

Appendix 1

The partitioning of the state space of $\epsilon(\tau)$:

$$\begin{aligned}
\Omega_1(\mathbf{K}) &\equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : x_1 < \frac{c_Q K_1^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T)}, x_2 < \frac{c_Q K_2^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_2(\tau, T)} \right\}, \\
\Omega_2(\mathbf{K}) &\equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : \frac{c_Q K_1^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T)} < x_1 < \frac{(c_Q + c_S) K_1^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T)}, x_2 < \frac{c_Q K_2^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_2(\tau, T)} \right\}, \\
\Omega_3(\mathbf{K}) &\equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : x_1 < \frac{c_Q K_1^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T)}, \frac{c_Q K_2^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_2(\tau, T)} < x_2 < \frac{(c_Q + c_S) K_2^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_2(\tau, T)} \right\}, \\
\Omega_4(\mathbf{K}) &\equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : x_1 > \frac{(c_Q + c_S) K_1^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T)}, \left(\frac{x_1}{c_Q + c_S} \right)^{-b} + \left(\frac{x_2}{c_Q} \right)^{-b} < \frac{K_1 + K_2}{((1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T))^{-b}} \right\}, \\
\Omega_5(\mathbf{K}) &\equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : x_2 > \frac{(c_Q + c_S) K_2^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_2(\tau, T)}, \left(\frac{x_1}{c_Q} \right)^{-b} + \left(\frac{x_2}{c_Q + c_S} \right)^{-b} < \frac{K_1 + K_2}{((1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T))^{-b}} \right\}, \\
\Omega_6(\mathbf{K}) &\equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : x_1 > \frac{c_Q K_1^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T)}, x_2 > \frac{c_Q K_2^{-1/b}}{(1+1/b) \mathbb{E}_0 \epsilon_2(\tau, T)}, \right. \\
&\quad \left. \left| x_1 K_1^{1/b} - x_2 K_2^{1/b} \right| < \frac{c_S}{(1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T)} \right\}, \\
\Omega_7(\mathbf{K}) &\equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : \left(\frac{x_1}{c_Q + c_S} \right)^{-b} + \left(\frac{x_2}{c_Q} \right)^{-b} > \frac{K_1 + K_2}{((1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T))^{-b}}, \right. \\
&\quad \left. x_1 K_1^{1/b} - x_2 K_2^{1/b} > \frac{c_S}{(1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T)} \right\}, \\
\Omega_8(\mathbf{K}) &\equiv \left\{ \mathbf{x} \in \mathbb{R}_+^2 : \left(\frac{x_1}{c_Q} \right)^{-b} + \left(\frac{x_2}{c_Q + c_S} \right)^{-b} > \frac{K_1 + K_2}{((1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T))^{-b}}, \right. \\
&\quad \left. x_2 K_2^{1/b} - x_1 K_1^{1/b} > \frac{c_S}{(1+1/b) \mathbb{E}_0 \epsilon_1(\tau, T)} \right\}.
\end{aligned}$$

The corresponding optimal output vector $\mathbf{Q}^*(\mathbf{K}, \epsilon(\tau))$:

$$\begin{aligned}
\text{If } \epsilon(\tau) \in \Omega_1(\mathbf{K}), \quad Q_i^* &= \left(\frac{(1+1/b) \mathbb{E}_\tau \epsilon_i(T)}{c_Q} \right)^{-b}, i = 1, 2. \\
\text{If } \epsilon(\tau) \in \Omega_2(\mathbf{K}), \quad Q_1^* &= K_1, Q_2^* = \left(\frac{(1+1/b) \mathbb{E}_\tau \epsilon_2(T)}{c_Q} \right)^{-b}. \\
\text{If } \epsilon(\tau) \in \Omega_3(\mathbf{K}), \quad Q_1^* &= \left(\frac{(1+1/b) \mathbb{E}_\tau \epsilon_1(T)}{c_Q} \right)^{-b}, Q_2^* = K_2.
\end{aligned}$$

