

## e-companion

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Electronic Companion—“Risk Mitigation in Newsvendor Networks:  
Resource Diversification, Flexibility, Sharing, and Hedging” by  
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## Online Appendix

### Technical Assumptions

Clearly,  $V(K, \omega)$  must have an expected value; that is, it is a measurable and integrable function of  $\omega$  for every  $K$ . Interchanging differentiation and integration requires conditions that bound the derivatives such as requiring that all functions are Lipschitz, as in the general conditions in Appendix A of Broadie and Glasserman (1996). In our setting all functions are concave and thus absolutely continuous on any compact subset of  $\mathbb{R}_+^n$ , where its right- and left-hand partial derivatives exist (and thus are finite) and are monotone increasing. Let  $\nabla f(K, \omega)$  denote this vector of right-hand partial derivatives of a concave function  $f$  with respect to  $K$ . An absolutely continuous function  $f$  satisfies a Lipschitz condition if and only if  $\|\nabla f\|$  is bounded. Clearly, for a concave function,  $\nabla f(K, \omega)$  is bounded, and thus Lipschitz, on any compact subset in  $\mathbb{R}_+^n$ . The only technical condition is to require finite derivatives also at 0 and  $\infty$  and require a Lipschitz condition on the open set  $\mathbb{R}_+^n$ :

**ASSUMPTION 1.** *The value function  $V(K, \omega)$  satisfies a Lipschitz condition on  $\mathbb{R}_+^n$  almost surely: there is a  $M_V(\omega)$  such that  $|V(K_1, \omega) - V(K_2, \omega)| \leq M_V(\omega)\|K_1 - K_2\|$  for all  $K_1, K_2 \in \mathbb{R}_+^n$ , where  $\mathbb{E}M_V(\omega) < \infty$ .*

**ASSUMPTION 2.** *The utility function  $u(x)$  satisfies a Lipschitz condition on  $\mathbb{R}$ : there is a  $M_u$  such that  $|u(x_1) - u(x_2)| \leq M_u|x_1 - x_2|$  for all  $x_1, x_2 \in \mathbb{R}$ .*

For newsvendor networks,  $\pi$  and thus also  $V$  are the solution of a linear program and have finite partial derivatives so that Assumption 1 is always satisfied. It is not unrealistic to assume that demands are bounded which then obviates Assumption 2.

**PROOF OF PROPOSITION 1.** The function  $V(\mathbf{K}, \omega)$  is concave in  $\mathbf{K}$  for any  $\omega$  as a sum of two concave functions. Because  $u(\cdot)$  is concave increasing, the scalar composition  $u(V(\mathbf{K}, \omega) + W)$  is also concave in  $\mathbf{K}$  for any  $\omega$  and  $W$ . (The latter is directly shown for twice differentiable functions, but also holds without assuming differentiability, see Boyd and Vandenberghe 2004, p. 84.) Finally, the expected utility function is concave as a linear combination of concave functions.  $\square$

**PROOF OF PROPOSITION 2.** Let  $f(\mathbf{K}, \omega) = u(V(\mathbf{K}, \omega) + W)$ , which is concave in  $K$  for any  $\omega$  and wealth  $W$  according to the proof of Proposition 1 so that  $\nabla U(\mathbf{K}^*) = 0$  is necessary and sufficient for an interior maximum.

Because  $f$  is concave, its right-hand partial gradient  $\nabla f(\mathbf{K}, \omega)$  exists for every  $\omega$ . Thus, for all  $\mathbf{K} \in \mathbb{R}_+^n$  and  $m > 0$ ,  $g_m = m(f(\mathbf{K} + m^{-1}\mathbf{e}_i, \omega) - f(\mathbf{K}, \omega)) \rightarrow_m \nabla_i f(\mathbf{K}, \omega)$ , where  $\mathbf{e}_i$  is the  $i$ th unit vector. Given that the Lipschitz property is preserved by composition, the technical assumptions guarantee the existence of Lipschitz modulus  $M(\omega)$  for  $f$  w.p. 1 that is integrable. Because  $|g_m(\mathbf{K}, \omega)| < M(\omega)$  with  $\mathbb{E}M(\omega) < \infty$ , the dominated convergence theorem shows that  $\lim_{m \rightarrow \infty} \mathbb{E}g_m = \mathbb{E}\lim_{m \rightarrow \infty} g_m$ . Thus, differentiation and integration interchange so that  $\nabla U(\mathbf{K}) = \mathbb{E}\nabla f = \mathbb{E}u'(V + W)\nabla V$ .  $\square$

