The Subjective State Space and
Multiple Subjective Weights

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Abstract

In a model of decision making over sets of alternatives, à la Dekel, Lipman and Rustichini (Econometrica, 2001), we consider the consequences of relaxing the Independence axiom. The representations we study involve an agent who, given a set of alternatives, conceives of the different utilities he will receive (depending on the state of mind he is in, his subjective state) when he finally makes a choice from the set. He weighs these different utilities which results in value being ascribed to the set of alternatives. Relaxing Independence means that the weights he attaches to the different utilities corresponding to his subjective states can vary with the set of alternatives in question. We are thus able to capture menu-dependent phenomena where more of a particular good can make the agent better or worse off, depending on the other elements of the menu. However, this weakening of Independence comes at the cost of our having to give up the notion of single, long-run self and the corresponding cost interpretations of self control that have found use in the literature. We also consider implications of requiring our preferences to be linear over singleton menus and characterise the sign of a state, which tells us the effect of a small change to a menu.

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§ I Motivation

An agent has to choose between two sets of alternatives in the morning, knowing that he will make a choice from the delineated set of alternatives in the afternoon. What he is unsure about, is how he will feel, and consequently, how he will make a final choice in the afternoon. More specifically, let us say that the agent conceives of all possible utilities he might achieve in the afternoon, depending on how he makes a choice, thereby leading to a collection of afternoon utilities for each choice procedure.

More concretely, let us suppose there is a finite set of prizes, $Z_1$ and the that a menu is a set of compact set of lotteries over the prizes. The agent knows that in the afternoon, a choice will be made according to a non-trivial expected utility function from the menu. Let $S^Z := \{ p \in \mathbb{R}^{|Z|} : \sum p_i = 0 \text{ and } \|p\| = 1 \}$ denote the space of (twice normalised) expected utility functions over $Z$. Each $s \in S^Z$ represents how the agent might actually feel in the afternoon and is therefore referred to as a subjective state. Thus, for each menu $A$, $u_A(s)$ is the maximum utility he expects to achieve when in subjective state $s$ in the afternoon. The question we pose in this paper is, How does the agent translate or aggregate this collection of ex post (afternoon) utilities into an ex ante (morning) utility, that will enable him to choose between menus?

A straightforward aggregator is a linear one, which can be represented by a signed Borel measure $\mu$ over $S^Z$, so that the utility of a menu $A$ (in the morning) is just $\int_{S^Z} u_A(s) \, d\mu(s)$. Such a representation was considered by Dekel, Lipman and Rustichini (2001) (hereafter referred to as DLR) who showed that for continuous preferences, such a representation completely characterises the Independence axiom (when applied to menus). One interpretation of the representation is that the agent attaches a weight to the utility he achieves in each state $s$ and aggregates by integrating over all states.

Notice that the representation suggests a strong menu/context independence, in that the weight the agent attaches to the utility in each state is constant across menus. Our main contention is that such a strong menu/context independence rules out some interesting menu dependent behaviour. There are two key ingredients to our model. The first is our axiom D-Independence, which is a relaxation of
Independence in that we only require it (ie Independence) to hold for two menus, wherein one menu always gives the agent 0 utility in every subjective state and the other menu always gives the agent a constant utility of \( R \) in every state. Our second ingredient is our axiom Pessimism, a kind of hedging axiom, which roughly says that the agent prefers a lower variation in ex post payoffs. In the presence of adequate continuity, this results in the following utility for a menu \( A \):

\[
U(A) := \min \left\{ \int_{S^Z} u_A(s) \, d\pi(s) : \pi \in \Pi \right\}
\]

where \( \Pi \) is a weak* compact, convex set of signed Borel measures on \( S^Z \). The \textit{subjective state space} is the set of states that the agent considers relevant in that he weighs them (positively or negatively) at some menu. We represent the subjective state space by \( S := \cup_{\pi \in \Pi} \text{supp} \pi \). We now consider two simple examples of menu dependent behaviour that can be captured in our model.

\textbf{I.1 Example} (Extreme Pessimism). Suppose the agent only considers the lowest utility he can get in some subset of states, \( S \), for each set of alternatives that he has. In other words,

\[
V(A) := \min \{ u_A(s) : s \in S \} = \min \left\{ \int_{S^Z} u_A(s) \, d\delta(s) \right\}
\]

where \( \delta(s) \) is the Dirac measure concentrated at \( s \).

A second example shows us another form of menu dependence and shows that preferences over singletons do not always give us the whole picture.

\textbf{I.2 Example}. The only prizes are servings of broccoli salad \( (b) \) and of chocolate cake \( (c) \). Servings are indivisible and come in different sizes and the servings of broccoli salad are represented by \( mb \) where a larger \( m \) represents a bigger serving. Our agent is a pessimist and the two options appeal to different sensibilities. Thus, preferences are such that \( \{b\} \sim \{c\} \). Preferences over menus also satisfy \( \{5b, 8c\} \succ \{4b, 8c\} \), ie if there are large amounts of chocolate cake in the menu, then increasing the amount of broccoli makes him better off. Similarly, if there are large amounts of chocolate cake in the menu, then more broccoli makes the agent better off. But,
if there is a lot of broccoli in the menu (as compared to chocolate cake), reducing
the amount of broccoli makes the agent worse off, e.g. \( \{5b, 2c\} \succ \{4b, 2c\} \).

Thus our agent is a pessimist in that he seems to focus on the serving which is
smaller. Such preferences can be rationalised as follows. Let the agent have two
subjective states, \( s_1 \) and \( s_2 \), and let \( u_A(s_i) := u^*_A \). For \( m \in \{0, 1, \ldots, N\} \), \( u^1(mb) =
\) \( u^2(mc) = m \) and \( u^1(mc) = u^2(mb) = 0 \). Also, let \( \Pi = \text{conv}\{\langle 2, -1\rangle, \langle -1, 2\rangle\} \). We
shall let \( V(A) := \min \{ \langle \pi, u_A \rangle : \pi \in \Pi \} \). Then, for \( A := \{mb, nc\} \), \( V(A) =
2u^1_A - u^2_A \) whenever \( n \geq m \) and \( V(A) = -u^1_A + 2u^2_A \) whenever \( n \leq m \). (Recall that
\( u^i_A = \max_{p \in A} u^i(p) \).) The comparative statics now follow immediately.

We emphasise that our model is one of preferences over menus of lotteries,
where the behaviour in the afternoon, the ostensible second period, is unobserved.
Our environment lends itself to many other consistent interpretations about sec-
ond period behaviour, a prime example of which is Gul and Pesendorfer (2001)
(for a discussion of which, see § II.4). We note that the interpretation of second
period behaviour is facilitated by the particular utility representation and that in
Gul and Pesendorfer (2001) and in other papers, that interpretation is facilitated
by the Independence axiom. One of our contributions is to clarify the rôle that
Independence plays in enabling these interpretations.

In particular, in a model of temptation, Gul and Pesendorfer (2001) argue that
preferences over singletons tell us about the agent’s normative or commitment
preferences and represent what he would like to choose, if given the chance to
commit to a single choice. This argument thereby provides us with unambiguous
measures of welfare. In § II.4, we highlight the rôle that Independence plays in this
argument, by showing that preferences over singletons do not necessarily carry a
lot of information and that welfare comparisons are more cumbersome. We argue
instead that the looking at the sign of a state, which in our model is a local property
of a state, provides a better measure of local welfare improvements.

WE now proceed as follows. We provide a view of the literature in § I.1 and
introduce the model and the main representation theorem in § II. We discuss notions
of comparative pessimism in § II.1 and provide a limited characterisation of the sign
of a state in § II.3. In § II.4, we consider some additional axioms and study the
more refined representations that result and conclude in § III. All proofs not found
in the text are in § IV.

§ I.1 Related Literature

That preferences over planning problems can be used to infer how an agent thinks he will behave was first pointed out by Kreps (1979). The construction of a unique subjective state space was carried out by DLR who also axiomatised the weak EU representation and the linear aggregator (which gives the additive EU representation). The subjective state space constructed by DLR is an improvement over Kreps in two ways. Firstly, it is essentially unique (while Kreps’s state space is not), and second, it does not require preferences to be monotonic. Preferences that are not monotonic form an important class and have been used to model, most notably, temptation by Gul and Pesendorfer (2001). However, this and subsequent papers, for instance Dekel, Lipman and Rustichini (2006) and Chatterjee and Krishna (2007), make heavy use of the Independence axiom. A study of temptation related problems while relaxing Independence is carried out by Noor (2006). In their model of self control Fudenberg and Levine (2006) argue that Independence is not an appropriate assumption for temptation and related phenomena.

From a formal viewpoint, our multi-weight representation is the subjective state equivalent of the multi-prior model of Gilboa and Schmeidler (1989). (Nevertheless, while the techniques are similar, our proof is substantially different.) An excellent textbook presentation of Gilboa-Schmeidler’s multi-prior model is in Ok (2007), who presents a finite dimensional version of the multi-prior model. A paper that looks at self control problems in the a Gilboa-Schmeidler framework is the paper by Epstein and Kopylov (2007). They draw on the aforementioned multi-prior model (and a variant of Set Betweenness, cf § II.4). In their environment, a menu is a collection of Anscombe-Aumann acts with a finite objective state space.

The paper that is closest to ours is the the paper by Epstein, Marinacci and Seo (2007). Their motivation is the study of agents who are not sure about the contingencies that may arise. Their environment is the same as ours and their representation is also very similar to ours. In particular, they find the value of a
menu to be

\[ W_{\text{ems}}(A) = \min \left\{ \int_{S^Z} \max_{p \in A} u(p, s) \, d\pi(s) : \pi \in \Pi \right\} \]

where \( \Pi \) is a weak* compact, convex set of probability measures on \( S^Z \). While this representation looks like our (★) multi-weight representation, there are some subtle qualifications to the above representation. The first is that since they assume preferences are monotone, each \( \pi \) is a probability measure. Moreover, they assume the existence of a prize \( z^* \) and define \( N := \{ s \in S^Z : u(z, s) \geq u(z^*, s) \text{ for all } z \in Z \} \). This results in a further restriction on \( \Pi \), namely that for each \( \pi \in \Pi \), \( \text{supp} \pi \subset N \). Moreover, the set \( \Pi \) is not unique, although there is a smallest such set (in terms of set inclusion) that satisfies the above.

To relate this to our representation, we first note that we do not require preferences to be monotone, so the measures we consider are not probability measures. However, if we assume preferences are monotone, then each measure is a probability measure. Moreover, we do not have the restriction that the support of each measure must lie in the set \( N \) and we can show that the set of probability measures, \( \Pi \), is unique. Some of this is due to the slightly more stringent non-triviality axiom that we assume.
§II Model and Results

Let $Z$ denote a finite set of prizes where the cardinality of $Z$ is $n+1$. We shall require that $n > 2$. The space of probability measures on $Z$ is denoted by $\Delta$ and $\mathcal{F}(\Delta)$ denotes the set of closed subsets of $\Delta$. When endowed with the Hausdorff metric, $\mathcal{F}(\Delta)$ becomes a compact metric space. Preferences are complete, transitive binary relations over $\mathcal{F}(\Delta)$, represented by $\succ \subset \mathcal{F}(\Delta) \times \mathcal{F}(\Delta)$. An important class of menus are the convex ones. We shall denote the set of all compact, convex subsets of $\Delta$ by $K(\Delta)$. Let $\{p^*\} = \left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right)$ be the uniform (probability) measure over the prizes and let $D_r := \{q \in \text{aff}(\Delta) : \|q - p^*\| \leq r\}$ be the closed disk of radius $r$ around the uniform measure, with $\text{aff}(\Delta)$ representing the affine hull of $\Delta$.

We now state our axioms.

**Axiom (Non-triviality*).** There exists $D_R \subset \Delta$ such that $D_R \not\sim \{p^*\}$.

This axiom says that preferences are non-trivial in a particular way. We should mention that weakening the axiom to requiring that there exist two menus that are not indifferent but allowing $D_R \sim \{p^*\}$ is a case we are not able to handle. Hence this assumption has behavioural content. The next axiom is the standard continuity axiom.

**Axiom (Continuity).** $\succ$ is continuous in the Hausdorff topology.

**Axiom (L Continuity).** There exists $N > 0$ such that for all $\varepsilon \in (0, 1/N)$, for all $A, B$ with $d_h(A, B) \leq \varepsilon$,

$$(1 - N\varepsilon)A + N\varepsilon A^* \succ (1 - N\varepsilon)B + N\varepsilon A_*$$

where $A^*, A_* \in \{D_r, \{p^*\}\}$ and $A^* \succ A_*$. We need the axiom above because Continuity by itself is not sufficient to give us the desired representation (even in the presence of other axioms). This axiom was first introduced in Dekel et al (2007), although our version is stronger in the sense that Dekel et al (2007) allow $A^*$ and $A_*$ to be arbitrary compact subsets of $\Delta$. As in that paper, one way to think about L Continuity is as follows: For each $A$
and $B$ with $B \succ A$, there exists a greatest $\lambda \in (0, 1)$ such that $\lambda A + (1 - \lambda)D_R \succneq \lambda B + (1 - \lambda)\{p^*\}$. As $d_h(A, B) \rightarrow 0$, $\lambda \rightarrow 0$. L Continuity says that $\lambda \rightarrow 0$ smoothly. We now introduce an axiom that is a very weak version of Independence.$^1$

**Axiom** (IR: Indifference to Randomisations$^2$). $A \sim \text{conv}(A)$ for all $A \in \mathcal{F}(\Delta)$.

In line with our interpretation of a menu as representing a set of alternatives from which a choice will be made in the future, we provide a justification of this axiom. Suppose the agent knows that in the future, he will be making a choice according to one of a given set of vN-M utility functions (which, taken together, represent his subjective state space), then the (ex post) utility in each state remains unchanged if we replace a menu by its convex hull. Thus, the agent is indifferent between a menu and its convex hull.

The classical *Independence* axiom adapted to the present setting was considered by DLR.

**Axiom** (Independence). $A \succ B$ implies $\lambda A + (1 - \lambda)C \succneq \lambda B + (1 - \lambda)C$ for all $C$, $\lambda \in (0, 1)$.

