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Multi-factor Dynamic Investment under Uncertainty*

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We characterize a firm's optimal factor adjustment when any number of factors face "kinked" linear adjustment costs so that all factor accumulation is costly to reverse. We first consider a general non-stationary case with a concave operating profit function, unrestricted form of uncertainty and a horizon of arbitrary length. We show that the optimal investment strategy follows a control limit policy at each point in time. The state space of the firm's problem is partitioned into various domains, including a continuation region where no adjustment should optimally be made to factor levels. We then consider two specific model classes and exploit their special structure to derive expressions for their continuation regions. *Journal of Economic Literature* Classification Numbers: D92, E22, E24.

1. INTRODUCTION

When it is costly to reverse investment in capital or labor, a firm's investment decisions governing factor use take on an important dynamic quality. This occurs, for example, when a firm faces labor firing costs or when it cannot recoup the acquisition price of capital when it is resold. In that case, the firm will use current information and its assessment of the future when deciding on the optimal stock level of each factor whose accumulation is costly to reverse. This paper presents a general model of dynamic investment in multiple factors (which may include various forms of labor and capital) under uncertainty. We show that the optimal investment strategy follows a control limit policy at each point in time, and then examine in more detail the implications of multi-factor costly reversibility in two specific classes of models.

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The possibility that investment may be costly to reverse has been recognized in the literature at least as far back as 1968, cf. Arrow [5], but has received more attention recently in a stochastic framework. Work by Bernanke [8], McDonald and Siegel [30] and Dixit [14] analyzes investment in projects or firms of discrete size. Later work by Pindyck [34] and Bertola [9] extends earlier models by allowing incremental investment. Most of this literatures studies investment in a single factor, implicitly assuming that no other factors are present or that they can be instantaneously and costlessly adjusted (and thus can be "maximized out" of the profit function). This literature emphasizes the finding that the marginal revenue product of capital sufficient to justify irreversible investment is greater than the Jorgensonian [23] user cost of capital; Abel and Eberly [2] show that in this case, the marginal revenue product of capital is equal to the user cost of capital appropriately calculated in the presence of costly reversibility. A similar focus on single factor investment is found in most of the operations research literature that studies capacity expansion, inspired by the seminal work of Manne [28, 29]. Luss [26] presents an extensive survey. More recent research has focused on expansion under uncertainty, *e.g.*, Davis, Dempster, Sethi and Vermes [13], Bean, Higle and Smith [6], and Paraskevopoulos, Karakitsos and Rustem [33].

While the literature above has taken labor to be perfectly flexible and has focused on investment in capital, costly changes in employment have recently brought more attention to investment in labor. Costly reversibility of labor investment has been recognized at least since the seminal work of Oi [32], but modeling costly reversibility by "kinked", linear costs of adjustment[1] was done only more recently. This allows the same analytical methods applied to investment in the papers above to be used to analyze employment decisions under uncertainty. Bentolila and Bertola [7] use this approach in a single factor (homogeneous labor) model to suggest that firing costs can rationalize the dynamics of European unemployment since the 1970s.

Multiple factor inputs have been considered in econometric models with quadratic costs of adjustment. These models typically focus on multiple types of capital inputs, as in Wildasin [40], Hayashi and Inoue [21] and Chirinko [12], although Nadiri and Rosen [31] and Galeotti and Schiantarelli [19] explicitly include costs of adjusting both capital and labor. The operations management literature has begun to investigate single-period, multi-factor models to study simultaneous investment in

[1] The cost to adjust a factor level either up or down is assumed to be linear (as modeled by equation (3) in the next section), but both directions may have different marginal costs, leading to a "kink" in the cost function or discontinuity in the slope of the cost function at zero adjustment.

flexible and dedicated manufacturing resources; see for example Fine and Freund [18]. Most closely related to our work is that of Dixit [15]. He assumes a linearly homogeneous, supermodular profit function to study the optimal investment dynamics in two factors, capital and labor, with kinked, linear adjustment costs in an infinite horizon model, which is a special case of the general framework we present here. Like Dixit, we show that the "flexibility" of a factor is determined endogenously, and in addition, we are able to derive the specific determinants of "flexibility" for a parametric model that satisfies Dixit's functional restrictions.

This paper presents a framework to study multi-factor investment under uncertainty[2] and has the following outline. The next section models the investment problem and formulates the research question in terms of our model primitives. Section 3 establishes minimal conditions on the operating profit function and the investment adjustment cost function for the existence of an optimal control limit investment policy, called an "Invest/Stay put/Disinvest (ISD)" policy, whose characteristic properties are presented. The state space of the dynamic investment problem is partitioned into various domains, including a "continuation region" where no adjustment need be made in the vector of current factor levels. From any one of the domains outside this continuation region, the optimal investment action is to adjust the vector of factor levels to a specified new point on the boundary of the continuation region. This control limit policy is optimal for any number of factors, with linear or convex adjustment cost functions, for concave operating profit functions, with non-stationary data, with either a finite or infinite planning horizon, and with essentially no restriction on the form or nature of uncertainty confronting the firm. Section 4 discusses the investment dynamics that result when optimal ISD policies are used.

As a first application of the general theory, Section 5 considers a stationary finite horizon model with IID periods and presents a closed form expression that characterizes the optimal investment strategy. More insights can be gained from a richer class of stationary infinite horizon models, called *stationary Markov models*, that place specific restrictions on the operating profit function and the stochastic process governing uncertainty. Section 6 shows that for these models the threshold for adjusting each factor depends positively on the levels of other factors, so that the more labor (for example) that the firm has, the more willing it is to invest in capital, or other factors. We also show that there exists an equilibrium

[2] Our model formulation also can be regarded as a multiperiod generalization of certain convex stochastic problems with linear constraints. As such, it is related to investment-consumption problems (e.g., Abrams and Karmarkar [3], Fama [17]), cash balance models (e.g., Eppen and Fama [16]), models of economic growth (e.g., Brock and Mirman [11]), and multidimensional stochastic production and inventory problems (e.g., see Karmarkar's [24, 25] models and extensive literature overview).

ordering of factor inputs that determines the frequency with which a factor is adjusted, or alternately, the "flexibility" of a factor. Section 7 studies a special subclass of stationary Markov models, called Brownian isoelastic models, whose thresholds that define the continuation region of the optimal investment strategy can be expressed in terms of the marginal profitability of a factor. In that case, the average profitability is proportional to marginal and may be used as a proxy. However, the presence of other factors that are costly to reverse reduces the importance of the costly reversibility of any single factor. In particular, the level of the marginal revenue product that justifies investing in a factor is decreased by the presence of those other factors, so that ignoring the presence of other costly reversible factors overestimates (by an arbitrarily large amount) the threshold associated with any single factor. Section 7 also derives the endogenous parameters that determine the equilibrium "flexibility" ordering of factor inputs and presents closed form solutions to the special case of irreversible investment. We offer concluding remarks in Section 8.

We conclude this section with some notational conventions. We will not distinguish in notation between scalars and vectors. All vectors are assumed to be column vectors, and primes denote transposes, so $u'v$ is the inner product of u and v. The ith component of an n-dimensional vector is denoted by v_i (or by $z_{t,i}$ for a time-indexed vector z_t). For $1 \leqslant i \leqslant j \leqslant n$, the subvector $(v_i, ..., v_j)$ is denoted by $v^{i:j}$; for $j=0$, we define $v^{i:0}=0$. Deleting the ith component in v yields the $n-1$ dimensional vector $v_{(i)} = (v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_n)'$. Vector inequalities and exponents should be interpreted componentwise, as well as $\max(0, u)$ and $\max(0, -u)$ which are denoted by u^+ and u^-, respectively. $\mathbb{N}$ is the set of non-negative integers and $\mathbb{R}$ is the real line. Finally, $\nabla g(x)$ denotes any subgradient of a concave function g at the point x, *i.e.*, $g(y) \leqslant g(x) + \nabla g(x)'(y-x)$ for all x, y in the domain of g.

2. NOTATION, MODEL PRIMITIVES, AND THE RESEARCH QUESTION

Consider a firm that employs n different factors of production and that has the option to change the stock level of each factor at the beginning of each period $t \in \{1, ..., T\}$ (for simplicity, a discrete-time model is adopted). At each such point in time, the firm will base its investment decision on the information then available and on its assessment of the uncertain future. Information availability and uncertainty, which are crucial to any investment strategy, are modeled by a standard probabilistic framework with a probability space $(\Omega, \mathscr{F}, P)$ and filtration $\mathbb{F} = \{\mathscr{F}_1, ..., \mathscr{F}_T\}$ as primitives. The filtration $\mathbb{F}$ is an increasing family of sub-σ-fields that shows how information arrives and uncertainty is resolved as time passes, with $\mathscr{F}_t$ representing the information available at the beginning of period t. We say that a process $Q=(Q_1, ..., Q_T)$ is *adapted* if Q_t is $\mathscr{F}_t$-measurable for all $t \in \{1, ..., T\}$ (informally, Q_t depends only on information available at the beginning of period t). Having observed that information and given its prior investment decision, the firm must choose a non-negative n-vector $K_t \in \mathbb{R}^n_+$ whose ith component represents the level of factor i to be made available for production during period t.

Given this choice of factor levels and given that state of the world $\omega \in \Omega$ obtains, the operating profit[3] earned in period t is $\pi_t(K_t, \omega)$. For each $t \in \{1, ..., T\}$, the function $\pi_t(\cdot, \cdot)$, also called the *operating profit function*, is assumed jointly measurable, $\pi_t(K_t, \cdot)$ is $\mathscr{F}_t$-measurable[4] for each $K_t \in \mathbb{R}^n_+$, and $\pi_t(\cdot, \omega)$ is *concave* for each $\omega \in \Omega$.

To illustrate this framework consider the following example. A firm sells m products in a competitive market where prices are uncertain. Let p_t represent the unit price vector for period t, which is observed at the beginning of the period *before* K_t is chosen. According to the philosophy of continuous improvement, the firm is improving its manufacturing technologies, but not in a deterministic fashion. The firm's technology matrix A_t for period t also is observed at the beginning of the period, before K_t is chosen. Assume for simplicity that production costs are zero. We let $\mathscr{F}_t$ be the σ-field generated by $\{p_1, A_1, ..., p_t, A_t\}$. Assuming that the firm's production quantities x_t are linearly constrained, the firm will set period t production according to the linear program:

$$\max_{x_t \in \mathbb{R}^m_+} \quad p_t(\omega)' \, x_t \tag{1}$$

$$\text{s.t.} \quad A_t(\omega) \, x_t \leqslant K_t. \tag{2}$$

The maximal value of the objective is the operating profit function $\pi_t(\cdot, \omega)$, which is indeed concave for each ω and t.

When changing investment levels from vector K_{t-1} to K_t, the firm incurs an adjustment cost $C_t(K_t - K_{t-1})$. We will focus on kinked piece-wise

[3] Thus, following Arrow [5, p. 2], we focus on any number of costly reversible factors and assume that "all other inputs and outputs to the production process are flows. Then for any fixed stock of [costly reversible factors], there is at any moment a most profitable current policy with regard to the flow variables; we assume the flow optimization to have taken place and therefore have defined a function relating operating profits (excess of sales over costs of flow inputs) as a function of the stock of [capital and labor]."

[4] It may seem restrictive to require that $\pi_t(K_t, \cdot)$ be $\mathscr{F}_t$-measurable, but starting with a more general formulation where the period t profit function depends on events observed during that period, one may simply define $\pi_t(K_t, \cdot)$ as a conditional expectation given $\mathscr{F}_t$, and our formulation then pertains.

linear convex functions C_t (noting that our results readily extent to functions C_t that are *jointly convex* in (K_{t-1}, K_t)):

$$C_t(x) = c_t' x^+ - r_t' x^-, \tag{3}$$

where the marginal investment costs c_t and disinvestment revenues r_t usually are positive. However, disinvestment in some factors (for example, labor) may be costly, or even prohibitively expensive, which means that investment is irreversible. Also, investment may be encouraged (for example, subsidized) and could conceivably generate a cash inflow. Thus, both c_t and r_t can have positive and negative components that can be infinite if investment or disinvestment is prohibitively expensive. We focus on factors that are costly to reverse,[5] such that $c_t - r_t > 0$. Also, we assume that the present value of a unit of used capacity cannot be higher than a new unit, i.e., $c_t \geqslant \delta^{\tau - t} r_\tau$ for $\tau = t, ..., T$, where $\delta > 0$ is the one-period discount factor (which also could be dynamic).

The final element of the formulation is a salvage value function $f(K, \omega)$ (mnemonic for final), where $f(\cdot, \cdot)$ is assumed jointly measurable and $f(\cdot, \omega)$ is *concave* for each $\omega \in \Omega$. We interpret $f(K, \omega)$ as the firm's final (salvage) value, possibly negative, for factor vector K given that state ω obtains.