**PROOF OF PROPOSITION 3.** Applying the definition of covariance and given that integration and differentiation can be interchanged, we have that:

$$\begin{aligned} \nabla \sigma^2(\mathbf{K}) &= \nabla_{\mathbf{K}}(\mathbb{E}\pi^2(\mathbf{K}, \mathbf{D}) - (\mathbb{E}\pi(\mathbf{K}, \mathbf{D}))^2) = \mathbb{E}\nabla_{\mathbf{K}}\pi^2(\mathbf{K}, \mathbf{D}) - \nabla_{\mathbf{K}}(\mathbb{E}\pi(\mathbf{K}, \mathbf{D}))^2 \\ &= 2\mathbb{E}[\pi(\mathbf{K}, \mathbf{D})\lambda(\mathbf{K}, \mathbf{D})] - 2(\mathbb{E}\pi(\mathbf{K}, \mathbf{D}))\mathbb{E}\lambda(\mathbf{K}, \mathbf{D}) = 2\text{Cov}(\lambda, \pi). \end{aligned}$$

For the second part, again applying the definition of covariance, we have that

$$\begin{aligned}\mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K}^n, \omega) - \mathbf{c})V(\mathbf{K}^n, \omega)] &= \text{Cov}(\boldsymbol{\lambda}(\mathbf{K}^n, \omega) - \mathbf{c}, V(\mathbf{K}^n, \omega)) + \mathbb{E}[\boldsymbol{\lambda}(\mathbf{K}^n, \omega) - \mathbf{c}]\mathbb{E}V(\mathbf{K}^n, \omega) \\ &= \text{Cov}(\boldsymbol{\lambda}(\mathbf{K}^n, \omega) - \mathbf{c}, \boldsymbol{\pi}(\mathbf{K}^n, \omega) - C(\mathbf{K}^n)) \quad (\text{second term} = 0 \text{ by (3)}) \\ &= \text{Cov}(\boldsymbol{\lambda}(\mathbf{K}^n, \omega), \boldsymbol{\pi}(\mathbf{K}^n, \omega)) \quad (\text{constants fall out}). \quad \square\end{aligned}$$

PROOF OF PROPOSITION 4. According to the implicit function theorem,  $\mathbf{K}(\gamma)$  is a continuous function of  $\gamma$  where  $(d/d\gamma)\mathbf{K}(\gamma)$  is found by differentiating the first-order condition:  $(d/d\gamma)\mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K}, \omega) - \mathbf{c}) \cdot \exp(-\gamma V(\mathbf{K}, \omega))] = 0$  or

$$\begin{aligned}\mathbb{E}[(\nabla_{\mathbf{K}}\boldsymbol{\lambda}(\mathbf{K}, \omega)) \exp(-\gamma V(\mathbf{K}, \omega)) + (\boldsymbol{\lambda}(\mathbf{K}, \omega) - \mathbf{c}) \exp(-\gamma V(\mathbf{K}, \omega))(-\gamma)\nabla_{\mathbf{K}}V] \frac{d}{d\gamma}\mathbf{K}(\gamma) \\ + \mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K}, \omega) - \mathbf{c}) \exp(-\gamma V(\mathbf{K}, \omega))(-V(\mathbf{K}, \omega))] = 0\end{aligned}$$

Recall that the Hessian  $H(\mathbf{K}^n) = \mathbb{E}[(\nabla_{\mathbf{K}}\boldsymbol{\lambda}(\mathbf{K}^n, \omega))]$  and evaluate at the risk-neutral case  $\gamma = 0$  to get:

$$H(\mathbf{K}^n) \frac{d}{d\gamma}\mathbf{K}(0) = \mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K}^n, \omega) - \mathbf{c})V(\mathbf{K}^n, \omega)].$$

Given that  $\Pi$  is concave, its Hessian  $H(\mathbf{K}^n)$  is negative-definite, and invertible.  $\square$