DLR (see also Dekel et al, 2007) define an *additive EU representation* of a preference $\succneq$ as a signed Borel measure $\mu$ such that $V : \mathcal{F}(\Delta) \rightarrow \mathbb{R}$ given by

$$V(A) := \int_{S^Z} \max_{p \in A} u(p, s) \, d\mu(s)$$

represents $\succneq$. (Here, $S^Z := \{p \in \mathbb{R}^{|Z|} : \sum p_i = 0$ and $\|p\| = 1\}$.) Moreover, the support of $\mu$ gives the *subjective state space*, $S := \text{supp} \mu$. The principal ingredient in this representation (other than the continuity axioms) is the above Independence axiom. We will be interested in a slight weakening of the above axiom.

$^1$This axiom was introduced by DLR. Dekel et al (2007) show that for a preference that satisfies vN-M continuity, Independence implies IR.

$^2$We should mention that IR rules out the following example, (taken from Dekel, Lipman and Rustichini, 2006). Suppose the agent is has to make a choice between a healthy dish and an unhealthy dish. To assuage feelings of guilt, he may prefer to add a randomisation of the two dishes to the menu. This is clearly a violation of IR.
In particular, we want to consider the possibility that the signed measure the agent uses to evaluate the utility of a menu depends on the menu at hand. Say that a preference $≽$ has a multi-weight EU representation if it has a representation of the form

$$V(A) = \min \{ \langle u_A, \pi \rangle : \pi \in \Pi \} \tag{★}$$

or

$$V(A) = \max \{ \langle u_A, \pi \rangle : \pi \in \Pi \} \tag{♠}$$

where $u_A(s) := \max_{p \in A} u(p, s)$ for each $s \in S^Z$, $\Pi$ is a weak* compact, convex set of signed measures on $S^Z$ and for all $\pi, \mu \in \Pi$, $\pi(S^Z) = \mu(S^Z)$. Also, $D_R \succ \{p^*\}$ implies $\pi(S^Z) = 1$ and $\{p^*\} \succ D_R$ implies $\pi(S^Z) = -1$.\(^3\) In the representations above, $\langle u_A, \pi \rangle := \int_{S^Z} u_A(s) \, d\pi(s)$ and the subjective state space is $S := \bigcup_{\pi \in \Pi} \text{supp} \, \pi$. Our first new axiom is a weakening of the Independence axiom in the sense that we shall require Independence to hold for only two sets.

**Axiom** (D-Independence\(^4\)). For all $A, B \in \mathcal{K}(\Delta)$ and all $\lambda \in (0, 1]$, $A \succ B$ implies (i) $\lambda A + (1 - \lambda)D_R \succ \lambda B + (1 - \lambda)D_R$ and (ii) $\lambda A + (1 - \lambda)\{p^*\} \succ \lambda B + (1 - \lambda)\{p^*\}$.

(The “D” in D-Independence stands for “disk”.) A heuristic interpretation of the axiom is as follows. Consider an agent whose preferences over menus admits a weak EU representation. Suppose $\{p^*\}$ is such that in each ex post state, the utility from $\{p^*\}$ is zero. For such an agent, $\lambda A + (1 - \lambda)\{p^*\}$ merely represents scaling ex post utilities by a factor of $\lambda$. Since $A \succ B$, and the ex post utilities of $B$ are scaled by the same factor, the agent ought to prefer $\lambda A + (1 - \lambda)\{p^*\}$ to $\lambda B + (1 - \lambda)\{p^*\}$.

\(^3\)In what follows, we will focus exclusively on multi-weight representations of the form (★). Multi-weight representations of the form $V(A) = \max \{ \langle \pi, u_A(s) \rangle : \pi \in \Pi \}$ admit similar results and, for expositional ease, are omitted.

\(^4\)We should remark that replacing $p^*$ with any $q \in \text{ri} \, \Delta$ and $D_R$ with $D \in \mathcal{K}(\Delta)$ where $D \ni q$, $D \subset \text{ri} \, \Delta$ and the boundary of $D$ is smooth delivers the same results.
Now suppose the agent gets the same utility \( R \) in every ex post state from the menu \( D_R \). Then, \( \lambda A + (1 - \lambda)D_R \) merely represents (or delivers) a convex combination of a vector of ex post utilities of \( A \) with \( R \). As in Gilboa and Schmeidler (1989), the idea is that the agent can easily visualise such mixtures, but may not find it so easy to visualise mixtures of more general menus (which do not have uniform payoffs in every subjective state).

We now introduce the behavioural axioms which capture optimism and pessimism. (Notice that the Pessimism axiom is our version of Gilboa-Schmeidler’s Uncertainty Aversion.) The intuition behind Pessimism is that the agent prefers greater uniformity in the ex post payoffs he receives.

Axiom (Pessimism). \( A \sim B \) implies \((1/2)A + (1/2)B \succ B\).

Axiom (Optimism). \( A \sim B \) implies \( B \succ (1/2)A + (1/2)B \).

We call the axioms Optimism and Pessimism because they correspond to exactly to the examples I.1 and ??. Indeed, consider an agent who is indifferent between two menus that give him ex post utilities of \((2, 0)\) and \((0, 2)\) respectively. (Thus, the agent has two subjective states.) Pessimism corresponds to the situation where the agent looks at the worst outcome (in terms of utility) and focuses only on that. Such an agent would strictly prefer the menu which gives him utility \(\frac{1}{2}(2, 0) + \frac{1}{2}(0, 2) = (1, 1)\). Conversely, Optimism corresponds to the situation who always looks at the best outcome. For such an agent, a menu which gives him ex post utilities of \((2, 0)\) (or \((0, 2)\)) is strictly better than the menu which gives ex post utilities \((1, 1)\). We are now ready to state our main representation theorem.

II.1 Theorem. A. The preference relation \(\succeq\) has a \((\star)\) multi-weight representation if and only if it satisfies Non-triviality*, Continuity, L Continuity and Pessimism.

B. The preference relation \(\succeq\) has a \((\spadesuit)\) multi-weight representation if and only if it satisfies Non-triviality*, Continuity, L Continuity and Optimism.

We now present a sketch of the proof.
Sketch of Proof. The first step in the proof is to embed the space of all convex subsets of \( \text{aff } \Delta \) as a cone \( K^* \) in the function space \( C(S^Z) \). (Recall that \( S^Z \) is isometrically homeomorphic to \( S^{n-1} \), the \((n-1)\)-dimensional sphere.) There is a standard embedding that can be found in Schneider (1993). In fact, a function \( f : S^Z \to \mathbb{R} \) is in \( K^* \) if and only if \( \bar{f} : \mathbb{R}^n \to \mathbb{R} \), which represents the unique extension of \( f \) to \( \mathbb{R}^n \) by positive homogeneity, is sublinear. Moreover, \( K^* - K^* \) is dense in \( C(S^{n-1}) \).

For the rest of the proof, we assume that \( D_R \succ \{p^*\} \). We now show that there exists a utility representation \( V \) of preferences that is \( D \)-linear, i.e., it satisfies (i) \( V(\lambda A + (1 - \lambda)\{p^*\}) = \lambda V(A) + (1 - \lambda)V(\{p^*\}) \) and (ii) \( V(\lambda A + (1 - \lambda)D_R) = \lambda V(A) + (1 - \lambda)V(D_R) \), with the normalisations that \( V(\{p^*\}) = 0 \) and \( V(D_R) = R \).

Using the first property, we can extend \( V \) to all compact convex subsets of \( \text{aff } \Delta \). This extension also preserves the second property. We also establish that \( V \) is Lipschitz continuous. This shows that an equivalent way of writing preferences is via a function \( \varphi : K^* \to \mathbb{R} \). It can be shown that such a function is superlinear (assuming preferences satisfy Pessimism), \( C^+ \)-additive, i.e., \( \varphi(v + \alpha 1) = \varphi(v) + \alpha \varphi(1) \), for \( \alpha \geq 0 \) and Lipschitz continuous (in the norm topology on \( C(S^Z) \)).

We now use the important observation that for any \( f \in C^2(S^Z) \), the space of twice continuously differentiable functions on \( S^Z \), there exists an \( \alpha > 0 \) such that \( f + \alpha 1 \in K_2^* := K^* \cap C^2(S^Z) \). This enables us to extend \( \varphi|_{K_2^*} \) to \( C^2(S^Z) \). There is a unique extension, \( \psi : C^2(S^Z) \to \mathbb{R} \) that is superlinear and satisfies \( C^+ \)-additivity. Moreover, \( \psi \) is Lipschitz continuous. We now use the fact that \( C^2(S^Z) \) is dense in \( C(S^Z) \) to uniquely extend \( \psi \) to \( C(S^Z) \). This extension is superlinear, satisfies \( C^+ \)-additivity and is Lipschitz continuous.

But superlinear functions that satisfy \( C^+ \)-additivity have the required characterisation, as the minimum of a family of linear functionals. (The Riesz Representation Theorem, turn, tells us that a linear functional on \( C(S^Z) \) can be written as the integral with respect to a unique signed Borel measure on \( S^Z \).) For preferences that satisfy Optimism, the representation follows immediately.

Finally, for preferences such that \( \{p^*\} \succ D_R \), we define the preference \( \succ^* \) as follows: \( A \succ^* B \) if and only if \( A \succ B \). We show that \( \succ^* \) satisfies Non-triviality*, D-Independence and Pessimism (resp Optimism) if and only if \( \succ \) satisfies Non-
triviality*, D-Independence and Optimism (resp Pessimism). Then, from the result above, $V^*$ represents $\succeq^*$ and defining $V := -V^*$, we see that $V$ represents $\succeq$ and has the desired functional form. \hfill\square

We end with the following observation.

\section*{II.2 Corollary} Let $\succeq$ be a preference relation that satisfies Non-triviality*. Then $\succeq$ also satisfies Continuity, L Continuity, D-Independence, Optimism and Pessimism if and only if $\succeq$ has an additive EU representation.

\section*{§ II.1 Comparative Pessimism}

Our representation allows us to compare degrees of pessimism. Roughly, one agent is more pessimistic than another if he desires a greater uniformity of payoffs across states. Before we proceed to the formalities, we recall an important theorem from DLR. Given preferences $\succeq_1$ and $\succeq_2$, say that agent 2 is more uncertain than agent 1 if $A \cup B \not\succeq_1 A$ implies $A \cup B \not\succeq_2 A$. Intuitively, agent 2 consider more ex post states possible than agent 1. This gives us the following characterisation.

\section*{II.3 Theorem} (Theorem 2, DLR). If $\succeq_2$ is more uncertain than $\succeq_1$, then $S_1 \subset S_2$.

Recall that one way to think about Pessimism is that the agent prefers uniform payoffs across states to variations in payoffs. For instance, an agent who compares menus that gives him ex post payoffs $(2,0)$ and $(0,2)$ and is indifferent between them prefers the menu with ex post payoffs $\frac{1}{2}(2,0) + \frac{1}{2}(0,2) = (1,1)$. More generally, the agent prefers smaller variations in ex post payoffs to larger ones and taking convex combinations makes this possible.

As before, the menus $D_R$ and $\{p^*\}$ play a special rôle in our model, ie $\{p^*\}$ represents the menu which gives the constant payoff of 0 in every subjective state and the menu $D_R$ gives the constant payoff of $R$ in every subjective state. Analogous, to the notion of the certainty equivalent of a gamble, we may define the subjective certainty equivalent of a menu $A$ where $D_R \succ A \succ \{p^*\}$ as the (unique) menu $\lambda_A D_R + (1 - \lambda_A)\{p^*\} \sim A$. Thus, the subjective certainty equivalent of a menu $A$
is the menu $D_{\lambda R} = \lambda A R + (1 - \lambda A)\{p^*\}$, that gives utility $\lambda A R$ in every state and is indifferent to $A$.

This gives us a natural way to operationalize the notion of greater pessimism. We say that $\succeq_2$ is more pessimistic than $\succeq_1$ if for each menu $A$ such that $D_R \succeq_1 A \succ_1 \{p^*\}$, $\lambda D_R + (1 - \lambda)\{p^*\} \sim_1 A$ implies $\lambda D_R + (1 - \lambda)\{p^*\} \succeq_2 A$. Intuitively, a more pessimistic agent has a smaller subjective certainty equivalent (in terms of set inclusion). Thus, the more pessimistic agent has a stronger preference for a smaller variation in ex post utilities. This leads us to the following characterisation.

**II.4 Theorem.** Suppose $\succeq_1$ and $\succeq_2$ both admit ($\star$) representations. Then, $\succeq_2$ is more pessimistic than $\succeq_1$ if and only if $\Pi_1 \subset \Pi_2$. Thus, if $\succeq_2$ is more pessimistic than $\succeq_1$, $S_1 \subset S_2$.

_Proof._ See Appendix.

As before, $\Pi_i$ is a weak* compact, convex set of Borel measures such that $V_i(A) := \min \{\int_{\mathcal{S}} \langle u_A, \pi \rangle : \pi \in \Pi_i\}$ represents $\succ_i$, for $i = 1, 2$. The subjective state space for $\succ_i$ is $S_i := \bigcup_{\pi \in \Pi_i} \text{supp} \pi$. It is worthwhile to observe that, in consonance with our interpretation, increasing pessimism also has the behavioural implication of an increased subjective state space (in terms of set inclusion).

**§ II.2 The Issue of Singleton Independence**

Notice that the multi-weight representations do not require that $V$ is linear over singletons. emphSingleton Independence (or Independence, as it is known in classical decision theory) is an attractive axiom from a normative point of view. In this section, we shall see the impact of assuming this property on preferences.

**Axiom** (Singleton Independence). For all $p, q, r \in \Delta$, $\{p\} \succ \{q\}$ implies $\lambda \{p\} + (1 - \lambda)\{r\} \succ \lambda \{q\} + (1 - \lambda)\{r\}$ for all $\lambda \in (0, 1]$.

Before stating our theorem, it is useful to introduce the notion of the dimension of the subjective state space. Recall that the subjective state space $S$ is a closed
subset of $S^Z (\simeq S^{n-1})$, the space of (twice normalised) vN-M utility functions on $\Delta$. Notice that $S^Z$ canonically lives in $n$-dimensional Euclidean space, so its elements are members of a vector space. Let us say that the subjective state $S$ is of dimension $k$ if $\dim \text{span} S = k$. In other words, there are $k$ linearly independent states in $S$. Clearly, $k \leq |S| = m$, its cardinality.