$$\text{If } \epsilon(\tau) \in \Omega_4(\mathbf{K}), Q_1^* = \left(\frac{(1+1/b) \mathbb{E}_\tau \epsilon_1(T)}{c_Q + c_S} \right)^{-b}, Q_2^* = \left(\frac{(1+1/b) \mathbb{E}_\tau \epsilon_2(T)}{c_Q} \right)^{-b}.$$

$$\text{If } \epsilon(\tau) \in \Omega_5(\mathbf{K}), Q_1^* = \left(\frac{(1+1/b) \mathbb{E}_\tau \epsilon_1(T)}{c_Q} \right)^{-b}, Q_2^* = \left(\frac{(1+1/b) \mathbb{E}_\tau \epsilon_2(T)}{c_Q + c_S} \right)^{-b}.$$

$$\text{If } \epsilon(\tau) \in \Omega_6(\mathbf{K}), \mathbf{Q}^* = \mathbf{K}.$$

$$\text{If } \epsilon(\tau) \in \Omega_7(\mathbf{K}), \mathbf{Q}^* \text{ is the unique solution to } Q_1 + Q_2 = K_1 + K_2 \text{ and}$$

$$\mathbb{E}_\tau \epsilon_1(T) Q_1^{1/b} - \mathbb{E}_\tau \epsilon_2(T) Q_2^{1/b} = \frac{c_S}{(1+1/b)}.$$

$$\text{If } \epsilon(\tau) \in \Omega_8(\mathbf{K}), \mathbf{Q}^* \text{ is the unique solution to } Q_1 + Q_2 = K_1 + K_2 \text{ and}$$

$$\mathbb{E}_\tau \epsilon_2(T) Q_2^{1/b} - \mathbb{E}_\tau \epsilon_1(T) Q_1^{1/b} = \frac{c_S}{(1+1/b)}.$$

Appendix 2

Proof of Proposition 1: For simplicity, we use \mathbf{p}^* and \mathbf{Q}^* as shorthand for $\mathbf{p}^*(\mathbf{Q}^*(\mathbf{K}))$ and $\mathbf{Q}^*(\mathbf{K})$, respectively, where \mathbf{p}^* is given by (5) and \mathbf{Q}^* is characterized in Appendix 1. Given the optimal pricing and output decisions, the firm value at time τ is

$$v(\tau; \mathbf{K}) = \sum_{i=1}^2 \left(\mathbb{E}_\tau \epsilon_i(T) (Q_i^*)^{1+1/b} - c_Q Q_i^* - c_S \max(Q_i^* - K_i, 0) - c_K K_i \right). \quad (9)$$

The Hessian matrix of (9) with respect to \mathbf{K} is

$$H_{\mathbf{K}}v(\tau; \mathbf{K}) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \epsilon(\tau) \in \Omega_{145}, \\ \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_1(T) K_1^{1/b-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \epsilon(\tau) \in \Omega_2, \\ \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_2(T) K_2^{1/b-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \epsilon(\tau) \in \Omega_3, \\ \begin{pmatrix} \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_1(T) K_1^{1/b-1} & 0 \\ 0 & \frac{1+b}{b^2} \mathbb{E}_\tau \epsilon_2(T) K_2^{1/b-1} \end{pmatrix} & \text{if } \epsilon(\tau) \in \Omega_6, \\ \frac{1+b}{b^2} \frac{\mathbb{E}_\tau \epsilon_1(T) \mathbb{E}_\tau \epsilon_2(T) (Q_1^*)^{1/b-1} (Q_2^*)^{1/b-1}}{\mathbb{E}_\tau \epsilon_1(T) (Q_1^*)^{1/b-1} + \mathbb{E}_\tau \epsilon_2(T) (Q_2^*)^{1/b-1}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \epsilon(\tau) \in \Omega_{78}. \end{cases}$$