PROOF OF (9). As illustrated in Figure 2, the MV-frontier function  $\mathcal{F}$  and the maximal utility  $U^{\text{MV}}(\gamma)$ , denoted as  $\mathcal{U}(\gamma)$ , are almost inverse functions in that they satisfy, except at possible inflection points:  $\mathcal{U}'(\mathcal{F}'(x)) = -x$  and thus  $\mathcal{U}''(\mathcal{F}'(x)) = -1/\mathcal{F}''(x)$ . Evaluating at  $x = \sigma^2(\mathbf{K}^n)$ , where  $z = \mathcal{F}'(x) = 0$ , directly yields  $\mathcal{U}'(0) = -\sigma^2(\mathbf{K}^n)$  and  $\mathcal{F}''(\sigma^2(\mathbf{K}^n)) = -1/\mathcal{U}''(0)$ . It only remains to find  $\mathcal{U}''(0)$ . Twice differentiate the defining condition of  $\mathbf{K}(z)$ :

$$\begin{aligned}\frac{d}{d\gamma}\mathcal{U}(\gamma) &= \nabla'\mu(\mathbf{K}(\gamma)) \frac{d}{d\gamma}\mathbf{K} - \frac{\gamma}{2}\nabla'\sigma^2(\mathbf{K}(\gamma)) \frac{d}{d\gamma}\mathbf{K} - \frac{1}{2}\sigma^2(\mathbf{K}(\gamma)), \\ \frac{d^2}{d\gamma^2}\mathcal{U}(\gamma) &= \left(H(\mathbf{K}(\gamma)) \frac{d}{d\gamma}\mathbf{K}\right)' \frac{d}{d\gamma}\mathbf{K} + \nabla'\mu(\mathbf{K}(\gamma)) \frac{d^2}{d\gamma^2}\mathbf{K} - \frac{1}{2}\nabla'\sigma^2(\mathbf{K}(\gamma)) \frac{d}{d\gamma}\mathbf{K} \\ &\quad - \frac{\gamma}{2} \frac{d}{d\gamma} \left( \nabla'\sigma^2(\mathbf{K}(\gamma)) \frac{d}{d\gamma}\mathbf{K} \right) - \frac{1}{2}\nabla'\sigma^2(\mathbf{K}(\gamma)) \frac{d}{d\gamma}\mathbf{K}.\end{aligned}$$

Evaluate at  $\gamma = 0$  and recall that  $\nabla\mu(\mathbf{K}(0)) = 0$  and  $H(\mathbf{K}(0))(d/d\gamma)\mathbf{K}(0) = \nabla\sigma^2(\mathbf{K}(0))$ :

$$\mathcal{U}''(0) = -\nabla'\sigma^2(\mathbf{K}(0))' \frac{d}{d\gamma}\mathbf{K}(0) = -\nabla'\sigma^2(\mathbf{K}^n)H^{-1}(\mathbf{K}^n)\nabla\sigma^2(\mathbf{K}^n). \quad \square$$

PROOF OF PROPERTY 1. Optimal activity  $x = \min(K, D)$  so that  $\lambda = v1_{\{D \geq K\}}$  with the familiar risk-neutral optimality condition  $\mathbb{E}(\lambda - c) = v(1 - F(K)) - c = 0$ . Using the standard normal pdf  $\phi$  and cdf  $\Phi$ ,  $v(1 - \Phi(z^n)) = c$  where  $z^n = (K^n - \mu_1)/\sigma_1$ . Hence,  $H(K^n) = (d/dK)\mathbb{E}\lambda = -(v/\sigma_1)\phi(z^n) < 0$  and

$$\begin{aligned}\mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K}^n, \omega) - \mathbf{c})V(\mathbf{K}^n, \omega)] &= \int_{-\infty}^{K^n} (-c)(vx - cK^n) dF + \int_{K^n}^{\infty} (v - c)(vK^n - cK^n) dF \\ &= -cv \underbrace{\int_{-\infty}^{K^n} x dF}_A + \underbrace{c^2K^n F(K^n) + (v - c)^2K^n(1 - F(K^n))}_B\end{aligned}$$

