Ambiguity and partially-revealing rational expectations equilibria*

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Abstract  
This paper demonstrates that when the concept of Rational Expectations Equilibrium (REE) is expanded to allow for agents whose preferences display ambiguity aversion, standard results on REE no longer hold. In particular, REE can be partially revealing over a set of parameters with positive Lebesgue measure. This finding illustrates that models with ambiguity averse investors provide a relatively tractable framework through which partial information revelation may be studied in a general equilibrium setting. Constructive examples provide further insight into the properties of these equilibria.

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1 Introduction

Market prices convey information. In every market equilibrium, positive prices at least inform participants that there is some demand for all goods. In many markets, prices convey much more information, including market participants’ privately held information about relevant economic variables. The concept of a rational expectations equilibrium (REE) formalized in Radner (1979) was formulated to study the dissemination of privately held information through market prices. One of the startling results of the pursuant literature is that when REE exist and the market has no “noise” (see for example Grossman and Stiglitz (1980)) equilibrium market prices almost surely reveal all agents’ private information.

The sharpness of this result proved to limit the application of REE as an equilibrium concept. Generally, one tends to think of the amount of private information held by individuals as both diverse (i.e. varying in quality and the set of markets to which it applies) and diffuse (i.e. held by many different participants in the market). The idea that all of this information is, almost surely, fully conveyed in a finite-dimensional market statistic like the equilibrium price proved too simplistic for many applications.

The aim of this paper is to demonstrate that the full-revelation property of the REE concept need not hold when the class of conceivable preferences for market participants is expanded beyond the standard subjective expected utility model of Savage (1954). Specifically, we demonstrate that when at least one investor has preferences that display ambiguity aversion (see Gilboa and Schmeidler (1989)) then the concept of REE (suitably generalized to allow for the generalization of preferences) permits robust equilibria that are partially revealing. We show explicitly that there exists a parameter set with positive measure in which partially-revealing REE exist.
Generally speaking, when applied to markets where investors may display ambiguity aversion, the REE concept permits a richer set of information revelation phenomena than the analogous model populated by only expected utility maximizers. These results also serve to highlight important additional differences between investors with preferences that display ambiguity aversion and investors who have subjective expected utility preferences.

It has been noted (e.g. Zambrano (2005)) in Walrasian equilibrium models, subjective expected utility places no restriction on observable market data. As such, in a Walrasian equilibrium one is never able to determine conclusively that a participant has preferences that are anything other than subjective expected utility. While this observational difficulty remains when the stricter concept of REE is employed (since only prices and not the price function is ever observed) this work highlights the fact that the mechanism by which ambiguity averse investors convey privately held information to the market differs from the way that subjective expected utility maximizing agents convey privately held information.

It is interesting to contrast the present examples with the examples of partially-revealing REE given in Grossman and Stiglitz (1980) and the expansive literature that followed. While that literature also gives examples of REE that are partially revealing, these seem to depend on the particular distribution of payoffs (normal) as well as the utility function of investors (constant absolute risk aversion). These models also usually make use of noise traders, or traders whose preferences are not explicitly modelled.

In the examples presented here however, no assumptions beyond those necessary to ensure the existence of equilibrium are placed on the von Neumann-Morganstern utility functions of these investors. Additionally, the preferences of all investors are specified ex ante and none of these investors is inherently “irra-
tional" in the sense that each has well specified preferences and makes decisions that are optimal given these preferences and the constraints faced.

The remainder of the paper proceeds as follows. Section 2 outlines the model and the definition of an REE while section 3 contains explicit examples of the nature of REE in a market with ambiguity aversion. Section 4 contains results for the more general case and section 5 concludes. The appendix provides analytical results on equilibria with ambiguity averse investors that are necessary but tangential to the discussion in the paper.

2 The model

The market is populated by a finite set $\mathcal{N} = \{1, \ldots, n, \ldots, N\}$ of investors who live for three periods labelled 0, 1 and 2. In period 2, one of a finite set $\Omega$ of possible states of nature is realized and investors in the economy consume. A typical element of $\Omega$ is denoted $\omega$.

In periods 0 and 1 some information is revealed to each investor. In period 0, investor $n \in \mathcal{N}$ receives a private signal $s^n$ that comes from a finite set $\mathcal{S} = \{1, \ldots, S\}$. Let the set of all possible collections of private information that might be available to the market be labelled $\Sigma = \mathcal{S}^N$ with representative element $\sigma$ and let $\mathcal{F}$ be the discrete $\sigma$-algebra over $\Sigma$. Investors’ private signals convey information about the relative likelihood of the elements in $\Omega$ being realized in period 2.

Each investor has an endowment $e^n \in \mathbb{R}^{[\Omega]}_+$ of a single consumption good and must choose a consumption allocation in $X = \mathbb{R}^{[\Omega]}_+$. This allocation is financed by trading contingent claims on consumption over $\Omega$ in the market during period 1.

The market opens in period 1 and in equilibrium, each investor derives information about the private signals of other investors by observing the prices of
the contingent claims that are traded in the market.

Let $P \subset \mathbb{R}^{[\Omega]}$ be the space of possible prices over contingent claims that can be purchased at time 1. The conditions imposed on preferences and endowments ensure that $P$ may be normalized so that it is equal to $\Delta^{[\Omega]}$ (the $|\Omega| - 1$-dimensional simplex). This normalization will be assumed throughout the paper but to avoid confusion, the notation $P$ will be used to refer to the price space and $\Delta^{[\Omega]}$ will refer to the space of probability distributions.

2.1 Preferences and beliefs

To isolate the role of ambiguity aversion in the partial-revelation properties of the proposed equilibria, we consider economies where all investors save one have preferences that can be represented by an expected utility functional.\(^1\) The first investor will differ from the others in that she will have preferences that display ambiguity aversion. Let $C(\Delta^{[\Omega]})$ be the set of non-empty, convex, closed subsets of $\Delta^{[\Omega]}$.

Let $\gamma : \Sigma \rightarrow C(\Delta^{[\Omega]})$ be a mapping that for each joint signal $\sigma$ assigns a set of probability distributions that is to be interpreted as the information that the signal $\sigma$ conveys to investor 1. The set $\gamma(\sigma)$ is the collection of probability distributions that the ambiguity averse investor believes may govern the resolution of uncertainty over $\Omega$ when she knows that the joint signal is $\sigma$. Let $\Gamma$ be the space of all such correspondences. These beliefs are only defined for events where investor 1 knows the exact joint signal. The equilibria that are discussed in this paper all have this property.

Investors 2, \ldots, $N$ have beliefs over the space $\Sigma \times \Omega$. These can be characterized for any investor $n$ by functions of the form $\pi^n_\sigma : \Sigma \rightarrow \Delta^{[\Omega]}$, that for each joint signal $\sigma$ assign a conditional probability distribution over $\Omega$ and

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\(^1\)This assumption allows for an easier exposition, and helps to get around some existence problems in REE that will be elaborated in what follows.
\[\pi^n_m : \Sigma \rightarrow [0, 1],\] a marginal distribution over the signal space \(\Sigma\). Denote the space of such beliefs \(\Pi\).

By an application of Bayes’ rule, the functions \(\pi^n_m\) and \(\pi^n_c\) can be used to uniquely define a conditional probability distribution \(\pi^n : \mathcal{F} \rightarrow \Delta^{[2]}\) over \(\Omega\) for any set of information \(f \in \mathcal{F}\). For example, suppose that investor \(n\) knows only that the joint signal is either \(\sigma'\) or \(\sigma''\). Then the conditional distribution \(\pi^n(\{\sigma', \sigma''\})\) describing his beliefs over \(\Omega\) is defined by

\[
\pi^n(\omega|\{\sigma', \sigma''\}) = \frac{\pi^n_c(\omega|\sigma')\pi^n_m(\sigma') + \pi^n_c(\omega|\sigma'')\pi^n_m(\sigma'')}{\pi^n_m(\sigma') + \pi^n_m(\sigma'')}.
\]

The probability measure \(\pi^n\) is defined for all sets \(f \in \mathcal{F}\) analogously.

Investors utilize information both from their private signal and from prices. Abusing notation slightly, we let \(f(s^n) \in \mathcal{F}\) be the set of joint signals that have \(\sigma(n) = s^n\), where \(\sigma(n)\) is the \(n\)th component of \(\sigma\). Each investor \(n\) knows by her private signal that \(\sigma \in f(s^n)\).

A price function \(\phi : \Sigma \rightarrow P\) defines a price for every joint signal \(\sigma\). In equilibrium, information is gathered from prices by using the equilibrium price function \(\phi\), so if the observed price is \(p\), then investor \(n\) knows that \(\sigma \in \phi^{-1}(p)\).

Combining the information derived from her personal signal and that inferred from prices, investor \(n\) in equilibrium knows that \(\sigma \in f(s^n) \cap \phi^{-1}(p)\). The next assumption describes the preferences of the investors in the economy.

