SORTING AND FACTOR INTENSITY: PRODUCTION AND UNEMPLOYMENT ACROSS SKILLS

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Motivation

- Many markets are characterized by sorting (e.g., production factors to workers)
- Many interesting implications: non-linear wage patterns, inequality,...
- Much of the existing work: one-to-one matching (Kontorovich 42, Shapley & Shubik 71, Becker 73,...)

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- Problem: How to capture factor intensity
- Example: Boom/bust in productivity (recession, globalization, trade...)
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Research Questions:

1. How to capture factor intensity in a tractable manner?
2. What are the sorting conditions?
3. What are the conditions for factor allocations?
4. How to tie it in with frictional theories of hiring?

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The existing one-on-one matching framework:

- $f(x, y)$ when firm hires worker
- tractable sorting condition: supermodularity
- trivial firm-worker ratio: unity; trivial assignment: $\mu(x) = x$

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Here: allowing for an intensive margin.

- $f(x, y, l)$ when firm hires l workers
- $F(x, y, l, r)$ when firm devotes fraction r of resources to l workers
- tractable sorting condition: cross-margin-supermodularity
  within-margin supermodularity larger than cross-margin supermodularity ($F_{12}F_{34} > F_{14}F_{23}$)
- capital-labor (worker-firm) ratio: type-dependent but tractable
- assignment: depends on how many workers each firm absorbs
- extensions: frictional hiring, mon. competition, general capital

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We consider competitive market. But welfare theorems hold. So consider planner's choices (two types):

- r_{ij} resources of firm type i devoted to worker type j, $r_{i1} + r_{i2} \leq h_i^f$
- l_{ji} labor of worker type j deployed at firm type i, $l_{j1} + l_{j2} \leq h_j^w$

worker / firms	h_1^f	h_2^f
h_1^w	$F(x_1, y_1, \quad , \quad)$	$F(x_1, y_2, \quad , \quad)$
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- Frictional Markets: one-on-one matching, but similar flavor under comp. search (Shimer-Smith 00, Atakan 06, Mortensen-Wright 03, Shi 02, Shimer 05, Eeckhout-Kircher 10)

Motivation

Characterize assignments when factor intensity choices are feasible.

Future:

1. How does the intensive margin adjust with economic conditions?
2. How does it integrate into macro/trade models?

The Model

- *Population*

- *Production of firm y*

- *Preferences*

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  - Workers of type $x \in X = [\underline{x}, \overline{x}]$, distribution $H^w(x)$
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    r_x fraction of firm's resources
  - F increasing in all arguments
  - F str. concave in each of the last two arguments
  - F constant returns to scale in last two arguments
  - Total output of the firm: $\int F(x, y, l_x, r_x)dx$
  - Production with one worker type: $f(x, y, l) = F(x, y, l, 1)$
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Different resource levels: $F(x, y, l, r) = \tilde{F}(x, y, l, rT(y))$.

Generic capital: $F(x, y, l, r) = max_k \tilde{F}(x, y, l, r, k) - ik$.

Competitive search: $F(x, y, l, r) = max_v \tilde{F}(x, y, vm(l/v), r) - vc$

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Hedonic wage schedule $w(x)$ taken as given.

- *Optimization*:

- *Feasible Resource Allocation*:

- *Equilibrium*

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  - Equivalent to: $\max_{r_x} \int r_x \max_{l_x} [F(x, y, \frac{l_x}{r_x}, 1) - w(x) \frac{l_x}{r_x}] dx$
  - Implies: $r_x > 0$ only if $(x, \frac{l_x}{r_x}) = \arg\max f(x, y, \theta) - \theta w(x) \quad (*)$
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  - $\mathcal{R}(x, y, \theta)$: resources to any $x' \leq x$ by any $y' \leq y$ with $\frac{l_{x'}}{r_{x'}} \leq \theta$.
    1. Firm scarcity: $\mathcal{R}(y|X, \Theta) \leq H^f(y)$ for all y.
    2. Worker scarcity: $\int_{\theta \in \Theta} \int_{x' \leq x} \theta d\mathcal{R}(\theta, x'|Y) \leq H^w(x)$ for all x.
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- *Equilibrium* is a tuple $(w, \mathcal{R})$ s.t.
  1. Optimality: $(x, y, \theta) \in \mathrm{supp}\mathcal{R}$ only if it satisfies $(*)$.
  2. Market Clearing: $\int \theta d\mathcal{R}(\theta|x, Y) \leq h^w(x)$, "$=$" if $w(x) > 0$.

Assortative Matching

Definition (Assortative Matching)

A resource allocation $\mathcal{R}$ entails sorting if its support only entails points $(x, \mu(x))$ for some monotone $\mu(x)$.