For the normal distribution, integration by parts yields  $A = \mu_1\Phi(z^n) - \sigma_1\phi(z^n) = \mu_1(1 - c/v) - \sigma_1\phi(z^n)$ . Moreover  $B = [c^2(1 - c/v) + (v - c)^2c/v]K^n = c(v - c)K^n$ . Putting this together yields

$$\begin{aligned}\mathbb{E}[(\boldsymbol{\lambda}(\mathbf{K}^n, \omega) - \mathbf{c})V(\mathbf{K}^n, \omega)] &= -cv(\mu_1(1 - c/v) - \sigma_1\phi) + c(v - c)(\mu_1 + z\sigma_1) \\ &= \sigma_1cv(\phi(z^n) + (1 - c/v)z^n),\end{aligned}$$

which is nonnegative because  $f(z) = \phi(z) + z\Phi(z)$  has  $f(-\infty) = 0$  and is nondecreasing ( $f'(z) = \Phi(z) \geq 0$ ).  $\square$

PROOF OF PROPERTY 3. Using the notation of the proof of Property 2:

$$\begin{aligned} \frac{1}{2}\nabla_2\sigma^2(\mathbf{K}^n) &= \mathbb{E}(\lambda_2(\mathbf{K}^n, \omega) - c_2)V(\mathbf{K}^n, \omega) \\ &= (v_2 - c_2)\mathbb{E}_{13}V(\mathbf{K}^n, \mathbf{D}) - c_2\mathbb{E}_{04}V(\mathbf{K}^n, \mathbf{D}) \\ &\leq V_1(\mathbf{K}^n)[(v_2 - c_2)P_1(\mathbf{K}^n) - c_2P_{04}(\mathbf{K}^n)] = 0. \end{aligned}$$

Also  $H_{12} = (\partial/\partial K_2)\mathbb{E}\lambda_1(\mathbf{K}^n, \omega) = (\partial/\partial K_2)v_1P(D_1 > K_1^n) = 0$ , so that

$$H^{-1}(\mathbf{K}^n) = \begin{bmatrix} \frac{-1}{v_1 f_1(K_1^n)} & 0 \\ 0 & \frac{-1}{v_2 f_2(K_2^n)} \end{bmatrix},$$

where  $f_i(\cdot)$  is the p.d.f. of  $D_i$ .  $\square$

PROOF OF PROPERTY 4 is similar to that of Property 2: The activity vector  $\mathbf{x}(\mathbf{K}, \mathbf{D})$  again is a simple greedy solution:  $x_1 = \min(D_1, K_1, K_3)$  and  $x_2 = \min(D_2, K_2, K_3 - x_1)$ . The optimal risk-neutral  $\mathbf{K}^n$  satisfies the optimality conditions  $\mathbb{E}\lambda(\mathbf{K}^n, \omega) = \mathbf{c}$ :

$$(v_1 - v_2)P_3(\mathbf{K}^n) + v_1P_4(\mathbf{K}^n) = c_1, \quad v_2P_1(\mathbf{K}^n) = c_2, \quad v_2P_{2+3}(\mathbf{K}^n) = c_3. \quad (\text{EC1})$$

According to Proposition 2, the (second) optimality condition for the risk-averse resource vector  $\mathbf{K}^u$  is:

$$0 = (v_2 - c_2)\mathbb{E}_1 u'(V(\mathbf{K}^u, \mathbf{D}) + W) - c_2\mathbb{E}_{0234} u'(V(\mathbf{K}^u, \mathbf{D}) + W).$$

Case 1.  $\sigma_1 \geq \sigma_2$  as shown in Figure EC.1. Notice that it is suboptimal for the point  $(K_1^u, K_3^u - K_1^u)$  to be above the demand line because reducing  $K_3^u$  by  $\epsilon$  would not change operating profits but reduce investment costs. Thus there are two possible cases: moderate  $c_3$  so that  $(K_3^n - K_2^n, K_2^n)$  falls above/at the demand line; with high  $c_3$  it falls below. The property applies in the former case. The proof is similar to that of Property 2: define the vector  $\mathbf{x}_1(\mathbf{K})$  and scalar  $k_1(\mathbf{K})$  and establish that, if  $k_1(\mathbf{K}^u) < K_1^u$ , then  $P_1(\mathbf{K}^u) \leq c_2/v_2 = P_1(\mathbf{K}^n)$ . Note that  $P_1(\mathbf{K}) = P(D_2 > K_2)$  so that  $K_2^u \geq K_2^n$ . The required conditions are (1) and (2) of the proof of Property 2 and (3) the point  $(K_3^n - K_2^n, K_2^n)$  falls above the demand line or  $z_1(K_3^n - K_2^n) + z_2(K_2^n) > 0$ . The increasing-in-risk aversion is proved similarly to the proof of Property 2.