**Assumption 1.** Given the joint signal \(\sigma \in \Sigma\),

1. Investor 1 has preferences that can be represented by the utility function

\[
U^1(x^1; \sigma) = \min_{\pi \in \gamma(\sigma)} E_{\pi} u^1(x^1),
\]

with \(\gamma(\sigma) \cap \text{bd} \Delta^{[2]} = \emptyset\) for all \(x^1 \in X\) and all \(\sigma \in \Sigma\).
2. For any information set \( f \in \mathcal{F} \), investor \( n \in \{2, \ldots, N\} \) has preferences given by

\[
U^n(x^n; f) = E_{\pi(f)} u^n(x^n),
\]

for all \( x^n \in X \) with \( \pi_m(f) > 0 \) and \( \pi_c(f) \notin \text{bd} \Delta^{[\Omega]} \) for all \( f \).

3. For all \( n \), the von Neumann-Morgenstern (vN-M) utility function \( u^n(\cdot) \) satisfies \( u^n \in C^2, u^n'(\cdot) > 0, u^{n''}(\cdot) < 0 \), and \( \lim_{c \to 0} u^n'(c) = \infty \).

It is useful to note that the definition of investor 1’s preferences includes as a special case the situation in which 1 is an expected utility maximizer.

From the characterization of preferences we now move on to the mechanics of the market.

For any price vector \( p \in P \), the set of feasible consumption bundles (or budget set) of investor \( n \) is the set

\[
F(e^n, p) = \{ x \in X : p(e^n - x^n) \geq 0 \}.
\]

**Definition 1.** A rational expectations equilibrium (REE) is a price function \( \phi : \Sigma \to P \) and an allocation \( x : \Sigma \to \mathbb{R}^N_{+} \) that for all \( n \) satisfies

1. \( x^n(\sigma) \in \arg \max U^n(x^n(\sigma); f(\sigma(n)) \cap \phi^{-1}(\phi(\sigma))) \) s.t. \( x^n \in F(e^n, \phi(\sigma)) \)

2. \( \sum_{n \in N} (e^n(\sigma) - x^n(\sigma)) = 0 \) for all \( \sigma \in \Sigma \)

Having fully described the model and defined equilibrium, the next section gives an example of the paper’s main results.
3 Examples

3.1 Existence and Robustness

Consider a pure exchange economy populated by two investors who will be denoted by $a$ (for ambiguity averse) and $e$ (for expected utility). There are two possible market environments (labelled 1 and 2) that can be realized in period 2. Investors may receive one of two possible signals that convey information on the environment that will prevail in this market. The signals are labelled $s_1$ and $s_2$.

The first investor will be denoted investor $a$. Her preferences, given the information that is available at time 1, can be described by a utility function of the form

$$U^a = \min_{\pi \in \gamma} E_\pi[u^a(x)] = \pi u^a(x(1)) + (1 - \pi)u^a(x(2))$$  \hspace{1cm} (3.1)

where $\gamma$ is a compact, convex subset of the unit interval denoting probability measures over the set of environments.

The second investor will be denoted investor $e$. His preferences, given the information available at time 1, can be represented by the utility function

$$U^e = E[u^e(x)] = \pi u^e(x(1)) + (1 - \pi)u^e(x(2))$$  \hspace{1cm} (3.2)

where $\pi \in (0, 1)$ denotes the probability of state 1.

Each investor has the endowment $\bar{e} \in \mathbb{R}_{++}$ in state 1 and state 2. Furthermore, assume that $u^e$ and $u^a$ are strictly increasing, strictly concave and that $\lim_{x \to 0} u^i(x) = \infty$ for $i \in \{e, a\}$.

For this example the set of possible signal profiles is

$$\Sigma = \{\sigma = (s^e, s^a) \in \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}\}.$$  \hspace{1cm} (3.3)
Let $\pi(s^e)$ (respectively $\gamma(s^a)$) represent investor $e$’s (a’s) beliefs about state one being realized given his (her) signal $s^e$ ($s^a$). Let $\pi(s_k)$ represent $e$’s distribution over states given that he knows only that $\sigma \in \{(s_k, s_1), (s_k, s_2)\}$. Suppose that the investors update their beliefs given their individual signals in such a way that the following condition holds

$$
\pi(s_1) \in \gamma(s_1, s_1) \cap \gamma(s_1, s_2)
$$

$$
\pi(s_2) \in \gamma(s_2, s_1) \cap \gamma(s_2, s_2)
$$

Then we have the following.

**Claim 1.** A partially revealing REE exists for the given economy.

**Proof.** Consider the allocation $x^e(\sigma) = x^a(\sigma) = \bar{e}$ for all $\sigma \in \Sigma$ and the price function satisfying $\phi((s_1, s_k)) = \pi(s_1)$ and $\phi((s_2, s_k)) = \pi(s_2)$ for $k \in \{1, 2\}$. First it is necessary to check that given the individual signals received by each investor the given allocation and prices are an Arrow-Debreu equilibrium. Then we verify that it is still an equilibrium given the information available in the price function.

First note that the allocation is market clearing. We now show that such an allocation is optimal given market prices. To do so, we investigate the necessary and sufficient conditions for an optimum for each investor. These conditions are demonstrated in the appendix for ambiguity averse investors.

Now, consider the first order conditions for investor $e$. Given the assumption of strict concavity these are necessary and sufficient to ensure that an allocation is optimal for investor $e$ under the given price. Given that investor $e$ has received
signal $s_k$, his first order conditions\(^2\) evaluated at the proposed equilibrium are

$$
\lambda^e \begin{pmatrix} \pi(s_k) \\ 1 - \pi(s_k) \end{pmatrix} = \begin{pmatrix} \pi(s_k)u_e(\bar{e}) \\ (1 - \pi(s_k))u_e(\bar{e}) \end{pmatrix}
$$

$$
\pi(s_k)\bar{e} + (1 - \pi(s_k))\bar{e} \leq \bar{e}
$$

These conditions are satisfied for the proposed equilibrium values given any signal $s_k$.

Before stating the related conditions for investor $a$, a bit of notation is required. Throughout the paper, if $x \in \mathbb{R}^m$ and $A \subseteq \mathbb{R}^m$ then the notation $x \cdot A$ is used to denote the set of vectors $B \subseteq \mathbb{R}^m$ given by $B = \{b : b = [x_1a_1 \cdots x_m a_m] \text{ for all } a \in A\}$. The first order conditions under the proposed equilibrium for investor $a$ (see Appendix ??) given that she has received the signal $s_l$ and investor $e$ has received the signal $s_k$ are

$$
\lambda^a \begin{pmatrix} \pi(s_k) \\ 1 - \pi(s_k) \end{pmatrix} \in u^{a'}(\bar{e}) \cdot \gamma(s_k, s_l)
$$

$$
\pi(s_k)\bar{e} + (1 - \pi(s_k))\bar{e} \leq \bar{e}
$$

Condition (3.6) is satisfied by letting $\lambda^a = u^{a'}(c)$ and then noting that (3.4) implies that $(\pi(s_k), (1 - \pi(s_k))) \in \gamma(s_l, s_k)$ for each $k$ and $l$. The resource constraint condition also holds.

Thus, if agents do not condition their behavior on market prices then this is an equilibrium. What remains is to verify that no additional information is available from market prices.

To see this, consider investor $e$. Knowing the market mechanism, he knows that investor $a$’s beliefs are such that regardless of the signals that either of them

\(^2\)Two additional “complementary slackness” conditions will also hold but given the assumption that $\pi(\sigma) \in (0, 1)$ for all $\sigma \in \Sigma$, the multipliers for these conditions will always be zero.
receive, \( a \) will fully-insure. Furthermore, he knows that she will do so at the equilibrium prices that clear markets. Thus, prices differ across the signals that investor \( e \) receives, but for any signal \( s_e \), prices do not differ across the signal that investor \( a \) receives and so investor \( e \) can garner no further information from market prices.

Now consider investor \( a \). She knows her own signal, and can also determine investor \( e \)'s signal from market prices. However, assumption (3.4) implies that even with knowledge of \( e \)'s signal, investor \( a \) still chooses to fully-insure so the given allocation is still an equilibrium when the information content of prices is considered.

The next result shows that in the space of updated beliefs, the set over which such a partially-revealing REE exists has positive Lebesgue measure. This result is the direct analog of the proposition in Radner (1979)[page 668]. First, we note that an updated belief for investor \( a \) given the signal \( \sigma \) can be characterized by two points (the lower and upper bound of probabilities of state 1 occurring given the joint signal \( \sigma \)). Thus, the set of possible updates that \( a \) may have is isomorphic to a subset of \([0, 1]^8\) (two points for each of the four possible joint signals). Label this set \( \Theta^a \). Likewise, the set of possible updates that \( e \) may have given the joint signals \( \sigma \) is isomorphic to a point in \([0, 1]^4\) (one point—representing the probability of state 1 occurring—for each of the four possible joint signals). This set is labelled \( \Theta^e \). To state the result, let \( \Theta = \Theta^e \times \Theta^a \) and endow this space with the usual subspace topology inherited from \( \mathbb{R}^{12} \).