Formally, an investment strategy is an adapted process $\mathscr{K} = (K_1, ..., K_T)$ such that (i) $0 \leqslant K_t < \infty$ a.s. for all $t \in \{1, ..., T\}$ and (ii) the discounted control costs are absolutely[6] summable a.s.,

$$\sum_{t=1}^{T} \delta^t (|r_t|' (K_{t-1} - K_t)^+ + |c_t|' (K_t - K_{t-1})^+) < \infty \qquad \text{a.s.} . \tag{4}$$

The initial factor levels vector $K_0 \in \mathbb{R}^n_+$ is given as problem data. The firm's expected net present value, evaluated at the beginning of period 1, under strategy $\mathscr{K}$ starting with initial levels K_0 is

$$v(K_0, \mathscr{K}) = \mathrm{E}\left[\sum_{t=1}^{T} \delta^{t-1} (\pi_t(K_t, \omega) - C_t(K_t - K_{t-1})) + \delta^T f(K_T, \omega) \right]. \tag{5}$$

We assume that the firm's objective is to maximize its expected net present value. Denoting the set of all strategies by $\mathbb{K}$, the research problem then is to determine a strategy $\mathscr{K} \in \mathbb{K}$ that maximizes $v(K_0, \mathscr{K})$.

[5] Optimal factor levels of costlessly reversible factors are found by optimizing the (extended) operating profit function. We assume that these factors have been maximized out and therefore don't appear in the operating profit function.

[6] Investment and disinvestment rates are not necessarily positive and may be infinite, in which case adjustment should not be allowed.

3. THE OPTIMAL INVESTMENT STRATEGY

3.1. *Optimality Equations*

Let $\mathscr{K}_t = (K_t, ..., K_T)$ for $t > 1$ denote a *partial* investment strategy, and set $\mathscr{K} = \mathscr{K}_1$ to simplify notation. In the usual way, we suppress dependence of π_t and f on ω, writing[7] $\pi_t(K_t)$ and $f(K_T)$ to mean $\pi_t(K_t, \cdot)$ and $f(K_T, \cdot)$. Let $v_t(K_{t-1}, \mathscr{K}_t, \omega)$ be the firm's expected net present value, evaluated at beginning of period t and conditioned on the available information, given that we start with factor levels K_{t-1} and strategy $\mathscr{K}_t$ is implemented:

$$v_t(K_{t-1}, \mathscr{K}_t, \omega) = \mathrm{E}\left[\sum_{\tau=t}^{T} \delta^{\tau-t}(\pi_\tau(K_\tau) - C_\tau(K_\tau - K_{\tau-1})) + \delta^{T+1-t} f(K_T) \,\middle|\, \mathscr{F}_t \right](\omega). \quad (6)$$

Denote the set of all partial investment strategies $\mathscr{K}_t$ by $\mathbb{K}_t$. The optimal value function when starting at the beginning of period t with factor levels vector K_{t-1} is

$$V_t(K_{t-1}, \omega) = \sup_{\mathscr{K}_t \in \mathbb{K}_t} v_t(K_{t-1}, \mathscr{K}_t, \omega). \quad (7)$$

The optimal value functions are adapted and satisfy the following recursive *optimality equations* for $t \in \{1, ..., T\}$:

$$V_t(K_{t-1}, \omega) = \sup_{\mathscr{K}_t \in \mathbb{K}_t} \{\pi_t(K_t, \omega) - C_t(K_t - K_{t-1}) + \delta \mathrm{E}[V_{t+1}(K_t) \mid \mathscr{F}_t](\omega)\} \quad (8)$$

$$V_{T+1}(K_T, \omega) = f(K_T, \omega). \quad (9)$$

We will assume that the supremum of (8) is attained by a unique admissible strategy, which we call the "optimal investment strategy". (Refer to Maitra [27] and Blackwell, Freedman and Orkin [10] for conditions guaranteeing measurability of the optimal value functions.) With convex adjustment costs, the concavity of the operating profit functions π_t is inherited by the optimal value functions:

THEOREM 1. *The optimal value function $V_t(\cdot, \omega)$ is concave for every $\omega \in \Omega$ and $t \in \{1, ..., T\}$.*

Proof. The terminal function f is concave by assumption. Now use induction on t and assume V_{t+1} is concave. We use a concavity preservation

[7] Notice that ω dependence is suppressed in two ways: $\pi_t(K_t(\omega), \omega)$.

lemma[8] from Heyman and Sobel [22, p. 525, Prop. B-4] on (8): $\{K_t \geqslant 0\} = \mathbb{R}^n_+$ is nonempty, $A = \mathbb{R}^{2n}_+$ is a convex set, and $\pi_t(K_t, \omega) - C_t(K_t - K_{t-1}) + \delta \mathrm{E}[V_{t+1}(K_t) \mid \mathscr{F}_t](\omega)$ is jointly concave in $(K_{t-1}, K_t) \in A$ as a sum of jointly concave functions.[9] Thus V_t is also concave in K_{t-1}. ▌

Corollary 1. *$V_t(\cdot, \omega)$ is continuous and has non-increasing left and right partial derivatives, which are equal almost everywhere, for every $\omega \in \Omega$ and $t \in \{1, ..., T\}$.* (Royden [38, p. 113])

Thus, the subgradient ∇V_t is unique and equal to the gradient of V_t, except on a set of Lebesgue measure zero.

3.2. *ISD Policies*

A policy for period t is a decision rule that prescribes a specific action at the beginning of period t for each $K_{t-1} \in \mathbb{R}^n_+$ and $\omega \in \Omega$. An investment strategy prescribes a policy for each period. Formally, a policy for period t is a function $\kappa_t: \mathbb{R}^n_+ \times \Omega \to \mathbb{R}^n_+: (K_{t-1}, \omega) \to K_t$ where $\kappa_t(\cdot, \cdot)$ is jointly measurable and $\kappa_t(K_{t-1}, \cdot)$ is $\mathscr{F}_t$-measurable for each $K_{t-1} \in \mathbb{R}^n_+$. An investment strategy is then $\mathscr{K} = (\kappa_1(K_0), \kappa_2(K_1), ..., \kappa_T(K_{T-1}))$, where $K_t = \kappa_t(K_{t-1})$ suppressing the ω dependence in the notation. The next section will show that it is optimal to invest according to a certain kind of control limit policy which we call an ISD (Invest/Stay put/Disinvest) policy. We define the ISD policy as follows:

Definition 1. A policy κ_t for period $t \in \{1, ..., T\}$ is an ISD policy if there exist two functions $K^L_t(K_{t-1}, \omega)$ and $K^H_t(K_{t-1}, \omega)$ mapping $\mathbb{R}^n_+ \times \Omega \to \mathbb{R}^n_+$ such that, for any $i \in \{1, ..., n\}$:

1. $K^L_t(\cdot, \cdot) \leqslant K^H_t(\cdot, \cdot)$.
2. the ith components, $K^L_{t,i}(K_{t-1}, \omega)$ and $K^H_{t,i}(K_{t-1}, \omega)$, depend on the pre-adjustment levels $K_{t-1,(i)}$ of the other factors, but not on $K_{t-1,i}$.
3.

$$\kappa_{t,i}(K_{t-1}, \omega) = \begin{cases} K^L_{t,i}(K_{t-1}, \omega) & \text{if } K_{t-1,i} < K^L_{t,i}(K_{t-1}, \omega), \\ K^H_{t,i}(K_{t-1}, \omega) & \text{if } K_{t-1,i} > K^H_{t,i}(K_{t-1}, \omega), \\ K_{t-1,i} & \text{otherwise.} \end{cases} \tag{10}$$

[8] Lemma 1 (Concavity Preservation). *If $Y(x)$ is a non-empty set for every $x \in X$, the set $A = \{(x, y): x \in X, y \in Y(x)\}$ is a convex set, and $g(x, y)$ is jointly concave on A, then $h(x) = \sup_{y \in Y(x)} g(x, y)$ is a concave function on any convex subset of $\{x \in X: h(x) > -\infty\}$.*

[9] The specific form $-C_t(K_t - K_{t-1}) = r'_t(K_{t-1} - K_t)^+ - c'_t(K_t - K_{t-1})^+$ is jointly concave as a sum of two halfplanes with $r_t \leqslant c_t$.

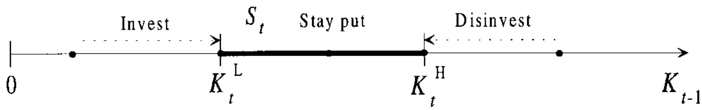

FIG. 1. Structure of a one-dimensional ($n=1$) ISD policy.

An ISD policy adjusts each factor level i according to a control limit policy: there are two critical numbers $K^L_{t,i} \leqslant K^H_{t,i}$ such that level i is increased to $K^L_{t,i}$ if $K_{t-1,i} < K^L_{t,i}$, decreased to $K^H_{t,i}$ if $K_{t-1,i} > K^H_{t,i}$, and not adjusted otherwise. An ISD policy is monotone in the action: the three possible actions occur in at most three connected zones that are ordered from invest $I \to$ stay put $S \to$ disinvest D along any coordinate axis of the state variable K. An ISD policy for the single factor case ($n=1$) has the structure shown in Fig. 1.

For ease of notation, we will state all results in this section in terms of a function $g_t \colon \mathbb{R}^n_+ \times \Omega \to \mathbb{R}$, where $g_t(\cdot, \omega)$ is *closed concave*[10] for each $\omega \in \Omega$. For example, one may think of $g_t(K_t, \omega)$ as the firm's expected net present value, evaluated at the beginning of period t and conditioned on the available information, given that factor levels have been adjusted to K_t and an optimal (partial) investment strategy is implemented:

$$g_t(K_t, \omega) = \pi_t(K_t, \omega) + \delta \mathrm{E}[V_{t+1}(K_t) \mid \mathscr{F}_t](\omega), \tag{11}$$

which is closed concave ($g_t(\cdot, \omega)$ is concave, continuous (and thus semi-continuous), and finite (since V_{t+1} is finite a.s.)). ISD policies are intimately connected to the concave optimality equations (8), which the firm has to solve under optimal investment decision-making at the beginning of each period:

THEOREM 2. *Consider $g_t \colon \mathbb{R}^n_+ \times \Omega \to \mathbb{R}$, with $g_t(\cdot, \omega)$ closed concave for each $\omega \in \Omega$ as above. If the solution $\kappa_t(K_{t-1}, \omega)$ to the following concave optimization problem*

$$G(K_{t-1}, \omega) = \sup_{K_t \in \mathbb{R}^n_+} \{ g_t(K_t, \omega) - C_t(K_t - K_{t-1}) \}, \tag{12}$$

where $K_{t\text{-}1} \in \mathbb{R}^n_+$ and $\omega \in \Omega$, is unique, then κ_t is an ISD policy.

Proof. We must show that the solution κ can be expressed in terms of two functions $K^L_t(K_{t-1}, \omega)$ and $K^H_t(K_{t-1}, \omega)$ mapping $\mathbb{R}^n_+ \times \Omega \to \mathbb{R}^n_+$,

[10] A function $h(\cdot)$ is closed concave if it is concave, finite, and lower semi-continuous on its domain (Rockafellar [37, p. 52]).

that satisfy the three conditions in Definition 1. For ease of notation, let $x = K_{t-1}$, $y = K_t$, and suppress the t and ω dependence in the notation.

Case I. $n = 1$. The optimization problem becomes

$$G(x) = \sup_{y \geqslant 0} \{r(x-y)^+ - c(y-x)^+ + g(y)\}.$$

With $r \leqslant c$, the objective function is concave and first order (sub)differential conditions are sufficient. For $x = 0$, an optimal value for y is

$$y^L = \sup\{\{0\} \cup \{y: \nabla g(y) \geqslant c\}\}. \tag{13}$$

Concavity yields that this *invest-up-to* level y^L is optimal for all $x \leqslant y^L$. For arbitrary large x, an optimal value for y is

$$y^H = \inf\{\{\infty\} \cup \{y: \nabla g(y) \leqslant r\}\}, \tag{14}$$

which is also optimal for all $x \geqslant y^H$. Concavity yields that it is optimal to "stay put" whenever $y^L \leqslant x \leqslant y^H$, i.e., the optimal value for y is x in that case. Thus, letting K^L and K^H be the constant functions y^L and y^H, respectively, we have that:

1. $K^L \leqslant K^H$.
2. K^L and K^H are independent of the pre-adjustment level x.
3. $$\kappa(x) = \begin{cases} K^L & \text{if} \quad x < K^L, \\ K^H & \text{if} \quad x > K^H, \\ x & \text{otherwise.} \end{cases} \tag{15}$$

Thus, the constant functions K^L and K^H define an ISD policy κ that solves the optimization problem G for $n = 1$.

