Dynamic Inventory Planning for Perishable Products with Censored Demand Data

Xiangwen Lu  
Department of IEEM, Hong Kong University of Science and Technology  
Clear Water Bay, Kowloon, Hong Kong  xwlu@ust.hk

Jing-Sheng Song  
Fuqua School of Business, Duke University  
Durham, NC 27708  jssong@duke.edu

Kaijie Zhu  
Department of IEEM, Hong Kong University of Science and Technology  
Clear Water Bay, Kowloon, Hong Kong  kzhu@ust.hk

May 29, 2005

1 Introduction

In many inventory systems, such as retail stores, demand in excess of inventory on hand is lost and cannot be observed or recorded. As a consequence, the historical sales data does not truly represent the demand and is termed censored data (which implies that the demand information is censored by the inventory availability). How to use censored data intelligently to determine the right inventory levels is therefore of great interest to practice.

Several authors have studied this issue. The literature focuses on multi-period models for perishable products, which means unobservable lost sales and no inventory carryover between periods. Although the type of demand distribution is known, its parameters are only subject to a prior belief, which can be updated periodically with the observed sales data based on the Bayesian approach. Harpaz, Lee, and Winkler (1982) analyze a general demand distribution and Lariviere and Porteus (1999) analyze a specific demand distribution that belongs to a family called newsvendor distributions (see Braden and Freimer 1991). These authors recognize the tradeoff between the inventory cost incurred in the current period and the future informational benefit. That is, while increasing inventory level increases the probability of overstocking, it reduces the probability of demand being censored and hence improves the accuracy of demand information. The improved informational accuracy can then bring the reduction of the future cost by improving the decision making for future periods (i.e., the informational benefit). Indeed, a major finding of all these
works is that the myopic solution, which minimizes the inventory cost in the current period but ignores the impact of learning from observed sales data on future periods, is a lower bound on the optimal inventory level. That is, to gain informational benefit, we should stock more than the myopic solution. In addition, Lariviere and Porteus obtain several other structural insights by solving the problem optimally considering one specific demand distribution.

However, the studies so far have only focused on gaining structural insights, and exactly how much the myopic solution is in short of the optimal solution has not been explored. In particular, it still remains a question exactly how increasing the inventory level in a period can improve demand forecasting and decision making for future periods. The purpose of our research is to understand these issues better.

2 Model and Assumptions

We consider an inventory-control problem for a perishable product over a $T$-period planning horizon. We assume that the demand is stochastic and apply the Bayesian approach to update the demand distribution over periods. We denote by $\xi_t$ the demand in period $t$. We assume that $\xi_1, \cdots, \xi_T$ follow the same distribution but are independent random variables. The distribution has a density function $f(\xi_t|\theta)$ and a cumulative density function $F(\xi_t|\theta)$, where $\theta \in \Theta$ itself is a random variable. At the beginning of period $t$, the prior distribution of $\theta$ is $\pi_t(\theta)$. With this prior distribution, we can compute the marginal density function and the marginal cumulative density function for $\xi_t$ as

$$m_t(\xi_t) = \int_{\Theta} f(\xi_t|\theta) \pi_t(\theta) \, d\theta,$$

$$M_t(\xi_t) = \int_{\Theta} F(\xi_t|\theta) \pi_t(\theta) \, d\theta.$$

We assume that the replenishment leadtime is zero (the analysis can be extended to the case with a constant replenishment leadtime). At the beginning of each period, based on the latest knowledge on the demand distribution, we decide on the ordering quantity and place an order accordingly. The order arrives immediately. During the period, demand occurs and is fulfilled entirely if there is enough inventory. Any inventory leftovers will be disposed with a salvage value at the end of period. If the demand exceeds the inventory level, however, the unsatisfied portion is lost and cannot be observed, which incurs a shortage-penalty cost. We use the observed sales data to update the demand distribution for the next period, which is discussed in detail next.

At the beginning of period $t$, our knowledge about the demand in that period is a prior distribution of $\theta$, denoted by $\pi_t$. Using this prior, we can derive the pdf $m_t$ and cdf $M_t$. The observed sales information in period $t$ consists of two parts, denoted by

$$o_t = (o_{t,1}, o_{t,2}),$$
where $o_{t,1}$ is the sales quantity and can take any nonnegative value, while $o_{t,2}$ is the observation status taking values of $e$ or $c$, representing that the observation of the demand is exact or censored, respectively.

For an illustration, suppose the stocked inventory is $y$ and the demand realization is $z_t$. If $y > z_t$, then the sales quantity is $z_t$ and the observation of the demand is exact. Otherwise, the sales quantity is $y$ and the observation is censored. Thus, $o_t$ is determined by $y$ and $z_t$, which we express as

$$o_t = y \otimes z_t \triangleq \begin{cases} 
(z_t, e) & \text{if } y > z_t \\
(y, c) & \text{if } y \leq z_t
\end{cases}.$$ 

For expositional simplicity, in the remainder of the paper, we use the short-hand notation $z_t^e$ and $y^c$ to represent $(z_t, e)$ and $(y, c)$, respectively.