**Claim 2.** The set \( \Theta \) for which assumption 3.4 holds has positive Lebesgue measure.

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\(^3\)The space of beliefs is characterized by a subset of this space because of the condition that given a particular joint signal \( \sigma \), the upper bound of the probability of state 1 occurring must be at least as great as the lower bound of the probability of state 1 occurring.
Proof. Let $\pi \in \Theta^e$ have the property that

$$\pi \notin \text{bd } \Theta^e$$  \hspace{1cm} (3.7)

and define $0 < \epsilon_\pi < \min_{x \in \text{bd } \Theta^e} ||x - \pi||$. Consider the set $\Theta^e \subseteq \Theta^c$ defined by

$$\Theta^e = \{ x \in \Theta^e : ||x - \pi|| < \epsilon_\pi \}$$ \hspace{1cm} (3.8)

The set $\Theta^e$ is open in $\Theta^e$.

Now consider the set $\Theta^a$. For convenience, let us adopt the convention that the coordinates $\{1, 2, 3, 4\}$ will refer to the lower bound of the probabilities of state 1 occurring given the four possible joint signals. Similarly, the coordinates $\{5, 6, 7, 8\}$ will refer to the upper bound of the probability of state 1 occurring given the four possible joint signals.

What remains to be shown is that there is a corresponding open set $\Theta^a \in \Theta^a$ that has positive measure (in $\Theta^a$) for which condition 3.4 holds. First we define the following sets.

$$\Theta^a = \Theta^a \cap \Theta^a^*$$

$$\Theta^a^* = \{ x \in \Theta^a : x(i) > \max_{y \in \Theta^e} y(i - 4) \text{ for } i \in \{5, 6, 7, 8\} \}$$  \hspace{1cm} (3.9)

Now define $\Theta^a = \Theta^a_a \cap \Theta^a^*$. By condition (3.7), $\Theta^a$ is non-empty and open. It can be seen that for all $y \in \Theta^e$, each set of beliefs represented by a point in $\Theta^a$ contains $y$ and as such the product $\Theta^a \times \Theta^e$ is non-empty, and hence (in conjunction with its openness) has positive Lebesgue measure in the space $\Theta$.

Then $\Theta^a \times \Theta^e$ will be the product of two open sets and hence open under the product topology. Since the product topology and the metric topology are equivalent on this space, $\Theta^a \times \Theta^e$ is open under the metric topology. $\blacksquare$
This example has shown that with ambiguity averse investors, standard REE may be partially revealing. It further shows explicitly the set of economies (paramaterized by beliefs) for which this partially-revealing equilibrium exists.

3.2 Comparative Statics

The next example illustrates some of the qualitative properties of these equilibria through comparative statics. Some simplifying assumptions are made so that salient points can be highlighted, but the qualitative properties of the example hold in a broader setting.

Suppose that there are two investors, an expected utility investor labelled $e$ and an ambiguity averse investors labelled $a$. There are two states of the world, labelled $\{1, 2\}$. Aggregate consumption in state 1 is given by $x + \epsilon$ and aggregate consumption in state 2 is $x$. The endowments of the investors are: $\bar{e}_e = (x, x)$ and $\bar{e}_a = (\epsilon, 0)$. Only investor $a$ receives a signal $s \in \{s_1, s_2\}$ that conveys information about the relative likelihood of each state occurring. The vN-M utility function for both investors is $\log(\cdot)$.

Investor $e$’s beliefs are denoted $\pi$. Without knowing $a$’s signal, investor $e$ believes that the probability of state 1 occurring is $\pi = \frac{1}{2}$. On the other hand, $e$’s conditional beliefs are given by $\pi(1|s_1) = \pi_1$ and $\pi(1|s_2) = \pi_2$. These beliefs are interpreted as the “correct” probabilities of state 1 occurring given the signal $s$. We assume that

$$\frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 = \frac{1}{2} = \pi, \text{ with } \pi_1 \leq \pi_2. \quad (3.10)$$

Investor $a$’s signal conveys (to him) information with some ambiguity. When he receives signal $s_k$ he believes that the true probability of state 1 occurring is in the set $[\pi_k - \alpha, \pi_k + \alpha]$.

The parameter $\alpha \geq 0$ can be thought of as describing the amount of ambi-
guity inherent in investor \( a \)'s preferences. In the limit as \( \alpha \) goes to zero, investor \( a \) becomes an expected utility maximizing investor. This comparative statics exercise will analyze the change in equilibrium price that arises from changes in \( \alpha \).

We will normalize the price of consumption in state 2 to be 1 and call the price of state 1 consumption \( p \).

Given beliefs \( \pi \), it can be seen that \( e \)'s demand for consumption in state 1 is

\[
x^e(p) = \frac{\pi(px + x)}{p}.
\]

Investor \( a \)'s demand for consumption in state 1 when the signal \( s_k \) has been received is

\[
x^a(p) = \begin{cases} 
\frac{(\pi_k + \alpha)p}{p} & p \geq \frac{\pi_k + \alpha}{1 - (\pi_k + \alpha)} \\
\frac{\pi_k - \alpha}{1 - (\pi_k - \alpha)} & \frac{\pi_k - \alpha}{1 - (\pi_k - \alpha)} \leq p \leq \frac{\pi_k + \alpha}{1 - (\pi_k + \alpha)} \\
\frac{(\pi_k - \alpha)p}{p} & p \leq \frac{\pi_k - \alpha}{1 - (\pi_k - \alpha)}
\end{cases}
\]

From these demands it can be shown that the equilibrium price when \( e \) knows only that \( s \in \{s_1, s_2\} \) and \( a \) fully insure is

\[
p_{pr} = \frac{\sqrt{x^2 + \epsilon^2} - \epsilon}{x}.
\]

Therefore, if \( \alpha \) is large enough that

\[
\frac{\sqrt{x^2 + \epsilon^2} - \epsilon}{x} \in \left[ \frac{\pi_1 - \alpha}{1 - (\pi_1 - \alpha)}, \frac{\pi_1 + \alpha}{1 - (\pi_1 + \alpha)} \right] \quad \text{and}
\]

\[
\frac{\sqrt{x^2 + \epsilon^2} - \epsilon}{x} \in \left[ \frac{\pi_2 - \alpha}{1 - (\pi_2 - \alpha)}, \frac{\pi_2 + \alpha}{1 - (\pi_2 + \alpha)} \right]
\]

\[\text{The fact that this price does not depend on } \pi (e's beliefs when he doesn't know a's signal) \text{ is an artifact of the assumption that } \pi = \frac{1}{2}. \text{ The qualitative properties of this equilibrium hold when } \pi \neq \frac{1}{2}, \text{ but the expressions become much more cumbersome.}\]
then there will exist a partially-revealing equilibrium with \( \phi(s_1) = \phi(s_2) = p_{pr} \). \(^5\)

However, as \( \alpha \) decreases toward zero, the condition (3.14) will fail to hold. At the point at which either one of the two inclusions in (3.14) do not hold, there is only a fully-revealing REE and it has the form

\[
\begin{align*}
p_{fr}(s_1) &= \frac{\pi_1 x}{(1 - \pi_1)x + (1 - (\pi_1 - \alpha)) \epsilon} \\
p_{fr}(s_2) &= \frac{\pi_2 x}{(1 - \pi_2)x + (1 - (\pi_2 - \alpha)) \epsilon}.
\end{align*}
\] (3.15)

Of note is that since in general \( p_{fr}(s_1) \neq p_{fr}(s_2) \neq p_{pr} \), as the amount of ambiguity (parametrized by \( \alpha \)) crosses the threshold at which either of the inclusions in (3.14) doesn’t hold (denoted by \( \bar{\alpha} \) in the diagram) then there is a discontinuous jump in the equilibrium prices away from the partially-revealing price \( p_{pr} \) to the two fully-revealing prices \( p_{fr}(s_1) \) and \( p_{fr}(s_2) \). This is demonstrated in the figure.

As noted earlier, there always exists a fully-revealing REE in this example and this REE’s price correspondence is continuous in \( \alpha \).

Having provided some intuition on the nature of equilibrium to be demonstrated, the next section gives some general results.

### 4 Partially-revealing REE

In this section the general existence of partially-revealing REE is demonstrated. It is then shown that the REE constructed in this way are robust to perturbations in the beliefs of the investors that are described by a point \((\gamma, \pi) \in \Gamma \times \Pi^{N-1}\).