Case II. $n > 1$. The n-dimensional functions K^L and K^H are defined componentwise as follows. Fix any $i \in \{1, 2, ..., n\}$. Recall that $v_{(i)}$ is the $n-1$ dimensional vector which obtains after deleting the ith component in $v \in \mathbb{R}^n$. For $x, y \in \mathbb{R}^n_+$, let

$$H(x_{(i)}, y_i) = \sup_{y_{(i)} \in \mathbb{R}^{n-1}_+} \{r'_{(i)}(x_{(i)} - y_{(i)})^+ - c'_{(i)}(y_{(i)} - x_{(i)})^+ + g(y)\}. \tag{16}$$

We can rewrite the original optimization problem G as:

$$G(x) = \sup_{y_i \geqslant 0} \{r_i(x_i - y_i)^+ - c_i(y_i - x_i)^+ + H(x_{(i)}, y_i)\}. \tag{17}$$

Because $H(x_{(i)}, \cdot)$ is concave according to Lemma 1, we can use the reasoning of Case I: Define the ith component of the n-dimensional function K^L and K^H as

$$K_i^L(x) = \sup\{\{0\} \cup \{y_i: \nabla_{y_i} H(x_{(i)}, y_i) \geqslant c\}\}, \tag{18}$$

$$K_i^H(x) = \inf\{\{\infty\} \cup \{y_i: \nabla_{y_i} H(x_{(i)}, y_i) \leqslant r\}\}. \tag{19}$$

It follows directly that

1. $K_i^L(\cdot) \leqslant K_i^H(\cdot)$.

2. $K_i^L(x)$ and $K_i^H(x)$ depend on the pre-adjustment levels $x_{(i)}$ of the other factors, but not on x_i.

3.

$$\kappa_i(x) = \begin{cases} K_i^L(x) & \text{if} \quad x_i < K_i^L(x), \\ K_i^H(x) & \text{if} \quad x_i > K_i^H(x), \\ x_i & \text{otherwise.} \end{cases} \tag{20}$$

Thus, for each factor *i*, there exists an optimal policy that is ISD for factor *i*. Now, since the solution to the optimization problem is unique, all solutions found by each of the n policies described above all lead to the same optimal point (the factor level after adjustment for all n factors). Thus, each policy found in each of the n separate problems must be the same, so that this unique policy is ISD w.r.t. all of the n different factors. ▮

Remark. The proof shows that the two n-dimensional functions K^L and K^H, that define the ISD policy, can be constructed from n parametric optimization problems of dimension $n-1$.

We will refer to an ISD policy that solves optimization problem (12) as *an ISD policy generated by* g_t. If the objective function in (12) is strictly concave, there is a unique optimal ISD policy that solves G. The concavity of g_t yields additional properties to such an ISD policy:

PROPERTY 1. *An ISD policy κ_t for period t generated by g_t is characterized by a connected set $S_t(\omega) \subset \mathbb{R}_+^n$ for each $\omega \in \Omega$, where*

$$S_t(\omega) = \{K \in \mathbb{R}_+^n : r_t \leqslant \nabla g_t(K, \omega) \leqslant c_t\}, \tag{21}$$

as follows:

1. *If $K_{t-1} \in S_t(\omega)$, no adjustments are made*: $\kappa_t(K_{t-1}, \omega) = K_{t-1}$.

2. *If $K_{t-1} \notin S_t(\omega)$, an adjustment to a point $\kappa_t(K_{t-1}, \omega)$ on the boundary of $S_t(\omega)$ is made.*

Proof. First notice that $S_t(\omega)$ is the $g_t(\cdot, \omega)$-subdifferential map from the convex set $N_t = \{v \in \mathbb{R}^n : r_t \leqslant v \leqslant c_t\}$. As such, $S_t(\omega)$ is a connected set because g_t is closed concave (Abrams and Karmarkar [3, p. 346].) We will prove the rest of the Property by induction on n using the notation and results in the proof of Theorem 2. The proof for $n=1$ follows directly from the proof of Theorem 2 where a one-dimensional ISD policy generated by g, which solves the optimization problem G, is characterized by

$$S = [K^L, K^H] = \{K \in \mathbb{R}^1_+ : r \leqslant \nabla g(K) \leqslant c\}. \tag{22}$$

Now assume the theorem is valid for $n-1 \geqslant 1$. We show that the theorem also holds for n. The induction hypothesis yields that an $n-1$ dimensional ISD policy in $x_{(i)}$ generated by $g(\cdot, y_i)$, which solves the optimization problem $H(\cdot, y_i)$, is characterized by

$$S^1(y_i) = \{K_{(i)} \in \mathbb{R}^{n-1}_+ : r_{(i)} \leqslant \nabla_{K_{(i)}} g(K_{(i)}, y_i) \leqslant c_{(i)}\}. \tag{23}$$

G is solved by a one-dimensional ISD policy in x_i generated by $H(x_{(i)}, \cdot)$ and characterized by

$$S^2(x_{(i)}) = \{K_i \in \mathbb{R}^1_+ : r_i \leqslant \nabla_{K_i} H(x_{(i)}, K_i) \leqslant c_i\}. \tag{24}$$

Now, only if $x_{(i)} \in S^1(y_i)$ and $x_i \in S^2(x_{(i)})$, then it is optimal not to adjust: $y_{(i)} = x_{(i)}$ and $y_i = x_i$, so that $y = x$. Thus, the set S in which the optimal n-dimensional ISD policy makes no adjustments is

$$S = \{x \in \mathbb{R}^n_+ : x_{(i)} \in S^1(y_i),\ x_i \in S^2(x_{(i)})\}, \tag{25}$$

$$= \{x \in \mathbb{R}^n_+ : x_{(i)} \in S^1(x_i),\ x_i \in S^2(x_{(i)})\}, \tag{26}$$

$$= \{x \in \mathbb{R}^n_+ : r_{(i)} \leqslant \nabla_{x_{(i)}} g(x_{(i)}, x_i) \leqslant c_{(i)},\ r_i \leqslant \nabla_{x_i} H(x_{(i)}, x_i) \leqslant c_i\}. \tag{27}$$

If $x \in S$, $H(x_{(i)}, x_i) = g(x_{(i)}, x_i)$ and (changing dummy variable to K):

$$S = \{K \in \mathbb{R}^n_+ : r \leqslant \nabla g(K) \leqslant c\}. \tag{28}$$

Finally, if $x \notin S$, either adjustment $y_{(i)}$ is on the boundary of $S^1(y_i)$ and/or y_i is on the boundary of $S^2(x_{(i)})$ so that the adjustment y is on the boundary of $S^1(y_i) \times S^2(x_{(i)}) = S$. ∎

Thus, if the factor levels vector K_{t-1} is within $S_t(\omega)$, it is optimal "to stay put" on all dimensions, i.e., to keep the same vector of investment levels for the next period ($K_t = K_{t-1}$). Therefore, the set S_t is also called the "region of inaction", "central domain", or "continuation region". (We will use these terms interchangeably.) The proof also shows a "dimensionality reduction" property of ISD policies generated by a concave function: If we fix $k \in \{1, ..., n-1\}$ components of K_{t-1} for $n > 1$, then the adjustments of the remaining $n-k$ components are governed by an $n-k$ dimensional ISD policy generated by the (concave) induced objective function that obtains after optimizing the k components in (17).

PROPERTY 2. *If $g_t(\cdot, \omega)$ is twice differentiable for each $\omega \in \Omega$, then the boundaries of $S_t(\omega)$ for factor k are increasing (decreasing) in direction j if the jk cross partial $g_t^{(jk)}(\cdot, \omega)$ is positive (negative).*

This result follows directly by differentiation in (21). Moreover, if $g_t(\cdot, \omega)$ has *continuous* second partial derivatives, $g_t^{(jk)}(\cdot, \omega) = g_t^{(kj)}(\cdot, \omega)$ and the control limits for factors k and j are either both increasing or both decreasing in direction j and k respectively. An operating profit function with positive (negative) cross partials is *supermodular* (*submodular*) and its inputs are economic complements (substitutes) so that a higher optimal investment threshold in factor k justifies a higher (lower) optimal investment threshold in factor j and vice versa, in agreement with Property 2. The following corollary generalizes the result of Dixit, who shows in [15] that the supermodularity of stationary operating profit functions is inherited by the optimal value functions $E[V_{t+1}(\cdot) \mid \mathscr{F}_t]$ in the infinite horizon case, so that the boundaries of the continuation region are increasing.

COROLLARY 2. *Let $g_t = \pi_t + \delta E[V_{t+1} \mid \mathscr{F}_t]$. If the operating profit functions $\pi_t(\cdot, \omega)$ and the salvage function $f(\cdot, \omega)$ are supermodular for each $\omega \in \Omega$ and $t \in \{1, ..., T\}$, then $g_t(\cdot, \omega)$ is also supermodular for each ω and t.*

Proof. We will show by induction on t that V_{t+1} is supermodular. The corollary is then immediate because a positive linear combination of the supermodular functions π_t and V_{t+1} is also supermodular. Clearly $V_{T+1} = f$ is supermodular. Now assume V_{t+1} is supermodular and follow the reasoning of Dixit [15, p. 11]: he shows that each term $c_{t,i}(K_{t,i} - K_{t-1,i})^+ - r_{t,i}(K_{t,i} - K_{t-1,i})^-$ of the adjustment cost function is *sub*modular (jointly in $K_{t,i}$ and $K_{t-1,i}$). Thus $-C_t(K_t - K_{t-1})$ is supermodular as a positive linear combination of supermodular functions. The maximand in the optimality equation 8 is also supermodular by positive linearity. Finally, maximizing over K_t preserves supermodularity in the remaining variable K_{t-1}, as shown by Topkis [39, p. 314], so that $V_t(\cdot, \omega)$ is supermodular for each ω. ∎

PROPERTY 3. *If the continuation region $S_t(\omega)$ is a proper subset of $\mathbb{R}_+^n$ with a non-empty interior, then the collection of points whose optimal level* *after adjustment is a point K on the boundary of $S_t(\omega)$ is $T_t(K, \omega) \bigcap \mathbb{R}^n_+$, where $T_t(K, \omega)$ is an affine convex cone:*[11]

$$T_t(K, \omega) = \{K + x: x \in \mathbb{R}^n, x'(v - \nabla g_t(K, \omega)) \geqslant 0 \text{ for all } v \in \mathbb{R}^n \text{ with } r_t \leqslant v \leqslant c_t\}. \quad (29)$$

Proof. Equation (29) clearly defines a convex affine cone and follows from the convex duality results of Karmarkar [24, pp. 340–342] (refer to Rockafellar [37] for a comprehensive treatment). Using Karmarkar's notation with $A = [1_n, -1_n]$, where 1_n is the n-dimensional identity matrix, we have that $D = \{v: \inf \nabla g < v < \sup \nabla g\}$ as the set of all vectors v for which $\inf_y \{v'y - g(y)\}$ is bounded, and $D' = \{v \in \mathbb{R}^n: r \leqslant v \leqslant c\}$. Thus, interior $D \cap$ interior $D' \neq \emptyset$ if ∇g is not constant (with sufficient condition that S be not empty and not equal to the positive octant $\mathbb{R}_+$) and interior $S \neq \emptyset$, in which case (29) holds. ∎

[$S_t = \emptyset$ corresponds to the uninteresting case where one will either disinvest in any factor $k \in \{1, 2, ..., n\}$ completely ($\sup \nabla_k g_t < r_t$) or invest up to arbitrarily high levels ($\inf \nabla_k g_t > c_t$).] Consider for example a one-dimensional ISD policy for period t generated by g_t which has two cones with vertices K_t^L and K_t^H. If $K_t^L > 0$, the infimum in (13) is attained because g_t is concave and

$$T_t(K_t^L) = \{K_t^L + K: K \in \mathbb{R} \text{ and } (v - c_t)\, K \geqslant 0 \text{ for all } v \in \mathbb{R} \text{ where } r_t \leqslant v \leqslant c_t\} \quad (30)$$

$$= (-\infty, K_t^L]. \quad (31)$$

If $K_t^L = 0$ and the infimum is not attained, (29) degenerates to a point: $K_t(0) = 0$. In any case, $T_t(K_t^L) \bigcap \mathbb{R}_+ = [0, K_t^L]$. Similarly, for a finite boundary point K_t^H, $T_t(K_t^H) = [K_t^H, \infty)$. The structure of an ISD policy generated by a (supermodular) concave function in $n = 2$ dimensions is shown in Fig. 2. The four corner points are the vertices of four convex cones. If we hold one component of K_{t-1} fixed, the adjustment for the other component is governed by a one-dimensional ISD policy. An n-dimensional ISD policy generated by g_t has a central domain with $2n$ faces, which partition $\mathbb{R}^n_+$ in 3^n regions of specific investment action.

PROPERTY 4. *Let, for each $\omega \in \Omega$, the equations $\nabla g_t(K, \omega) = c_t$ and $\nabla g_t(K, \omega) = r_t$ have unique solutions in $K \in \mathbb{R}^n_+$, denoted $K_t^I(\omega)$ and $K_t^D(\omega)$*

[11] $T(K)$ is an affine coine if $T(K) - K$ is a cone. Thus, if $y \in T(K)$ then $K + \alpha(y - K) \in T(K)$ for all $\alpha \in \mathbb{R}_+$.