Thus, $H_{\mathbf{K}}v(\tau; \mathbf{K})$ is negative definite if $\epsilon(\tau) \in \Omega_6$ and negative semidefinite otherwise. Therefore, $v(\tau; \mathbf{K})$ is concave in \mathbf{K} for any $\epsilon(\tau)$ and the concavity is strict if $\epsilon(\tau) \in \Omega_6$. This means that $v(0; \mathbf{K}) = \mathbb{E}_0 v(\tau; \mathbf{K})$ is concave in \mathbf{K} and the first-order optimality condition $\nabla_{\mathbf{K}} v(0; \mathbf{K}) = \mathbf{0}$ is sufficient. Furthermore, if $c_S > 0$, then $\Pr(\Omega_6(\mathbf{K})) > 0$ and the concavity is strict, implying that \mathbf{K}^* is unique. The uniqueness of \mathbf{K}^* together with the symmetry of all parameters implies that $K_1^* = K_2^*$. Taking the derivative of $v(0; \mathbf{K})$ with respect to K_1 yields

$$\frac{\partial v(0; \mathbf{K})}{\partial K_1} = \frac{\partial}{\partial K_1} \mathbb{E}_0 v(T; \mathbf{K}) = \frac{\partial}{\partial K_1} \sum_{i=1}^8 \Pr(\Omega_i(\mathbf{K})) \mathbb{E}_0(v(T; \mathbf{K}) | \Omega_i(\mathbf{K})), \quad (10)$$

where the firm terminal value, given the optimal pricing and output decisions, is

$$v(T; \mathbf{K}) = \sum_{i=1}^2 (Q_i^* p_i^* - c_P Q_i^* - c_S \max(Q_i^* - K_i, 0) - c_K K_i).$$

Note that $v(T; \mathbf{K})$ is continuous in $\epsilon(\tau)$ and, therefore, the terms from differentiating the boundaries of $\Omega_1, \dots, \Omega_8$ with respect to K_1 in (10) cancel out. This leaves us with

$$\frac{\partial v(0; \mathbf{K})}{\partial K_1} = \sum_{i=1}^8 \Pr(\Omega_i(\mathbf{K})) \mathbb{E}_0 \left(\frac{\partial v(T; \mathbf{K})}{\partial K_1} \middle| \Omega_i(\mathbf{K}) \right).$$

Differentiating $v(T; \mathbf{K})$ with respect to K_1 and setting $\partial v(0; \mathbf{K}) / \partial K_1 = 0$ results in (7). \square

Proof of Corollary 1: The result follows from Proposition 1 with $\tau = 0$, $c_K = \tilde{c}_K$ and $c_Q = \tilde{c}_Q$. \square

Proof of Lemma 1: It follows from Proposition 1 that if $c_S = c_Q = 0$, the optimal total capacity and firm value are, respectively,

$$K_1^* + K_2^* = \left[\frac{1 + 1/b}{c_K} \mathbb{E}_0 \left(\left(\mathbb{E}_\tau^{-b} \epsilon_1(T) + \mathbb{E}_\tau^{-b} \epsilon_2(T) \right)^{-1/b} \right) \right]^{-b} \quad \text{and} \quad v^*(0) = \frac{c_K}{|1+b|} (K_1^* + K_2^*).$$

This together with Corollary 1 gives the desired result. \square

Proof of Lemma 2: To simplify the notation, we normalize $T = 1$ and $\epsilon(0) = \mathbf{1}$. To prove the desired results, it is sufficient to show that $\frac{\partial}{\partial \tau} \|\epsilon(1)\|_\tau \geq 0$. Recall that $\|\epsilon(1)\|_\tau = \mathbb{E}_0 \left\{ \left(\frac{\mathbb{E}_\tau^{-b} \epsilon_1(1) + \mathbb{E}_\tau^{-b} \epsilon_2(1)}{2} \right)^{-1/b} \right\}$

and $\ln \epsilon(t) \sim N(\ln \epsilon(0), t\Sigma)$. Using the fact that $\mathbb{E}_\tau \epsilon_i(1) = \epsilon_i(\tau) \exp(\frac{1}{2}\sigma^2(1-\tau))$, we can write

$$\begin{aligned} \|\epsilon(1)\|_\tau &= \mathbb{E}_0 \left[\left(\frac{(\epsilon_1(\tau) \exp(\frac{1}{2}\sigma^2(1-\tau)))^{-b} + (\epsilon_2(\tau) \exp(\frac{1}{2}\sigma^2(1-\tau)))^{-b}}{2} \right)^{-1/b} \right] \\ &= 2^{1/b} \exp\left(\frac{1}{2}\sigma^2(1-\tau)\right) \mathbb{E}_0 \left[\left(\epsilon_1^{-b}(\tau) + \epsilon_2^{-b}(\tau) \right)^{-1/b} \right]. \end{aligned}$$