Sorting is positive if $\mu' > 0$, it is negative if $\mu' < 0$.

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Proposition (Condition for Assortative Matching)

A necessary condition for positive assortative matching in equilibrium is

$$F_{12}F_{34} \geq F_{23}F_{14}$$

along the equilibrium path. The opposite inequality is necessary for negative assortative matching.

Next: Proof, Examples, Graph, Resource Allocation

Proof of Assortative Matching Condition

Assume assortative matching on $(x, \mu(x))$ with associated $\theta(x)$. Must be optimal, i.e., maximizes:

$$\max_{x,\theta} f(x, \mu(x), \theta) - \theta w(x).$$

First order conditions:

$$\begin{aligned} f_\theta(x, \mu(x), \theta(x)) - w(x) &= 0 \quad (1) \\ f_x(x, \mu\left(x\right), \theta(x)) - \theta(x) w'(x) &= 0, \quad (2) \end{aligned}$$

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The Hessian is

$$Hess = \begin{pmatrix} f_{\theta\theta} & f_{x\theta} - w'(x) \\ f_{x\theta} - w'(x) & f_{xx} - \theta w''(x) \end{pmatrix}.$$

Second order condition requires $|Hess| \geq 0$:

$$f_{\theta\theta}[f_{xx} - \theta w''(x)] - (f_{x\theta} - w'(x))^2 \geq 0. \qquad (3)$$

Differentiate (1) and (2) with respect to x, substitute:

$$-\mu'(x)[f_{\theta\theta} f_{xy} - f_{y\theta} f_{x\theta} + f_{y\theta} f_x / \theta] \quad \geq \quad 0$$

Positive sorting means $\mu'(x) > 0$, requiring $[...] < 0$ and after rearranging:

$$F_{12} F_{34} \geq F_{23} F_{14}. \qquad (4)$$

Special Cases

Efficiency Units of Labor

- Skill equivalently to quantity: $F(x, y, l, r) = \tilde{F}(y, xl, r)$
- In this case no sorting: $F_{12}F_{34} = F_{23}F_{14}$

Multiplicative Separability

- $F(x, y, l, r) = A(x, y)B(l, r)$. Sorting: $[AA_{12}/(A_1A_2)][BB_{12}/(B_1B_2)] \geq 1$
- If B is CES with substitution ϵ: $[AA_{12}/(A_1A_2)] \geq \epsilon$.
- Implies that root-supermodularity in qualities needed (Eeckhout-Kircher 10).

Becker's one-on-one matching

- $F(x, y, \min\{l, r\}, \min\{r, l\}) = F(x, y, 1, 1) \min\{l, r\}$,
- Like inelastic CES ($\epsilon \to 0$), so sorting if $F_{12} \geq 0$

Sattinger's span of control model

- $F(x, y, l, r) = \min\{\frac{r}{t(x,y)}, l\}$,
- Write as CES between both arguments
- Our condition converges for inelastic case to log-supermod. in qualities

Illustration of Strength of Sorting

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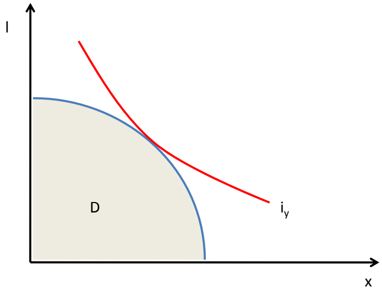
Example: $F(x, y, l, r) = A(x, y)B(l, r)$

Budget Set: $D = \{(x, l) | lw(x) \leq M\}$

Isoprofit Curve: $i_y = \{(x, l) | A(x, y)B(l, r) = \Pi\}$



Slope of Isoprofit Curve: $\frac{\partial l}{\partial x} = -\frac{A_x(x,y)B(l,1)}{A(x,y)B_1(l,1)}$.

If $A_{xy} = 0$: higher y has flatter slope as only denominator moves.

If $A_{xy} > 0$: higher y can have steeper slope.

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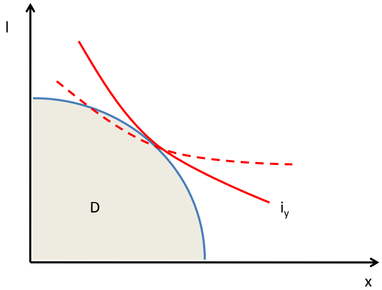

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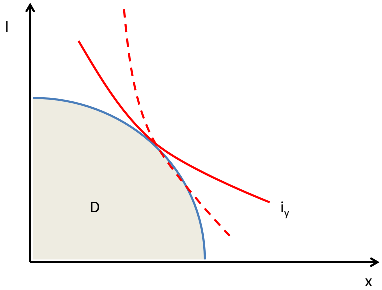