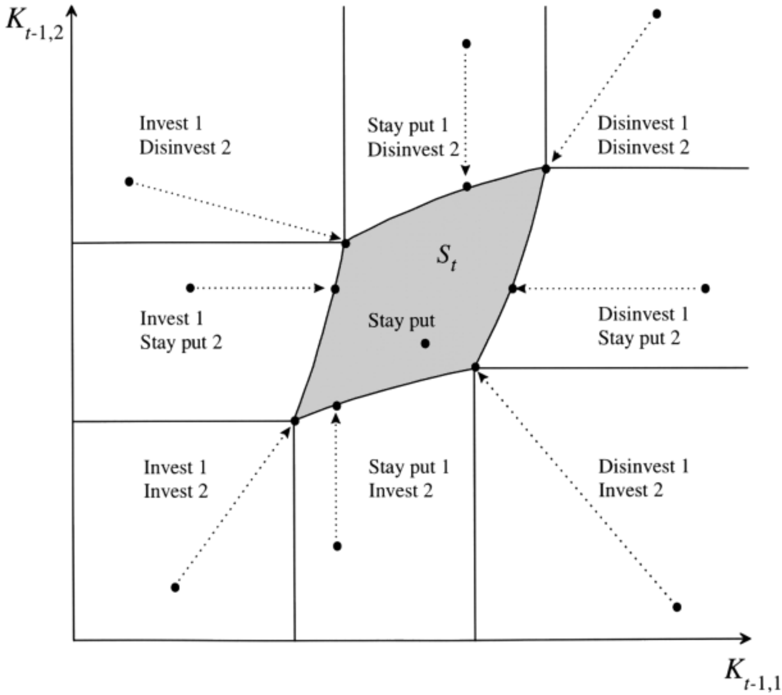

FIG. 2. Structure of a two-dimensional ISD policy generated by a supermodular concave function.

respectively. Then K_t^I *is the* invest-up-to *level for all* $K_{t-1} \leqslant K_t^I$ *and* K_t^D *is the* disinvest-down-to *level for all* $K_{t-1} \geqslant K_t^D$.

Proof. This *base-stock* level result follows directly from (29) which yields the cones $T_t(K_t^I, \omega) = \{K_t^I(\omega) - x: x \in \mathbb{R}_+^n\}$ and $T_t(K_t^D, \omega) = \{K_t^D(\omega) + x: x \in \mathbb{R}_+^n\}$. ∎

Remark. If g_t is strictly concave, then K_t^I and K_t^D are unique if they exist.

3.3. *Optimality of ISD Policies*

THEOREM 3. *An ISD policy generated by* $\pi_t(\cdot, \omega) + \delta E[V_{t+1}(\cdot) \mid \mathscr{F}_t](\omega)$ *is optimal for* t, *for every* $t \in \{1, ..., T\}$ *and* $\omega \in \Omega$.

Proof. Follows from Porteus' framework [35, 36] of preservation and attainment of structured policies and optimal functions. The set of structured policies consists of all ISD policies generated by a closed concave function, and let G^* be the set of functions $g(\cdot, \omega)$ that are closed concave on $\mathbb{R}_+^n$ for each ω. The optimality equations preserve concavity according to Theorem 1, and Theorem 2 shows that the optimal value function is attained by an ISD policy generated by g for any $g \in G^*$, because we have assumed that the optimal investment strategy is unique. ▮

All results essentially apply to the infinite horizon case ($T \to \infty$) under mild technical conditions. A variety of conditions are available from the dynamic programming literature to insure the existence of a solution to the optimality equations. We shall assume the simple but realistic assumption that factor levels are finite and bounded and that the discount factor is strictly smaller than 1, acknowledging that the results hold under much more general[12] conditions. Summarizing, the *Infinite Horizon Assumptions* are: (i) Factor levels are (uniformly) bounded, i.e., there exists a real number M such that $K_t < M$ for all $t \in \mathbb{N}$, and (ii) the discount factor is strictly smaller than one, i.e., $\delta < 1$.

THEOREM 4. *If the additional infinite horizon assumptions hold, then the optimal value functions $V_t(\cdot, \omega)$ are concave for every $\omega \in \Omega$ and every $t \in \mathbb{N}$, and an ISD policy generated by $\pi_t(\cdot, \omega) + \delta E[V_{t+1}(\cdot) \mid \mathscr{F}_t](\omega)$ is optimal for t, for every $t \in \mathbb{N}$ and $\omega \in \Omega$.*

Proof. Again use Porteus' framework [36] of structured policies and optimal functions: Let Δ^* be the set of ISD policies generated by a closed concave function and let $Z^0 = Z^*$ be the set of functions $z(\cdot, \omega)$ that are closed concave on the compact set $[0, M]^n$ for each $\omega \in \Omega$. Because we have shown optimality under finite horizon, we only need to check that the infinite horizon assumption IH in [36, p. 425] holds:

(a) The set of closed concave functions is complete on a compact set under the L^1 (supremum) norm.

(b) Using Porteus' notation, we have that $\rho(H_t^n(v)\, u, H_t^n(v)\, z) < N$ for all structured policies v and regular functions $u, z \in Z_{n+1}^0 = Z^0$, because all operating profit functions, values, and terminal functions are bounded (for each ω) on the compact set because they are concave.

(c) $H_t^m(v)$ is δ-Lipschitzian on $Z_{m+1} = Z$ (the set of all functions $z(\cdot, \omega)$ that are continuous on $[0, M]^n$ for each ω) for all structured policies v and $m \geqslant t$ and $\delta^m M \to 0$ as $m \to \infty$ because $\delta < 1$.

Although not strictly necessary, Porteus' Assumption RP also holds, and invoking Porteus' Theorems 5.1. and 5.2. ends this proof. ▮

[12] E.g., the boundedness assumption can be relaxed with the assumption that total discounted expected operating profits $\sum_t \delta^t \mathrm{E}_t\, |\pi_t(\cdot, \omega)|$ are finite and convergent under the supremum L^1 norm. More relaxation involves increasingly more technical conditions for which we refer to the dynamic programming literature, e.g. Porteus [36].

4. INVESTMENT DYNAMICS

A continuation region with a non-empty interior stems from investment that is costly to reverse: If investment in factor k is costly to reverse, its marginal adjustment costs $r_{t,k} < c_{t,k}$ create a nonlinearity in the adjustment cost function C_t. It is this nonlinearity that generates the stay-put or "hysteresis" zone in the kth coordinate direction whose width is an increasing function of the degree of irreversibility, $c_{t,k} - r_{t,k}$. *Hysteresis* is the failure of an effect to reverse itself as its underlying cause is reversed. Similar phenomena, such as imperfect mechanic elasticity under tension or compression and remaining magnetism in magnetic materials, were originally found in many physical materials. A firm that increases investment in all factors up to the "invest-up-to" level K^L when its current level K is smaller than K^L, does not disinvest in all factors when the cause is reversed (current level larger than K^L), but only when its factor level is larger than the "disinvest-down-to" level K^H (and $K^L < K^H$ if $r < c$). Moreover, if investment in factor k is irreversible, $K_k^H = +\infty$, and its investment can never be reversed. On the other hand, if the investment is reversible or "frictionless", $r_{t,k} = c_{t,k}$, there is no hysteresis zone in coordinate direction k (if the generating function g_t is strictly concave), and one will make adjustments almost always.

An ISD policy is another instance of the principle in economics whereby marginal revenue equals marginal cost with optimal decision making. Imagine that we make an infinitesimal adjustment $\mathrm{d}z_i$ to the level z_i of factor i at time t. If the adjustment is positive, the investment cost is $c_{t,i}\,\mathrm{d}z_i$ and the corresponding increase in the expected net present value of total future returns is $\mathrm{d}\pi_t + \delta\,\mathrm{dE}[V_{t+1} \mid \mathcal{F}_t] \geqslant \nabla_i \pi_t(z)\,\mathrm{d}z_i + \delta \nabla_i \mathrm{E}[V_{t+1}(z) \mid \mathcal{F}_t]\,\mathrm{d}z_i$, suppressing the ω dependence in the notation (the $\geqslant$ sign is needed at non-differentiable points z). Clearly, it is optimal to invest whenever $c_{t,i} \leqslant \nabla_i \pi_t(z) + \delta \nabla_i \mathrm{E}[V_{t+1}(z) \mid \mathcal{F}_t]$. Analogously, if the adjustment is negative, the disinvestment revenue is $-r_{t,i}\,\mathrm{d}z_i$ and the *decrease* in future returns is not larger than $-\nabla_i \pi_t(z)\,\mathrm{d}z_i - \delta \nabla_i \mathrm{E}[V_{t+1}(z) \mid \mathcal{F}_t]\,\mathrm{d}z_i$. Thus it is optimal to disinvest if $r_{t,i} \geqslant \nabla_i \pi_t(z) + \delta \nabla_i \mathrm{E}[V_{t+1}(z) \mid \mathcal{F}_t]$. Theorem 3 shows that these two conditions are disjoint and sufficient because V is concave. If neither of the two conditions holds, it is optimal to make no adjustment.

It is insightful to relate the dynamics of optimal investment according to ISD policies to the research on Brownian control theory, promulgated by Harrison [20]. When the time period becomes arbitrarily small, one can instantaneously adjust investment levels. If we assume enough regularity, a control problem in continuous time is obtained, where the central region S_t may move continuously over time. Only the initial investment adjustment may represent an "impulse" control to the boundary of S_0 if the initial state K_0 is outside S_0. All subsequent optimal controls are the multidimensional generalization of what Harrison calls "barrier" or "instantaneous" controls. No control is needed as long as K_t remains inside S_t. When K_t "hits" the boundary of S_t, the minimal amount of control is exercised so as to prevent K_t from "leaving" S_t. Thus, the optimal investment dynamics are similar to the dynamics of a point K_t floating inside a (possibly moving) domain S_t, being reflected on its boundaries.

In the remaining sections of the paper we study two specific classes of investment models. The first class of models has an IID structure, which makes the model very amenable to analysis and allows us to derive the central region S_t in closed form. The second class of models has uncertainty modeled by a stationary Markov process and assumes supermodular, linearly homogeneous operating profit functions. For this class, we show that factors are always adjusted in a fixed sequence. This fixed sequence relates to the *flexibility* of a factor, which is endogenously specified in our model. Finally, Section 7 studies a subset of the second class and considers a geometric Brownian motion as the generator of uncertainty and an isoelastic, Cobb–Douglas-like operating profit function. Assuming arbitrarily small time periods and sufficient regularity, we obtain a rich class of models that are still amenable to analysis and provide more detailed information on the optimal investment dynamics and factor flexibility.

5. OPTIMAL INVESTMENT IN AN IID STRUCTURE

In general, the continuation region $S_t(\omega)$ depends on the period and on the information available at the beginning of that period. We say that *the periods are independent* if $\mathscr{F}_t = \bigcup_{i=1}^{t} \mathscr{G}_i$ for all $t \in \{1, ..., T\}$, where the σ-fields $\mathscr{G}_1, ..., \mathscr{G}_T$ are independent, and if the operating profit functions do not depend on previous information, i.e., there exist deterministic functions Π_t such that $\pi_t(\cdot, \omega) = \Pi_t(\cdot)$. In that special case, conditional expectations reduce to unconditional expectation, $\mathrm{E}[V_{t+1}(z) \mid \mathscr{F}_t](\omega) = \mathrm{E}V_{t+1}(z)$, and the continuation region depends only on the period but not on the information available at the beginning of that period. The probabilistic ω-dependence modeling information flows no longer is needed and the model greatly simplifies to an exercise in real (deterministic) analysis.