The normal vector $\ln \epsilon(\tau)$ can be rewritten in terms of two independent standard normal random variables as $\ln \epsilon(\tau) = \sqrt{\tau\Sigma}\mathbf{Z}$, where $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$ and \mathbf{I} is a 2×2 identity matrix. Since $\tau\Sigma$ is positive definite, $\sqrt{\tau\Sigma}$ exists and can be obtained using eigenvector decomposition, $\sqrt{\tau\Sigma} =$

$\sqrt{\tau}\sigma \begin{pmatrix} \sqrt{(1-\rho)/2} & \sqrt{(1+\rho)/2} \\ -\sqrt{(1-\rho)/2} & \sqrt{(1+\rho)/2} \end{pmatrix}$. Using this transformation, we obtain

$$\|\epsilon(1)\|_{\tau} = 2^{1/b} \exp\left(\frac{1}{2}\sigma^2(1-\tau)\right) \times \mathbb{E}_0 \left[\exp\left(\sqrt{\tau}\sigma\sqrt{(1+\rho)/2}Z_2\right) \left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) \right)^{-1/b} \right].$$

Since Z_1 and Z_2 are independent, we can further simplify

$$\|\epsilon(1)\|_{\tau} = 2^{1/b} \exp\left(\frac{1}{2}\sigma^2 - \frac{1}{4}\tau\sigma^2 + \frac{1}{4}\tau\sigma^2\rho\right) \times \mathbb{E}_0 \left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) \right)^{-1/b} \right]. \quad (11)$$

Next, we take the derivative of (11) with respect to τ . After some algebra, we obtain

$$\frac{\partial}{\partial\tau} \|\epsilon(1)\|_{\tau} = \frac{1}{4}\sigma^2(\rho-1)\|\epsilon(1)\|_{\tau} + 2^{1/b} \exp\left(\frac{1}{2}\sigma^2 - \frac{1}{4}\tau\sigma^2 + \frac{1}{4}\tau\sigma^2\rho\right) \frac{\sigma\sqrt{(1-\rho)/2}}{\sqrt{\tau}} \times \mathbb{E}_0 \left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) \right)^{-1/b-1} \exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) Z_1 \right]. \quad (12)$$

To evaluate (12), we make use of the fact that for a differentiable function g and a standard normal random variable Z_1 , $\mathbb{E}(g(Z_1)Z_1) = \mathbb{E}g'(Z_1)$ (Rubinstein 1976). Applying this result and some

algebra to (12), we obtain

$$\begin{aligned}
\frac{\partial}{\partial \tau} \|\epsilon(1)\|_{\tau} &= -\sigma^2 (1 - \rho) (1 + b) 2^{1/b} \exp\left(\frac{1}{2}\sigma^2 - \frac{1}{4}\tau\sigma^2 + \frac{1}{4}\tau\sigma^2\rho\right) \times \\
&\quad \mathbb{E}_0 \left[\left(\exp\left(-b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) + \exp\left(b\sqrt{\tau}\sigma\sqrt{(1-\rho)/2}Z_1\right) \right)^{-1/b-2} \right] \\
&= -\sigma^2 (1 - \rho) (1 + b) 2^{1/b} \mathbb{E}_0 \left[\left(\prod_{i=1,2} \mathbb{E}_{\tau}^{-b}\epsilon_i(1) \right) \left(\sum_{i=1,2} \mathbb{E}_{\tau}^{-b}\epsilon_i(1) \right)^{-1/b-2} \right] \\
&\geq 0. \square
\end{aligned}$$

Proof of Lemma 3: The proof is similar to the proof of Lemma 2 and is omitted. \square

Proof of Lemma 4: The proof is similar to the proof of Lemma 2 and is omitted. \square