A Periodic Review Inventory Model with Two Freight Modes and Continuous Cost Accounting

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A majority of products are sourced from producers in low cost countries, at the expense of less responsiveness through long lead-times. One possibility to mitigate the loss of responsiveness over sourcing locally, is the use of fast, but more expensive, freight. This paper considers a firm which sources from a make-to-order manufacturer. After the completion of manufacturing, the firm can choose how to have the order shipped – by slow freight, by fast freight, or by a combination of the two. The presence of such a dual freight policy will inherently affect the inventory level. Hence, its usage should be considered in combination with the ordering policy itself. Based on the optimal use of freight modes, we formulate a dynamic program to study optimal ordering, which is in the form of a state dependent threshold policy. Further, we study the use of the simpler, non-state dependent, \((s, S)\) policy and compare it to the use of single freight modes.

In studies of periodic review inventory models, it is commonly assumed that the inventory holding and back-order penalty costs are incurred on inventory levels at the end of each period. Rudi, Groenevelt and Randall (2004) argues that the inventory costs are predominantly incurred continuously in time, and show that employing end-of-period cost accounting may lead to large errors on the inventory policy and the resulting costs. Further, this assumption restricts the analysis to lead-times which are integer multiples of the review period length. In this paper we account for costs continuously in time and allow for general values of lead-times.

In our model, at each review epoch, the inventory position (sum of inventory level, which represents on-hand inventory when positive and back-orders when negative, and outstanding orders) is assessed, and the decision of whether to and how much to order is made. Orders are placed with a make-to-order supplier. Irrespective of its size, placing an order incurs fixed cost \(K_1\) and its manufacturing takes time \(L_1\). At the completion of manufacturing, two options are available for shipping: Regular freight mode, which takes time \(L_2\) and has fixed cost \(K_2\), and express freight mode, which takes time \(l_2\) (< \(L_2\)), has fixed cost \(k_2\) and
additional variable cost $c_f$ per unit shipped. An order can be split across the two freight modes in any proportion. We assume that the length of review period $T$ is larger than the lead-time difference between the freight modes, i.e., $T > (L_2 - l_2)$, implying that all the units ordered in the previous review periods arrive before the units shipped via express freight mode in the current review period. For each unit of physical inventory, the system incurs holding costs $h$ per time unit and for each unit of demand back-ordered, it incurs penalty cost $p$ per time unit. We denote the random demand experienced in time interval $(t_1, t_2]$ by $D_{(t_1,t_2]}$. We assume a stationary demand process with independent and non-negative increments. For simplicity, the analysis is performed assuming that the demand process has continuous increments, nevertheless, our results are applicable to demand processes with discrete increments. For initial supply $x$ and a random cumulatively demand $d$, define the cost rate function $G(x, d) = E(h(x - d)^+ + p(d - x)^+)$. For notational simplicity we denote $l = L_1 + l_2$ and $L = L_1 + L_2$.

In the setting described above, consider the problem of optimal replenishment decisions in a finite horizon consisting of $N$ review epochs, counted backwards. The manufacturing order decision at a review epoch does not affect the inventory level, and hence, the cost until time $l$, which is the earliest time at which any of the ordered units can be received. Thus, at a review epoch, for the purpose of determining the optimal replenishment decision, only the cost incurred from time $l$ to the end of the horizon has to be considered. Let $V_n(x)$ denote the relevant optimal expected cost at the $n$th review epoch, as a function of inventory position $x$ prior to placing an order. Without loss of generality, we assume that the horizon ends at $l + T$ time after review epoch 1 (which is the last review epoch in the horizon) and each unit of residual physical inventory and back-ordered demand at that point costs $h_0$ and $p_0$, respectively. Thus, $V_n(x)$ is the optimal cost of the inventory system incurred in the time interval $(l, l + nT]$ and $V_0(x) = h_0x^+ + p_0x^-$. $V_n(x)$ can be determined by solving the following dynamic program

$$V_n(x) = \min_{y \geq x} \left\{ \Theta(x, y) + EV_{n-1}(y - D_{(0,T]}) \right\} \forall n = N \ldots 1,$$

where $\Theta(x, y)$ is the expected cost incurred in the next review period, in which an order is placed to raise the inventory position from $x$ to $y \geq x$, and the order is shipped by optimally combining the two freight modes. $\Theta(x, y)$ consists of the expected inventory cost incurred in interval $(l, l + T]$ and the costs of ordering and shipping. Further, given an order size, the freight mode decision does not affect the inventory levels and the cost incurred after the
arrival of regular freight. This implies that the relevant costs for the optimal freight mode
decision is the inventory cost incurred in the time interval \([l, L]\) and the costs of freight
modes. Also, the freight mode decision is made after the completion of manufacturing, i.e.,
\(L_1\) time after the review epoch, and the information of the demand incurred meanwhile
is used in making the decision. Using these facts, the optimal value of the expected cost
relevant for the freight mode decision is,

\[\theta(x, y) = \mathbb{E}\left[\min_{0 \leq q \leq y-x} \left\{ K_2 \mathbb{I}_{q < y-x} + k_2 \mathbb{I}_{q \geq 0} + c_f q + \int_l^L G(x + q - D_{(0, L_1]}, D_{(L_1, t)}) \, dt\right\}\right].\]  

Thus, \(\Theta(x, y) = K_1 \mathbb{I}_{y>x} + \theta(x, y) + \int_l^{l+T} G(y, D_{(0, t)}) \, dt\), where the last term is the expected
inventory cost incurred in time interval \([L, l + T]\).