As a first example of the results that can be obtained with the general theory, consider a class of models with the simplest imaginable dynamic structure: an IID structure. In addition to (1) independent periods, an IID structure requires that (2) the probability measures for ω_i and ω_j, where $\omega = \omega_1 \omega_2 \cdots \omega_T$, are identical, and (3) the operating profit function and adjustment cost function are stationary, i.e., $\pi_t = \pi$, $c_t = c$, $r_t = r$ for all $t \in \{1, ..., T\}$. The IID structure is the most amenable to analysis. Information flow is at its bare minimum (the ω-dependence can be dropped), and the multi-period dynamics essentially collapse to a single period model:

PROPOSITION 1. *Let the structure be IID such that $\pi_t(\cdot, \omega_t) = \Pi(\cdot)$ for all t and ω, and the salvage function be identical to the disinvestment function, $f(K, \omega) = r'K$. Starting from any initial investment level $K_0 \in \mathbb{R}^n_+$, the optimal investment strategy makes no factor level adjustments after the beginning of period 1 (that is, $K_1 = K_2 = \cdots = K_T$ under the optimal strategy). Moreover, the stay-put region, or continuation region, characterizing the optimal ISD policy for period 1 is given by*:

$$S_1 = \left\{ K \in \mathbb{R}^n_+ : (1-\delta)\, r \leqslant \nabla \Pi(K) \leqslant \frac{1-\delta}{1-\delta^T} (c - \delta^T r) \right\}. \tag{32}$$

Proof. The kernel of the proof is that the continuation regions are nested, i.e., $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_T$. We prove this by induction. It is immediate that

$$S_T = \{ K \in \mathbb{R}^n_+ : (1-\delta)\, r \leqslant \nabla \Pi(K) \leqslant c - \delta r \}, \tag{33}$$

and because the continuation regions are deterministic, either S_T contains S_{T-1} or it does not. Assume $S_{T-1} \subseteq S_T$, so that $K_{T-1} = K_T \in S_{T-1}$ and $V_T(K_{T-1}) = \Pi(K_{T-1}) + \delta r' K_{T-1}$. Thus,

$$S_{T-1} = \left\{ K \in \mathbb{R}^n_+ : (1-\delta)\, r \leqslant \nabla \Pi(K) \leqslant \frac{c - \delta^2 r}{1+\delta} \right\}, \tag{34}$$

which is indeed contained in S_T because $c \geqslant r$. Now assume that $S_{t+1} \subseteq S_{t+2} \subseteq \cdots \subseteq S_T$ so that $K_{t+1} = \cdots = K_T \in S_{t+1}$ and

$$V_{t+2}(K_{t+1}) = (1 + \delta + \cdots + \delta^{T-t-2})\, \Pi(K_{t+1}) + \delta^{T-t-1} r' K_{t+1}, \tag{35}$$

and

$$S_{t+1} = \left\{ K \in \mathbb{R}^n_+ : (1-\delta)\, r \leqslant \nabla \Pi(K) \leqslant \frac{1-\delta}{1-\delta^{T-t}} (c - \delta^{T-t} r) \right\}. \tag{36}$$

Again, either S_{t+1} contains S_t or it does not. Assume $S_t \subseteq S_{t+1}$, so that $K_t = K_{t+1} \in S_t$ and the same reasoning yields

$$S_t = \left\{ K \in \mathbb{R}^n_+ : (1-\delta)\, r \leqslant \nabla \Pi(K) \leqslant \frac{1-\delta}{1-\delta^{T+1-t}} (c - \delta^{T+1-t} r) \right\}. \tag{37}$$

Because $c \geqslant r$, $[(1-\delta)/(1-\delta^{T+1-t})](c - \delta^{T+1-t} r) \leqslant [(1-\delta)/(1-\delta^{T-t})](c - \delta^{T-t} r)$, so that S_{t+1} indeed contains S_t which proves the theorem. ∎

Thus, the IID structure allows us to characterize the optimal investment strategy in analytic closed form. Moreover, this formulation is stated only in terms of the marginal revenue product $\nabla \Pi(K)$ of the factors and the marginal adjustment costs. It is interesting that the disinvest control surfaces are constant, but the invest control surfaces depend on the time horizon T. The continuation region is contracting as T increases and converges geometrically to a fixed region for the infinite horizon case, $(T \to \infty)$. The optimal investment-up-to points on the invest control surface are increasing in T.

6. INVESTMENT AND FLEXIBILITY IN LINEARLY HOMOGENEOUS STATIONARY MARKOV MODELS

In this section we consider a second class of investment models, called *linearly homogeneous stationary Markov models*, that is characterized as follows. In addition to our general model assumptions, these models study a stationary, infinite horizon setting, where the form of uncertainty is defined by a one-dimensional Markov process $X = \{X_t(\omega) \in \mathbb{R}_+ : \omega \in \Omega, t = 0, 1, 2, ...\}$ with given initial value $X_0(\omega)$. Also, at any time $t \geqslant 0$, the stationary concave operating profit function only depends on current factor levels K_t and on information modeled by X_t (and not on earlier information). Therefore, since X (hereafter, suppressing dependence on ω) is a Markov process, we can write the operating profit function as $\pi(K_t, X_t)$. Finally, we assume that $\pi(K, X)$ is increasing in X and supermodular and linearly homogeneous in X and K. Investment dynamics for this class of models with two factors are discussed in detail by Dixit [15], and this section generalizes his results to an arbitrary number of factors.

Because all model primitives are stationary, the optimal value function is also stationary and, given the Markov nature of the uncertainty, we can write $V(K_t, X_t)$.

PROPOSITION 2. *In linearly homogeneous stationary Markov models, $\nabla_j V(K_t, X_t)$ is homogeneous of degree zero and increasing in $K_{t,i}$ ($\forall i \neq j$) and X_t. Thus, the continuation region can be specified as*

$$S(X_t) = \{K \in \mathbb{R}^n_+ : r \leqslant \nabla V(K/X_t, 1) \leqslant c\}. \tag{38}$$

Proof. Because we have a stationary, infinite horizon setting, with the operating function π and the adjustment cost function C being supermodular and linearly homogeneous in X and K, the optimal value function V is also supermodular and linearly homogeneous in X and K. (Corollary 2 shows the preservation of supermodularity in a finite horizon, non-stationary setting. Dixit [15] shows the preservation of supermodularity and homogeneity in a two-factor, infinite horizon, stationary setting, but his argument directly generalizes to an arbitrary number of dimensions.) Therefore, its partial derivatives are each increasing in X and K_i, $\forall i \neq j$, and homogeneous of degree zero. ▌

The first implication of Proposition 2 is that factor j will not always be adjusted when it reaches a fixed level (relative to the stochastic term X), as in the single factor case. Rather, the threshold for increasing factor j depends positively on the amount of other factors held by the firm. Second, Proposition 2 shows that stationary Markov models as defined here have a *fixed* central region in a K_t/X_t-axis system. To appreciate the significance of this result, it is useful to first make a transformation of variables $k_i = \log(K_i/X)$. In this axis-system, the central domain S is a time-independent volume as shown in Fig. 3 for a stationary Markov model with three factors. (Fig. 3 shows S as a parallelepiped for expository simplicity. Generally, stationary Markov models will have a central region that is a "curvi-linear parallelepiped").

Assume an initial state k_0 in the interior of the region of inaction (otherwise an initial impulse control will adjust k_0 to the boundary of S). Depending on whether the stochastic term X_t increases or decreases, the

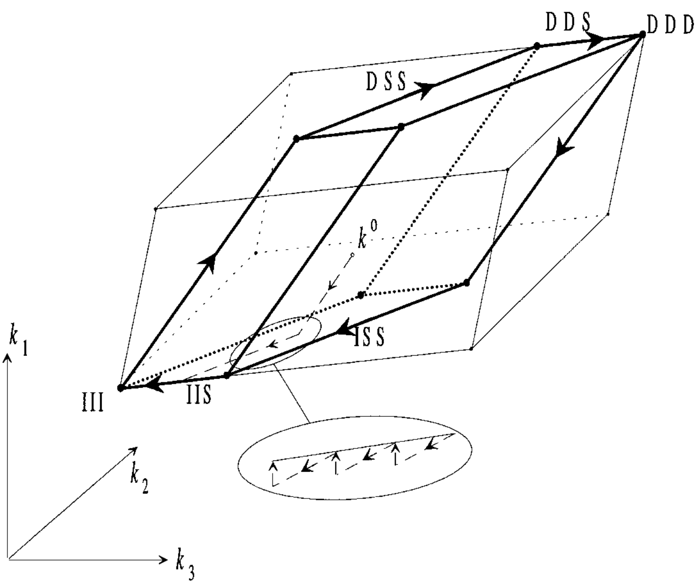

FIG. 3. Investment dynamics for a stationary Markov model with three factors.

state vector k_t will move down or up on a straight line parallel to the vector (1, 1, 1) when not adjusted. Say, X increases. Then, k_t will eventually hit the lower "ISS (Invest, Stay put, Stay put)" boundary of S, and investment in the first factor will prevent k_t from leaving S. (For clarification, this process is magnified assuming a small discrete time step; assuming sufficient regularity, a limiting argument would show that in continuous time, k_t never leaves S.) As long as X decreases, scaled factor levels k_t will move on a curve on the ISS boundary face (the projection of the original (1, 1, 1) line onto that boundary face) and will eventually intersect with the front boundary face. At that time, simultaneous investment in factors 1 and 2 will prevent k_t from leaving S. The trajectory can be continued and the result is that in equilibrium, the scaled factor level vector k_t "lives" inside the (thick lined) parallelepiped—note that at any time X_t can change direction so that k_t will move into this parallelepiped on a (1, 1, 1) line. (Before an equilibrium is reached, variations in relative factor levels arising from initial conditions may cause the sequence of adjustment to differ, but once all factors have been adjusted at least once, this possibility is eliminated.) The following proposition formalizes this feature of the optimal investment strategy.

PROPOSITION 3. *For linearly homogeneous stationary Markov models, there exists an equilibrium weak ordering (or labeling) of the n factors such that, for each $i \in \{1, 2, ..., n\}$, factor $j > i$ will only be adjusted after factor i has been adjusted, and when adjusted, factor j will be adjusted simultaneously with and in the same direction as factor i.*

Remark. The ordering can be weak in the sense that it can be that factors i and $i+1$ are always adjusted simultaneously. In Fig. 3 that would happen for factors 2 and 3 if the points III and DDD are positioned such that the thick lined equilibrium parallelepiped would be a planar parallelogram.

Proof. The proposition follows directly from the graphical interpretation of the investment dynamics in $k = \log(K/X)$ space provided S is a (curvi-linear) parallelepiped that "contains displacements along (1, 1, ..., 1)" in the sense that for at least one $k \in S$ and an $\varepsilon > 0$, the displacement vector $k + \varepsilon(1, 1, ..., 1) \in S$. We can generalize Dixit's [15] proof to n dimensions as follows. From the supermodularity of V, it follows that all boundary faces of S are increasing and thus monotone. Therefore, it suffices to show that factor i's boundary surfaces have slopes $\leqslant 1$, while all other surfaces have slopes $\geqslant 1$, in any (k_i, k_j) subspace with $i \neq j$ and the other $k_{l \neq i, j}$ and X considered fixed.

Fix any i $(1 \leqslant i \leqslant n)$ and totally differentiate factor i's boundary conditions $\nabla_i V(K, X) = c_i$ (r_i) which yields that the two factor i (dis)investment surfaces have normal vector $\nabla\nabla_i V(K,X)=\nabla V_i(K,X)$. Thus, the slope of the (dis)investment boundary of factor i in any $(k_{i'},k_j)$ subspace with $i'\neq j$ and the other $k_{l\neq i',j}$ and X considered fixed is

$$\frac{dk_{i'}}{dk_j}=\frac{dK_{i'}/K_{i'}}{dK_j/K_j}=-\frac{K_j V_{ij}(K,X)}{K_{i'}V_{ii'}(K,X)} \qquad (\forall i',j \text{ with } i'\neq j).$$

Concavity of V yields that $V_{i,i}\leqslant 0$, while supermodularity yields $V_{i,j\neq i}\geqslant 0$, so that $dk_{i'}/dk_j\leqslant 0$ if $i\neq i'\neq j$ and non-negative if either i' or j equals i. In addition, Euler's identity for V_i which is homogeneous of degree zero yields

$$XV_{i,X}(K,X)+\sum_{j=1}^{n}K_jV_{i,j}(K,X)=0\Rightarrow -K_iV_{i,i}\geqslant K_jV_{i,j} \qquad (\forall j\neq i).$$

Therefore, the slope of the factor i (dis)investment boundary with $i\neq i'\neq j$ satisfies

$$\frac{dk_{i'}}{dk_j}\leqslant 0\leqslant\frac{dk_i}{dk_j}\leqslant 1\leqslant\frac{dk_{i'}}{dk_i} \qquad (\forall i',j \text{ with } i'\neq j\neq i).$$

Thus, in any (k_i,k_j) subspace $(\forall j\neq i)$, the (dis)investment boundary of factor i has all its slopes non-negative but not larger than 1, while all other increasing boundaries have slopes not smaller than 1. ▮

Thus, after all factors have been adjusted at least once, equilibrium is reached and only a few of the $2n$ faces of the inaction region are ever hit and always in the same sequence. The equilibrium ordering of the factors in Fig. 3 is 1, 2, 3: factor 3 will only be adjusted after factor 2 has been adjusted, and factor 2 will follow factor 1. Moreover, adjustment in factor 2 and/or 3 is simultaneous and in same direction as adjustment in factor 1. Finally, factor 1 will be adjusted most often and can therefore be called the "most flexible" factor. In general then, some factors will be systematically adjusted more often than other factors, and thus be systematically "more flexible" than other factors. Specifically, any factor will always be adjusted along with "more flexible" factors and prior to "less flexible" factors. Proposition 3 guarantees the existence of an equilibrium ordering, and shows that in our model the flexibility of factors is determined endogenously. From our argument, it is clear that the ordering follows the ordering of the width of S measured along the coordinate directions (due to homogeneity, the ordering of widths is preserved when measured at different positions). Our general results indicate that the width of the region of inaction is increasing in the difference between the acquisition and resale prices of capital. In the next section, we will show that for specific forms of the Markov process X and the operating profit function π, it is the fractions r_i/c_i that endogenously determine factor flexibility.