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Proposition (Factor Intensity and Assignment)

If sorting condition holds, then the equilibrium assignment and factor intensity are determined by the system of differential equations:

$$\mu'(x) = \frac{h_w(x)}{\theta(x)h_f(x)}, \quad \theta'(x) = \frac{1}{f_{\theta\theta}}\left[\frac{1}{\theta}f_x - \frac{h_w}{\theta h_f}f_{y\theta} - f_{x\theta}\right]$$

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Proof: μ' from market clearing: $H_w(\overline{x}) - H_w(x) = \int_{\mu(x)}^{\overline{y}} \theta(\tilde{x})h_f(\tilde{x})dx$

θ' from FOC: $f_\theta = w(x)$ and $f_x/\theta = w'$, diff. and subst. μ'.

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$$\theta'(x) = \frac{(1-\alpha)A_2(x,\mu(x)) - \alpha A_1(x,\mu(x))\theta^{1-\gamma}}{A(x,\mu(x))[1+\theta^\gamma][1-\gamma]} \;\; ; \;\; \mu'(x) = \frac{1}{\theta(x)}.$$

- symmetry A and $\alpha = 1/2$: then $\theta(x) = 1$ and $\mu(x) = x$
- symmetric A but $\alpha < 1/2$: then $\theta' > 0$
- non-symmetry but inelastic limit (Becker): $\theta(x) = 1$ and $\mu(x) = x$

Additional Extensions

Competitive Search with Large Firms

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Two equivalent formulations:

1 $\max_{s_x, r_x} \int [G(x, y, s_x, r_x) - w(x)s_x]dx$, where

$G(x, y, s_x, r_x) = \max_{v_x}[F(x, y, v_x m(s_x/v_x), r_x) - v_x c]$.

2 $\max_{r_x, l_x, v_x} \int [F(x, y, l_x, r_x) - C(x, l_x)]dx$, where

$C(x, l_x) = \min_{v_x, q_x} cv_x + q_x v_x w(x)$ s.t. $l_x = v_x m(q_x)$.

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From 1.: check sorting, compute $w(x)$ as in previous part.

From 2.: determine unemployment. FOC (Cobb-Douglas Matching, coefficient α) :

$$w(x)q_x = \frac{1-\alpha}{\alpha}c$$
$$\Rightarrow \textit{Unemployment}: \; m(q_x)/q_x = q_x^{-\alpha} = \left[\frac{\alpha}{(1-\alpha)c} w(x)\right]^{\alpha}$$

ADDITIONAL EXTENSIONS

GENERAL CAPITAL, MONOPOLISTIC COMPETITION

General Capital:

- $F(x, y, l, r) = \max_k \hat{F}(x, y, l, r, k) - ik$ (CRS in quantities)
- sorting condition: $\hat{F}_{12}\hat{F}_{34}\hat{F}_{55} - \hat{F}_{12}\hat{F}_{35}\hat{F}_{45} - \hat{F}_{15}\hat{F}_{25}\hat{F}_{34} \geq$
  $\hat{F}_{14}\hat{F}_{23}\hat{F}_{55} - \hat{F}_{14}\hat{F}_{25}\hat{F}_{35} - \hat{F}_{15}\hat{F}_{23}\hat{F}_{45}$.

Additional Extensions

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Monopolistic Competition:

- consumers have CES preferences with substitution ρ
- sales revenue of firm y: $\chi F(x, y, l, 1)^\rho$
- Sorting condition

$$\left[\rho\tilde{F}_{12} + (1-\rho)(\tilde{F})\frac{\partial^2 \ln \tilde{F}}{\partial x \partial y}\right]\left[\rho\tilde{F}_{34} - (1-\rho)l\tilde{F}\frac{\partial^2 \ln \tilde{F}}{\partial l^2}\right]$$
$$\geq \left[\rho\tilde{F}_{23} + (1-\rho)\tilde{F}\frac{\partial^2 \ln \tilde{F}}{\partial y \partial l}\right]\left[\rho\tilde{F}_{14} + (1-\rho)\left(l\tilde{F}_{13} - l\tilde{F}\frac{\partial^2 \ln \tilde{F}}{\partial x \partial r}\right)\right].$$

- independent of χ
- our condition under $\rho = 1$, log-sm when production linear in l.

Conclusion

This work:

- Lay out a tractable sorting model with factor intensity
- Derive tractable sorting condition ($F_{12}F_{34} \geq F_{14}F_{23}$)
- Characterize equilibrium factor intensity and assignment
- Extend to frictional market with sorting and large firms
- Various other extensions (general capital, monop. comp.)

Future:

- Generate more work on sorting on the intensive market
- Comparative statics on consequences of aggregate changes
- Applications in trade/macro/...