = u_i(\theta, x) + \sum_t p_{it}$
- Assumption: $F_{it}(\theta_{it}|\theta^{t-1}, (x^{t-1}, p^{t-1})) = F_{it}(\theta_{it}|\theta_i^{t-1}, x_i^{t-1})$
- *Independent Types*: $\tilde{\theta}_{i,t} \sim F_{i,t}(\cdot|\theta_i^{t-1}, x_i^{t-1})$, independent across $i$
- BNE
- Revelation Principle: *truthful* + *minimal disclosure*
  - postponed payments
- Deterministic direct mechanisms: $\langle \chi_t : \Theta^t \to X_t \rangle_{t=1}^T \quad \psi : \Theta \to \mathbb{R}^N$
- $\mu_i[\chi, \psi]|(\theta_i^s, m_i^t, x_i^u)$: process as viewed by $i$

# Payoff Equivalence

- IC-FOC: For all $t$, all $h_i^{t-1} = (\theta_i^{t-1}, \theta_i^{t-1}, x_i^{t-1})$

$$\frac{\partial V_i(\theta_{it}, h_i^{t-1})}{\partial \theta_{it}} = \mathbb{E}^{\mu_i[\chi,\psi]|(\theta_{it}, h_i^{t-1})}\left[\sum_{\tau=t}^{T} J_{it}^{\tau}(\tilde{\theta}, \tilde{x}) \frac{\partial u_i(\tilde{\theta}, \tilde{x})}{\partial \theta_{i\tau}}\right]$$

- Pins down $V_i(\theta_{it}, h_i^{t-1})$ as function of $\chi$ and $\theta_{it}$ up to $K_i(h_i^{t-1})$
- Iterated expectations $\longrightarrow$ get rid of dependence of $K_i(h_i^{t-1})$ on $h_i^{t-1}$

### Theorem

*Let $(\chi, \psi)$ and $(\chi, \hat{\psi})$ be any two ex-ante IC mechanisms that implement same $\chi$. For all $t, i$, with prob. 1,*

$$\mathbb{E}^{\mu[\chi,\psi]}[U_i(\tilde{\theta}, \tilde{y}) \mid \theta_i^t] - \mathbb{E}^{\mu[\chi,\hat{\psi}]}[U_i(\tilde{\theta}, \tilde{y}) \mid \theta_i^t] = K_i$$

- Single agent $\Rightarrow$ $\chi$ pins down payoff and transfer
- Many agents $\Rightarrow$ expectation of payoff and transfer over others' types pinned down as function of own type
- E.g., different dynamic mechanisms implementing efficiency (Athey-Segal, Bergemann-Valimaki,...) are "equivalent" in this sense

# Participation Constraint and Relaxed Problem

- Agents can quit in any period
- Agents can post bonds $\Rightarrow$ only $1^{st}$-period participation constraints bind:
$$V_i(\theta_{i1}) \geq 0 \qquad (\text{IR}_i(\theta_{i1}))$$
- "Relaxed Program": max profits subject to IC-FOC and $\text{IR}_i(\underline{\theta}_{i1})$
  - Sufficient conditions for "IC-FOC + $\text{IR}_i(\underline{\theta}_{i1}) \Rightarrow \text{IR}_i$":
    $\partial u_i(\theta, x) / \partial \theta_{it} \geq 0$ and $I_{it}^{\tau}(\theta, x) \geq 0$ ($\Rightarrow J_{it}^{\tau}(\theta, x) \geq 0$)
    $\Rightarrow$ by IC-FOC, $\partial V_i(\theta_{i1}) / \partial \theta_{i1} \geq 0$
    $\Rightarrow$ only $\text{IR}_i(\underline{\theta}_{i1})$ binds
  - Sufficient conditions for "IC-FOC $\Rightarrow$ IC" — later

# Information Rents

- Let

$$\eta_{i1}\left(\theta_{i1}\right) \equiv \frac{f_{i1}\left(\theta_{i1}\right)}{1 - F_{i1}\left(\theta_{i1}\right)}$$