7. INVESTMENT IN BROWNIAN ISOELASTIC MODELS

7.1. *Model Specification and Assumptions*

In this section we consider a subset of the stationary Markov models of the previous section with specific functional forms and arbitrarily small time step so that the firm operates in continuous time and discounts future profits at a constant rate $\rho > 0$ over an infinite horizon. Specifically, we assume that the Markov process X is a one-dimensional geometric Brownian motion with instantaneous drift μ, where $-\infty < \mu < \rho$, and standard deviation $\sigma > 0$, and that the linearly homogeneous, supermodular operating profit function has the specific form (suppressing time subscripts wherever possible hereafter):

$$\pi(K, X) = hX^{\theta}\left(\prod_{i=1}^{n} K_i^{\gamma_i}\right)^{1-\theta} \qquad \text{where} \quad h > 0, \sum_{i=1}^{n} \gamma_i = 1 \text{ and } 0 < \theta, \gamma_i < 1. \tag{39}$$

This operating profit function can be derived from a constant returns-to-scale Cobb-Douglas production function and a constant elasticity demand function.[13] Finally, we now allow factors to depreciate at rate $\lambda \geqslant 0$ and study the equilibrium regime. We call a linearly homogeneous stationary Markov model that satisfies these assumptions a *Brownian isoelastic model.* For these models, we will express ∇V in terms of the marginal operating profit $\nabla\pi$. For now we will assume that the factors are labelled such that the equilibrium factor adjustment sequence of Proposition 3 is $1, 2, ..., n$, and later we will show how to construct the sequence from the problem data.

Because the form (39) for $\pi(K, X)$ satisfies the requirements of Section 2, a limiting argument on the time-step shows that the optimal policy is of the ISD form. In continuous time, the optimality equation (8) takes the form of the Bellman equation

$$\begin{aligned} \rho V(K, X) &= \pi(K, X) + \mathrm{E}[DV(K, X)]/dt \\ &= hX^{\theta} \prod_{i=1}^{n} K_i^{\gamma_i(1-\theta)} + \mu X V_X + \tfrac{1}{2}\sigma^2 X^2 V_{XX} - \lambda(\nabla V)' K. \end{aligned} \tag{40}$$

where D is the total derivative and ∇V is the vector of partial derivatives with respect to the elements of K (when differentiating with respect to the stochastic process X, we will write V_X explicitly). The left-hand side of the Bellman equation can be interpreted as the required rate of return on

[13] This is the same function used by Bertola [9] and Dixit [14], generalized to allow for multiple factor inputs.

the firm, $\rho V(K, X)$, which must equal net profits (the dividend) plus the expected capital gain. Because the Bellman equation must hold identically in the range of inaction (when none of the factors is adjusted), it can be differentiated with respect to each factor. The marginal value of an additional unit of each type of capital, $q_j \equiv V_{K_j}$, therefore obeys the partial differential equation for each $j = 1, ..., n$:

$$\begin{aligned}(\rho+\lambda)\, q_j(K, X) &= \pi_{K_j}(K, X) + \mathrm{E}[Dq_j(K, X)]/dt \\ &= \gamma_j(1-\theta)\, hK_j^{-1}X^{\theta} \prod_{i=1}^{n} K_i^{\gamma_i(1-\theta)} + \mu X q_{j,X} \\ &\quad + \tfrac{1}{2}\sigma^2 X^2 q_{j,XX} - \lambda(\nabla q_j)'\, K. \end{aligned} \tag{41}$$

Section 3 shows that the marginal value function $q = \nabla V$ equals the marginal cost c at the investment boundaries and the marginal revenue r at the disinvestment boundaries. This boundary condition is known as "smooth pasting" in the control literature. One additional boundary condition, known as "high contact", is a direct generalization of one-dimensional control problems and requires that the marginal benefit and marginal cost of adjusting be equalized both before and after the infinitesimal adjustment. This condition requires that at the investment and disinvestment boundaries for factor j,

$$\text{High Contact Condition: } q_{j,K_i}(K, X) = 0, \qquad \forall i \leqslant j. \tag{42}$$

7.2. *Analytic Expressions for the Optimal Investment Strategy*

This section presents analytic expressions for the marginal value function $q = \nabla V$ and the continuation region S of the optimal investment strategy for a Brownian isoelastic model. It will be useful to introduce the following quantities. Let $\xi^+ > 1$ and $\xi^- < 0$ be the two roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2\xi^2 + (\mu + \lambda - \tfrac{1}{2}\sigma^2)\,\xi - (\rho + \lambda) = 0, \tag{43}$$

and define the scalar $R(\theta) > 0$ as

$$R(\theta) \equiv \rho + \lambda - \theta(\mu + \lambda) + \tfrac{1}{2}\sigma^2\theta(1-\theta). \tag{44}$$

Also, let e denote a vector of ones of variable length and denote the n-vector that is composed of e and the $(n-i+1)$-vector $K^{i:n}$ by

$$(\mathrm{e}, K^{i:n}) \equiv (1, 1, ...1, K_i, ..., K_n). \tag{45}$$

PROPOSITION 4. *For a Brownian isoelastic model in equilibrium, the function $q^{\star}(K, X)$ defined componentwise as*

$$q_j^{\star}(K, X) = R^{-1}(\theta)\,\pi_{K_j}(K, X) + K_j^{-1} \sum_{i=1}^{j} A_{ji} K_i^{1+\alpha \sum_{k=0}^{i-1} \gamma_k (1-\theta)} \pi_{K_i}^{\alpha}((\mathrm{e}, K^{i:n}), X) + K_j^{-1} \sum_{i=1}^{j} B_{ji} K_i^{1+\beta \sum_{k=0}^{i-1} \gamma_k (1-\theta)} \pi_{K_i}^{\beta}((\mathrm{e}, K^{i:n}), X), \tag{46}$$

where A_j and B_j are constant vectors, $\gamma_0 = 0$, and $\alpha \equiv \xi^{+}/\theta > 1$ and $\beta \equiv \xi^{-}/\theta < 0$, satisfies the optimality equation (41) *and defines constant vectors $\bar{r} \leqslant \bar{c}$ which specify an equilibrium ISD investment strategy with central domain*

$$S(X_t) = \{K \in \mathbb{R}_{+}^{n} : \bar{r} \leqslant \nabla \pi(K, X_t) \leqslant \bar{c}\}. \tag{47}$$

Proof. For any $j = 1, .., n$, the partial differential equation (41) has particular solution $R(\theta)^{-1}\,\pi_{K_j}(K, X)$, where $R(\theta)$ is as defined in (44), and a continuum of homogeneous solutions of the form $C_j X^{\vartheta_0} \prod_{i=1}^{n} K_i^{\vartheta_i}$, where C_j is an arbitrary constant and $\sum_{i=0}^{n} \vartheta_i = 0$ (because $q_j(K, X)$ is homogeneous of degree zero in its arguments). The expression (46) is among this class of solutions and it is verified by direct substitution that this candidate solution satisfies the partial differential equation (41).

The system of equations defined by (46) has a triangular form. Beginning with factor 1, (46) becomes

$$q_1^{\star}(K, X) = \frac{\pi_{K_1}(K, X)}{R(\theta)} + A_{11} \pi_{K_1}^{\alpha}(K, X) + B_{11} \pi_{K_1}^{\beta}(K, X), \tag{48}$$

which is monotone increasing in $\pi_{K_1}(K, X)$. From Proposition 2, we can rewrite the first two inequalities that define the boundaries of the range of inaction in terms of two constants $\bar{r}_1 \leqslant \bar{c}_1$ as $\bar{r}_1 \leqslant \pi_{K_1}(K, X) \leqslant \bar{c}_1$. For factor 2, (46) becomes

$$q_2^{\star}(K, X) = \frac{\pi_{K_2}(K, X)}{R(\theta)} + \frac{K_1}{K_2} \{A_{21} \pi_{K_1}^{\alpha}(K, X) + B_{21} \pi_{K_1}^{\beta}(K, X)\} + A_{22} [K_2^{\gamma_1(1-\theta)} \pi_{K_2}((\mathrm{e}, K^{2:n}), X)]^{\alpha} + B_{22} [K_2^{\gamma_1(1-\theta)} \pi_{K_2}((\mathrm{e}, K^{2:n}), X)]^{\beta}. \tag{49}$$

From Proposition 3, when in equilibrium factor 2 is adjusted factor 1 is also adjusted simultaneously and in the same direction, so when factor 2 is adjusted upward $\pi_{K_1}(K, X) = \bar{r}_1$ and when factor 2 is adjusted downward, $\pi_{K_1}(K, X) = \bar{c}_1$. Using this fact and the fact that from the form of the operating profit function, $\pi_{K_2}(K, X) = \pi_{K_1}(K, X)(\gamma_2 K_1/\gamma_1 K_2)$, the expression in (49) can be rewritten and evaluated at the investment boundary to obtain

$$\begin{aligned} q_2^\star(K, X) = & \frac{\gamma_2 \bar{r}_1}{\gamma_1 R(\theta)} \frac{K_1}{K_2} + \frac{K_1}{K_2} \{A_{21} \bar{r}_1^\alpha + B_{21} \bar{r}_1^\beta\} \\ & + A_{22} \left[\frac{\gamma_2 \bar{r}_1}{\gamma_1} \left(\frac{K_1}{K_2} \right)^{1-\gamma_1(1-\theta)} \right]^\alpha + B_{22} \left[\frac{\gamma_2 \bar{r}_1}{\gamma_1} \left(\frac{K_1}{K_2} \right)^{1-\gamma_1(1-\theta)} \right]^\beta \\ = & c_2. \end{aligned} \tag{50}$$

Thus, at the simultaneous investment boundary for factors 1 and 2, the ratio K_1/K_2 always equals a constant. Since $\pi_{K_1}(K, X) = \bar{r}_1$ at the equilibrium investment boundary for factor 2, and given the form of the operating profit function, this implies that $\pi_{K_2}(K, X)$ always equals a constant at the investment boundary for factor 2. A similar argument holds at the disinvestment boundary for factor 2, so that the boundaries of the continuation region may be written as $\bar{r}_2 \leqslant \pi_{K_2}(K, X) \leqslant \bar{c}_2$. Continuing, using the triangular form of the system of equations defined by (46), we find that $\pi_{K_j}(K, X)$ always equals a constant value when factor j is adjusted upward, and similarly when adjusted downward. Thus, in equilibrium, the central domain may be written in terms of the marginal profitability of capital, yielding (47). ▮

Proposition 4 proposes a solution $q^\star$ for q that satisfies the partial differential equation (41) and determines an ISD investment strategy by its central region which satisfies the boundary conditions. The next proposition shows that the same expression $q^\star$ that is proposed for q in (46) obtains when calculating the expected present value of marginal revenue products directly using the unique ISD strategy with central domain specified by (47).

PROPOSITION 5. *For a Brownian isoelastic model in equilibrium, the ISD investment strategy with central domain $S(X_t)$ specified by* (47) *yields the marginal value of an additional unit of a factor $j = 1, ..., n$ given by*

$$\nabla_j V(K_t, X_t) = q_j^\star(K_t, X_t), \tag{51}$$

where $q_j^\star(K_t, X_t)$ is defined by (46).

Proof. See Appendix A. ▮

Because the operating profit function (39) is strictly concave, V is strictly concave inside the region of inaction and the Brownian isoelastic investment model has a unique optimal ISD investment strategy. Therefore, Propositions 4 and 5 together establish that the proposed solution $q^\star$ given by (46) indeed is the marginal value of an additional unit of a factor $q = \nabla V$ of that unique optimal ISD strategy, and that the unique optimal ISD policy has central domain (47). Thus, even though the Brownian isoelastic model has dynamic persistence, the thresholds for optimal investment can be written in terms of a factor's current marginal profitability (as in the earlier IID model of Section 5) because of its Markovian structure. That is, when current marginal profits reach an upper threshold investment occurs in a factor, and similarly, when marginal profits reach a lower threshold, disinvestment occurs in that factor.

7.3. *Endogenous Characterization of Factor Flexibility*

Proposition 3 guarantees the existence of an equilibrium factor adjustment sequence. The following proposition shows how this sequence can be constructed. Specifically, in a symmetric version of the Brownian isoelastic case where $\gamma_i = \gamma_j \ \forall i, j$, it is the ratio of the acquisition prices to the resale prices, c/r, that determines the relative degree of flexibility of the factors.