To begin, a few preliminary results are needed. In what follows, let \( f \) be

\(^5\)One may also construct a fully-revealing equilibrium by assuming that a’s signal is common knowledge. The dotted lines in the diagram show the signal-dependent prices corresponding to this equilibrium.
Figure 1: The solid lines give the equilibrium prices for a partially-revealing REE as a function of the parameter $\alpha$. 
a set of signals representing the information available to some investor and let \( \sigma \in f \). Given assumption 2 any conditional belief \( \gamma(\sigma) \) can be represented by its extreme points. This set of extreme points is a point in \( \Delta^{[\Omega]} L \). So, given conditional beliefs \( (\gamma(\sigma), \pi(f)) \in \Delta^{[\Omega]} L \times \Delta^{[\Omega]} N \), let \( W(\gamma(\sigma), \pi(f)) \) be the set of Arrow-Debreu equilibria when all investors hold these updated beliefs.

**Proposition 1.** Let \( \{\pi^2(f), \ldots, \pi^N(f)\} \) be conditional beliefs for investors \( 2, \ldots, N \) when each investor knows that the joint signal \( \sigma \) is in \( f \). There exists a probability distribution \( \bar{\pi} \in \Delta^{[\Omega]} \) such that if investor 1’s beliefs satisfy \( \bar{\pi} \in \gamma(\sigma) \) then all equilibria in \( W(\gamma(\sigma), \pi(f)) \) satisfy \( x^1(\omega) = x^1(\omega') \) for all distinct \( \omega, \omega' \in \Omega \).

**Proof.** Consider a hypothetical economy in which investor 1 has Leontief preferences of the form \( U^1(x) = \min_{\omega \in \Omega} x(\omega) \) and all other investors have expected utility preferences with beliefs given by \( \{\pi^2(f), \ldots, \pi^N(f)\} \) respectively.

For any strictly positive price vector \( p \in P \), investor 1’s excess demand is

\[
x^1 = p e^1 - e^1
\]

where \( e^1 \) is the \( |\Omega| \)-dimensional vector of 1’s.

An Arrow-Debreu equilibrium exists for this economy. The proof of existence of the Arrow-Debreu equilibrium is essentially identical to that presented in Mas-Colell, Whinston, and Green (1995)[Proposition 17.C.1]. Let \( Z(p) \) be the excess demand correspondence for this economy. It must be shown that

1. \( Z(p) \) continuous in \( p \).
2. \( Z(p) \) is homogeneous of degree zero in \( p \).
3. \( pZ(p) = 0 \) for all \( p \).
4. There is a \( b > 0 \) such that \( Z_\omega > -b \) for all \( \omega \in \Omega \) and all \( p \).
5. If $p^k$ is a sequence of prices converging to $p$ where $p(\omega) = 0$ for some $\omega$ then $\max\{Z(1; p^k), \ldots, Z(\Omega; p^k)\} \to \infty$ as $p^k \to p$.

The first four items follow from the fact that each of the preferences for investors in this economy are continuous, strictly convex and locally non-satiated. The final property holds because $x^n(p^k) \to \infty$ for all investors but 1 and 1's demand $x^1(\omega)$ is bounded below by 0, so excess demand is bounded below by $-\max_{\omega \in \Omega} e^1(\omega)$. Thus, there is an equilibrium $(x, p)$ for which $x^1$ satisfies (4.1).

From the equilibrium in this economy one can derive conditions on the beliefs of the ambiguity averse investor that ensure that her demand is equal to (4.1) around the equilibrium point. From the first order conditions given in (A.38) it can be seen that if for the equilibrium price $p$, beliefs satisfy $p \in \gamma(\sigma)$ then the full-insurance allocation $x^1$ will be optimal for investor 1. Therefore, any beliefs $\gamma$ that satisfy this condition will guarantee that investor 1 fully insures in this economy. 

The previous result has demonstrated that for any set of updated beliefs for the expected utility maximizers in the economy, there is always a set of beliefs for investor 1 that have the property that if 1 holds these beliefs then she will hold a riskless portfolio in equilibrium.

Since expected utility maximizers are a special case of the investors whose preferences are studied here, the following result is immediate from the rational expectations literature.\(^6\)

**Lemma 4.1.** For some $\gamma, \pi \in \Gamma \times \Pi^{N-1}$ there exists an REE that is fully-revealing.

The next task is to demonstrate that from any fully-revealing REE one can construct a partially-revealing REE by allowing a single investor’s preferences to demonstrate ambiguity aversion.

\(^6\)See Allen (1984) for one proof.
Theorem 1. Let \((x, \phi)\) be a fully revealing REE under the beliefs \(\{\gamma, \pi^2, \ldots, \pi^N\}\). Then there exists a belief \(\gamma'\) for which there is a partially-revealing REE under beliefs \(\{\gamma', \pi^2, \ldots, \pi^N\}\).

Proof. The proof is constructive. Let \(\sigma', \sigma'' \in \Sigma\) be two joint signals that differ only in 1’s private signal. As before, denote by \((\bar{x}, \bar{\phi})\) our candidate partially-revealing REE. For all \(\sigma / \in \{\sigma', \sigma''\}\), let \((\bar{x}(\sigma), \bar{\phi}(\sigma)) = (x(\sigma), \phi(\sigma))\). Denote by
\[
\{\pi^2(\{\sigma', \sigma''\}), \ldots, \pi^N(\{\sigma', \sigma''\})\}
\] (4.2)
the updated beliefs for investors 2, \ldots, \(N\) when they know only that \(\sigma / \in \{\sigma', \sigma''\}\).

In this equilibrium investors 2, \ldots, \(N\) are unable to precisely distinguish the signal investor 1 has received. Let \((\bar{x}(\sigma'), \bar{\phi}(\sigma'))\) \(\in W(\gamma'(\sigma'), \pi^2(\{\sigma', \sigma''\}), \ldots, \pi^N(\{\sigma', \sigma''\}))\) where \(\gamma'\) satisfies
\[
\bar{\phi}(\sigma') \in \gamma'(\sigma').
\] (4.3)
This implies that \(\bar{x}^1(\omega; \sigma') = \bar{x}^1(\omega'; \sigma')\) for all \(\omega, \omega' \in \Omega\). By Proposition 1 such beliefs exist. Define \((\bar{x}(\sigma''), \bar{\phi}(\sigma''))\) = \((\bar{x}(\sigma'), \bar{\phi}(\sigma'))\). The allocation and price functions \((\bar{x}, \bar{\phi})\) are a partially-revealing REE if \(\gamma'\) satisfies
\[
\bar{\phi}(\sigma') = \bar{\phi}(\sigma'') \in \gamma'(\sigma') \cap \gamma'(\sigma'').
\] (4.4)

Note that prices are the same across joint signals \(\sigma'\) and \(\sigma''\). Since these prices constitute an Arrow-Debreu equilibrium given the beliefs of each trader 2, \ldots, \(N\) when they know only that \(\sigma / \in \{\sigma', \sigma''\}\), their behavior is optimal under the price \(\bar{\phi}(\sigma') = \bar{\phi}(\sigma'')\). Since by construction this is an Arrow-Debreu equilibrium for each joint signal, this is an REE and because \(\bar{\phi}(\sigma') = \bar{\phi}(\sigma'')\), this is a partially-revealing REE. ■

Nothing in the previous theorem requires that investors 2, \ldots, \(N\) be expected utility maximizers. The only constraint is that there exist a fully-revealing
REE in the economy. Thus, if the preferences of all investors were generalized to display ambiguity aversion and it were known that a fully-revealing REE existed, then the method of constructing partially-revealing REE from fully-revealing REE used to prove Theorem 1 would apply equally well.

The inclusion (4.4) has economic content. For any price vector \( \phi(\sigma) \in P \), if an investor’s beliefs \( \gamma \) satisfy \( \phi(\sigma) \in \gamma(\sigma) \) then the investor believes that it is possible that the price vector represents the true probability distribution over states in \( \Omega \). That is, the investor believes that the market may have assigned relative prices to assets in these states that are exactly the likelihood of these assets paying off. Since investor 1 is ambiguity averse she has a desire to fully-insure as long as prices do not differ from what she thinks might be the true probabilities over states. The condition (4.4) implies that the information that she receives in her own private signal under both \( \sigma' \) and \( \sigma'' \) does not give her any reason to bet against the odds that the market presents her. This happens even though it can be the case that the set of distributions that she believes to be possible given \( \sigma' \) and the set she believes to be possible under \( \sigma'' \) differ greatly. The only requirement is that both \( \gamma(\sigma) \) and \( \gamma(\sigma') \) have the price vector \( \phi(\sigma) \) as a common, interior element.