II.5 Theorem. Suppose $V : \mathcal{H}(\Delta) \to \mathbb{R}$ has a (⋆) representation and satisfies Singleton Independence. If $S$ is linearly independent in $\text{span} S^Z$, then $\Pi$ is a singleton.

Sketch of proof. The main idea behind this theorem is that $\text{aff} \Delta$ is a finite dimensional subspace of $K^*$. If $S$ is linearly independent, then it must be finite, so that all the essential information about ex post utilities is in a cone $K$ which lives in an $|S|$-dimensional Euclidean space and contains $F$. Our axiom says that $V$ is linear on this space, so that $V$ must be linear everywhere. \qed

Notice that since $\Pi$ is a weak* compact set, $\text{ri} \Pi = \emptyset$. But in the event that $S$ is finite, we can think of $\Pi$ as being a (compact, convex) subset of an $(|S| - 1)$-dimensional affine manifold. In general, it is not the case that $\text{ri} \Pi = \emptyset$, but this changes if Singleton Independence is assumed.

II.6 Corollary. Suppose $V : \mathcal{H}(\Delta) \to \mathbb{R}$ has a (★) multi-weight representation and $V$ satisfies Singleton Independence. Then, $\text{ri} \Pi = \emptyset$.

We emphasise that the conclusion above always holds if $S$ is infinite. The observation above is non-trivial because it also holds if $S$ is finite (under the appropriate hypotheses).

§ II.3 Sign of the State

In this section, we discuss how a subjective state (and the concomitant multi-weight representation) determine the impact (on the agent) of making local changes to a menu. To do this, we shall recall some ideas from DLR.
In their discussion on weak EU representations (of which the multi-weight representations are a special case), DLR show that it is possible to attach a sign to a state. A state \( s \in S^Z \) is said to be positive if for each neighbourhood \( N \) of \( s \), there exist menus \( A \) and \( B \) with \( A \subset B \) and \( B \succ A \), wherein \( u_A(s') = u_B(s') \) for all \( s' \in S^Z \setminus N \). The idea is that increasing the ex-post utility only in some parts of a neighbourhood state \( s \) of the menu \( A \) to give the menu \( B \) makes \( B \) more valuable than \( A \). Similarly, a state \( s \in S^Z \) is negative if for each neighbourhood \( N \) of \( s \), there exist menus \( A \) and \( B \) with \( A \subset B \) and \( A \succ B \), wherein \( u_A(s') = u_B(s') \) for all \( s' \in S^Z \setminus N \). A curious property of multi-weight representations (and weak EU representations in general) is that it is possible for a state to be both positive and negative. To understand why this might be so, recall that the sign of a state is a local property, in that we are concerned with small perturbations to ex-post utility in the neighbourhood of a state and the effect that has on the menu’s ex-ante value. As is commonly known, local phenomena don’t usually translate into global properties.

Before proceeding to the characterisation of the sign of a state, we mention a peculiar occurrence. In what follows, we will have to restrict attention to multi-weight representations that have a finite subjective state space. The reason is that if the state space is infinite, then even under the stronger assumption of Independence, it may be that a state is both positive and negative.\(^5\)

We shall now demonstrate that the multi-weight representation above provides an easy characterisation of the sign of a state. Toward this end, we shall need some additional notation and definitions. First, let \( S := \{s_1, \ldots, s_m\} \) be a finite state space. Then, \( \Pi \subset \pm \text{aff } \Delta^{m-1} \) (where \( \Delta^{m-1} \) is the space of probability measures on \( S \)). Let \( \Pi_i := \text{proj}_i \Pi \) for \( i = 1, \ldots, m \), be the projection of \( \Pi \) onto the \( i \)-th axis. Say that \( \Pi_i \) is negative if \( (-\infty, 0) \cap \Pi_i \neq \emptyset \) and that \( \Pi_i \) is positive if \( (0, \infty) \cap \Pi_i \neq \emptyset \). We begin with two important characterisations.

**II.7 Proposition.** Let state \( s_i \) be positive (resp negative). Then, \( \Pi_i \) is positive (resp negative).

\(^5\)DLR claim (p 912) that if Independence holds, a state is either positive or negative, but not both. Unfortunately, this is not true in general. It is, however, true if the state space is finite.
II.8 Proposition. Let $\pi = (\pi^1, \ldots, \pi^i, \ldots, \pi^m) \in \text{ext } \Pi$ be such that $\pi^i \not< (\text{resp } \not>) 0$. Then, state $s_i$ is positive (resp negative).

We have thus proved the main theorem in this section.

II.9 Theorem. State $s_i$ is positive if and only if $\Pi_i$ is positive and is negative if and only if $\Pi_i$ is negative.

It is also useful to know if a state with a particular sign exists. The following theorem, which follows immediately from Proposition II.8, provides us with this characterisation.

II.10 Theorem. There exists a positive state if $\mathcal{D}_R \succ \{p^*\}$. There exists a negative state if $\{p^*\} \succ \mathcal{D}_R$.

§ II.4 Special Cases

Here we shall consider some additional axioms on preferences. These additional axioms have been useful in characterising certain kinds of behaviour. In what follows, we shall assume that preferences have a multi-weight representation with a finite state space.

II.11 Definition. A preference $\succ$ is upward monotonic if $A \supset B$ implies $A \succ B$. It is downward monotonic if $A \supset B$ implies $B \succ A$.

Monotonic preferences have an easy characterisation (that we state without proof).

II.12 Proposition. Suppose $V : \mathcal{K}(\Delta) \rightarrow \mathbb{R}$ has a finite state (★) representation where $|S| = m$. Then,

(i) $V$ is upward monotonic if and only if $\Pi \subset \text{aff } \Delta^{m-1}$, and
(ii) $V$ is downward monotonic if and only if $-\Pi \subset \text{aff } \Delta^{m-1}$.
Notice that if preferences are upward monotonic, then all states are positive and if preferences are downward monotonic, then all states are negative. We now consider the axiom Set Betweenness introduced by Gul and Pesendorfer (2001) and the following weakenings of Set Betweenness, introduced by Dekel, Lipman and Rustichini (2006).

**Axiom** (Positive Set Betweenness). $A \succ B$ implies $A \succ A \cup B$.

**Axiom** (Negative Set Betweenness). $A \succ B$ implies $A \cup B \succ B$.

Introducing these axioms imposes a lot of structure on $\Pi$, as we shall below.

**II.13 Proposition.** Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite state ($\star$) representation and satisfies Positive Set Betweenness. Then, for each $\pi = (\pi^1, \ldots, \pi^i, \ldots, \pi^m) \in \Pi$, there exists exactly one $i$ such that $\pi^i > 0$. In other words, for any menu $A$, we have

$$V(A) = \min\{U_\pi(A) : \pi \in \Pi\}$$

where $U_\pi$ is a finite additive EU representation with exactly one positive state.

The fact that a finite additive EU representation satisfies Positive Set Betweenness if and only if it has exactly one positive state is Lemma 1 in Dekel, Lipman and Rustichini (2006). The effect of imposing Negative Set Betweenness is similar. Again, Lemma 2 from Dekel, Lipman and Rustichini (2006) is the fact that a finite additive EU representation satisfies Negative Set Betweenness if and only if it has exactly one negative state.

**II.14 Proposition.** Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite state ($\star$) representation and satisfies Negative Set Betweenness. Then, for each $\pi = (\pi^1, \ldots, \pi^i, \ldots, \pi^m) \in \Pi$, there exists exactly one $i$ such that $\pi^i < 0$. In other words, for any menu $A$, we have

$$V(A) = \min\{U_\pi(A) : \pi \in \Pi\}$$

where $U_\pi$ is a finite additive EU representation with exactly one negative state.
This brings us to the following axiom from Gul and Pesendorfer (2001).

**Axiom** (Set Betweenness). $A \succ B$ implies $A \succ A \cup B \succ B$.

We first begin with an easy fact that is interesting to record. It is a corollary of Theorem II.5.

**II.15 Proposition.** Suppose $V : \mathcal{K}(\Delta) \rightarrow \mathbb{R}$ has a finite state ($\star$) representation, satisfies Singleton Independence and there are two subjective states, ie $|S| = 2$, then exactly one of the following holds:
(i) $V$ is monotonic, or
(ii) $V$ satisfies Set Betweenness.
Moreover, in either case, $V$ is a finite additive EU representation.

This brings us to the following important characterisation.

**II.16 Theorem.** Suppose $V : \mathcal{K}(\Delta) \rightarrow \mathbb{R}$ has a finite state ($\star$) representation and satisfies Set Betweenness, then there are at most two subjective states, ie $|S| \leq 2$. If $|S| = 2$, one state is positive (but not negative) and the other state is negative (but not positive). If $|S| = 1$, then that state is either positive or negative, but not both.

It follows from Propositions II.13 and II.14 that if $|S| = 2$ and preferences satisfy Set Betweenness, then each $\piext \Pi$ can have only two components, one of which is positive (resp negative) and one is non-positive (resp non-negative). It does not follow from this that a state cannot be both positive and negative. That this cannot be so is demonstrated in the appendix.

In particular, if we have 2 states and have $D_R \succ \{p^s\}$. Then, $V$ takes the following form:

$$V(A) := \min \{ \langle \pi, u_A \rangle, \langle \nu, u_A \rangle \}$$

where $\pi^1 + \pi^2 = \nu^1 + \nu^2 = 1$ and $\text{sgn}(\pi^s) = \text{sgn}(\nu^s)$. To get at an interesting interpretation of this representation, let us first assume $\pi = \nu$ (and, say, $\pi^1 =$
1 − π^2 > 1), so that $V(A)$ can be rewritten as

$$V(A) = \max_{p \in A} v(p) - \max_{p \in A} w(p)$$

$$= \max_{p \in A} [v(p) - c(p; A)]$$

where $w(p) := |\pi^2| u_2(p)$, $(v + w)(p) := \pi^1 u_1(p)$ and $c(p; A) = \max_{p \in A} w(p) - w(p)$. Notice that for singletons, we now have $V(\{p\}) = v(p)$. We can interpret $c(p; A)$ as a cost function and the interpretation is that the agent believes in the second period when he makes a choice, we will incur a self-control cost which, in turn, informs the agent’s choices between menus. (This cost interpretation is central to the analysis of Gul and Pesendorfer (2001), Dekel, Lipman and Rustichini (2006) and Noor (2006). Notice that requiring a multi-weight representation to also satisfy Singleton Independence is sufficient to ensure that $\pi = \nu$.)

Now consider the case where $\pi \neq \nu$. Then, we can write

$$V(A) = \min \left\{ \max_{p \in A} [v_1(p) - c_1(p; A)], \max_{p \in A} [v_2(p) - c_2(p; A)] \right\},$$

and $c_2(p, A) = \frac{\nu^2}{\pi^2} c_1(p, A)$.

In other words, the agent can interpreted as being unsure about two things: firstly, his objective (or normative) preferences, given by his preferences over singletons, and second the cost function he will face. This brings into stark relief, the crucial rôle that Independence plays in the point of view that normative welfare judgements can be made by looking at preferences over singletons.
§ III Discussion and Conclusion

In this paper, we provide a tractable class of aggregators for the weak EU representation over subjective states introduced by Dekel, Lipman and Rustichini (2001). The aggregators can capture psychological phenomena such as optimism, pessimism, cognitive dissonance and anticipatory feelings of virtuousness and guilt. The aggregators are obtained by weakening the Independence axiom and is the formal equivalent (in the environment of preferences over menus) of Gilboa-Schmeidler’s maxmin (or multi-prior) utility.

We characterise the sign of the state, ie whether a subjective state is positive or negative at a menu. This is a local characterisation in the sense that at a particular menu, a state can be positive and at another menu, the state can be negative. We end by imposing looking at some special cases of preferences that have found use in the literature. We now discuss some of the methodological implications of our model.

A particularly influential point of view is that in our environment, preferences over singletons represent the agent’s objective or normative preferences. Advocates of this view are, among others, Gul and Pesendorfer (2001) and Chatterjee and Krishna (2007). Indeed, the representations obtained in these papers suggests that there is a long-run self who anticipates the possibility of a suboptimal choice being made from the menu. As we demonstrate (cf Examples I.2 and ??), one consequence of weakening Independence is that it is not possible to make statements about what is normatively desirable by looking at preferences over singletons. In other words, the ability to look at the agent’s preferences over singleton menus and treat them as the agent’s normative preferences depends, to a large extent, on the Independence axiom.

Finally, we come to, what is in essence, a matter of interpretation. More specifically, our environment consists of looking at preferences over menus of lotteries. We interpret this as looking at the behaviour of someone who will eventually make a choice from the menu, which leads to our characterisation. An equally acceptable interpretation (of the environment) is one which views a menu of lotteries as representing objective ambiguity about the nature of uncertainty. Such a model is discussed in Stinchcombe (2007). Another interpretation, subtly different from
DLR’s (and hence ours) is that of Gul and Pesendorfer (2001), who assume that the choice in the (unmodelled) second period is *always* made according to a utility function that is a weighted sum of two utility functions. (Thus, in our terminology, they have a subjective state space of cardinality at most 2.) In § II.4, we show that this interpretation is greatly facilitated by assuming *Independence*. Yet another interpretation is offered by Chatterjee and Krishna (2007), who assume that the choice in the second period is always according to one of (possibly) two utility functions. This is because in that model, the agent does not care about the utility he will receive in the second period, but is assumed to value the choice made in the second period according to his preferences over singletons.
§IV Proofs

§IV.1 The Cone of Ex-post utilities

Recall that $S^{n-1}$ is the canonical $(n-1)$-dimensional sphere and let $\mathscr{X}^n$ denote the space of all compact convex subsets of $\text{aff} \, \Delta$ (which is essentially $\mathbb{R}^n$). It is easy to show that $\mathscr{X}^n$ is a complete metric space when endowed with the Hausdorff metric, $d_h$. Let $S^Z := \{ p \in \mathbb{R}^{|Z|} : \sum p_i = 0 \text{ and } \|p\| = 1 \}$. Notice that $S^Z$ is isometrically homeomorphic to $S^{n-1}$. For any $A \in \mathscr{X}^n$, its support function $h_A : S^Z \to \mathbb{R}$ is given by $h_A(s) := \max_{p \in A} \langle p - p^*, s \rangle$, where $p^* := (\frac{1}{n+1}, \ldots, \frac{1}{n+1})$. Notice that for each $A \in \mathscr{X}^n$, $h_A$ is a continuous function on $S^Z$. This definition of a support function has some immediate consequences. Firstly, $h_{\{p^*\}} = 0$ and second, for $D_r := \{ p \in \text{aff} \, \Delta : \|p - p^*\| \leq r > 0 \}$, $h_{D_r}(s) = r$ for each $s \in S$.