- Agent $i$'s ex-ante expected information rent (using IC-FOC)

$$\begin{aligned} \mathbb{E}\left[V_i(\tilde{\theta}_{i1})\right] &= \mathbb{E}\left[\frac{1}{\eta_{i1}\left(\tilde{\theta}_{i1}\right)} \frac{\partial V_i(\tilde{\theta}_{i1})}{\partial \theta_{i1}}\right] \\ &= \mathbb{E}\left[\frac{1}{\eta_{i1}\left(\tilde{\theta}_{i1}\right)} \sum_{\tau=1}^{T} J_{i1}^{\tau}(\tilde{\theta}, \tilde{x}) \frac{\partial u_i(\tilde{\theta}, \tilde{x})}{\partial \theta_{i\tau}}\right] \end{aligned}$$

# Profit-Maximizing Multi-Agent Mechanisms

- Principal $\longrightarrow$ agent $0$

**Theorem**

*Let $\mathcal{X}^*$ denote set of allocation rules that maximize "expected virtual surplus"*

$$\mathbb{E}\left[\underbrace{\sum_{i=0}^{N} u_i(\tilde{\theta}, \chi(\tilde{\theta}))}_{\text{Total Expected Surplus}} - \sum_{i=1}^{N} \underbrace{\frac{1}{\eta_{i1}\left(\tilde{\theta}_{i1}\right)} \sum_{t=1}^{T} J_{i1}^{t}(\tilde{\theta}_i, \chi(\tilde{\theta})) \frac{\partial u_i(\tilde{\theta}, \chi(\tilde{\theta}))}{\partial \theta_{it}}}_{\text{Captures agent } i\text{'s information rents}}\right],$$

*<u>and</u> arise in an IC and IR mechanism $(\chi, \psi)$. If $\mathcal{X}^*$ is non-empty, then $\mathcal{X}^*$ is set of profit-maximizing allocation rules.*

# Intuition for Allocative Distortions

- Assume $N = 1$, $u_i(\theta, x) = \sum_t u_{it}(\theta_t, x_t)$, $i = 0, 1$, $J_1^t(\theta)$
- $\Rightarrow$ Maximize virtual surplus for each $t, \theta$ :

$$\max_{x_t \in X_t} \left[ \underbrace{u_{0t}(\theta_t, x_t) + u_{1t}(\theta_t, x_t)}_{\text{Total Surplus in } t} - \underbrace{\frac{J_1^t(\theta)}{\eta_1(\theta_1)} \frac{\partial u_{1t}(\theta_t, x_t)}{\partial \theta_t}}_{\text{Agent's information rent in } t} \right],$$

- Distort $x_t$ to reduce info. rents based on $\theta_1$ and its effect on period $t$
- E.g., for $t > 1$: If $\theta_t = \bar{\theta}_t$ or $= \underline{\theta}_t$, then $F_t(\theta_t|\theta^{t-1}) \equiv 1$ or $\equiv 0$ $\Rightarrow J_1^t(\theta) \equiv 0 \Rightarrow$ implement efficient $x_t$
- $F_\tau(\theta_\tau|\theta^{\tau-1}, x^{\tau-1})$ decreasing in $\theta^{\tau-1}$ (FOSD) $\Rightarrow I_t^\tau, J_t^\tau \geq 0 \Rightarrow$ distort $x_t$ to reduce $\partial u_{1t}(\theta_t, x_t) / \partial \theta_\tau$
  - E.g. $\frac{\partial^2 u_{1t}(\theta_t, x_t)}{\partial \theta_t \partial x_t} > 0$ (SCP) $\Rightarrow$ distort $x_t$ below efficient level
- Note: distortion in $x_t$ is nonmonotonic in $\theta_t$ for $t > 1$ (unlike in static model, or in Battaglini)

# Conditions for Downward Distortions

- $\mathcal{X}$ : set of all (measurable) allocation rules. $\mathcal{X}^0$ : set of allocation rules solving Relaxed Program. $\mathcal{X}^E$ : set of allocation rules maximizing expected total surplus.