While it seems that (4.4) can be interpreted as a restriction on the informativeness of investor 1’s signal this is somewhat misleading. In fact, (4.4) says nothing about the absolute informativeness of 1’s portion of the joint signals \( \sigma' \) and \( \sigma'' \). It is entirely consistent with the previous result to assume that if some other investor knew 1’s private signal he would hold beliefs that are very different than he does when he cannot determine the signal. That is, generically in \( \Pi \), \( \pi^n(\sigma') \neq \pi^n(\sigma'') \neq \pi^n(\{\sigma', \sigma''\}) \), meaning that beliefs under these three different states of information are likely to vary.

It can be seen that the existence of these partially-revealing REE does not
fall out of the realm of the results established by Radner (1979). In particular, investor 1 could be an expected utility maximizing investor who happens to believe that $\sigma'$ and $\sigma''$ convey the same information to her. Then her conditional beliefs over $\Omega$ will be the same under these two signals. However, it is apparent that in a world in which all investors maximize expected utility this phenomenon is not robust. That is, perturbing 1’s beliefs slightly must then induce a fully-revealing REE. In a world in which investors are ambiguity averse however, perturbation of beliefs does not necessarily remove the possibility of a partially-revealing REE.

Before establishing this claim, we restrict beliefs for the ambiguity averse investor further to aid in the robustness result of the paper. Prior to stating the assumption, the following definition is needed.

**Definition 2.** The set $A \in C(\Delta^{[\Omega]})$ is said to be finitely-generated if it can be represented as the convex hull of a finite number of points (or distributions) in $\Delta^{[\Omega]}$.

**Assumption 2.** For any joint signal $\sigma$, the conditional beliefs $\gamma(\sigma)$ of investor 1 are finitely-generated by at most $L < \infty$ extreme points.

This assumption is made to provide a finite-dimensional space for beliefs so that standard measure-theoretic tools may be used to discuss robustness. Existence results (principally Theorem 1) are obtained without this assumption and it is possible that there exist suitable extensions for Theorem 2 on robustness to more general spaces.

The set of all functions $\gamma$ that map from joint signals into finitely-generated, compact, convex sets of probability distributions can be represented as a subset of a finite-dimensional Euclidean space. Accordingly, when speaking of a set of beliefs that have positive measure, instead of investigating the function space of which $\gamma$ is an element, we may investigate the isomorphic Euclidean space.
and impose the usual Lebesgue measure on this space. The construction of this space is left to the appendix. Assumption 2 is used to give meaning to the following lemma, on which Theorem 2 is built.

**Lemma 4.2.** Suppose that for some \( p \in P \), \( p \in \text{int}[\gamma(\sigma) \cap \gamma(\sigma')] \) for some \( \gamma \in \Gamma \) and some \( \sigma, \sigma' \in \Sigma \) with \( \sigma \neq \sigma' \). Then there exists an open neighborhood \( B_{\gamma} \subset \Gamma \) of beliefs with positive Lebesgue measure in \( \Gamma \) and an open neighborhood \( B_{p} \subset P \) with positive Lebesgue measure for which \( B_{p} \in \gamma'(\sigma) \cap \gamma'(\sigma') \) for all \( \gamma' \in B_{\gamma} \).

**Proof.** This is a direct corollary of Lemma A.11 in the appendix. \( \blacksquare \)

We now show that partial revelation is a robust property when there is an ambiguity-averse investor present.

**Theorem 2.** For almost all endowments there exists a set of beliefs \( A \subseteq \Gamma \times \Pi^{N-1} \) with positive measure for which there is a partially-revealing REE.

**Proof.** Let \( (\gamma, \pi) \in \Gamma \times \Pi^{N-1} \) permit a partially-revealing REE. By Theorem 1 such a \( (\gamma, \pi) \) exists. The proof will show that there exists an \( \epsilon > 0 \) such that for all \( (\gamma', \pi') \) that satisfy \( ||(\gamma', \pi') - (\gamma, \pi)|| < \epsilon \), there is a partially-revealing REE.

From the proof of Theorem 1 it can be seen that a sufficient condition for a partially-revealing REE to exist is condition (4.4). It will be shown that such a condition holds locally around \( (\gamma, \pi) \) by showing that both sides of the inclusion are “continuous” in \( (\gamma, \pi) \) in a specific sense.

First it is demonstrated that the equilibrium price function \( \phi(\sigma') = \phi(\sigma'') \) is locally continuous around \( (\gamma, \pi) \). This is done using the generic determinacy results of Debreu (1970). After having established that, an application of the implicit function theorem provides that locally there exists a continuous function \( \hat{\phi}(\gamma, \pi) \) such that \( Z(\hat{\phi}(\gamma', \pi'); \gamma', \pi') = 0 \) for all \( (\gamma', \pi') \) in a neighborhood of \( (\gamma, \pi) \). Finally, it will be shown that due to the local continuity of \( \phi(\gamma, \cdot) \) and \( \gamma \), the conclusions of the theorem hold.
Consider the hypothetical economy constructed in Proposition 1 where investor 1 has Leontief preferences given some joint signal. Each investor 2, ..., N behaves according to a utility function that is $C^2$ and satisfies the requirements necessary for the differentiability of demand (see Mas-Colell, Whinston, and Green (1995)[Chapter 3, Appendix A]). Thus, $x^n(p; \pi^n)$ is differentiable in $p$ for each $n \in \{2, \ldots, N\}$. Application of the implicit function theorem to the first order conditions of these investors also shows that these demands are differentiable in $\pi^n$ (and hence differentiable in the larger space $\Pi$). Locally (around $(\gamma, \pi)$), demand for investor 1 is given by $x^1(p; \gamma) = p e^1_1$ which is also differentiable in both $p$ and $\gamma$ (trivially). Since demand is continuously differentiable around the equilibrium point for all $n$ the theorem in Debreu (1970) shows that the set of endowments at which this economy is not locally determinate has zero measure. Therefore, the economy with the given demand functions will be determinate almost surely (in endowments).

Since the economy is almost surely determinate, for almost all endowments

$$\det D_p Z(p; \gamma, \pi) \neq 0.$$  \hspace{2cm} (4.5)

The rest of the discussion will apply to these determinate economies.

Since $Z(p; \gamma, \pi)$ is locally continuously differentiable in $p$ and $(\gamma, \pi)$, the implicit function theorem then gives that there exists a continuous function $\hat{\phi}(\gamma, \pi)$ such that $Z(\hat{\phi}(\gamma', \pi'); \gamma', \pi') = 0$ for all $(\gamma', \pi')$ in some neighborhood $B$ of $(\gamma, \pi)$.

Now, by Lemma 4.2 we may choose neighborhoods $B_\phi$ around $\phi(\sigma)$ and $B_\gamma$ around $\gamma$ for which $B_\phi \in \text{int} \gamma'(\sigma') \cap \gamma'(\sigma'')$ for all $\gamma' \in B_\gamma$. Since $\phi(\gamma, \pi)$ is continuous in $\gamma, \pi$, there exists a neighborhood $B((\gamma, \pi), \epsilon_{\gamma, \pi})$ such that $\phi(\gamma', \pi') \in B_\phi$ for all $(\gamma', \pi') \in B((\gamma, \pi), \epsilon_{\gamma, \pi})$.

Combining the continuity of $\phi$ with Lemma 4.2, choose $\epsilon$ so that $B((\gamma, \pi), \epsilon) \cap$
$B_{\gamma} \subset B_{\gamma}$ and $B((\gamma, \pi), \epsilon) \cap B((\gamma, \pi), \epsilon_{\gamma, \pi}) \subset B((\gamma, \pi), \epsilon_{\gamma, \pi})$. This implies that for all $(\gamma', \pi') \in B((\gamma, \pi), \epsilon)$, $\phi(\gamma', \pi') \in \gamma'(\sigma') \cap \gamma'(\sigma'')$ which proves the result.

Since the set of beliefs $(\gamma, \pi)$ for which such a partially-revealing REE exists has positive Lebesgue measure, it is not an artifact of carefully chosen model parameters. In this sense, the inclusion of ambiguity averse investors in a traditional REE framework provides for fundamentally different equilibria than those robustly found when all investors have expected utility preferences.

5 Conclusion

This paper has shown that when the REE concept is extended to include investors whose preferences are not of the expected utility form that the information content of prices can vary drastically. It is clear that the implications of the presence of ambiguity averse agents in financial, insurance or other information-centric markets are yet to be fully understood. Models that explicitly allow for the presence of asymmetric information and ambiguity aversion in such markets may yield results that differ from those of more “traditional” models. In such markets it is not always (or even almost always) true that all privately held information is revealed through market prices, a result that stands in sharp contrast to Radner’s original work.

A Appendix

The goal of this appendix is to collect several results which are used in the central results of the paper. Some of the results may be found in Aubin (1979) and in cases where the proof is not expositionally important and can be found elsewhere I have cited an appropriate reference.
For what follows, let $X = \mathbb{R}_+^W$ and let $f : X \rightarrow R$ be a continuous and concave function. The set $\partial f(x_0)$ for $x_0 \in X$ is defined to be

$$\partial f(x_0) = \{ p \in R^W : f(x) - f(x_0) \leq p \cdot (x - x_0) \text{ for all } x \in X \} \quad (A.1)$$

is called the superdifferential of $f$ at $x_0$.