Case 2.  $\sigma_1 < \sigma_2$  proceeds similarly but uses the point  $\mathbf{x}_1(\mathbf{K}) = (K_3 - K_2, K_2)$ . The required conditions are (1) of the proof of Property 2; (2)  $k_1(\mathbf{K}^n) < K_1^n$  or  $K_3^n < K_1^n + (1 - v_2/v_1)K_2^n$ ; and (3) the point  $(K_1^n, K_3^n - K_1^n)$  falls above the demand line or  $z_1(K_1^n) + z_2(K_3^n - K_1^n) > 0$ .  $\square$

PROOF OF PROPERTY 5. Use the notation of the proof of Property 4.

$$\begin{aligned} \frac{1}{2}\nabla_2\sigma^2(\mathbf{K}^n) &= \mathbb{E}(\lambda_2(\mathbf{K}^n, \omega) - c_2)V(\mathbf{K}^n, \omega) \\ &= (v_2 - c_2)\mathbb{E}_1 V(\mathbf{K}^n, \mathbf{D}) - c_2\mathbb{E}_{0234} V(\mathbf{K}^n, \mathbf{D}) \\ &\leq V_1(\mathbf{K}^n)[(v_2 - c_2)P_1(\mathbf{K}^n) - c_2P_{0234}(\mathbf{K}^n)] = 0. \end{aligned}$$

Figure EC.1 The Activity Vector  $\mathbf{x}$  for the Serial Network When  $\sigma_1 \geq \sigma_2$  and  $\rho = -1$

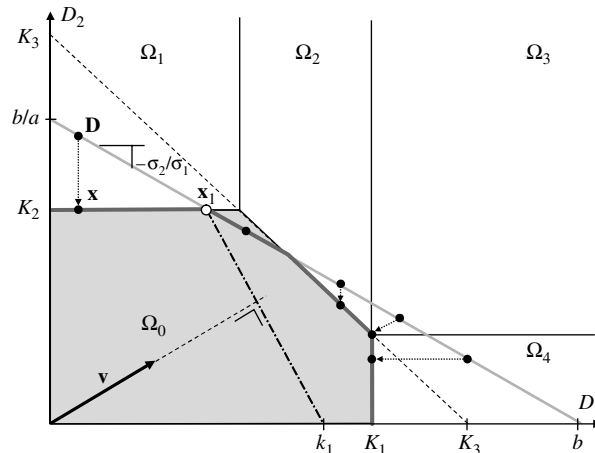
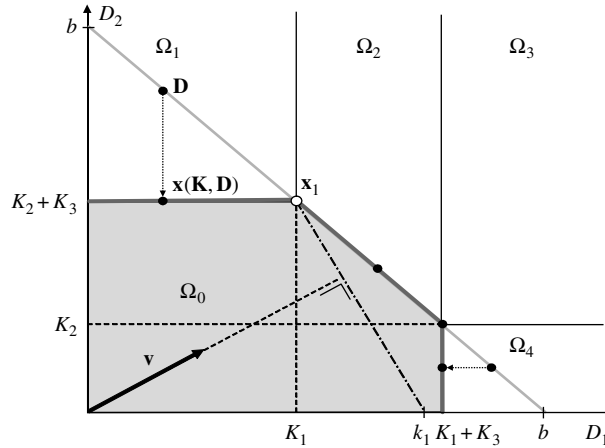


Figure EC.2 The Activity Vector  $\mathbf{x}$  for the Parallel Network When  $\sigma_1 = \sigma_2$  and  $\rho = -1$



It is also easily verified that

$$H^{-1}(\mathbf{K}^n) = \frac{1}{|H|} \begin{bmatrix} \dots & 0 & \dots \\ 0 & \dots & 0 \\ \dots & 0 & \dots \end{bmatrix},$$

where all nonzero elements (denoted by  $\dots$ ) are positive and  $|H| < 0$ . Similarly, if  $v_1 = v_2$ :

$$\begin{aligned} \frac{1}{2} \nabla_3 \sigma^2(\mathbf{K}^n) &= \mathbb{E}(\lambda_3(\mathbf{K}^n, \omega) - c)V(\mathbf{K}^n, \omega) \\ &= (v_2 - c_3)\mathbb{E}_{23}V(\mathbf{K}^n, \mathbf{D}) - c_3\mathbb{E}_{014}V(\mathbf{K}^n, \mathbf{D}) \\ &\geq V_3(\mathbf{K}^n)[(v_2 - c_3)P_{23}(\mathbf{K}^n) - c_3P_{014}(\mathbf{K}^n)] = 0, \end{aligned}$$

where  $V_3(\mathbf{K}^n) = V(\mathbf{K}^n, \mathbf{x}_3(\mathbf{K}^n))$  where  $\mathbf{x}_3(\mathbf{K}) = (K_1, K_3 - K_1)$ .  $\square$