It should be noted that for any $\lambda \in [0, 1) \cup (1, \infty)$, $\lambda A \notin \mathscr{X}^n$. So, in what follows, when we write $\lambda A$ where $\lambda \geq 0$, we shall actually mean $\lambda(A - p^*) + p^*$. Similarly, for $A, B \in \mathscr{X}^n$, $A + B$ shall mean $(A - p^*) + (B - p^*) + p^*$, which is a compact, convex subset of $\text{aff} \, \Delta$. Notice that for any $p \in \text{aff} \, \Delta$ and $\lambda \geq 0$, $\lambda(p - p^*) + p^* = \lambda p - (\lambda - 1)p^* \in \text{aff} \, \Delta$. Thus, $\lambda A$ as defined above is also a subset of $\text{aff} \, \Delta$. Moreover, for $A, B \in \mathscr{X}^n$ and $\lambda \in (0, 1)$, $\lambda A + (1 - \lambda)B \in \mathscr{X}^n$. We begin with some technical properties of support functions.

IV.1 Proposition. The support function has the following properties:

(i) $h_{A+B} = h_A + h_B$,
(ii) $h_{\lambda A} = \lambda h_A$ for all $\lambda \geq 0$,
(iii) $h_A \land h_B = h_{A \land B}$,
(iv) $h_A \lor h_B = h_{\text{conv}(A \lor B)}$, and
(v) $\|h_A - h_B\|_\infty = d_h(A, B)$.
(vi) The (unique) extension $\tilde{h}$ of a support function $h$ to span $S^Z$ by positive homogeneity is sublinear and the restriction to $S^Z$ of a sublinear function on span $S^Z$ is a support function.
(vi) Moreover, the following duality relation holds: $h_{K_h} = h$ and $K_{h_{K_h}}$.

Proof. See §§ 5.18 and 5.19 of Aliprantis and Border (1999). □

Thus, there is an isometry between the space of compact, convex sets of $\mathbb{R}^n$ ($\simeq \text{span} \, S^Z$) and the space of sublinear functions on $\mathbb{R}^n$. Let $K^* \subset C(S^Z)$ de-
note the cone of functions on \( S^Z \) whose unique extensions to span \( S^Z \) by positive homogeneity are sublinear. Then, \( K^* \simeq \mathcal{H}^n \).

We shall be interested in the embedding of \( \text{aff} \Delta \) in \( K^* \). Let \( F := \{ h(p) : p \in \text{aff} \Delta \} \). Thus, \( F \) represents the space of support functions of singletons (which are also compact, convex). We show below that \( F \) is an \( n \)-dimensional subspace of \( K^* \).

**IV.2 Proposition.** Let \( S_0 \subset S^Z \) be a linearly independent set. Then, \( F_0 := \left\{ (h(p))(s) : s \in S_0, p \in \text{aff} \Delta \right\} \) is a (closed) \( |S_0| \)-dimensional subspace of \( K^* \). Therefore, if \( S_0 \) is a maximal linearly independent set, then \( F_0 = F \).

**Proof.** Let \( S_0 \) be a linearly independent subset of \( S^Z \). Notice first that for any \( p \in \text{aff} \Delta \), there exists (a unique) \( p' \in \text{aff} \Delta \) such that \((p + p')/2 = p^*\). Therefore \( p - p^* = -(p' - p^*)\), so that \( h(p)(s) = (p - p^*, s) = -(p' - p^*, s) = -h(p')(s) \) for all \( s \in S_0 \). This shows that for any \( p \in \text{aff} \Delta \) and any \( \lambda \in \mathbb{R} \), \( \lambda h(p)(s) = h_{\lambda p}(s) \) for all \( s \in S_0 \) (where \( \lambda \{p\} \) is actually \( \lambda(p - p^*) + p^* \)). Also, for any \( p, q \in \text{aff} \Delta \), \( h_{\{p\} + \{q\}}(s) = h_{\{p\}}(s) + h_{\{q\}}(s) \) for all \( s \in S_0 \). Thus, \( F_0 \) is a vector subspace of \( K^* \) and it is easily seen to be of dimension \( |S_0| \). We can similarly show that \( F \) is a vector subspace of \( K^* \) from which it follows that \( F_0 \) is a vector subspace of \( F \).

Suppose now that \( S_0 \) spans \( S^Z \). To show that \( F = F_0 \), notice that for any \( s \in S^Z \setminus S_0 \), there exist (unique numbers) \( \lambda_1, \ldots, \lambda_k \) such that \( s = \sum_{i=1}^k \lambda_i s_i \). Then, \( h_{\{p\}}(s) = \left( p - p^*, \sum_{i=1}^k \lambda_i s_i \right) = \sum_{i=1}^k \lambda_i (p - p^*, s_i) = \sum_{i=1}^k \lambda_i h_{\{p\}}(s_i) \). Thus, \( F = F_0 \). \( \square \)

As mentioned in the text, the subjective state space can be interpreted as the set of (expected) utility functions that the agent believes can be used to make a choice from a set in the second stage. For a given menu, he will receive some utility in each state. If the subjective state space is finite, then we can say a little more about the space defined above. The following proposition is stated without proof.

**IV.3 Proposition.** Let \( S \subset S^{n-1} \) be finite with cardinality \( m \). Then, \( K := \{(h_A(s))_{s \in S} : A \in \mathcal{H}^n\} \) is a closed convex cone that spans \( \mathbb{R}^m \).

In general, we cannot guarantee that \( K \) is a vector space. To see this, suppose \( S = \{u, -u\} \). Then, for any \( x > 0 \), \( (-x, -x) \notin K \). We now show that \( K \) has a non-empty interior.
**IV.4 Proposition.** $K$ has a topological interior. Therefore, span $K = \mathbb{R}^m$.

*Proof.* Let $r > 0$ be such that $D_r \subset \text{ri} \Delta$. Since $S$ is finite, there exists $\varepsilon > 0$ such that for any $s \in S$, there exist $A_s^*$ and $B_s^*$, compact convex subsets of (the relative interior of) $\Delta$ such that (i) $h_{A_s^*}(s) = r - \varepsilon$ and $h_{B_s^*}(s) = r + \varepsilon$ and (ii) $h_{A_s^*}(s') = r$ and $h_{B_s^*}(s') = r$ for any $s' \neq s$. Thus, there is neighbourhood of $r1$ in $K$, so $K$ has an interior. Finally, since $0 \in K$, span $K = \mathbb{R}^m$.

The following corollary is immediate.

**IV.5 Corollary.** $1 \in \text{int} K$. Therefore, for each $w \in \mathbb{R}^m$, there exists $\lambda_w > 0$ such that $w + \lambda w1 \in \text{int} K$.

*Proof.* Since $K$ is a convex cone, we see that if $w \in \text{int} K$, $\lambda w \in \text{int} K$ for all $\lambda > 0$. Therefore, $1 \in \text{int} K$. Now take $w \in \mathbb{R}^m$. Then, there exists $\mu > 0$ such that $\mu w + (1 - \mu)1 \in \text{int} K$. This implies $\frac{1}{\mu}(\mu w + (1 - \mu)1) =: w + \lambda w1 \in \text{int} K$. 

§ IV.2 **A Utility Representation**

We shall now construct a utility representation for our preferences. Recall that one of our main axioms is D-Independence. Now consider the following strengthening of D-Independence.

**Axiom** (Strong D-Independence). $A \succ B$ implies $\lambda A + (1 - \lambda)D_r \succ \lambda B + (1 - \lambda)D_r$ for all $\lambda \in (0, 1)$ and $r > 0$ and $D_r \subset \Delta$.

It is clear that Strong D-Independence implies D-Independence. It is useful to note that the two are actually equivalent. This is shown below.

**IV.6 Proposition.** For a preference relation $\succ \subset \Delta \times \Delta$, strong D-Independence is equivalent to D-Independence.

*Proof.* As noted above, it follows from the definitions that Strong D-Independence implies D-Independence. To prove the converse, suppose $A \succ B$. We will first prove that for any $r \in (0, R)$ and any $\lambda \in (0, 1]$, $\lambda A + (1 - \lambda)D_r \succ \lambda B + (1 - \lambda)D_r$. Fix such an $r$ and $\lambda$ and let $\mu := r/R$, so that $D_r = \mu D_R + (1 - \mu)\{p^*\}$. Then, $\frac{\lambda}{\lambda + (1 - \lambda)\mu^*}A +$
\[(1 - \lambda)\mu_d \succ \frac{\lambda}{\lambda + (1 - \lambda)\mu} B + \frac{(1 - \lambda)\mu}{\lambda + (1 - \lambda)\mu} D_R.\] This implies 
\[\gamma \left[ \frac{\lambda}{\lambda + (1 - \lambda)\mu} A + \frac{(1 - \lambda)\mu}{\lambda + (1 - \lambda)\mu} D_R \right] + (1 - \gamma)\{p^*\} \succ \gamma \left[ \frac{\lambda}{\lambda + (1 - \lambda)\mu} B + \frac{(1 - \lambda)\mu}{\lambda + (1 - \lambda)\mu} D_R \right] + (1 - \gamma)\{p^*\} \] where \(\gamma := \lambda + (1 - \lambda)\mu\). Notice that \(1 - \gamma = 1 - \lambda - (1 - \lambda)\mu = (1 - \lambda)(1 - \mu)\), so that we can rewrite the above relations as \(\lambda A + (1 - \lambda)[\mu D_R + (1 - \mu)\{p^*\}] \succ \lambda B + (1 - \lambda)[\mu D_R + (1 - \mu)\{p^*\}]\), i.e. \(\lambda A + (1 - \lambda)D_r \succ \lambda B + (1 - \lambda)D_r\) as required.

The case where \(r > R\) and \(D_r \subset \Delta\) is similar. Fix such an \(r > R\) and notice that there exists \(\nu > 0\) such that \(\nu r < R\). Also fix a \(\lambda \in (0, 1] \). By D-Independence, \(\nu A + (1 - \nu)\{p^*\} \succ \nu B + (1 - \nu)\{p^*\}\). Notice also that \(D_{\nu r} := \nu D_r + (1 - \nu)\{p^*\}\). By what we have proved in the paragraph above, we get \(\lambda[\nu A + (1 - \nu)\{p^*\}] + (1 - \lambda)D_{\nu r} \succ \lambda[\nu B + (1 - \nu)\{p^*\}] + (1 - \lambda)D_{\nu r}\) which can be rewritten as \(\nu \lambda A + (1 - \nu)\lambda\{p^*\} + (1 - \lambda)[\nu D_r + (1 - \nu)\{p^*\}] \succ \nu \lambda B + (1 - \nu)\lambda\{p^*\}\) which in turn can be written as \(\nu[\lambda A + (1 - \lambda)D_r] + (1 - \nu)\{p^*\} \succ \nu[\lambda B + (1 - \lambda)D_r] + (1 - \nu)\{p^*\}\), which holds if and only if \(\lambda A + (1 - \lambda)D_r \succ \lambda B + (1 - \lambda)D_r\), which is what we wanted to prove. \(\square\)

By dlr’s Theorem 1.A, there exists a closed subset \(S \subset S^Z\) and a utility function \(V : \mathcal{K}(\Delta) \rightarrow \mathbb{R}\) such that \(V\) is continuous and
\[V(A) := U(u_A).\]
This \(V\) can be chosen such that \(V(p^*) = 0\) (where \(p^* = (1/n+1, \ldots, 1/n+1)\) is the uniform lottery). Notice that since \(V\) represents preferences, and \(V(D_R)\) is positive or negative if (and only if) \(D_R \succ \{p^*\}\) or \(\{p^*\} \succ D_R\). Let \(D_1\) be the unit disk in \(\mathcal{K}^n\) with unit radius. Our second normalisation will set \(V(D_1) \in \{-1, 1\}\). Unfortunately, even with these normalisations, we do not have a utility representation with any structure. In the rest of this subsection, we shall construct a utility function \(V\) which satisfies (i) \(V(\lambda A + (1 - \lambda)\{p^*\}) = \lambda V(A)\) and (ii) \(V(\lambda A + (1 - \lambda)D_R) = \lambda V(A) + (1 - \lambda)V(D_R)\), where \(V(D_R) = RV(D_1)\). For the rest of this section (and the remainder of the proof), we shall assume that \(D_R \succ \{p^*\}\), so that \(V(D_R) = R\). (The other case will be derived from a simple duality result.) We shall proceed in a number of simple steps. We shall first show that there is a useful class of sets in our domain that form a mixture space.

For each \(A \in \mathcal{K}(\Delta)\), let \(\Xi_A := \{\lambda A + (1 - \lambda)\{p^*\} : \lambda \in [0, 1]\}\). Notice that each \(\Xi_A\) is a mixture space. To see this, let \(\alpha, \beta, \lambda \in [0, 1]\) and define \(A_\alpha := \alpha A + (1 - \alpha))A\).
\( \alpha \{p^*\} \). Then, \( \lambda A_\alpha + (1 - \lambda) A_\beta = (\lambda \alpha + (1 - \lambda) \beta) A + (\lambda(1 - \alpha) + (1 - \lambda)(1 - \beta)) \{p^*\} \in \Xi_A \). The following proposition is useful to record.

**IV.7 Proposition.** If \( A \succ \{p^*\} \), then for all \( \mu \in (0, 1) \), \( \mu A + (1 - \mu) \{p^*\} \succ A \).

Similarly, if \( \{p^*\} \succ A \), then for all \( \mu \in (0, 1) \), \( A \succ \mu A + (1 - \mu) \{p^*\} \).

**Proof.** We shall only prove the first part. Suppose, by way of contradiction, that \( A_1 := \mu A + (1 - \mu) \{p^*\} \succcurlyeq A \). Then, by D-Independence, \( A_2 := \mu A_1 + (1 - \mu) \{p^*\} \succcurlyeq \mu A + (1 - \mu) \{p^*\} = A_1 \). Defining \( A_n \) inductively, we see that \( A_n \succcurlyeq A \). But \( A_n \to \{p^*\} \), which contradicts the continuity of \( \succcurlyeq \). The case where \( \{p^*\} \succ A \) is proved similarly.