**Theorem**

*Suppose each $X_t$ is lattice and*

*(i) decisions don't affect types: $F_{i,t}\left(\theta_{it}|\theta_i^{t-1}\right)$*

*(ii) FOSD: $F_{i,t}\left(\theta_{it}|\theta_i^{t-1}\right)$ nondecreasing in $\theta_i^{t-1}$*

*(iii) SCP: $u_i\left(\theta, x\right)$ supermodular in $\left(x, \theta_i\right)$*

*(iv) $u_i\left(\theta, x\right)$ supermodular in $x$*

*(v) $\frac{\partial u_i(\theta,x)}{\partial \theta_{it}}$ submodular in $x$*

*Then $\mathcal{X}^0 \leq \mathcal{X}^E$ in strong set order.*

- Proof: Topkis Thm applied to

$$g(\chi, z) \equiv \mathbb{E}\left[\sum_{i=0}^{N} u_i(\tilde{\theta}, \chi(\tilde{\theta})) + z \sum_{i=1}^{N} \frac{1}{\eta_{i1}(\tilde{\theta}_{i1})} \sum_{t=1}^{T} J_{i1}^{t}(\tilde{\theta}_i) \frac{\partial u_i(\tilde{\theta}, \chi(\tilde{\theta}))}{\partial \theta_{it}}\right]$$

# Sufficient Condition for Implementable Allocation Rules

- Characterization hard due to multidimensional strategies, decisions

**Theorem**

*Suppose mechanism $(\chi, \psi)$ is IC at any (possibly non-truthful) period $t+1$ history. If for all $i$, all $(\theta_i^t, x_i^{t-1})$*

$$\mathbb{E}^{\mu_i[\chi,\psi]|\theta_i^t,(\theta_i^{t-1},m_{it}),x_i^{t-1}} \left[ \sum_{\tau=t}^{T} J_{it}^{\tau}(\tilde{\theta}, \tilde{x}_i) \frac{\partial u_i(\tilde{\theta}_i, \tilde{x}_i)}{\partial \theta_{i\tau}} \right].$$

*is nondecreasing in $m_{it}$, then there exists transfer rule $\hat{\psi}$ s.t. mechanism $(\chi, \hat{\psi})$ is IC at (a) any truthful period-$t$ history, (b) at any period $t+1$ history.*

- Markov process: IC at truthful histories $\Leftrightarrow$ IC at all histories,
  - can iterate backward to show that $\chi$ is implementable in mechanism that is IC at *all* histories
  - truthful strategies form weak PBE (with beliefs that other agents are truthful at all histories)

# Sufficient Condition - Intuition

- IC at all period $t+1$ histories $\Rightarrow$ suffices to prevent single lie $m_{it}$
- $\Psi_t\left(\theta_{it}, m_{it}\right)$ : agent $i$'s expected utility at history $\left(\theta_i^{t-1}, \theta_i^{t-1}, x_i^{t-1}\right)$
- Think of $m_{it}$ as 1-dimensional "allocation" chosen by agent $i$
- Condition says that $\partial \Psi_t\left(\theta_{it}, m_{it}\right) / \partial \theta_{it}$ (evaluated using IC-FOC at period $t+1$ histories) is nondecreasing in $m_{it}$, $\longrightarrow$ i.e., $\Psi_t$ has SCP
- $\Rightarrow$ monotonic "allocation rule" $m_{it}\left(\theta_{it}\right)$ is implementable (using transfers constructed from IC-FOC)

# A Set of (Stronger) Sufficient Conditions

1. Decisions don't affect types: $F(\theta_{it}|\theta_i^{t-1})$
2. FOSD: $F(\theta_{it}|\theta_i^{t-1})$ is nonincreasing in $\theta_i^{t-1}$ ($\Rightarrow J_{it}^{\tau}(\theta) \geq 0$)
3. SCP: $\partial u_i(\theta, x_i)/\partial\theta_{it}$ nondecreasing in $x_i$
4. $\chi_i(\theta)$ nondecreasing in $\theta_i$ ("Strong" Monotonicity)