At the end of period $t$, based on the observed sales information $o_t$, we update $\pi_t$ to $\pi_{t+1}$, the posterior distribution of $\theta$. We define

$$l(o_t|\theta) = \begin{cases} 
f(o_{t,1}|\theta) & \text{if } o_{t,2} = e, \\
1 - F(o_{t,1}|\theta) & \text{if } o_{t,2} = c.
\end{cases}$$

as the likelihood function of the sales information $o_t$ for any given $\theta$. Based on the Bayes formula, we have

$$\pi_{t+1}(\theta|o_t) = \frac{l(o_t|\theta)\pi_t(\theta)}{\int_{\Theta} l(o_t|\theta')\pi_t(\theta') d\theta'}.$$  

We denote by $c$, $h$, and $b$ the unit ordering cost, the unit salvage value, and the unit shortage-penalty cost in each period, respectively (the analysis can be extended to the case where these values vary from period to period). We assume $b > c > h$. The objective is to minimize the total expected cost over the $T$-period planning horizon.

We can formulate the problem as a dynamic program. We denote by $V_t(\pi_t)$ the optimal expected cost from period $t$ to the end of the planning horizon, given that $\pi_t$ is the prior distribution of $\theta$ at the beginning of period $t$. Thus, we obtain the following optimality equations:

$$V_{T+1}(\pi_{T+1}) = 0$$

$$V_t(\pi_t) = \min_{y \geq 0} G_t(y), \quad 1 \leq t \leq T,$$

where

$$G_t(y) = \int_0^{+\infty} r(y, \xi_t) m_t(\xi_t) d\xi_t + E[V_{t+1}(\pi_{t+1})],$$

$$r(y, \xi_t) = cy - h(y - \xi_t)^+ + b(\xi_t - y)^+$$

3
3 First Order Condition

In this section, we derive an explicit expression of the first derivative function for determining the optimal inventory level, which drastically increases the transparency of exactly how increasing the inventory level in a period improves demand information and brings the corresponding benefit in future periods.

we define

\[ C_{t+i}(y, o_t, \ldots, o_{t+i-1}) = \int_0^{+\infty} r(y, \xi_{t+i}) m_{t+i}(\xi_{t+i}|o_t, \ldots, o_{t+i-1}) d\xi_{t+i}, \quad 0 \leq i \leq T-t, \]
\[ G_{t+i}(y, o_t, \ldots, o_{t+i-1}) = C_{t+i}(y, o_t, \ldots, o_{t+i-1}) + E[V_{t+i+1}(\pi_{t+i+1})], \quad 0 \leq i \leq T-t. \]

Based on the above notation, we can rewrite \( V_{t+i}(\pi_{t+i}) \) as

\[ V_{t+i}(\pi_{t+i}) = \min_{y \geq 0} G_{t+i}(y, o_t, \ldots, o_{t+i-1}) = G_{t+i}(S^*_{t+i}(o_t, \ldots, o_{t+i-1}), o_t, \ldots, o_{t+i-1}), \]

where \( S^*_{t+i}(o_t, \ldots, o_{t+i-1}) \) represents the optimal inventory level in period \( t+i \). To highlight the dependence of \( S^*_{t+i} \) on \( o_t \), we denote recursively

\[ S^*_{t+i}(o_t) = S^*_{t+i+1}(o_t, \pi^*_{t+1}(o_t) \land \xi_{t+1}, \ldots, S^*_{t+i-1}(o_t) \land \xi_{t+i-1}), 1 \leq i \leq T-t. \tag{4} \]

**Theorem 1** The first derivative function of \( G_t(y) \) is

\[ G'_t(y) = C'_t(y) + \left[ G_{t+1}(S^*_{t+1}(y^e), y^e) - \tilde{G}_{t+1}(S^*_{t+1}(y^e), y^e) \right] m_t(y), \tag{5} \]

where

\[ G_{t+1}(S^*_{t+1}(y^e), y^e) = \sum_{i=1}^{T-t} EC_{t+i}(S^*_{t+i}(y^e), y^e, S^*_{t+1}(y^e) \land \xi_{t+1}, \ldots, S^*_{t+i-1}(y^e) \land \xi_{t+i-1}), \]
\[ \tilde{G}_{t+1}(S^*_{t+1}(y^e), y^e) = \sum_{i=1}^{T-t} EC_{t+i}(S^*_{t+i}(y^e), y^e, S^*_{t+1}(y^e) \land \xi_{t+1}, \ldots, S^*_{t+i-1}(y^e) \land \xi_{t+i-1}), \]

and \( S^*_{t+i}(y^e) \) and \( S^*_{t+i}(y^e) \) are defined according to (4).