The proposition above can be restated as follows.

**IV.8 Corollary.** Let \( 1 \geq \alpha > \beta > 0 \). If \( A \succ \{p^*\} \), then \( \alpha A + (1 - \alpha) \{p^*\} \succ \beta A + (1 - \beta) \{p^*\} \). If \( \{p^*\} \succ A \), then \( \beta A + (1 - \beta) \{p^*\} \succ \alpha A + (1 - \alpha) \{p^*\} \).

We shall now show that preferences restricted to each \( \Xi_A \) satisfy not only D-Independence, but also Independence.

**IV.9 Proposition.** The naturally induced preference over the mixture space \( \Xi_A \) satisfies Independence.

**Proof.** Let \( A_\alpha, A_\beta, A_\gamma \in \Xi_A \), where \( \alpha A + (1 - \alpha) \{p^*\} =: A_\alpha \succ A_\beta \) and let \( \lambda \in (0, 1] \). Notice that \( \lambda A_\alpha + (1 - \lambda) A_\gamma = (\lambda \alpha + (1 - \lambda) \gamma) A + (\lambda(1 - \alpha) + (1 - \lambda)(1 - \gamma)) \{p^*\} \) and \( \lambda A_\beta + (1 - \lambda) A_\gamma = (\lambda \beta + (1 - \lambda) \gamma) A + (\lambda(1 - \beta) + (1 - \lambda)(1 - \gamma)) \{p^*\} \). If \( A \succ \{p^*\} \), it must be that \( \alpha > \beta \), so that (by the Corollary above) \( \lambda A_\alpha + (1 - \lambda) \{p^*\} \succ \lambda A_\beta + (1 - \lambda) \{p^*\} \) as desired. If, on the other hand, \( \{p^*\} \succ A \), then \( \alpha < \beta \), so that \( \lambda A_\alpha + (1 - \lambda) \{p^*\} \succ \lambda A_\beta + (1 - \lambda) \{p^*\} \). The proof is completed with the observation that if \( A \sim \{p^*\} \), then \( A_\alpha \sim \{p^*\} \) for each \( A_\alpha \in \Xi_A \). (This follows from D-Independence.)

We now proceed to constructing the desired utility representation. Recall that we are only considering the case where \( D_R \succ \{p^*\} \). We shall proceed via a number of simple steps.
**Step 1.** For all $D_r \subset \Delta$, $V(D_r) = r$. This represents preferences since $\Xi_{D_r}$ is a mixture space for each $D_r$, and preferences on this domain satisfy Independence.

**Step 2.** Now suppose $A \succ \{p^*\}$. We claim that if $A \succ D_r$ (for some $D_r$, and since preferences are continuous, such a $D_r$ must exist), there exists a unique $\lambda > 0$ such that $\lambda A + (1 - \lambda) \{p^*\} \sim D_r$. To see that this is the case, notice that since preferences are continuous, there exists at least one $\lambda$ such that $\lambda A + (1 - \lambda) \{p^*\} \sim D_r$. So suppose there are $\lambda_1 > \lambda_2 > 0$ such that $\lambda_i A + (1 - \lambda_i) \{p^*\} \sim D_r$ for $i = 1, 2$. Define $\mu := \lambda_2 / \lambda_1 < 1$. By hypothesis, $\lambda_1 A + (1 - \lambda_1) \{p^*\} \sim D_r$. Also, $D_r \sim \lambda_2 A + (1 - \lambda_2) \{p^*\} = \mu(\lambda A + (1 - \lambda) \{p^*\}) + (1 - \mu) \{p^*\} \sim \mu D_r + (1 - \mu) \{p^*\}$ which contradicts Step 1. Hence the $\lambda$ must be unique. For such an $A$ with $\lambda A + (1 - \lambda) \{p^*\} \sim D_r$, let $V(A) = r / \lambda$. Moreover, for $A_\alpha \in \Xi_A$, let $V(A_\alpha) = \alpha V(A)$.

**Step 3.** For $A \sim \{p^*\}$, let $V(A) = 0$ and also $V(A_\alpha) = 0$, where $A_\alpha := \lambda A + (1 - \lambda) \{p^*\}$.

**Step 4.** Suppose $\{p^*\} \succ A$. We claim that there exists a unique $\lambda \in (0, 1)$ such that $\lambda A + (1 - \lambda) D_r \sim \{p^*\}$. If this is true, define $V(A)$ such that $\lambda V(A) + (1 - \lambda) V(D_r) = V(\{p^*\}) = 0$, i.e., $V(A) = -r \frac{1 - \lambda}{\lambda}$. Now extend $V$ linearly to all $\Xi_B$ that contain $A$. Clearly, this represents preferences restricted to $\Xi_B$.

To see that the claim is true, notice first that, since preferences are continuous, there exists a $\lambda \in (0, 1)$ such that $\lambda A + (1 - \lambda) D_r \sim \{p^*\}$. Suppose $\lambda$ is not unique, assume there exist $\lambda_1, \lambda_2$ such that $\lambda_i A + (1 - \lambda_i) D_r \sim \{p^*\}$, with $\lambda_1 > \lambda_2$. As before, let $\mu := \lambda_2 / \lambda_1 < 1$. Then, by D-Independence, $\{p^*\} \sim \lambda_2 A + (1 - \lambda_2) D_r = \mu(\lambda_1 A + (1 - \lambda_1) D_r) + (1 - \mu) D_r \sim \mu \{p^*\} + (1 - \mu) D_r$, which contradicts Step 1.

We shall now show that $V$ is well defined. Consider first the case where $A \succ \{p^*\}$. Then, for some $D_r$, $\lambda A + (1 - \lambda) \{p^*\} \sim D_r$, giving us $V(A) = r / \lambda$. We need to show that $V(A)$ is independent of the $D_r$ chosen. Suppose $D_s \subset D_r$, so that $D_r \succ D_s$. Letting $\eta := s / r$, we see that $\eta[\lambda A + (1 - \lambda) \{p^*\}] \sim \eta D_r + (1 - \eta) \{p^*\} = D_s$. Consistency would require that $s / \eta \lambda = r / \lambda$ which is definitely the case. Similar arguments show this for the case where $s > r$, and also show that the value of $V(A)$ is independent of the $\Xi_B$ it is contained in.

Now to the case where $\{p^*\} \succ A$. Recall that by finding $V(A)$, we can assign values to all $A'$ in any $\Xi_B$ that contains $A$. So suppose $\lambda A + (1 - \lambda) D_r \sim \{p^*\}$ for some $\lambda \in (0, 1)$, $D_r$, giving us $V(A) = -r(1 - \lambda) / \lambda$. Suppose $D_s$ is such that $s < r$. Then, by D-Independence, $\{p^*\} \sim \lambda_2 A + (1 - \lambda_2) D_r = \mu(\lambda_1 A + (1 - \lambda_1) D_r) + (1 - \mu) D_r \sim \mu \{p^*\} + (1 - \mu) D_r$, which contradicts Step 1.
and let $\eta := s/r$ so that $A' := \eta A + (1 - \eta)\{p^*\}$ and $V(A') = \eta V(A)$. It will suffice to show that $\lambda A' + (1 - \lambda)D_r \sim \{p^*\}$. To see that this is the case, notice that $\{p^*\} \sim \eta[\lambda A + (1 - \lambda)D_r] + (1 - \eta)\{p^*\} = \lambda[\eta A + (1 - \eta)\{p^*\}] + (1 - \lambda)[\eta D_r + (1 - \eta)\{p^*\}] = \lambda A' + (1 - \lambda)D_r$, as required.

We have thus shown that $V$ is well defined. We shall now check that $V(\lambda A + (1 - \lambda)D_r) = \lambda V(A) + (1 - \lambda)V(D_r)$ for all $A$, $D_r$ and $\lambda \in (0, 1)$. Assume first that $A \succ \{p^*\}$ and assume, without loss of generality, that $A \sim D_t$. Then, $\lambda A + (1 - \lambda)D_r \sim \lambda D_t + (1 - \lambda)D_r$ so that $V(\lambda A + (1 - \lambda)D_r) = \lambda t + (1 - \lambda)r = \lambda V(A) + (1 - \lambda)V(D_r)$.

Now for the case where $\{p^*\} \succ A$. For each $D_r$, there exists a $\lambda \in (0, 1)$ such that $\lambda A + (1 - \lambda)D_r \sim \{p^*\}$. Consider first the case where $1 > \mu > \lambda$, so that $\eta := \lambda/\mu$. Then, $\eta[\mu A + (1 - \mu)D_r] + (1 - \eta)D_r = \lambda A + (1 - \lambda)D_r \sim \{p^*\}$ which means $V(\mu A + (1 - \mu)D_r) = -r(1 - \eta)/\eta = -r\frac{\mu - \lambda}{\lambda}$. But $\mu V(A) + (1 - \mu)r = -\mu r\frac{1 - \lambda}{\lambda} + (1 - \mu)r = -r\frac{\mu - \lambda}{\lambda} = V(\mu A + (1 - \mu)D_r)$, as desired. Finally consider the case where $\mu < \lambda$ and once again let $\eta := \mu/\mu_a$. Recalling that $\lambda A + (1 - \lambda)D_r \sim \{p^*\}$ which means that $V(A) = -r\frac{1 - \lambda}{\mu_a}$, we see that $\mu A + (1 - \mu)D_r = \eta[\lambda A + (1 - \lambda)D_r] + (1 - \eta)D_r \sim \eta\{p^*\} + (1 - \eta)D_r$, which implies $V(\mu A + (1 - \mu)D_r) = (1 - \eta)r = r\frac{\lambda - \mu}{\lambda} = \mu V(A) + (1 - \mu)V(D_r)$.

In sum, we have constructed a function $V : \mathcal{K}(\Delta) \to \mathbb{R}$ that represents $\succ$ and satisfies (i) $V(\lambda A + (1 - \lambda)\{p^*\}) = \lambda V(A)$ and (ii) $V(\lambda A + (1 - \lambda)D_r) = \lambda V(A) + (1 - \lambda)V(D_r)$, where $V(\{p^*\}) = 0$ and $V(D_r) = r$. We shall call such a function D-linear. We shall now show that if $\succ$ also satisfies L continuity, $V$ is actually Lipschitz continuous. It should be noted that the proof of the following lemma exactly the same as the proof of Lemma 1 in Dekel et al (2007). The key difference is that we have a stronger version of L continuity in that we require $A^*$ and $A_*$ to be one of $D_R$ and $\{p^*\}$. Thus, the linearity that is so crucial in the proof of the lemma in Dekel et al (2007), is still present in our setting since $V$ is D-linear.

**IV.10 Definition.** A function $V : \mathcal{K}(\Delta) \to \mathbb{R}$ is Lipschitz continuous if there is an $\tilde{N}$ such that $V(B) - V(A) \leq \tilde{N} d_h(A, B)$ for all $A, B$.

**IV.11 Lemma.** Let $\succ$ have a D-linear representation $V$ where $V(D_R) \neq V(\{p^*\})$. Then $V$ is Lipschitz continuous if and only if $\succ$ satisfies L continuity.

As mentioned above, the proof is essentially the same as the proof of Lemma
1 in Dekel et al (2007). Therefore, we only provide a sketch with some details missing, for which the reader is referred to the original.

**Sketch of Proof.** Suppose \( \succcurlyeq \) satisfies L continuity. Fix an \( N > 0 \) and \( M \in (0,1/N) \) and fix \( A, B \) such that \( \delta := d_h(A, B) \leq M \). Suppose \( \delta > 0 \) (if not, we are done) so that L continuity implies

\[
(1 - N\delta)A + N\delta A^\ast \succcurlyeq (1 - N\delta)B + N\delta A_\ast.
\]

Using the D-linearity of \( V \), we can rewrite this as

\[
V(B) - V(A) \leq \frac{N}{1 - N\delta} [V(A^\ast) - V(A_\ast)] d_h(A, B).
\]

Let \( \bar{N} := \frac{N}{1 - N\delta} [V(A^\ast) - V(A_\ast)] \), so that for all \( A, B \) with \( d_h(A, B) \leq M \), we have

\[
V(B) - V(A) \leq \bar{N} d_h(A, B).
\]

To show the same is true for arbitrary \( A \) and \( B \), fix an \( A \) and \( B \) and a sequence

\[
0 = \lambda_0 < \lambda_1 < \cdots < \lambda_J < \lambda_{J+1} = 1
\]

such that \( (\lambda_{j+1} - \lambda_j)d_h(A, B) \leq M \). Defining

\[
B_j := \lambda_j B + (1 - \lambda_j)A,
\]

so that \( d_h(B_{j+1}, B_j) = (\lambda_{j+1} - \lambda_j)d_h(A, B) \). This implies,

\[
V(B_{j+1}) - V(B_j) \leq \bar{N} (\lambda_{j+1} - \lambda_j)d_h(A, B)
\]

and summing over \( j \) from 0 to \( J \) gives

\[
V(B) - V(A) \leq \bar{N} d_h(A, B), \quad \text{i.e., Lipschitz continuity of } V.
\]

To see the converse, suppose there is an \( \bar{N} \) such that \( V(B) - V(A) \leq \bar{N} d_h(A, B) \) for all \( A, B \). Let \( N := \bar{N} / |V(D_R) - V({p^*})| \). Thus, for all \( A, B \) with \( d_h(A, B) \leq 1/N \) and for every \( \varepsilon \in [d_h(A, B), 1/N) \),

\[
V(B) - V(A) \leq \frac{N\varepsilon}{1 - N\varepsilon} |V(D_R) - V({p^*})|.
\]

Rearranging terms gives us the L continuity of \( \succcurlyeq \).