PROPOSITION 6. *In a Brownian isoelastic model, factor i is more flexible than (that is, is adjusted more often and before) factor j if and only if* $(\bar{c}_i/\bar{r}_i) < (\bar{c}_i/\bar{r}_i)$. *If the model is symmetric so that factors are otherwise identical,* ($\gamma_i = \gamma_j$ *and either* $c_i = c_j$ *or* $r_i = r_j$, $\forall i, j$), *then*

$$\frac{\bar{c}_i}{\bar{r}_i} < \frac{\bar{c}_j}{\bar{r}_j} \Leftrightarrow \frac{c_i}{r_i} < \frac{c_j}{r_j}. \tag{52}$$

Proof. From Proposition 3, in linearly homogeneous Markov models, the ordering of the factors (from most flexible to least flexible) follows the ordering of the width of S measured along the coordinate directions. Our general results indicate that the width of the continuation region is increasing in the difference $c_i - r_i$. If factors are identical except for the secondary market discount $c_i - r_i$, then $\gamma_i = \gamma_j$ and *either* $c_i = c_j$ or $r_i = r_j$, $\forall i, j$. Thus, for otherwise identical factors, the width of the continuation region is increasing in c_i/r_i. ∎

7.4. *Optimal Irreversible Investment*

If $r = 0$ in a Brownian isoelastic model, investment in all factors is irreversible[14] and a closed form solution is obtained for the investment threshold for each factor.

[14] Because Brownian isoelastic models have positive marginal operating profits $\nabla\pi$, the expected present value q of marginal operating profits is also positive. Therefore, the firm would never disinvest in factor j (thus making factor j irreversible) if its resale value of capital, r_j, were negative or zero.

PROPOSITION 7. *For irreversible investment* $(r=0)$ *in a Brownian isoelastic model, the central domain is*

$$S(X_t)=\left\{K\in\mathbb{R}^n_+ : \nabla\pi(K, X_t)\leqslant \bar{c}=\frac{\alpha(\theta)\,R(\theta)}{\alpha(\theta)-1}\,c\right\} \tag{53}$$

Proof. With irreversible investment the marginal profitability of any factor j can become arbitrarily close to zero. Therefore, to prevent marginal profit raised to the power $\beta<0$ from becoming infinite, the constants B_{ji} in (46) must be zero. Totally differentiate $q_j(K, X)$ and impose the high contact condition to obtain the condition $\sum_{i>j} q_{jK_i}(K, X)\,dK_i+q_{jX}(K, X)\,dX=0$, remembering that $q_j(K, X)$ is homogeneous of degree zero. From Proposition 3, no adjustment is made to factor $i>j$, so $dK_i=0$ in all the terms in the summation, yielding $q_{jX}(K, X)\,dX=0$, which requires $q_{jX}(K, X)=0$ along the boundaries for adjusting factor j. Using the expression for $q_j(K, X)$ in (46), differentiating and multiplying by X yields for all $j\in\{1, ..., n\}$:

$$q_{jX}(K, X)\,X=\frac{\theta\pi_{K_j}(K, X)}{R(\theta)}+\alpha\theta\sum_{i=1}^{j}\frac{K_i}{K_j}\,A_{ji}K_i^{\alpha\sum_{k=0}^{i-1}\gamma_k(1-\theta)}\pi^{\alpha}_{K_i}((\mathrm{e}, K^{i:n}), X)=0.$$

Using this expression to eliminate the summation terms in expression (46) for $q_j(K, X)$ and imposing the boundary condition $q_j=c_j$, we obtain

$$q_j(K, X)=\frac{\pi_{K_j}(K, X)}{R(\theta)}\left[1-\frac{1}{\alpha}\right]=c_j.$$

At the investment boundary the threshold value of the marginal profitability of factor j that justifies investment in factor j is thus given by

$$\bar{c}_j=\frac{\alpha(\theta)\,R(\theta)}{\alpha(\theta)-1}\,c_j. \quad \blacksquare \tag{54}$$

PROPOSITION 8. *If in a Brownian isoelastic model factors* $1, ..., l$ *are irreversible and the remaining* $n-l$ *factors are costlessly reversible* ($r^{1:l}=0$ *and* $r^{l+1:n}=c^{l+1:n}$), *then the central domain is*

$$S(X_t)=\left\{K\in\mathbb{R}^n_+ : \nabla^{1:l}\pi(K, X_t)\leqslant \bar{c}(l)\equiv\frac{\alpha(\eta)\,R(\eta)}{\alpha(\eta)-1}\,c^{1:l}\right.$$
$$\left.\text{and } \nabla^{l+1:n}\pi(K, X_t)=(\rho+\lambda)\,c\right\}, \tag{55}$$

where

$$\eta = \frac{\theta}{1-(1-\theta)\sum_{i=l+1}^{n}\gamma_i} \geqslant \theta, \tag{56}$$

(*with strict inequality when* $l<n$), *and the irreversible investment boundaries are strictly decreasing in the total share*, $\sum_{i=1}^{l}\gamma_i$, *of the irreversible factors.*

Proof. If factors $l+1, ..., n$ are costlessly adjustable, their optimal levels are found by maximizing the operating profit function π of the Brownian isoelastic model; the marginal operating profit of each costlessly adjustable factor is set equal to the Jorgensonian user cost, $(\rho+\lambda)\,c_j$. The effective operating profit function becomes a function $\hat{\pi}(K_t^{1:l}, X_t)$ of the irreversible factor levels only:

$$\hat{\pi}(K_t^{1:l}, X_t) = \hat{h} X_t^{\eta} \prod_{i=1}^{l} K_{t,i}^{\gamma_i(1-\eta)}, \tag{57}$$

where

$$\hat{h} = \left(h \prod_{i=l}^{n}\left[\frac{\gamma_i(1-\theta)}{w_i}\right]^{\gamma_i(1-\theta)}\right)^{\eta/\theta} \qquad \text{and} \qquad \eta \equiv \frac{\theta}{1-\sum_{i=l+1}^{n}\gamma_i(1-\theta)} \geqslant \theta. \tag{58}$$

Using the maximized operating profit function $\hat{\pi}(K_t^{1:l}, X_t)$, Proposition 7 applied to this equivalent model with l irreversible factors gives the threshold for investing in irreversible factor $j \leqslant l$ is given by

$$\bar{c}_j(l) = \frac{\alpha(\eta)\,R(\eta)}{\alpha(\eta)-1}\,c_j. \tag{59}$$

Differentiation of (59) yields

$$\frac{\partial \bar{c}_j(l)}{\partial \eta} = \frac{1}{2}\sigma^2 \xi^{+} c_j > 0. \tag{60}$$

Letting $\Gamma \equiv \sum_{i=1}^{l}\gamma_i$, we have $\sum_{i=l+1}^{n}\gamma_i = 1-\sum_{i=1}^{l}\gamma_i = 1-\Gamma$, so that $\partial\eta/\partial\Gamma < 0$. Together we have

$$\frac{\partial \bar{c}_j}{\partial \Gamma} = \frac{\partial \bar{c}_j}{\partial \eta}\frac{\partial \eta}{\partial \Gamma} < 0. \quad \blacksquare \tag{61}$$

Although expression (55) is similar in form to (53), notice that both the root α and the discount factor R differ in the two expressions and

$$\bar{c}_j(l) > \bar{c}_j(n) = \bar{c}_j, \qquad \text{for} \quad j \leqslant l < n, \tag{62}$$

which implies that the investment threshold decreases in l. Thus, the fewer factors are irreversible, the higher is the investment threshold and the smaller is the region of inaction.

The magnitude of the effect of overall irreversibility on factor j's threshold can be obtained from the derivative in (61). The effect of the additional irreversible factors on the threshold is increasing in σ^2, which has a positive effect on the threshold for investment. This result implies that as γ_1 becomes small, the threshold as calculated in the single-irreversible factor case ($l = 1$) becomes large relative to the multiple factor case, and this effect is exacerbated the more the threshold is driven up by uncertainty.

We can now compare optimal investment for a Brownian isoelastic model with two[15] factors under three scenarios: Scenario I assumes that investment in both factors is frictionless ($c = r$), a situation in which the marginal operating profit of each factor is always equal to Jorgensonian user cost. Scenario II assumes that investment in factor 2 is frictionless and in factor 1 irreversible ($c_2 = r_2$ and $r_1 = 0$). Finally, scenario III assumes that investment in both factors is irreversible ($r = 0$). Under scenario I, the optimal choice of relative factor levels occurs at the intersection of the two arcs equating the marginal operating profit of each type of capital to its Jorgensonian user cost. This investment point is labeled as point J in Fig. 4 and relative factor use K_1/K_2 is given by the slope of the line from the origin through point J. Under scenario II, the dashed (concave) arc containing point I gives the optimal threshold for adjusting the level K_1 of the irreversible first factor for every value of K_2. Since K_2 is continuously optimally chosen, however, it is still chosen along the convex arc containing point J, the Jorgensonian factor choice. The firm exists along the arc that contains IJ, and increases K_1 only when K_1/X falls sufficiently that the firm reaches point I. At this point, relative factor use K_1/K_2 is lower than in the Jorgensonian case, but may be higher as K_1/X rises beyond point J. Under scenario III with both factors being irreversible, the region of inaction S is unbounded (the disinvestment boundaries of Fig. 4 are at $+\infty$) and the firm exists in the shaded area beyond the intersection at point M. Observe that point M is above point I, implying that the firm chooses a higher value of K_1/K_2 when both factors are irreversible than when only

[15] The two factor case is used for geometric simplicity, but the results hold (and can in fact be strengthened) with any number of irreversible factors.

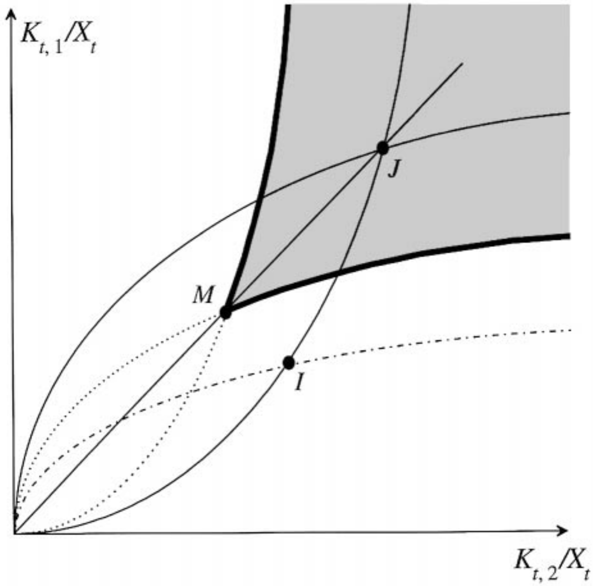

FIG. 4. Comparison of optimal investment with two irreversible factors (M), one irreversible factor and one frictionless factor (I), and two frictionless factors (J).

factor 1 is irreversible. This occurs because the firm prefers to use the flexible factor 2, increasing its relative investment, when factor 1 is the only irreversible factor. When this relative advantage is removed and both factors are irreversible, the distortion of factor use is eliminated. The point M where investment in both factors occurs is along the line from the origin through point J, so that desired factor shares are not affected by the irreversibility constraint.[16] In this case, the marginal revenue product that justifies investment is above the Jorgensonian level, but below the single-irreversible-factor level.[17]

8. CONCLUSIONS

This paper has studied optimal choice of many factors with costly reversibility. Costly reversibility of factors arises when a firm faces labor firing costs or when it cannot recoup the price of capital when it is resold.

[16] This result relies on the homotheticity of the production function, so that optimal factor shares are not affected by the scale of production.

[17] This result relies on decreasing returns in the profit function, either from physical decreasing returns to factor inputs or from market power. If the profit function exhibits constant returns to scale, then points J, I, and M will all lie at the same point.

Because not all of the expected present value of future payments is recovered when factor use is reduced, the firm will be more reluctant to change factor levels. This reluctance is described by a region of inaction that characterizes optimal investment in our general investment model. Within this region it is optimal not to adjust factor use, but from anywhere outside the region or at its boundaries, it is optimal to adjust factors to a specific point on the boundary of the range. This paper presents various properties of the general optimal strategy.

Additional insights are obtained by applying the general framework to two specific model classes. Investment models with an IID structure allow an analytic, closed form expression for the entire investment strategy, even under a finite horizon. The second class of models assumes a one-dimensional stochastic process as the generator of uncertainty, and a supermodular, linearly homogeneous operating profit function. The familiar Cobb-Douglas related operating profit function belongs to this class. A key result is that it is possible to characterize the entire investment dynamics and endogenously infer from that a "flexibility" ordering of the factors. This implies an attractive model feature that is potentially empirically observable: costly reversibility implies that investment and hiring, for example, will exhibit periods of inactivity and episodes of accumulation or decumulation. We also show that it is possible to express investment threshold rules in terms of the marginal profitability of a factor, which may expedite empirical analysis of optimal factor investment.

A. PROOF OF PROPOSITION 5

A.1. *Preliminary Properties*

In this section, we first derive some properties of the ISD investment strategy with central domain $S(X_t)$ specified by (47) in terms of the marginal profitability $\nabla\pi$. (Hereafter we will refer to this specific ISD strategy as "the proposed ISD strategy".) The Laplace transform of the hitting times under the proposed ISD strategy is evaluated using the following Lemma (cf. [20] for a proof in the case of a standard Brownian motion and [9] for the extension to the geometric case).