It can be seen that if $x_0$ maximizes the function $f$ over the set $X$ and $x_0 \in \text{int} X$ then

$$0 \in \partial f(x_0). \quad (A.2)$$

If $f$ is a strictly concave function and has a maximum in $\text{int} X$ then the condition (A.2) is both necessary and sufficient for $x_0$ to be a maximum.

The rest of the results are applied specifically to the utility functions of interest, i.e. they apply for functions of the form $U(x) = \min_{\pi \in \gamma} E_\pi u(x) = \min_{\pi \in \gamma} \sum_{\omega \in \Omega} \gamma(\omega) u(x(\omega))$.

**Lemma A.1.** If $u(\cdot)$ is strictly increasing and strictly concave then $U(x)$ is increasing in each of its arguments and strictly concave. Furthermore, if $\gamma \cap \text{bd} \Delta^{[\Omega]} = \emptyset$, then $U(x)$ is strictly increasing in each of its arguments.

**Proof.** Let $x, x'$ and $\omega'$ be such that for all $\omega \neq \omega'$, $x(\omega) = x'(\omega)$ but $x(\omega') < x'(\omega')$. Suppose that $U(x') < U(x)$ and let $\pi^* \in \arg \min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega) u(x'(\omega))$.

Then

$$\min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega) u(x'(\omega)) < \min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega) u(x(\omega)) \leq \sum_{\omega \in \Omega} \pi^*(\omega) u(x(\omega)).$$

This last inequality implies that

$$0 < \sum_{\omega \in \Omega} \pi^*(\omega) (u(x(\omega)) - u(x'(\omega))) \quad (A.3)$$
which with the definition of $\pi^*$ gives

$$0 < \pi^*(\omega')(u(x(\omega')) - u(x'(\omega'))) \quad (A.4)$$

which is a contradiction since $u(\cdot)$ is strictly increasing. Notice that if $\gamma \cap \text{bd } \Delta^{[\Omega]} = \emptyset$ and we start with the weak inequality $U(x') \leq U(x)$ then all of the strict inequalities become weak and the final inequality can be satisfied only for $\pi^*(\omega) = 0$ which is again a contradiction.

To show strict concavity, notice that for each $\pi \in \gamma$, the function $\sum_{\omega \in \Omega} \pi(\omega)u(x(\omega))$ is strictly concave. Define $\pi^*(x) \in \arg \min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega)u(x(\omega))$.

Let $x, x' \in X$. Then for $\alpha \in (0, 1)$ we have that

$$\alpha U(x) + (1 - \alpha) U(x') \quad (A.5)$$

$$= \alpha \min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega)u(x(\omega)) + (1 - \alpha) \min_{\pi \in \gamma} \sum_{\omega \in \Omega} \pi(\omega)u(x'(\omega)) \quad (A.6)$$

$$\leq \min_{\pi \in \gamma} \left\{ \alpha \sum_{\omega \in \Omega} \pi(\omega)u(x(\omega)) + (1 - \alpha) \sum_{\omega \in \Omega} \pi(\omega)u(x'(\omega)) \right\} \quad (A.7)$$

$$= \min_{\pi \in \gamma} \left\{ \sum_{\omega \in \Omega} \pi(\omega) [\alpha u(x(\omega)) + (1 - \alpha) u(x'(\omega))] \right\} \quad (A.8)$$

$$< \min_{\pi \in \gamma} \left\{ \sum_{\omega \in \Omega} \pi(\omega)u(\alpha x(\omega) + (1 - \alpha) x'(\omega)) \right\} \quad (A.9)$$

$$= U(\alpha x + (1 - \alpha)x'). \quad (A.10)$$

Equation (A.6) holds by definition. Equations (A.7) and (A.8) follow from the definition of the minimum and an algebraic manipulation. (A.9) follows from the strict concavity of $u(\cdot)$ and (A.10) again holds by definition.

For the remainder of the appendix we will assume that the conditions present in assumption 1 hold. The next lemma is a direct result of these assumptions.

**Lemma A.2.** Assume that for investor $n$, assumption 1 holds. Then for any
price vector \( p \in \text{int } X \), if

\[
x \in \arg \max_{x \in B(e^n, p)} U^n(x)
\]  \hfill (A.11)

then \( x \in \text{int } X \).

Proof. Assumption 1 ensures that the Inada condition at 0 holds for every possible state \( \omega \). \( \blacksquare \)

**Lemma A.3.** For the strictly concave function \( U \) and a compact, convex set \( Y \subset X \) that is defined by the system of linear inequalities \( g(x) \geq 0 \) there exists a Lagrange multiplier \( \lambda \in \mathbb{R}_+ \) such that

\[
\arg \max_{x \in X} L(x, \lambda) = \max_{x \in Y} U^n(x) + \lambda g(x) = \arg \max_{x \in Y} U^n(x).
\]  \hfill (A.12)

In particular, for each investor \( n \) there exists a Lagrange multiplier \( \lambda^n \) such that

\[
\max_{x \in B(e^n, p)} U^n(x) = \max_{x \in X} U^n(x) + \lambda^n p(e^n - x) = L^n(x, \lambda^n)
\]  \hfill (A.13)

Furthermore, if \( u^n(\cdot) \) is strictly concave then \( L(x, \lambda) \) is also strictly concave in \( x \).

Proof. \( U^n \) is continuous and concave and meets the necessary constraint qualification in (Aubin 1979)[5.3.1, Theorem 1] \( \blacksquare \)

Let \( L(x, \lambda) = U(x) + \lambda p(e - x) \). By the strict concavity of \( U(x) \) and the linearity of the constraint it can be seen that \( L(x, \lambda) \) is strictly concave in \( x \). As such a necessary condition for a solution to max \( L(x, \lambda) \) for a given \( \lambda \) is that

\[
0 \in \partial L(x, \lambda)
\]  \hfill (A.14)

To derive the first order conditions for an ambiguity averse investor the following lemma is needed.
Lemma A.4. Let $f : X \to \mathbb{R}$ be superdifferentiable and $g : X \to \mathbb{R}$ be affine (and thus Gateaux differentiable). If $L = f + g$ then

$$\partial L(x) = \partial f(x) + \partial g(x) = \partial f(x) + g'(x). \quad (A.15)$$

Proof. It is straightforward to show that $\partial f(x) + \partial g(x) \subseteq \partial L(x)$. What remains to be shown is that $\partial L(x) \subseteq \partial f(x) + \partial g(x)$.

Suppose that there exists $p \in \partial L(x)$ such that $p \notin \partial f(x) + \partial g(x)$. By assumption

$$f(y) + g(y) - f(x) - g(x) \leq p(y - x) \text{ for all } y \in X. \quad (A.16)$$

Since $g$ is Gateaux differentiable, $\partial g(x) = \{g'(x)\}$ (i.e. it is a single vector in $\mathbb{R}^W$ labelled $g'(x)$.) Without loss of generality one may define $p = g'(x) + p'$. Since $p' + g'(x) \notin \partial f(x) + \partial g(x) = \partial f(x) + g'(x)$ this implies that $p' \notin \partial f(x)$.

Therefore, there exists a $y^* \in X$ such that

$$f(y^*) - f(x) > p'(y^* - x). \quad (A.17)$$

Since $g(x)$ is affine,

$$g(y^*) - g(x) = g'(x)(y^* - x). \quad (A.18)$$

Summing (A.17) and (A.18) yields

$$f(y^*) + g(y^*) - f(x) - g(x) > (p' + g'(x))(y^* - x) \quad (A.19)$$

which contradicts (A.16). ■

Of note in the previous lemma is the fact that it is not true in general that $\partial L(x) = \partial f(x) + \partial g(x)$ when $g$ is not affine.
Applying the lemma A.4 implies the following necessary condition for ambiguity averse investors.

\[ 0 \in \partial U(x_0) - \lambda p. \]  

(A.20)

Assumption 1 ensures that all optimal allocations are interior. This fact, combined with the previous lemma implies the following result.

**Proposition 2.** Let \( x_0 \) be a solution to the problem

\[ \max_{x \in B(e,p)} U(x) \]  

(A.21)

If \( x_0 \in \text{int} \ X \) then

\[ \lambda p \in \partial U(x_0) \]  

(A.22)

\[ p(e - x) = 0 \]

The condition (A.22) are necessary for utility maximization. If \( U(\cdot) \) is strictly concave then these will also be sufficient.

The next corollary provides the intuition behind the robustness of the partially-revealing REE discussed in the paper. Essentially, it gives a sufficient condition for two investors to behave identically in equilibrium.