It is easy to see that extending \( V \) to \( \mathcal{K}(\text{aff } \Delta) \) by positive homogeneity preserves the above properties. Abusing notation, let the extension be denoted by \( V : \mathcal{K}(\text{aff } \Delta) \to \mathbb{R} \). We shall now use the embedding of aff \( \Delta \) in \( C(S^Z) \) and extend \( V \) to a dense linear subspace of \( C(S^Z) \). It is easy to see that the extension is also Lipschitz continuous.
§IV.3  Extension to a Dense Linear Subspace

Recall that $S^Z := \{ p \in \mathbb{R}^{|Z|} : \sum p_i = 0 \text{ and } ||p|| = 1 \}$ is isometrically homeomorphic to $S^{n-1}$ and that $\mathcal{K}(\text{aff } \Delta)$ is isometrically embedded as $K^* \subset C(S^Z)$ (which has the sup norm). For each $A \in \mathcal{K}(\text{aff } \Delta)$, $h_A \in K^*$. (Also, a function $f \in K^*$ if and only if the unique extension of $f$ to $\mathbb{R}^n$, which is isometrically homeomorphic to aff $\Delta$, by positive homogeneity is superlinear. Thus, for each $f \in K^*$, there exists an $A \in \mathcal{K}(\text{aff } \Delta)$ such that $f = h_A$.) Let $C^2(S^Z)$ be the space of all twice continuously differentiable functions on $S^Z$ viewed as a subspace of $C(S^Z)$ and define $K^*_2 := K \cap C^2(S^Z)$. We begin by defining a functional on $K^*$.

Define $\varphi : K^* \to \mathbb{R}$ as $\varphi(h_A) = V(A)$. Notice that since $V$ is Lipschitz continuous, $\varphi$ is too. The following property is a weakening of C-additivity introduced by Gilboa and Schmeidler (1989).

**IV.12 Definition.** Let $K \subset C(S^Z)$ be a cone such that $1 \in K$. A function $\varphi : K \to \mathbb{R}$ is $C^+$-additive if $\varphi(f + \alpha \mathbf{1}) = \varphi(f) + \varphi(\alpha \mathbf{1})$ for all $f \in K$ and $\alpha \in \mathbb{R}_+$.

Notice that $\varphi$ is, by definition, positively homogeneous. We shall now show that $\varphi$ is $C^+$-additive and superlinear.$^6$

**IV.13 Lemma.** Let $\varphi : K \to \mathbb{R}$ be defined as above. Then $\varphi$ is $C^+$-additive.

**Proof.** Recall that if $v \in K$, $v + \alpha \mathbf{1} \in K$ for all $\alpha \in \mathbb{R}_+$. Now consider $\alpha \in [0, 1)$ and let $w$ such that $(1 - \alpha)w = v$. Then,

\[
\varphi(v + \alpha \mathbf{1}) = \varphi((1 - \alpha)w + \alpha \mathbf{1}) = V((1 - \alpha)A_w + \alpha D)
\]

\[
= V((1 - \alpha)A_w) + V(\alpha D)
\]

\[
= \varphi((1 - \alpha)w) + \varphi(\alpha \mathbf{1})
\]

\[
= \varphi(v) + \varphi(\alpha \mathbf{1})
\]

where $A_w \in \mathcal{K}^n$ is such that $w = u_A$. To prove the general case, we shall use the fact that $\varphi$ is positively homogeneous. Now let $\alpha \in [1, \infty)$. For any such $\alpha$, there exists $\lambda > 0$ such that $\lambda \alpha < 1$. Then, $\varphi(v + \alpha \mathbf{1}) = (1/\lambda)\varphi(\lambda v + \lambda \alpha \mathbf{1}) = (1/\lambda)(\varphi(\lambda v) + \lambda \varphi(\alpha \mathbf{1})) = \varphi(v) + \varphi(\alpha \mathbf{1})$ which gives us the desired result. \hfill \Box

$^6$Recall that a function is superlinear if it positively homogeneous and concave.
And now to finally show that $\phi$ is superlinear.

**IV.14 Lemma.** The functions $\phi$ as defined above is superlinear.

*Proof.* We have already established that $\phi$ is positively homogeneous. It only remains to establish that $\phi$ is concave. To see this, suppose $v, w \in K$ such that $\phi(v) = \phi(w)$. Then, there exist $A_v, A_w \in \mathcal{K}$ such that $V(A_v) = V(A_w)$ and $u_{A_v} = v$ and $u_{A_w} = w$. Then, $V(A_v + A_w) = 2V\left(\frac{1}{2}(A_v + A_w)\right) \geq 2V(A_v)$ by Pessimism. Also $V(A_v) = \frac{1}{2}(V(A_v) + V(A_w))$ so that $V(A_v + A_w) \geq V(A_v) + V(A_w)$. Thus, $\phi(v + w) \geq \phi(v) + \phi(w)$.

If $v, w \in K$ are such that $\phi(v) \neq \phi(w)$, let us suppose $\phi(v) > \phi(w)$. Let $\alpha := (\phi(v) - \phi(w))/\phi(1)$ and define $z := w + \alpha 1$. Then $\phi(z) = \phi(w + \alpha 1) = \phi(w) + \alpha \phi(1) = \phi(w) + \phi(v) - \phi(w)$ by the $C^+$-additivity of $\phi$. Using the $C^+$-additivity again, we see that $\phi(v + w) + \phi(\alpha 1) = \phi(v + w + \alpha 1) = \phi(v + z) \geq \phi(v) + \phi(z) = \phi(v) + \phi(w) + \phi(\alpha 1)$, which gives us $\phi(v + w) \geq \phi(v) + \phi(w)$, the desired result. \hfill \Box

We now extend the function to $C^2(S^2)$ in such a way that it preserves $C^+$-additivity and superlinearity. Define $\psi : K^*_2 \to \mathbb{R}$ as $\psi(f) := \phi(f)$ for all $f \in K^*_2$. The first step we need to take is to observe that $\text{span } K^*_2 = C^2(S^2)$. Let us state this formally.

**IV.15 Theorem.** $\text{span } K^*_2 = C^2(S^2)$. In particular, for each $f \in C^2(S^2)$, there exists a convex body $K$ and $r > 0$ such that $f = h_K - h_{D_r}$.

This is merely Lemma 1.7.9 from Schneider (1993). Another way of stating this is that for each $f \in C^2(S^2)$, there exists $\alpha > 0$ such that $f + \alpha 1 \in K^*_2$. Moreover, for each $\alpha' > \alpha$, $f + \alpha' 1 \in K^*_2$. This is because $f + \alpha' 1 = (f + \alpha 1) + (\alpha' - \alpha) 1$, the sum of two convex sets, which is also convex.

Now, extend $\psi$ to $C^2(S^2)$ in the natural way. First, define $\psi(-1) := -\psi(1)$. For $\alpha > 0$, define $\psi(-\alpha 1) := \alpha \psi(-1)$. More generally (from the theorem above), for each $f \in C^2(S^2)$, there exists $\alpha > 0$ such that $f + \alpha 1 \in K^*_2$. Then define,

$$
\psi(f) := \psi(f + \alpha 1) - \alpha \psi(1).
$$

---

7Here, 1 is the constant function with value 1 everywhere.
Now extend $\psi$ by positive homogeneity. To see that $\psi$ is still well defined, let $\lambda > 0$, and notice that $\lambda f + \lambda \alpha 1 \in K_2^*$ and that $\psi(\lambda f + \lambda \alpha 1) = \lambda \psi(f + \alpha 1)$.

We now show that the definition of $\psi$ is independent of the particular $\alpha$ chosen for each $f$. Suppose $\alpha_1 > 0$ is such that $g + \alpha_1 1 \in K_2^*$. As before, define $\psi(g) := \psi(g + \alpha_1 1) - \alpha_1 \psi(1)$. Then, for $\alpha_2 > \alpha_1$, we have $\psi(g + \alpha_2 1) - \alpha_2 \psi(1) = \psi(g + \alpha_1 1 + (\alpha_2 - \alpha_1) 1) - \alpha_2 \psi(1) = \psi(g + \alpha_1 1) + (\alpha_2 - \alpha_1) \psi(1) - \alpha_2 \psi(1) = \psi(g)$ (where we have used the fact that $\psi$ is $C^+$-linear on $K^*$). Thus, the definition of $\psi$ doesn’t depend on the choice of $\alpha > 0$.

We also want to check that $\psi$ is $C^+$-additive everywhere on its domain. To see this, let $f \in C^2(S^Z)$ and let $\beta > 0$ be such that $f + \beta 1 \in K_2^*$. Clearly, for all $\alpha \geq \beta$, $\psi(f + \alpha 1) = \psi(f) + \alpha \psi(1)$. Now suppose $\alpha \in (0, \beta)$. As before, $\psi(f) = \psi(f + 1) - \beta \psi(1)$. Thus, $\psi(f + \alpha 1) = \psi(f + 1) - (\beta - \alpha) \psi(1) = \psi(f) + \alpha \psi(1)$, which shows that $\psi$ is $C^+$-additive.

Finally, we check that $\psi$ is superlinear. It is clear that $\varphi : K_2^* \to \mathbb{R}$ is superlinear, hence for all $f, g \in K_2^*$, $\psi(f + g) \geq \psi(f) + \psi(g)$. Suppose $f, g \in C^2(S^Z)$ and $\psi(f) = \psi(g)$. Then there exists an $\alpha > 0$ such that $f + \alpha 1, g + \alpha 1 \in K_2^*$. Then $\psi(f + g) = \psi(f + g + 2\alpha 1) - 2\alpha \psi(1) \geq \psi(f + \alpha 1) - \alpha \psi(1) + \psi(g + \alpha 1) - \alpha \psi(1) = \psi(f) + \psi(g)$.

Now suppose $\psi(f) > \psi(g)$. Let $\alpha \psi(1) := \psi(f) - \psi(g)$ and define $\tilde{f} := g + \alpha \psi(1)$. Then, $\psi(\tilde{f}) = \psi(g + \alpha 1) = \psi(g) + \alpha \psi(1) = \psi(f)$. Therefore, $\psi(f + g) + \alpha \psi(1) = \psi(f + g + \alpha 1) = \psi(f + \tilde{f}) \geq \psi(f) + \psi(\tilde{f}) = \psi(f) + \psi(g) + \alpha \psi(1)$, ie $\psi(f + g) \geq \psi(f) + \psi(g)$. Thus, $\psi$ is superlinear.

We conclude with the simple observation that since $\varphi$ is Lipschitz continuous, so is $\psi$. Thus, there is a unique extension of $\psi$ to $C(S^Z)$ that we shall also denote $\psi$. Moreover, this extension is also superlinear and $C^+$-additive.

§ IV.4 Proof of Theorem II.1

Now that we constructed a D-linear utility representation for our preference and extended this utility representation to a Banach space, we are ready to proceed to the final steps of our proof.
We start with the observation that since $\psi : \mathcal{C}(S^Z) \to \mathbb{R}$ is a superlinear function, there exists a nonempty convex set $\mathcal{L}$ of continuous linear functions on $\mathcal{C}(S^Z)$ such that

$$\psi(f) = \min \{L(f) : L \in \mathcal{L}\} \text{ for all } f \in \mathcal{C}(S^Z).$$

By the Riesz Representation Theorem, for each continuous linear functional $L$, there exists a regular, countably additive, signed measure $\pi$ such that

$$L(f) = \langle f, \pi \rangle.$$

Thus, we can represent $\psi$ as

$$\psi(f) = \min \{\langle f, \pi \rangle : \pi \in \Pi\}$$

where $\Pi$ is a weak* compact, convex set in $M(S^Z)$, the space of all signed Borel measures on $S^Z$. It is possible to show directly that $\Pi$ is unique, but we shall employ the following indirect method which will be useful in the sequel.

Recall that $\mathcal{C}(S^Z)$ is a Banach space and $M(S^Z)$, the space of signed (finite) Borel measures on $S^Z$ is its dual. Indeed, $\langle \mathcal{C}(S^Z), M(S^Z) \rangle$ is a dual pair. By Theorem 6.27 of Aliprantis and Border (1999), we know that a linear functional on $\mathcal{C}(S^Z)$ is continuous if and only if it is Mackey continuous. Also, by Lemma 2.38 of Aliprantis and Border (1999), we know that the pointwise infimum of a family of Mackey continuous linear functionals is Mackey continuous. This establishes the Mackey continuity of $\psi$. Finally, by Theorem 5.102 of Aliprantis and Border (1999), we see that there is a bijection between the space of Mackey continuous sublinear functions on $\mathcal{C}(S^Z)$ and the space of weak* compact, convex subsets of $M(S^Z)$, so that $\Pi$ is unique. Indeed, $\psi$ is the support function of $\Pi$.

The hypograph of a function $\psi$ is given by $\{(f, t) \in \mathcal{C}(S^Z) \times \mathbb{R} : \psi(f) \geq t\}$. It is easy to see that since $\psi$ is superlinear, the hypograph has a nonempty interior, so that for each $f \in \mathcal{C}(S^Z)$, there exists $\pi_f \in \Pi$ such that $\psi(f) = \langle f, \pi_f \rangle$. Moreover, for each $\pi \in \text{ext} \Pi$ (where $\text{ext} \Pi$ represents the set of extreme points of $\Pi$), there exists an $f_\pi \in \mathcal{C}(S^Z)$ such that $\psi(f_\pi) = \langle f_\pi, \pi \rangle$.

Recall that $\psi$ is $\mathcal{C}^+$-additive. Then, for any $\pi \in \mathcal{C}(S^Z)$, let $f_\pi =: g + \alpha \mathbf{1}$, where $\alpha > 0$ and $f_\pi$ is such that $\psi(f_\pi) = \langle f, \pi \rangle$. Then, $\psi(g + \alpha \mathbf{1}) = \psi(g) + \alpha \psi(\mathbf{1}) = \psi(f) = \langle f, \pi \rangle = \langle g, \pi \rangle + \alpha \langle \mathbf{1}, \pi \rangle$. Since $\psi(g) \leq \langle g, \pi \rangle$ and $\psi(\mathbf{1}) \leq \langle \mathbf{1}, \pi \rangle$, it must be the case that $\psi(\mathbf{1}) = \langle \mathbf{1}, \pi \rangle$. 

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Since we have assumed $D_R \succ \{p^*\}$, we can set $\varphi(1) = 1$. With the observation that assuming Optimism instead of Pessimism leads to a change in the direction of some inequalities above, we get both the (★) and (♠) representations for this case.