- (1)-(4) imply monotonicity condition in theorem
- $\chi$ implementable with mechanism that is IC even if $i$ is shown $\theta_{-i}$ (both past and future)

# Application: Linear AR(k) values

$$u_i(\theta, x) = \sum_{t=1}^{T} \theta_{it} x_{it} - c_i\left(x_i^T\right); \qquad X_t \subset \mathbb{R}^N;$$

$$\theta_{it} = \sum_{l=1}^{k} \phi_{il} \theta_{i,t-l} + \varepsilon_{it} \text{ for } t > 1.$$

- Total information indices $J_{i1}^t(\theta, x) = J_{i1}^t$ "impulse responses constant"
- Expected virtual surplus:

$$\mathbb{E}\left[u_0\left(\tilde{\theta}, x\right) - \sum_{i=1}^{N} \underbrace{\sum_{t=1}^{T} J_{i1}^t \eta_{i1}^{-1}(\tilde{\theta}_{i1}) x_{it}}_{\text{Agent } i\text{'s "info rents"}} + \sum_{i=1}^{N} u_i\left(\tilde{\theta}, x\right)\right]$$

- Optimal mechanism: **"Handicapped" efficient mechanism** (with extra costs $J_{i1}^t \eta_{i1}^{-1}(\theta_{i1})$ of giving objects to agents)
- Incentives from $t = 2$ onward ensured using e.g. "Team Transfers" (Athey-Segal) following truthtelling in $t = 1$
- Incentives at $t = 1$ must be checked application-by-application

# Auctions with AR(k) values

- Time-separable payoffs: $u_i(\theta, x) = \sum_{t=1}^{T} \theta_{it} x_{it}$ (thus $c_i(x_i) \equiv 0$)
- Can maximize virtual surplus separately for each $t, \theta$:

$$\chi_t(\theta) \in \arg\max_{x \in X_t} \left[ \theta_{0t} x_{0t} + \sum_{i=1}^{N} \left( \theta_{it} - J_{i1}^{t} / \eta_{i1}(\theta_{i1}) \right) x_{it} \right]$$

- $\chi_t(\theta)$ depends only on $1^{st}$-period types and current types!
- Implementation: Each $i$ makes a $1^{st}$-period payment determining his "handicap." Then each period, a "handicapped" VCG auction is played
- Truthtelling is IC at any $h_i^t$, $\quad t \geq 2$ (actually ex post IC)
- Assume $\phi_{il} \geq 0$ ($\Rightarrow J_{i1}^{t} \geq 0$) and $\eta'_{i1}(\cdot) \geq 0 \Rightarrow \chi_{it}(\theta)$ nondecreasing in $\theta_{i1} \Rightarrow$ IC at $t = 1$ as well

# Other Applications

- Agents learn values by consuming – experimentation
- Principal or agents have intertemporal costs/capacity constraints
  - In all these settings profit-maximizing mechanisms can again be viewed as "handicapped" version of corresponding efficient mechanism
- Non-quasilinear payoffs: wealth effects, cash constraints, or intertemporal consumption smoothing/risk sharing
  - "Bonding" is not optimal/feasible $\Rightarrow$ participation constraints may bind in all periods
  - $\Rightarrow 1^{st}$ period is not as prominent $\longrightarrow$ analysis more difficult
  - cf. Hendel-Lizzeri paper on optimal long-term life insurance contracts with consumption smoothing

# Summary

- Methodological contributions:
  - "Smoothness" conditions for environment (not mechanisms)
  - Formula for payoffs via IC-FOC from incentive compatibility
  - Revenue equivalence
  - Profit-maximizing mechanisms
- Sufficient conditions for IC
- Applications
  - Handicapped-efficient mechanisms
  - Optimal sequential auctions
  - Experimentation