**Corollary 1.** Suppose that for two investors \( m \) and \( n \), \( e^m = e^n \) and that for the allocation \( x_0 \), \( \partial U^m(x_0) \subseteq \partial U^n(x_0) \). Then if \( x_0 \) is a solution to

\[ \max_{x \in B(e^m,p)} U^m(x) \]  

(A.23)

then it is also a solution to

\[ \max_{x \in B(e^n,p)} U^n(x) \]  

(A.24)

**Proof.** If (A.22) holds for investor \( m \) then it must hold for investor \( n \) as well since the budget constraints are the same and the Euler condition A.20 is a weaker condition for investor \( n \). ■
To obtain an idea of the geometry of the set $\partial U(c)$ it is helpful to characterize it in terms of quantities that are somewhat easier to calculate.

**Definition 3.** For the continuous function $U(x)$ define the derivative from the right of $U$ at $x_0$ in the direction $y$ to be

$$DU(x_0)(y) = \lim_{\alpha \to 0^+} \frac{U(x_0 + \alpha y) - U(x_0)}{\alpha} \tag{A.25}$$

Of note in the preceding definition is that $y$ need not be in $X$, although for sufficiently small $\alpha$ it must be true that $x_0 + \alpha y \in X$ since $U$ is only defined over $X$.

**Lemma A.5.**

$$DU(x_0)(y) = \min_{p \in \partial U(x_0)} py \tag{A.26}$$

*Proof.* Aubin (1979)[Section 4.3.2, proposition 4]

Therefore we have that for $x_0$

$$\partial U(x_0) = \{ p \in \mathbb{R}^W_+ : py \geq D(x_0)(y) \text{ for all } y \in \mathbb{R}^W \} \tag{A.27}$$

The function $DU(x)(\cdot)$ is the support function of the $\partial U(x)$. Since compact and convex sets may be completely characterized as the intersection of all of the half spaces in which they are contained and the support function characterizes such half spaces, we turn to the function $DU(x)(\cdot)$ to better understand $\partial U(x)$.

**Lemma A.6.** Let $\gamma_0(x) = \{ \pi \in \gamma : U(x) = E_\pi u(x) \}$. Then

$$DU(x_0)(y) = \min_{\pi \in \gamma_0(x)} DE_\pi u(x)(y) \tag{A.28}$$

*Proof.* Aubin (1979)[Section 4.3.3, proposition 6]
Standard calculus shows that
\[
DE_{x}u(x)(y) = \lim_{\alpha \to 0^+} \frac{E_{x}(u(x + \alpha y)) - E_{x}(u(x))}{\alpha}
\]
\[
= \left. \frac{d}{d\alpha} E_{x}(u(x + \alpha y)) \right|_{\alpha = 0^+}
\]
\[
= E_{x}u'(x)y
\]  
\hfill (A.29)

Applying this to the definition of \( \partial U(x_0) \) gives
\[
\partial U(x_0) = \{ p \in R^W_+ : py \geq \min_{\pi \in \gamma_0(x)} E_{\pi}u'(x)(y) \text{ for all } y \in R^W \}. \tag{A.30}
\]

From this, the superdifferential for some allocations can be calculated directly.

**Lemma A.7.** Consider an investor with beliefs given by \( \gamma \subseteq \Delta^{[\Omega]} \). Let \( x \) be an allocation such that \( x(\omega) = x(\omega') \) for all \( \omega, \omega' \in \Omega \).

\[
\partial U(x) = u'(x)\gamma.
\]  
\hfill (A.31)

**Proof.** Noting that \( u'(x(\omega)) = u'(x(\omega')) \) for all \( \omega, \omega' \) and applying equation (A.30) gives
\[
\partial U(x) = \{ p \in R^W_+ : py \geq \min_{\pi \in \gamma}(x) E_{\pi}y \text{ for all } y \in R^W \}. \tag{A.32}
\]

As defined in lemma A.6, \( \gamma_0 = \gamma \) for the allocation \( x \) since all probability distributions in \( \gamma \) give the same expected value for the constant random variable \( u(x) \). By definition, \( E_{\pi}y = \pi y \) so one can rewrite equation (A.32) as
\[
\partial U(x) = \{ p \in R^W_+ : \frac{py}{u'(x)} \geq \min_{\pi \in \gamma} \pi y \text{ for all } y \in R^W \}. \tag{A.33}
\]

But the right hand side of the inequality in the definition is the support function
for the set $\gamma$. As such, is equivalent to $pu'(x) \in \gamma$. □

Alternatively, if for a particular allocation $x$ the set $\gamma_0(x)$ is a singleton, the equation (A.30) reduces to the single vector

$$\partial U(x) = \{[\pi(\omega)u'(x(\omega))]_{\omega \in \Omega}\}. \quad (A.34)$$

The general case follows.

**Lemma A.8.** The superdifferential of $U$ at the point $x$ is given by

$$\partial U(x) = u'(x) \cdot \gamma_0(x) \quad (A.35)$$

**Proof.** The following manipulation of the definition of $\partial U(x)$ gives the result.

$$\partial U(x) = \{p \in \mathbb{R}_+^W : py \geq \min_{\pi \in \gamma_0(x)} E_{\pi}u'(x)y \text{ for all } y \in \mathbb{R}_+^W\}$$

$$= \left\{ p \in \mathbb{R}_+^W : py \geq \min_{\pi \in \gamma_0(x)} [\pi \cdot u'(x)]y \text{ for all } y \in \mathbb{R}_+^W \right\} \quad (A.36)$$

$$= \left\{ p \in \mathbb{R}_+^W : py \geq \min_{q \in u'(x) \cdot \gamma_0(x)} qy \text{ for all } y \in \mathbb{R}_+^W \right\}$$

□

**Corollary 2.** Let $n$ be an investor whose preferences satisfy Assumption 1. An allocation $x_0 \in X$ is a solution to the problem

$$\max U^n(x) \text{ s.t. } x \in B(e^n, p) \quad (A.37)$$

if and only if

$$\lambda p \in u'(x_0) \gamma_0(x_0)$$

$$p(e - x_0) = 0 \quad (A.38)$$

**Corollary 3.** Let $n$ and $m$ be two different investors having beliefs $\gamma^m$ and $\gamma^n$, but identical vN-M utility functions $u(\cdot)$ and equal endowments ($e^m = e^n$). If
$\gamma_n^0(x) \subseteq \gamma_0^n(x)$ then if $x \in X$ solves

$$\max U^m(x) \text{ s.t. } x \in B(e^m, p) \quad \text{(A.39)}$$

then $x$ also solves

$$\max U^n(x) \text{ s.t. } x \in B(e^n, p) \quad \text{(A.40)}$$

**Proof.** Inspecting equations (A.22) shows that any solution to these equations for beliefs $\gamma^m$ is also a solution for beliefs $\gamma^n$. ■

**Proposition 3.** Let $(x, p) \in X^N \times P$ be an Arrow-Debreu equilibrium for the economy characterized by $g = (\gamma^1, \ldots, \gamma^N) \in \Gamma^N$. If $\hat{g} = (\hat{\gamma}^1, \ldots, \hat{\gamma}^N)$ satisfies $\gamma_n^0(x^n) \subseteq \hat{\gamma}_0^n(x^n)$ then $(x, p)$ is an equilibrium for the economy $\hat{g}$ as well.

**Proof.** Applying corollary 3 for each investor shows that the allocation $x$ continues to be utility maximizing for all investors. Markets must also clear since $(x, p)$ is an equilibrium for the economy $g$. ■

**Lemma A.9.** Assume that $e^n \gg 0$ for all $n$, and that each investor satisfies assumption 1. Let $g = \{\gamma^1, \ldots, \gamma^n\} \in G^n$. Let $E(g)$ be an economy where beliefs for each investor $n$ are given by $g(n)$. Then a Walrasian equilibrium exists for each economy $E(g)$ for all $g \in G^n$.

**Proof.** It suffices to check that the conditions presented in (Debreu 1959)[Section 5.7] are satisfied for every economy $E(g)$. It is straightforward to show that

1. $X$ is closed, convex and has a lower bound for each investors’ preference ordering

2. No investor is satiated in $X$

3. For each $x' \in X$ the sets $\{x \in X | U^m(x) \geq U^m(x')\}$ and $\{x \in X | U^n(x) \leq U^n(x')\}$ are closed for all investors $n$. 

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4. If $U^n(x') \geq U^n(x)$ then $U^n(tx' + (1-t)x) \geq U^n(x)$ for $t \in (0,1)$.

5. There exists $x \in X$ such that $x \ll e^n$ for all $n$ (by assumption).

For an endowment economy these conditions are sufficient to guarantee the existence of equilibrium. ■

**Lemma A.10.** For any economy $E(g)$ let $E : G \to P$ represent the equilibrium price correspondence. Then $E(g)$ is upper-hemicontinuous in $g$.