PROOF OF PROPERTY 6 is similar to that of Property 2: Notice that  $K_1^u + K_2^u + K_3^u > b$  is suboptimal because reducing  $K_3^u$  by  $\epsilon$  and increasing  $K_1^u$  by  $\epsilon$  would not change operating profits but would decrease investment cost by  $(c_3 - c_1)\epsilon > 0$ . Thus there are two possible cases:  $K_1^u + K_2^u + K_3^u < b$  if  $c_3$  is high and  $K_1^u + K_2^u + K_3^u = b$  otherwise. The property applies in the latter boundary case which is shown in Figure EC.2.

Let  $\mathbf{K}_{1:2} = (K_1, K_2)$  be the independent variable for this boundary case where  $K_3 = b - K_1 - K_2$ . The associated two-dimensional shadow vector on this boundary has components  $\lambda_1^b(\mathbf{K}_{1:2}^u, \mathbf{D}) = -v_2 1_{\{\mathbf{D} \in \Omega_1(\mathbf{K}_{1:2}^u)\}}$  and  $\lambda_2^b(\mathbf{K}_{1:2}^u, \mathbf{D}) = -v_1 1_{\{\mathbf{D} \in \Omega_4(\mathbf{K}_{1:2}^u)\}}$  with effective marginal cost  $c^b = (c_1 - c_3, c_2 - c_3) < 0$ . The risk-neutral boundary solution satisfies  $\mathbb{E}\lambda^b(\mathbf{K}^n, \mathbf{D}) = c^b$  so that  $P_1(\mathbf{K}^n) = (c_3 - c_1)/v_2$  and  $P_4(\mathbf{K}^n) = (c_3 - c_1)/v_2$ . Define the vector  $\mathbf{x}_1(\mathbf{K}) = (K_1, K_2 + K_3)$  to partition and bound marginal utilities similar to the proof of Property 2: The optimality conditions for  $\mathbf{K}^u$  include

$$\begin{aligned} 0 &= \mathbb{E}(\lambda_1^b(\mathbf{K}^u, \mathbf{D}) - c_1^b)u'(V(\mathbf{K}^u, \mathbf{D}) + W) \\ &= (-v_2 - c_1 + c_3)\mathbb{E}_1 u'(V(\mathbf{K}^u, \mathbf{D}) + W) + (-c_1 + c_3)\mathbb{E}_{24} u'(V(\mathbf{K}^u, \mathbf{D}) + W) \\ &\leq u'(V_1(\mathbf{K}^u))[-v_2 P_1(\mathbf{K}^u) - c_1 + c_3] \Rightarrow P_1(\mathbf{K}^u) \leq (c_3 - c_1)/v_2. \end{aligned}$$

Thus,  $P_1(\mathbf{K}^u) \leq P_1(\mathbf{K}^n)$  so that  $K_1^u \leq K_1^n$  and  $K_2^u + K_3^u \geq K_2^n + K_3^n$ . The increase in risk aversion is proved similarly to the proof of Property 2. The required conditions are (1)  $v_1 > v_2$  and  $\sigma_1 = \sigma_2$ ; (2)  $k_1(\mathbf{K}^n) < K_1^n + K_3^n$  or  $v_2 K_2^n < (v_1 - v_2)K_3^n$ ; (3)  $K_1^n + K_2^n + K_3^n = b$  or conditions (c) of Proposition 7 of Van Mieghem (1998, Proposition 7).  $\square$

## References

See references list in the main paper.

Boyd, S., L. Vandenberghe. 2004. *Convex Optimization*. Cambridge University Press, Cambridge, UK.

Broadie, M., P. Glasserman. 1996. Estimating security price derivatives using simulation. *Management Sci.* 42(2) 269–285.