The optimal policy for combining the two freight modes is the solution \(q^*\) to the minimization problem in (2). Given the optimal policy for combining the freight modes, the
function \(\Theta(x, y)\) in general can not be expressed as a sum of a function of \(x\) and a function
of \(y\). This leads to an optimal ordering policy which is not necessarily a simple \((s, S)\) policy.
In fact the optimal policy is a state dependent threshold policy: At review epoch \(n\), given
inventory position \(x\) prior to the ordering decision, it is optimal to place an order-up-to \(S_n(x)\)
if and only if \(x\) is less than \(s_n\). An \((s, S)\) policy is a special case of this threshold policy, with
\(S_n(x)\) being a constant. When the fixed costs of freight modes \(K_2\) and \(k_2\) are small relative
to the fixed cost of placing an order, the optimal policy is not state dependent, i.e., it is an
\((s, S)\) policy.

The optimal state dependent threshold policy for placing orders for the dual freight model
is complex to compute. We carry out an infinite horizon stationary analysis of the model
assuming an \((s, S)\) policy.

Let \(M(\Delta)\) denote the renewal function associated with \(D_{(0, T]}\), the demand incurred in a
review period. Starting a review period at inventory position which is \(\Delta\) above the reorder
threshold value \(s\), let \(\kappa(s, \Delta)\) denote the inventory holding and back-order penalty cost
experienced until the next order is placed. Denote \(\pi(y) = \int_l^{l+T} G(y, D_{(0, t)}) \, dt\), the expected
inventory cost incurred in time interval \([l, l + T]\) after a review epoch, as a function of
inventory position \(y\) at the review epoch. Then using the results of renewal theory, the
expected number of review periods between two consecutive orders is given by \(1 + M(\Delta)\),
and the expression for the long run average cost per review period of a stationary \((s, S)\)
policy for the dual freight model is

$$C(s, \Delta) = \frac{E(\Theta(X(s,\Delta), s + \Delta) - \pi(s + \Delta) + \kappa(s,\Delta))}{1 + M(\Delta)},$$  \hspace{1cm} (3)

where $\Delta = S - s$ is the minimum order quantity for the $(s, S)$ policy, and $X(s,\Delta)$ is the random variable representing the inventory position on the review epochs at which orders are placed. Note that the stationary probability distribution of $X(s,\Delta)$ depends on $s$ and $\Delta$. In order to keep notation simple, we suppress the dependence of $X(s,\Delta)$ on $s$ and $\Delta$.

Let $s^*(\Delta)$ denote the optimal reorder threshold value for an arbitrary value of $\Delta$. And let $I(s,\Delta)$ denote the average inventory level with the $(s, S)$ policy. Then, $s^*(\Delta)$ is a solution to

$$P(I(s,\Delta) > 0) = \frac{p}{h + p}. \hspace{1cm} (4)$$

The left-hand side of (4) is the probability that a demand is immediately satisfied on its arrival to the system. This term is commonly referred to as the fill rate. Thus, the optimality condition (4) states that at the optimal value of $s$ is set such that the fill rate is equal to right-hand side of (4).

Let $s^R(\Delta)$ ($s^E(\Delta)$) denote the optimal reorder threshold reorder value for the given $\Delta$ in the model with only regular (express) freight mode. Note that for given values of $s$ and $\Delta$, the average inventory level of the dual freight model is higher than that of the model with only regular freight mode, and is smaller than that of the model with only express freight mode. The optimality condition (4), however, holds for all three models. This leads to the result, $s^E(\Delta) \leq s(\Delta) \leq s^R(\Delta)$.

Let $\Delta^*$ denote the optimal value of the minimum order quantity $S - s$ for the dual freight model. Then $\Delta^*$ satisfies

$$C(s^*(\Delta), \Delta) = \pi(s^*(\Delta)) + E(\Theta(s^*(\Delta) - D_{[0,T]}, s^*(\Delta) + \Delta) - \Theta(s^*(\Delta), s^*(\Delta) + \Delta). \hspace{1cm} (5)$$

For the stationary $(s, S)$ policy, the expected number of review periods between two consecutive orders is an increasing function of $\Delta$. This implies that determining the optimal $\Delta$ is equivalent to deciding the optimal expected length of replenishment cycle. In optimality condition (5), the right-hand side is the marginal cost of delaying the ordering decision by one period, evaluated at the reorder threshold value. Thus, (5) states that at optimality this marginal cost is equal to the average cost per review period. Let $\Delta^R$ ($\Delta^E$) denote the optimal value of the minimum order quantity in the model with only regular (express) freight mode.
Then, for the special case $K_2 = k_2 = 0$ and $c_f < (L - l)$, it follows that $\Delta^* \geq \max \{ \Delta^s, \Delta^f \}$, which implies that the optimal average order size of the dual freight model is larger than the optimal average order size of the single freight models. This result also holds for non-zero $K_2$ and $k_2$ as long as the inventory costs are large relative to the fixed costs of freight modes.

References