The Poisson and Exponential Distributions

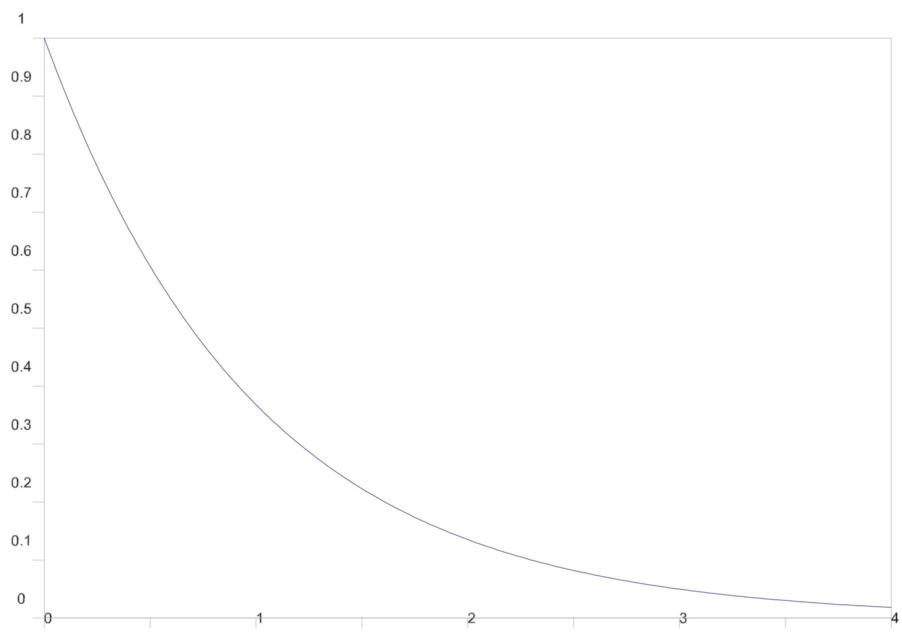
The binomial distribution deals with the number of successes in a fixed number of independent trials, and the geometric distribution deals with the time between successes in a series of independent trials.

Just so, the *Poisson distribution* deals with the number of occurrences in a fixed period of time, and the *exponential distribution* deals with the time between occurrences of successive events as time flows by continuously. Obviously, there's a relationship here.

Continuous Random Variables

A *continuous random variable* is a random variable which can take any value in some interval. A continuous random variable is characterized by its *probability density function*, a graph which has a total area of 1 beneath it: The probability of the random variable taking values in any interval is simply the area under the curve over that interval (and the probability of the random variable taking any one specific value is essentially 0).

The *exponential distribution*: Consider the time between successive incoming calls at a switchboard, or between successive patrons entering a store. These “interarrival” times are typically exponentially distributed. If the mean interarrival time is $1/\lambda$ (so λ is the mean arrival *rate* per unit time), then the variance will be $1/\lambda^2$ (and the standard deviation will be $1/\lambda$). The graph below displays the graph of the exponential density function when $\lambda = 1$. Generally, if X is exponentially distributed, then $\Pr(s < X \leq t) = e^{-\lambda s} - e^{-\lambda t}$ (where $e \approx 2.71828$) .



The exponential distribution fits the examples cited above because it is the only distribution with the “lack-of-memory” property: If X is exponentially distributed, then $\Pr(X \leq s+t \mid X > s) = \Pr(X \leq t)$. (After waiting a minute without a call, the probability of a call arriving in the next two minutes is the same as was the probability (a minute ago) of getting a call in the following two minutes. As you continue to wait, the chance of something happening “soon” neither increases nor decreases.) Note that, among discrete distributions, the geometric distribution is the only one with the lack-of-memory property; indeed, the exponential and geometric distributions are analogues of one another.

Let the time between successive arrivals into some system be exponentially distributed, and let N be the number of arrivals in a fixed interval of time of length t. Then N (a discrete random variable) has the *Poisson distribution*, and

$$\Pr(N = k) = e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!} .$$

$E[N] = \lambda t$, and $Var[N] = \lambda t$ as well. The exponential and Poisson distributions arise frequently in the study of queuing, and of process quality. An interesting (and sometimes useful) fact is that the minimum of two independent, identically-distributed exponential random variables is a new random variable, also exponentially distributed and with a mean precisely half as large as the original mean(s).

Approximations: Rule of thumb: If $n > 20$ and $p < 0.05$, then a binomial random variable with parameters (n, p) has a probability distribution very similar to that of a Poisson random variable with parameters $\lambda = np$ and $t = 1$. (Think of dividing one interval of time into n subintervals, and having a probability p of an arrival in each subinterval. That's very much like having a rate of np arrivals (on average) per unit time.)