**Proof.** Let $g^k$ be a sequence of economies converging to $g$ and let $p^k$ be the equilibrium price vector for economy $g^k$. Then $Z(p^k, g^k) = 0$ for all $k$. By the continuity of $Z(\cdot, g^k)$ if $p^k$ converges then its limit is also a market clearing price. ■

From general equilibrium results we now move on to more specific results on the space of beliefs for both ambiguity averse and expected utility investors. In order to analyze the size of the set of economies for which an REE exists it is necessary to investigate the properties of the space of economies that are considered in this paper. The fact that partially-revealing REE occur for a set of beliefs that has positive measure requires that this set of beliefs be explicitly described.

First a characterization of the beliefs of the ambiguity averse investor is given and it is shown that these beliefs can be embedded in a finite-dimensional Euclidean space. This embedding allows us to use Lebesgue measure on the space of beliefs. After characterizing the beliefs of the ambiguity averse investor the space of beliefs for expected utility maximizing investors will be characterized.

As noted in Assumption 2 the conditional beliefs of investor 1 can always be represented as the convex hull of a finite set of extreme points. Therefore, if the beliefs of investor 1 are given by $\{\gamma(\sigma)\}_{\sigma \in \Sigma}$, each of these conditional beliefs can be characterized by a set of $L$ probability distributions in $\Delta(\Omega)$. So for any
joint signal $\sigma$, beliefs for the investor can be identified with an element of the space $\Delta^{[\Omega|L]}$. Since the cardinality of the set of joint signals is $|\Sigma|$, the beliefs for investor 1 can be characterized by an element of the space $\Delta^{[\Omega|L]|\Sigma}$. It is clear that any point in $\Delta^{[\Omega|L]|\Sigma}$ represents a set of beliefs that meet our assumptions for investor 1. However, since the order of the distributions $(\pi_1, \ldots, \pi_L)$ that define $\gamma(\sigma)$ doesn’t matter when generating the convex hull of these points, there are always multiple elements of $\Delta^{[\Omega|L]|\Sigma}$ that represent the same beliefs over $\Omega$. The fact that the map from points in $\Delta^{[\Omega|L]|\Sigma}$ into beliefs on $\Sigma$ is not injective will not matter for the applications in this paper. To ease the exposition, for this appendix we will abuse notation slightly and let $\Gamma = \Delta^{[\Omega|L]|\Sigma}$ keeping in mind that $\Gamma$ as defined in the body is the function space of beliefs related to the set $\Delta^{[\Omega|L]|\Sigma}$.

This formulation of beliefs is used because it allows for the discussion of convergence of beliefs in the traditional sense. A sequence $\gamma^k \subseteq \Gamma$ is said to converge to some element $\gamma \in \Gamma$ if it converges in the standard (Euclidean, metric) sense.

To obtain the robustness results of Lemma 4.2 which is in turned used to prove Theorem 2, we need the following lemma.

**Lemma A.11.** Let $\gamma$ be any set of beliefs in $\Gamma$. Let $\pi \in \text{int } \gamma(\sigma) \cap \gamma(\sigma')$ for some $\sigma, \sigma' \in \Sigma$. Then there exist open neighborhoods $B(\gamma, \epsilon_\gamma) \subset \Gamma$ and $B(\pi, \epsilon_\pi) \subset \Delta^{[\Omega]}$ such that for all $\gamma' \in B(\gamma, \epsilon_\gamma)$, $B(\pi, \epsilon_\pi) \subset \text{int } \gamma'(\sigma) \cap \gamma'(\sigma')$.

**Proof.** Define

$$d(\gamma(\sigma), B(\pi, \epsilon_\pi^1)) = \inf_{\pi' \in \partial \gamma(\sigma), \pi \in B(\pi, \epsilon_\pi^1)} ||\pi' - \pi||. \quad (A.41)$$

Without loss of generality we may select $\epsilon_\pi^1$ so that $d(\gamma(\sigma), B(\pi, \epsilon_\pi^1)) > 0$. This implies that if $B(\pi, \epsilon_\pi^1)$ is the closure of $B(\pi, \epsilon_\pi^1)$ then $B(\pi, \epsilon_\pi^1) \in \text{int } \gamma(\sigma)$. 
It can be seen that \( d(\gamma(\sigma), B(\pi, \epsilon^1_\pi)) \geq d(\gamma(\sigma), \bar{B}(\pi, \epsilon^1_\pi)) \).

The function \( d(\gamma(\sigma), B) \) is continuous in \( \gamma \) if \( B \) is a compact set.\(^7\) To see this, let \( \{\pi_1, \ldots, \pi_L\} \) be the extreme points of the set \( \gamma(\sigma) \) and \( Q \) be the finite set of facets (\( L - 1 \) dimensional faces) of the convex polytope \( \gamma(\sigma) \), equation (A.41) can be rewritten as

\[
d(\gamma(\sigma), B) = \min_{\pi' \in \partial \gamma(\sigma), \pi \in B} \|\pi' - \pi\|
\]

\[
= \min_{q \in Q, \pi \in B} \|q - \pi\|
\]

\[
= \min_{q \in Q} \min_{\pi' \in q, \pi \in B} \|\pi' - \pi\|
\]

\[
= \min_{q \in Q} \min_{\alpha \in \Delta^L} \min_{\pi \in B} \left\| \sum_l \alpha_l \pi_l - \pi \right\|
\]

The facets of \( \gamma(\sigma) \) are found by taking convex combinations of sets of \( (L - 1) \) vertices of \( \gamma(\sigma) \). For each facet \( q \), the function \( \min_{\pi' \in q, \pi \in B} \|\pi' - \pi\| = \min_{\alpha \in \Delta^L} \min_{\pi \in B} \left\| \sum_l \alpha_l \pi_l - \pi \right\| \) is absolutely continuous over \( q \) in its vertices (since \( q \) is a compact set). Therefore the distance \( \min_{q \in Q} \|q - \pi\| \) is the minimum of a finite set of absolutely continuous functions and is therefore absolutely continuous in the set of vertices.

Since \( d(\gamma(\sigma), B) \) is continuous in \( \gamma \), there exists a neighborhood \( B(\gamma, \epsilon^1_\gamma) \) such that for all \( \gamma' \in B(\gamma, \epsilon^1_\gamma) \), \( d(\gamma'(\sigma), B(\pi, \epsilon^1_\pi)) > 0 \). From this it can be shown that if \( \gamma' \in B(\gamma, \epsilon^1_\gamma) \), then \( B(\pi, \epsilon^1_\pi) \subset \text{int} \gamma'(\sigma) \).

This process is then repeated for \( \gamma(\sigma') \) to show that there exists an \( \epsilon^2_\pi \) and \( \epsilon^2_\gamma \) such that \( B(\pi, \epsilon^2_\pi) \subset \text{int} \gamma'(\sigma') \) for all \( \gamma' \in B(\gamma, \epsilon^2_\gamma) \).

Then, choosing \( \epsilon_\pi \) and \( \epsilon_\gamma \), so that \( B(\pi, \epsilon_\pi) \subset B(\pi, \epsilon^1_\pi) \cap B(\pi, \epsilon^2_\pi) \) and \( B(\gamma, \epsilon_\gamma) \subset B(\gamma, \epsilon^1_\gamma) \cap B(\gamma, \epsilon^2_\gamma) \) it is seen that for any \( \gamma' \in B(\gamma, \epsilon_\gamma), B(\pi, \epsilon_\pi) \subset \gamma'(\sigma) \cap \gamma'(\sigma') \).

\(^7\)Recall that \( \gamma \) is an element of the finite-dimensional space \( \Gamma \) meaning that \( \gamma(\sigma) \) and \( \gamma(\sigma') \) belong to the finite dimensional space \( \Delta^{\mid\Omega\mid}_{L} \).
Characterizing the space of beliefs for investors $2, \ldots, N$ is more intuitive. Beliefs for these investors are given by $\pi_m, \pi_c \in \Delta^{[\Sigma]} \times \Delta^{[\Omega|\Sigma]}$. The following result is used implicitly in Theorem 2.

**Lemma A.12.** Let $\Pi$ be the space of beliefs for investor $n \geq 2$. For any $\epsilon > 0$ there exists a $\delta > 0$ such that if $||\pi'' - \pi|| \leq \delta$ with $\pi' \in \Pi$ then $||\pi'(\{\sigma, \sigma'\}) - \pi(\{\sigma, \sigma'\})|| < \epsilon$.

**Proof.** By Bayes rule, given any beliefs $\pi$,

$$
\pi(\{\sigma, \sigma'\}) = \frac{\pi_m(\sigma)\pi_c(\sigma) + \pi_m(\sigma')\pi_c(\sigma')}{\pi_m(\sigma) + \pi_m(\sigma')} \quad (A.42)
$$

In light of Assumption 1, it is seen that $\pi(\{\sigma, \sigma'\})$ is continuous in $\pi_c$ and $\pi_m$ which proves the lemma. ■

**References**


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