We now consider the case where $\{p^*\} \succ D_R$. Define the preference relation $\succ^* \subset \mathcal{F}(\Delta) \times \mathcal{F}(\Delta)$ as follows: $A \succ^* B$ if and only if $B \succ A$. Then, $D_R \succ^* \{p^*\}$ and $\succ^*$ satisfies IR and is continuous. Now suppose $A \succ^* B$, so that $B \succ A$. Then, by D-Independence, for all $\lambda \in (0, 1]$, $\lambda A + (1 - \lambda)D_R \succ \lambda B + (1 - \lambda)D_R$, ie, $\lambda B + (1 - \lambda)D_R \succ^* \lambda A + (1 - \lambda)D_R$, with a similar condition holding for the second part of D-Independence. Thus, $\succ^*$ satisfies D-Independence. Finally, notice that $\succ$ satisfies Optimism (resp Pessimism) if and only if $\succ^*$ satisfies Pessimism (resp Optimism).

Thus, given preference relation $\succ$ that satisfies IR, Continuity and L-Continuity, D-Independence and Optimism (resp Pessimism) and has $\{p^*\} \succ D_R$, there exists a preference relation $\succ^*$ that satisfies IR, D-Independence and Pessimism (resp Optimism) and has $D_R \succ \{p^*\}$ with the property that $A \succ B$ if and only if $B \succ^* A$. But by the proof above, we know that $\succ^*$ has a utility representation $V^*: \mathcal{F}(\Delta) \to \mathbb{R}$ that admits a (★) representation:

$$V^*(A) := \min \{ \langle u_A, \pi \rangle : \pi \in \Pi \}.$$  

Now define, $V : \mathcal{F}(\Delta) \to \mathbb{R}$ as follows: $V(A) = -V^*(A)$. Then, $V$ represents $\succ$, and can be written as

$$V(A) := \max \{ \langle u_A, \mu \rangle : \mu \in -\Pi \},$$

which is the (♠) representation for $\succ$. Similarly, if $\succ$ satisfies Pessimism, using the arguments above we can show it has a (★) representation. This concludes the proof of Theorem II.1.

§ IV.5 Proof of Theorem II.4

As in the text, say that $\succ_2$ is more pessimistic than $\succ_1$ if, for each $A$ where $D_R \succ_1 A \succ_1 \{p^*\}$, $\lambda D_R + (1 - \lambda)\{p^*\} \sim_1 A$ implies $\lambda D_R + (1 - \lambda)\{p^*\} \succ_2 A$. 

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Let $V_i$ represent $\succeq_i$ and be such that $V_i(\{p^*\}) = 0$ and $V(D_R) = R$. Clearly, $V_i$ is D-linear. We shall first show that $\succ_2$ is more pessimistic than $\succ_1$ if and only if for each menu $A$, $V_1(A) \succeq V_2(A)$.

**IV.16 Proposition.** The preference $\succ_2$ is more pessimistic than $\succ_1$ if and only $V_1(A) \succeq V_1(A)$ for each menu $A$.

*Proof.* Notice that for any $A$ such that $D_R \succ_1 A \succ_1 \{p^*\}$, $\succ_2$ is more pessimistic than $\succ_1$ if and only if $V_1(A) \succeq V_2(A)$, since $V_1(A) = V_1(\lambda A D_R + (1 - \lambda A)\{p^*\}) = V_2(\lambda A D_R + (1 - \lambda A)\{p^*\}) \succeq V_2(A)$.

Now consider an $A$ where $\{p^*\} \succ_1 A$. For such an $A$, there exists a $\mu$ such that $D_R \succ_1 \mu A + (1 - \mu)D_R \succ_1 \{p^*\}$ and for such a $\mu$, $V_1(\mu A + (1 - \mu)D_R) \succeq V_2(\mu A + (1 - \mu)D_R)$ which is true if and only if $V_1(A) \succeq V_2(A)$ (using the D-linearity of $V_i$).

Similarly, for a menu $A$ with $A \succ_1 D_R$, there exists a $\mu$ such that $D_R \succ_1 \mu A + (1 - \mu)\{p^*\} \succ_1 \{p^*\}$. Once again using the D-linearity of the $V_i$, we see that $V_1(A) \succeq V_2(A)$ and that this is equivalent to saying that $\succ_2$ is more pessimistic than $\succ_1$.

Recall from the proof of theorem II.1, the existence (and construction) of a unique function $\psi : C(S^Z) \to \mathbb{R}$ such that for any menu $A$, $\psi(h_A) = V(A)$. Moreover, $\psi$ is $C^+$-additive, sublinear and continuous (and so is Lipschitz continuous). We now show that if $\succ_2$ is more pessimistic than $\succ_1$, it is the case that $\psi_1 \succeq \psi_2$. To see this, let $f \in C^2(S^Z)$. Therefore, there is an $\alpha > 0$ such that $f + \alpha 1$ is sublinear and therefore the support function of some compact, convex set. Thus, for some $\lambda > 0$, there exists a menu $A$ with support function $h_A = (1/\lambda)(f + \alpha 1)$. Using the $C^+$-additivity of the $\psi_i$’s, we see that $\psi_1(f) \succeq \psi_2(f)$. Using the density of $C^2(S^Z)$ and the continuity of $\psi_i$, we reach the desired conclusion.

As established in § IV.4, $\psi_i$ is Mackey continuous and sublinear, hence $\psi_1 \succeq \psi_2$ implies $\Pi_1 \subset \Pi_2$ (see § 5.19, Aliprantis and Border, 1999). It follows immediately that $S_1 \subset S_2$. We have thus shown that if $\succ_2$ is more pessimistic than $\succ_1$, then $\Pi_1 \subset \Pi_2$. The converse follows from the fact that $\Pi_1 \subset \Pi_2$ implies $\psi_1 \succeq \psi_2$ so that $V_1(A) \succeq V_2(A)$ for any menu $A$. 

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§ IV.6 Proof of Theorem II.5

We begin with a simple fact from linear algebra. Let \( X \) be a vector space and \( F \subset X \) a (proper) subspace. Denote by \( X' \), the space of linear functionals on \( X \) and fix \( g \in X' \). Then, \( G_F := \{ f \in X' : f(x) = g(x) \text{ for all } x \in F \} \) is an affine subset of \( X' \). To see this, notice that, by definition, \( G_F - g := \{ f \in X' : f|_F = 0 \} \).

Also, \( 0 \in G_F \) (since \( 0 = g - g \) and \( g \in G_F \)). Then, for any \( \lambda \in \mathbb{R} \) and \( f, h \in G_F - g \), \( \lambda f + h \in G_F - g \) (since \( (\lambda f + h)|_F = 0 \)). Thus, \( G_F - g \) is a subspace of \( X' \) and so \( G_F \) is an affine subset of \( X' \).

Another way of saying this is the following: Define the equivalence relation \( \sim_F \) on \( X' \) as \( f \sim_F h \) if (and only if) \( f|_F = h|_F = 0 \). Then, for any \( g \in X' \), \( G_F - g = X'/ \sim_F \) is the quotient space so that \( \text{codim dim} \ G_F = \dim F \).

We can now proceed to the proof of the theorem.

Proof of Theorem II.5. Recall that the space of ex post utilities for singletons, \( F \), is a (proper) subspace of \( K^* \). If Singleton Independence holds, then \( V|_F \) is linear.

In other words, for all \( \pi, \pi' \in \Pi \), it is the case that \( \pi(u) = \pi'(u) \) for all \( u \in F \).

Now suppose \( \dim F > 1 \). Then, for a given \( \pi \in \Pi \), the set of all \( \pi' \in \Pi \) such that \( \pi|_F = \pi'|_F \) is a (compact, convex) set of codimension at most \( \dim F \) and is therefore a nowhere dense subset of \( M(Z) \).

If \( \dim F = 1 \), it must be the case that \( |S| = 2 \), and the states are antipodal points on \( S^Z \). In this case, it is easy to see that \( F = \{(u^1, u^2) \in \mathbb{R}^2 : u^1 + u^2 = 0\} \).

Fix \( \pi' \in \Pi \). Then, \( G_F := \{ \pi \in \mathbb{R}^2 : \langle \pi, u \rangle = \langle \pi', u \rangle \text{ for all } u \in F \} \) is clearly not \( \text{aff } \Delta^1 \) and so \( G_F \cap \text{aff } \Delta^1 \) is a singleton. Therefore, if there are only two states, then Singleton Independence always implies that \( \Pi \) is a singleton.

§ IV.7 Proofs of Theorems II.9 and II.10

Recall that a state \( s_i \in S \) is negative if there exist menus \( A, B \) where \( \max_{p \in A} u(p, s_i) < \max_{p \in B} u(p, s_i) \) and \( \max_{p \in A} u(p, s_j) = \max_{p \in B} u(p, s_j) \) for all \( j \neq i \) such that \( V(A) > V(B) \) and positive if there exist menus \( A, B \) where \( \max_{p \in A} u(p, s_i) > \max_{p \in B} u(p, s_i) \) and \( \max_{p \in A} u(p, s_j) = \max_{p \in B} u(p, s_j) \) for all \( j \neq i \) such that \( V(A) > V(B) \). Also, \( \Pi_i := \text{proj}_i \Pi \) for \( i = 1, \ldots, m \), is the projection of \( \Pi \) onto the \( i \)-th axis. For menu \( A \), let \( u_A := (\max_{p \in A} u(p, s_i))_{i=1}^m = (u_A^1, \ldots, u_A^m) \) and let \( e_i := (0, \ldots, 1, \ldots, 0) \) with 1 being the \( i \)-th entry.
For any menu $A$, let $\pi_A \in \Pi$ satisfy $V(A) = \langle \pi_A, u_A \rangle$ and $\text{ext } \Pi$ is the set of extreme points of $\Pi$. In proving the theorem, it is useful to recall Corollary ??, which states that for any $\pi \in \Pi$ which is an extreme point, there exists a menu $A$ such that $V(A) = \langle u_A, \pi \rangle$. In other words, every such $\pi$ is the minimum weight for some menu. Moreover, the menu can be taken to have a relative interior.

**IV.17 Proposition.** The mapping $A \mapsto \pi_A$ is upperhemicontinuous.

*Proof.* This is an easy consequence of the Maximum Theorem of Berge. 

As noted in the text, it will suffice to prove Propositions II.7 and II.8. We shall prove these in turn.

*Proof of II.7.* (i) Suppose state $s_i$ is positive. Then there exist menus $A, B$ such that $u_A - u_B = (u_A^i - u_B^i)e_i > 0$ (where $u_A^i - u_B^i > 0$) and $V(A) > V(B)$. Suppose $\Pi_i$ is not positive, so $\Pi_i \subset (-\infty, 0]$. Then, $\langle \pi_B, u_A - u_B \rangle \leq 0$ since $\pi_B^i \leq 0$ and $u_A^i - u_B^i > 0$. Thus, $\langle \pi_B, u_A \rangle \leq \langle \pi_B, u_B \rangle$. Since $V(B) < V(A)$, we also have $\langle \pi_B, u_B \rangle < \langle \pi_A, u_A \rangle$. By the definition of $\pi_A$ and $V(A)$, it must be that $V(A) = \langle \pi_A, u_A \rangle \leq \langle \pi_B, u_A \rangle$. Combining these inequalities, we get a contradiction.

Similarly, suppose state $s_i$ is negative and by way of contradiction, $\Pi_i$ is not negative, so that $\Pi_i \subset [0, \infty)$. Since $s_i$ is negative, there exist menus $A, B$ such that $u_A - u_B = (u_A^i - u_B^i)e_i > 0$ (where $u_A^i - u_B^i > 0$) and $V(A) < V(B)$. Since $\pi_A^j > 0$, we have $\langle \pi_A, u_A - u_B \rangle \geq 0$, so that $\langle \pi_A, u_A \rangle \geq \langle \pi_A, u_B \rangle$. By the definition of $\pi_B$ and $V(B)$, $\langle \pi_A, u_B \rangle \geq \langle \pi_B, u_B \rangle$. Finally, since $V(B) > V(A)$, we have $\langle \pi_B, u_B \rangle > \langle \pi_A, u_A \rangle$. Combining these inequalities, we get a contradiction which concludes the proof.

The proof of Proposition II.8 is similar.

*Proof of Proposition II.8.* Suppose $\pi_A \in \text{ext } \Pi$ is such that $\pi_A^i > 0$. By Corollary ??, there exists a menu $A$ that has a relative interior and has $V(A) = \langle \pi_A, u_A \rangle$. Let $B$ be a menu such that $u_A - u_B = \varepsilon e_i$, $\varepsilon > 0$. (Such a menu exists since $A$ has a relative interior.) We claim that $V(A) > V(B)$. To see this, notice that $V(B) - V(A) = \langle \pi_B, u_B \rangle - \langle \pi_A, u_A \rangle \leq \langle \pi_B, u_B \rangle - \langle \pi_A, u_A \rangle = \langle \pi_A, u_B - u_A \rangle = \langle \pi_A, -\varepsilon e_i \rangle < 0$. Thus, state $s_i$ is a positive state.
Now suppose \( \pi_A \in \text{ext } \Pi \) is such that \( \pi_A^i < 0 \). By Corollary ??, there exists a menu \( A \) that has a relative interior and has \( V(A) = \langle \pi_A, u_A \rangle \). Let \( B \) be a menu such that \( u_B - u_A = \varepsilon e_i \). Thus, \( \langle \pi_A, u_B - u_A \rangle = \langle \pi_A, \varepsilon e_i \rangle < 0 \). This means that \( V(B) = \langle \pi_B, u_B \rangle \leq \langle \pi_A, u_B \rangle < \langle \pi_A, u_A \rangle = V(A) \), so that \( s_i \) is a negative state. \( \square \)

Recall that, by our normalisations, \( V(D_1) = 1 \) if \( D_R \succ \{p^*\} \) and \( V(D_1) = -1 \) if \( \{p^*\} \succ D_R \). Moreover, for any \( \pi \in \Pi \), \( \sum_{i=1}^{m} \pi^i = V(D_1) \) (this is Theorem II.1). The proof of theorem II.10 now follows from Proposition II.8 and from this normalisation of utilities.

§ IV.8  Special Cases

Here we shall prove Propositions II.13 and II.14 and Theorem II.16.