Here, \( G_{t+1}(S^*_{t+1}(y^e), y^e) \) is the total expected cost from period \( t+1 \) till \( T \) if the optimal policy \( S^*_{t+i}(y^e) \) is followed \((i = t+1, \ldots, T)\), provided \( o_t = y^e \). \( \tilde{G}_{t+1}(S^*_{t+1}(y^e), y^e) \) is the total expected cost from period \( t+1 \) till \( T \) if \( S^*_{t+i}(y^e) \) is followed \((i = t+1, \ldots, T)\) given that \( o_t = y^e \). In addition, since \( G_{t+1}(S^*_{t+1}(y^e), y^e) \) is the cost resulted from the optimal policy, we have

\[ \tilde{G}_{t+1}(S^*_{t+1}(y^e), y^e) \geq G_{t+1}(S^*_{t+1}(y^e), y^e). \tag{6} \]

From equations (5) and (6), we obtain

\[ G'_t(y) \leq C'_t(y). \]
Thus, the myopic solution, which solves $C'_t(y) = 0$, is no more than the optimal inventory level. The same result has been shown under different settings in the literature (see Harpaz, Lee, and Winkler 1982, Lariviere and Porteus 1999, and Ding, Puterman, and Bisi 2002).

Note that the first derivative function $G'_0(t)(y)$ measures the marginal cost of having additional inventory $dy$ at the beginning of period $t$. Theorem 1 tells us that this marginal cost can be decomposed into two parts. The first term on the right hand side of equation (5) is the marginal inventory cost in the current period. The second term is the marginal cost in the future. Because the second term is negative, its absolute value actually represents the marginal informational benefit resulted from having additional inventory $dy$.

We now show that this second term is precisely the marginal benefit of having additional inventory $dy$ for the case $\xi_t = y$. To see this, note that $o_t = y^c$ if the initial inventory is $y$. Based on this information, the optimal decision is $S^*_{t+1}(y^c)$, and the resulting cost is $\tilde{G}_{t+1} (S^*_{t+1}(y^c), y^c)$. On the other hand, if there is additional inventory $dy$, then $o_t = y^e$. With this information, the optimal decision $S^*_{t+1}(y^e)$ is followed, and the resulting cost is $G_{t+1} (S^*_{t+1}(y^e), y^e)$. The probability of $\xi_t$ being $y$ is $m_t(y) dy$. Thus, the resulting reduction of the future cost beyond period $t$ has a magnitude of $\left[ \tilde{G}_{t+1} (S^*_{t+1}(y^e), y^e) - G_{t+1} (S^*_{t+1}(y^e), y^e) \right] m_t(y) dy$, which is the second term on the right hand side of equation (5).

This fact implies that the additional inventory $dy$ only brings negligible informational benefit for the cases $\xi_t < y$ and $\xi_t > y$. In the first case, $o_t = \xi_t < y$. So, it is obvious that having additional inventory $dy$ will not add any information. In the second case, $o_t = (y + dy)^c$. That is, having additional inventory $dy$ still misses the true demand. Theorem 1 suggests that the extra information gained does not add much value.

4 Approximation and Evaluation

The optimal policy is state dependent policy. The state includes all the sales data obtained previously. As a result, the dimensionality of the state space is high. Although the state-space reduction technique could reduce the dimensionality and thus make the dynamic program numerically tractable, the literature shows that the applicability of this technique to censored data is restricted to only newsvendor distributions (see Lariviere and Porteus 1999). For a general demand distribution, high dimensionality of the state space can make the computation of the optimal inventory level difficult and, more importantly, impractical.

We show the finiteness of the optimal inventory level and develop both lower and upper bounds on it. We then generate a general class of heuristics by averaging the lower and upper bounds on the optimal inventory level with weights. To our knowledge, no prior study has established upper bounds on the optimal inventory level, and even the finiteness of the optimal inventory level has
remained to be an open question (see Ding, Puterman, and Bisi 2002). Finally, we develop an analytical approach to evaluate the performance of any heuristic by developing an upper bound on the relative cost error of the heuristic to the optimal policy. This is done by performing the worst-case analysis in a sample-path approach (e.g., Lu, Song and Regan, 2003).

5 Conclusion Remarks

In this paper, we address demand forecasting and inventory management in the presence of censored data. The contribution of this paper is following: On structural insights, we derive an explicit expression of the first derivative function for determining the optimal inventory level in each period. From this result, we show explicitly how increasing the inventory level improves demand forecasting and decision making for future periods. We also find that a celebrated result in the existing literature, that the myopic solution is a lower bound on the optimal inventory level, becomes a by-product of the first derivative function. On operational practices, we prove the finiteness of the optimal inventory level and then develop both lower and upper bounds on it. We construct a general class of heuristics by averaging the lower and upper bounds with weights and develop an analytical approach to evaluate the performance of such heuristics. Our research appears to be the first attempt in developing upper bounds on the optimal inventory level in inventory models with censored data. We hope our results will inspire more effort in this direction.

References