Proof of Proposition II.13. Suppose not, so suppose there is \( \pi_A \in \text{ext } \Pi \) such that \( \pi_A^1, \pi_A^2 > 0 \). Then, by Corollary ??, there exists a menu \( A \) with a relative interior such that \( V(A) = \langle \pi_A, u_A \rangle \). Now, for \( j = 1, 2 \), define \( B_j \) such that \( u_{B_j}^i = u_A^i \) for \( i \neq j \) and \( u_{B_j}^j = u_A^j - \varepsilon > 0 \) for some appropriate \( \varepsilon > 0 \). Then, \( V(B_j) = \langle \pi_{B_j}, u_{B_j} \rangle \leq \langle \pi_A, u_A \rangle = V(A) \). Thus, \( V(A) = V(B_1 \cup B_2) > V(B_1), V(B_2) \), contradicting Positive Set Betweenness. \( \square \)

The proof of Proposition II.14 is similar, but requires a little more subtlety. In particular, we shall refer to the following facts about the differentiability of concave functions defined on an open interval. Let \( I \subset \mathbb{R} \) be an open interval and for \( f : I \to \mathbb{R} \), let \( f'_- \) and \( f'_+ \) be the left and right derivatives of \( f \). The following theorem is standard and can be found, for instance, as Theorems 1.6 and 1.8, van Tiel (1984).

IV.18 Theorem. Let \( f : I \to \mathbb{R} \) be a concave function and \( I \subset \mathbb{R} \) an open interval. Then, \( f \) has right and left derivatives everywhere on \( I \), \( f'_+ \) and \( f'_- \) are non-decreasing and for any \( c \in I \),

\[
f'_-(c) \geq f'_+(c).
\]

Moreover, \( f'_- \) is left continuous and \( f'_+ \) is right continuous.
Thus, $f'_-$ is upper semicontinuous on the interior of any interval. We can now proceed to the proof.

Proof of Proposition II.14. Suppose not, so suppose there is $\pi \in \text{ext } \Pi^\circ$ such that $\pi^1, \pi^2 < 0$. Then, by Corollary ??, there exists a menu $B_1$ with a relative interior such that $V(B_1) = \langle \pi, u_{B_1} \rangle$. Moreover, we can take $B_1$ such that $u_{B_1} > 0$ and $u_{B_1} \in \text{int } K$. Define $I_\varepsilon := \{u \in \text{int } K : u^j = u^j_{B_1} \text{ for } j \neq 1 \text{ and } |u^1 - u^1_{B_1}| < \varepsilon \}$ and $f : I_\varepsilon \to \mathbb{R}$ as $f := \varphi|_{I_\varepsilon}$. (Recall that for any menu $A$, $V(A) := \varphi(u_A)$.) Also, $\varphi$ is differentiable at $B_1$, so that $f$ is differentiable at $u_{B_1}$. This means $f'_+(u_{B_1}) = f'_-(u_{B_1})$ and since $f'_-$ is upper semicontinuous, there exists $\varepsilon > 0$ such that for some $u_B \in I_\varepsilon$, $f'_-(u_B) = f'_+(u_B) < 0$, ie $f$ is differentiable at $u_B$. Let $B$ be a menu with ex post utilities given by $u_B$. Now, choose $B_2$ such that $u^2_B < u^2_{B_2}$ and $u^j_B = u^j_{B_2}$ for $j \neq 2$.

Since $\pi^1 < 0$, $V(B) \leq \langle \pi, u_B \rangle < \langle \pi, u_{B_1} \rangle = V(B_1)$. By continuity of $V$, we can choose $B_2$ such that $B_2 \succ B_1$. Notice that, by construction, $B = B_1 \cap B_2$. Now define $A := \text{conv}(B_1 \cup B_2)$ so that $u_A := u_{B_1} \lor u_{B_2}$.

We claim that $B_1 \succ A$. To see this, notice that $u_A - u_{B_1} = (u_B - u_{B_2})e_2 =: \delta e_2 > 0$, which implies $\langle \pi, u_A - u_{B_1} \rangle = \pi^2 \delta < 0$. Thus, $V(A) \leq \langle \pi, u_A \rangle < \langle \pi, u_{B_1} \rangle = V(B_1)$, which contradicts Negative Set Betweenness. □

Proof of Theorem II.16. The above proofs tell give us part of the characterisation theorem II.16. To show that a state cannot be both positive and negative, we construct a counterexample.

Notice that if preferences satisfy Set Betweenness, it must be the case that $|S| = 2$. Now suppose a state is both positive and negative. In other words, $\Pi = \text{conv}\{\pi, \mu\}$ where $\pi^1 + \pi^2 = \mu^1 + \mu^2 \in \{-1, 0, 1\}$. For simplicity, let us assume that $\pi^1 + \pi^2 = 1$. Also, instead of proving the general case, we shall provide a simple example. The general case is easily seen to follow. Suppose, $\pi = (-1, 2)$ and $\mu = (2, -1)$ and suppose utilities to two singletons are as given below:

<table>
<thead>
<tr>
<th></th>
<th>$u^1$</th>
<th>$u^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>$c$</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

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It is easily seen that for any menu $A$, $\langle \pi, u_A \rangle \leq \langle \mu, u_A \rangle$ if and only if $(\pi^1 - \mu^1)(u^1_A - u^2_A) \leq 0$. Therefore, $V(\{s\}) = \langle \pi, u_{\{s\}} \rangle$, $V(\{c\}) = \langle \mu, u_{\{c\}} \rangle$ and $V(\{s, c\}) = \langle \pi, u_{\{s, c\}} \rangle > V(\{s\}), V(\{c\})$.

To prove the general case, notice that the menu we have constructed is generic, in that we can find a menu with such ex post utilities for any subjective state space of cardinality 2. The construction now follows easily.

§ IV.9  Finite Subjective State Space

DLR’s IR axiom for non-trivial continuous preferences ensures the presence of a unique subjective state space. However, this state space can have any cardinality between 2 and $2^{\aleph_0}$. In this section, we describe the axioms we must impose on preferences to ensure the existence of a finite subjective state space. The first axiom that we shall assume throughout this section is D-Independence. The second axiom, Finiteness, is introduced towards the end of the section. Recall that the space of all closed convex subsets of $\Delta$ is denoted by $\mathcal{K}(\Delta)$ and $N_\varepsilon(s)$ is the $\varepsilon$-neighbourhood of $s \in \mathbb{R}^n$.

IV.19 Definition. A point $s \in S^Z$ is strongly relevant if for every $\varepsilon > 0$ there exist $A, A' \in \mathcal{K}(\Delta)$ such that $A \sim A'$ and $h_A(s) = h_{A'}(s)$ for all $s \in S^Z \setminus N_\varepsilon(s)$.

Notice that we can take, without loss of generality, $A \subset A'$. This is because $\text{conv}(A \cup A')$ is not indifferent to either $A$ or $A'$ and $A, A' \subset \text{conv}(A \cup A')$. This follows from the fact that $h_{\text{conv}(A \cup A')} = h_A \lor h_{A'} = \max\{h_A, h_{A'}\}$. In the presence of D-Independence, we can say even more about $A$ and $A'$.

IV.20 Proposition. Let $D_r(p^*) \subset \text{ri } \Delta$. Then, we can take $A, A'$ such that for all $\delta > 0$, $d_h(A, D_r) < \delta$ and $A' \subset \text{ri } D_r$.

Proof. First note that, without loss of generality, we can take $A, A' \subset \text{ri } D_r$. To see this, take $\varepsilon > 0$ and since $s$ is strongly relevant, there exist $A, A'$ such that $h_A(s) = h_{A'}(s)$ for all $s \in S^Z \setminus N_\varepsilon(s)$ and $A \sim A'$. As noted above, we can assume $A \subset A'$. Now, by D-Independence, $\lambda A + (1 - \lambda)p^* \sim \lambda A' + (1 - \lambda)p^*$ for all $\lambda \in (0, 1)$. For $\lambda$ sufficiently small, $\lambda A' + (1 - \lambda)p^* \subset D_r$. Moreover, since
\( h_{\lambda A^{+}(1-\lambda)\{\cdot\}^{*}}(s) = \lambda h_A(s) + (1-\lambda)h_{\{\cdot\}^{*}}(s) \) and \( h_{\{\cdot\}^{*}}(s) = 0 \) for all \( s \in S^Z \), it follows that \( h_{\lambda A^{+}(1-\lambda)\{\cdot\}^{*}}(s) = h_{\lambda A^{+}(1-\lambda)\{\cdot\}^{*}}(s) \) for all \( s \in S^Z \setminus N_{\varepsilon}(s) \).

Since \( A \subset A' \subset ri D_r \), it follows that \( \lambda A + (1-\lambda)D_r \subset \lambda A' + (1-\lambda)D_r \subset ri D_r \) for all \( \lambda \in (0,1) \). Moreover, for any \( \delta > 0 \), there exists \( \lambda \in (0,1) \) sufficiently small such that \( d_h(\lambda A + (1-\lambda)D_r) < \delta \). Finally, by D-Independence, \( \lambda A + (1-\lambda)D_r \approx \lambda A' + (1-\lambda)D_r \) for all \( \lambda \in (0,1) \). Since, \( h_{D_r}(s) = r \) for all \( s \in S^Z \), it follows that \( h_{\lambda A^{+}(1-\lambda)D_r}(s) = h_{\lambda A^{+}(1-\lambda)D_r}(s) \) for all \( s \in S^Z \setminus N_{\varepsilon}(s) \). \( \Box \)

Notice that for any \( r > 0 \) \( D_r \subset ri \Delta \) (where \( D_r := D_r(p^*) \)), \( D_r \) is isometrically homeomorphic to \( r \text{ conv } S^Z \). Of course, in a very obvious sense, \( D_1 \) is conv \( S^Z \).

Therefore, for any \( s \in S^Z \), there exists a unique \( p_s \in D_r \) such that \( h_{D_r}(s) = \langle (p_s-p^*)/r, p_s - p^* \rangle = r \). Clearly, \( p_s \in bd D_r \). Conversely, for any point \( p \in bd D_r \), there exists a unique \( s_p \in S^Z \) such that \( h_{D_r}(s_p) = \langle (p-p^*)/r, p - p^* \rangle = r \). Moreover, for any \( s \in S^Z \), \( s_p = s \) and for any \( p \in D_r \), \( p_s = p \).

Recall that \( S^{n-1} := \{ x \in \mathbb{R}^n : \|x\| = 1 \} \) and \( e_n := (0, \ldots, 1) \). If \( x = (x_1, \ldots, x_{n-1}, 1-\varepsilon) \in S^{n-1}, \langle x, e_n \rangle = 1 - \varepsilon \) and \( x_1^2 + \cdots + x_{n-1}^2 = 1 - (1-\varepsilon)^2 = \varepsilon(2-\varepsilon) \). This implies \( \|x - e_n\|^2 = x_1^2 + \cdots + x_{n-1}^2 + (1-\varepsilon - 1)^2 = \varepsilon(2-\varepsilon) + \varepsilon^2 = 2\varepsilon \), so that \( \|x - e_n\| = \sqrt{2\varepsilon} \).

For any point \( p \in bd D_r \), define the function \( \psi_p : \Delta \rightarrow \mathbb{R} \) as follows:

\[
\psi_p(q) := \frac{\langle p - p^*, q - p^* \rangle}{r^2}.
\]

Clearly, \( \psi_p(p^*) = 0 \) and \( \psi_p(p) = 1 \). Also, the range of \( \psi_p \) for any \( p \in bd D_r \) is a compact set. For any \( K \in \mathcal{K}(\Delta) \), we may consider \( \max_{q \in K} \psi_p(q) \), the largest height in the direction of \( p - p^* \) of all the lotteries in the menu \( K \), ie the height of the menu \( K \) in the direction \( p - p^* \). For instance, \( h_{D_r}(s) = \max_{q \in D_r} \psi_{p_s}(q) = \max_{q \in D_r} \psi_{p_s}(q) \).

We will be interested in the directions where increasing the height of a menu can change the value of the menu. The following definition is the central idea in this section.

**IV.21 Definition.** A point \( p \in bd D_r \) is **critical to** \( \varepsilon \) if for every \( \varepsilon > 0 \), there exist \( A, A' \in \mathcal{K}(\Delta) \) with \( A \approx A' \) such that for all \( \bar{p} \in bd D_r \setminus N_{\varepsilon}(\bar{p}) \), \( \max_{q \in A} \psi_{\bar{p}}(q) = \max_{q \in A'} \psi_{\bar{p}}(q) \).
The intuitive idea is that $p$ is critical if there exist menus who have equal heights in all directions $\tilde{p} - p^*$ (where $\tilde{p} \in D_r$) save a set of directions $p' - p^*$ where $p'$ is an arbitrarily small neighbourhood of $p$. The following proposition makes this clear.

**IV.22 Proposition.** If $s \in S^{n-1}$ is strongly relevant, $p_s$ is critical to $\succ$. Conversely, if $p$ is critical to $\succ$, $s_p$ is strongly relevant.

**Proof.** First, let $s \in S^{n-1}$ be strongly relevant and fix $\varepsilon > 0$. Notice also that for any other $t \in S^{n-1}$, $\|t - s\| = \|p_t - p_s\|/r$ so that $\|s - t\| > \varepsilon/r$ if and only if $\|p_t - p_s\| > \varepsilon$. Since $s$ is strongly relevant, for such a $t \in S^{n-1} \setminus N_{\varepsilon/r}(s)$ there exist $A, A' \in \mathcal{K}(\Delta)$ with $A \sim A'$ such that $h_A(t) = h_{A'}(t)$. But this is the same as saying $\max_{q \in A} \psi_{p_t}(q) = \max_{q \in A'} \psi_{p_t}(q)$ so that $p_s$ is critical to $\succ$. The converse is proved similarly.

This brings us to the following axiom and result.

**Axiom** (Finiteness). The set of points critical to $\succ$ is finite.

**IV.23 Theorem.** A continuous preference relation $\succ$ satisfies IR and Finiteness if and only if it has a weak EU representation with a finite state space.

**Proof.** Sufficiency is clear. To see necessity, assume that $\succ$ has a weak EU representation $V$ and a subjective state space $S$ that is not finite. Then the set of points critical to $\succ$ is also not finite, a contradiction.

It should be pointed out that our axiom for ensuring the finiteness is rather stringent, especially when compared to the axiom used by Dekel, Lipman and Rustichini (2006) to ensure the finiteness of the state space. The reason for this is that in the presentation above, we are working with continuous preferences that only satisfy IR. Dekel, Lipman and Rustichini (2006) on the other hand need Independence to prove the finiteness of the state space, and noted in DLR Independence is strictly stronger than IR.
References