LEMMA 2. *Let x_t follow a geometric Brownian motion so that $dx_t = mx\,dt + sx\,dz$, where dz is the increment of a standard Brownian motion. If τ is defined as the length of the interval until the first hitting time for the process x to either a or b, where $a < x_t < b$, then*

$$\mathrm{E}_t e^{-\nu\tau} = \mathrm{E}_t[e^{-\nu\tau}; x_{t+\tau} = a] + \mathrm{E}_t[e^{-\nu\tau}; x_{t+\tau} = b] = \left(\frac{x_t}{a}\right)^{\zeta^-} + \left(\frac{x_t}{b}\right)^{\zeta^+}, \quad \text{(A1)}$$

where E_t *is the expectations operator taken at time* t, $\zeta^- < 0 < 1 < \zeta^+$ *are the roots of the quadratic equation* $0 = -v + (m - \frac{1}{2}s^2)\,\zeta + \frac{1}{2}s^2\zeta^2$, *and we follow* [20] *in using the notation* $\mathrm{E}_t[z; A]$ *to denote the product of the conditional expectation and probability,* $\mathrm{E}_t[z \mid A]\, P_t(A) = \int_A z\, dP_t$.

To evaluate the hitting time T_i, let[18] $x_t = K_{i,t}^{\sum_{k=0}^{i-1} \gamma_k (1-\theta)} \pi_{K_i}((\mathrm{e}, K_t^{i:n}), X_t)$ and $v = \rho + \lambda$, so that $m = \theta[\mu + \lambda - \frac{1}{2}\sigma^2(1-\theta)]$ and $s^2 = \theta^2\sigma^2$.[19] The quadratic equation, written in terms of $\zeta\theta$, is the same as the quadratic equation in (43), so that the roots $\xi^+ = \zeta^+\theta$ and $\xi^- = \zeta^-\theta$. In terms of the notation in the text, we obtain $\beta = \xi^-/\theta$ and $\alpha = \xi^+/\theta$. The properties of the roots ($\alpha > 1$ and $\beta < 0$) of (43) are verified by observing that the $f''(\xi) > 0$ and using the properties $f(0) = -(\rho + \lambda) < 0$ and $f(1) = -\rho + \mu < 0$.

Consider the following n "restricted ISD strategies": under the ith restricted strategy, the first i factors[20] $K^{1:i}$ are adjusted according to the (restricted) proposed ISD strategy with domain $\{K^{1:i} \in \mathbb{R}^i_+ : \bar{r}^{1:i} \leqslant \nabla^{1:i}\pi(K, X_t) \leqslant \bar{c}^{1:i}\}$, while the last $n - i$ factors $K^{i+1:n}$ remain unadjusted (but do depreciate at rate λ). Denote the expected present value of marginal and total profits from factor i under the ith restricted strategy by y_i and Y_i, respectively:

$$y_{i,t} \equiv y_i(K_t, X_t)$$

$$\equiv \mathrm{E}_t \int_0^\infty \pi_{K_i}([K_{t+s}^{1:i}, K_t^{i+1:n} e^{-\lambda s}], X_{t+s})\, e^{-(\rho+\lambda)s}\, ds, \tag{A2}$$

$$Y_{i,t} \equiv Y_i(K_t, X_t)$$

$$\equiv \mathrm{E}_t \int_0^\infty \pi([K_{t+s}^{1:i}, K_t^{i+1:n} e^{-\lambda s}], X_{t+s})\, e^{-\rho s}\, ds. \tag{A3}$$

Six useful results regarding these present value calculations follow immediately:

LEMMA 3. *Under the* i*th restricted strategy, we have that for any* $j > i$:

$$\mathrm{E}_t \int_0^\infty \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds = (1-\theta)\, \gamma_j \frac{Y_{i,t}}{K_{j,t}} \tag{A4}$$

[18] Note that e denotes Euler's number, whereas e represents a vector of all ones.

[19] Equivalently, we could define $x_t = \pi_{K_i}(K_t, X_t)$ and using Lemma 8 obtain the definition of x_t above. The dynamic properties of x_t are identical in both cases.

[20] Recall the notation for subvectors: for $i < n$, we have $K = [K^{1:i}, K^{i+1:n}]$ where $K^{1:i} = (K_1, ..., K_i)$ and $K^{i+1:n} \equiv (K_{i+1}, ..., K_n)$. For $i = n$, let $K^{i+1:n} \equiv 1$.

Proof. K_j is a fixed factor for $j > i$ (and thus $K_{t+s}^{i+1:n} = K_t^{i+1:n} e^{-\lambda s}$). The lemma is immediate from the fact that

$$\pi_{K_j}([K_{t+s}^{1:i}, K_t^{i+1:n} e^{-\lambda s}], X_{t+s}) = \frac{\gamma_j(1-\theta)}{K_{j,t}} \pi([K_{t+s}^{1:i}, K_t^{i+1:n} e^{-\lambda s}], X_{t+s})\, e^{\lambda s}, \quad \text{(A5)}$$

using the homogeneous form of the operating profit function. ∎

LEMMA 4. *If no factors are ever adjusted (the "0th restricted strategy"), we have that*

$$Y_{0,t} \equiv E_t \int_0^\infty \pi(K_t e^{-\lambda s}, X_{t+s})\, e^{-\rho s}\, ds = \frac{\pi(K_t, X_t)}{R(\theta)}. \quad \text{(A6)}$$

Proof. Because all factors 1, ..., n are assumed to depreciate geometrically with no other adjustments, $Y_{0,t}$ is the expected discounted present value of a process following a geometric Brownian motion. ∎

LEMMA 5. *If only factor* 1 *is ever adjusted (the first restricted strategy), we have that*

$$y_{1,t} = \frac{\pi_{K_1}(K_t, X_t)}{R(\theta)} + a\pi_{K_1}^{\alpha}(K_t, X_t) + b\pi_{K_1}^{\beta}(K_t, X_t), \quad \text{(A7)}$$

where a and b are constants and α and β solve the quadratic (43) *as previously.*

Proof. Because factors 2, ..., n are assumed fixed, Lemma 5 follows directly from a single factor investment problem with costly reversibility, as in [9]. ∎

LEMMA 6. *Under the ith restricted strategy, $y_{i,t}$ is constant on the investment and disinvestment boundaries for factor i: there exist constant vectors $\bar{\varphi}$ and $\underline{\varphi}$ such that*

$$y_{i,t} = \bar{\varphi}_i \qquad \text{if} \quad \pi_{K_i}(K, X_t) = \bar{c}_i,$$

$$y_{i,t} = \underline{\varphi}_i \qquad \text{if} \quad \pi_{K_i}(K, X_t) = \bar{r}_i.$$

Proof. $y_{i,t}$ can be calculated by recognizing that $y_{i,t} \equiv q_{i,t}$ in the special case where $r_j = 0$ and $c_j \to \infty$ for $j > i$; in this case, factors $j \ldots n$ are "irreversible" and "non-expandable" in the terminology of [1]. Now, for *any*

ISD policy characterized by a continuation region S, the marginal value $q = \nabla V$ is constant outside and on the boundary of S. Indeed, V is linear outside S with $q_i = \nabla_i V$ equal to c_i or r_i, depending on whether the ISD rule prescribes investment or disinvestment to bring the firm to the boundary of S. That is, any ISD policy (optimal or not—note that the optimality of an ISD policy depends on its choice of S) satisfies the smooth pasting conditions. Therefore, at the factor i boundaries of the ith restricted ISD policy, this "special" $q_{i,t}$ is constant and thus so is $y_{i,t}$. ∎

Consider the proposed ISD policy and fix any time $t > 0$. Define T_1 as the time when any element of K is next adjusted, so that $T_1 - t$ is the interval until the first factor adjustment, $T_2 - T_1$ is the interval between the first and second adjustment, and so on. The proposed ISD policy then yields two useful properties of factor ratios:

LEMMA 7. *Under the proposed ISD policy, we have that* $K_{i,T_i}/K_{j,T_i} = K_{i,t}/K_{j,t}$ $\forall i < j$.

Proof. The central domain (47) of the proposed ISD policy has non-curvilinear boundaries in $\log(K/X)$-space. The discussion leading to Proposition 3 (and appropriate labeling of the factors) then shows that factor $j > i$ will only be adjusted after factor i, and factor i is not adjusted before its first hitting time, T_i, so the ratio of factor stocks is unchanged between time t and time T_i. ∎

LEMMA 8. *Under the proposed ISD policy, factor proportions are constant on the investment and disinvestment boundaries for factor j:*

$$K_j/K_i = \frac{\bar{c}_i \gamma_j}{\bar{c}_j \gamma_i} \qquad \textit{for all} \quad i<j \quad \textit{if} \quad \pi_{K_j}(K, X_t) = \bar{c}_j,$$

$$K_j/K_i = \frac{\bar{r}_i \gamma_j}{\bar{r}_j \gamma_i} \qquad \textit{for all} \quad i<j \quad \textit{if} \quad \pi_{K_j}(K, X_t) = \bar{r}_j.$$

Proof. From the discussion leading to Proposition 3, when factor $j > i$ is adjusted, it will be adjusted simultaneously and in the same direction as factor i. The central domain (47) of the proposed ISD policy shows that, along the investment boundary for any factor j, $\pi_{K_j}(K, X) = \bar{c}_j$, and along the disinvestment boundary for any factor j, $\pi_{K_j}(K, X) = \bar{r}_j$. Applying this condition to both factors i and j, using the form of the operating profit function, and taking the ratio we obtain the two constant ratios K_j/K_i along the investment and disinvestment boundaries, respectively. ∎

LEMMA 9. *Under the proposed ISD policy, $Y_{i,t}$ is proportional to $K_{i,t}$ on the investment and disinvestment boundaries for factor i: there exist constant vectors $\bar{\zeta}_i$ and $\underline{\zeta}_i$ such that*

$$Y_{i,t} = \bar{\zeta}_i K_{i,t} \qquad \text{if} \quad \pi_{K_i}(K_t, X_t) = \bar{c}_i,$$

$$Y_{i,t} = \underline{\zeta}_i K_{i,t} \qquad \text{if} \quad \pi_{K_i}(K_t, X_t) = \bar{r}_i.$$

Proof. Under the ith restricted policy, factors $i+1:n$ are not adjusted, so the operating profit function $\pi(K_t, X_t)$ may be equivalently written as a function of the exogenous process $X_t^{\theta} \prod_{l=i+1}^{n} K_{l,t}^{\gamma_l(1-\theta)}$ and the adjustable factor vector as $K_t^{1:i}$. Then the operating profit function and $Y_{i,t}$ are linearly homogeneous in the composite $\bar{X}_{i,t} \equiv [X_t^{\theta} \prod_{l=i+1}^{n} K_{l,t}^{\gamma_l(1-\theta)}]^{1/[\theta + (1-\theta)\sum_{l=i+1}^{n} \gamma_l]}$ and the vector $K_t^{1:i}$. Under the proposed ISD policy, when factor i is adjusted, $\pi_{K_i}(K_t, X_t) = \bar{c}_i$ (if at the investment boundary) or $\pi_{K_i}(K_t, X_t) = \bar{r}_i$ (if at the disinvestment boundary). Using the form of the operating profit function and the fact that (from Lemma 8) $K_{l,t}$ is proportional to $K_{i,t}$, for $l < i$, along the investment and disinvestment boundaries we have that $X_t^{\theta} \prod_{l=i+1}^{n} K_{l,t}^{\gamma_l(1-\theta)}$ is proportional to $K_{i,t}^{1-\sum_{l=1}^{i} \gamma_l(1-\theta)}$. Since $\sum_{l=1}^{n} \gamma_l = 1$, this implies that $\bar{X}_{i,t}$ is proportional to $K_{i,t}$.

Therefore, since $Y_{i,t}$ is linearly homogeneous in the composite $\bar{X}_{i,t}$ and $K_t^{1:i}$, and all of these elements are proportional to $K_{i,t}$, then $Y_{i,t}$ is proportional to $K_{i,t}$. ∎

A.2. *Calculating q of the Proposed ISD Strategy by Decomposition*

First, as argued in the proof of Lemma 6, the marginal value $q = \nabla V$ obtained under the proposed ISD strategy with central domain $S(X_t)$ specified by (47) is constant outside and on the boundary of S, satisfying the smooth pasting conditions. Therefore, it only remains to be shown that the marginal value q obtained under the proposed strategy for a point inside of S equals q^*.

Bertola [9] shows that in the single factor case, the marginal value of an additional unit of capital, $q_j(K_t, X_t)$, equals the expected present value of marginal operating profits. His argument generalizes directly to an arbitrary number of factors in the environment we consider and we can write the marginal value of an additional unit of any factor j as

$$q_j(K_t, X_t) \equiv V_{K_j}(K_t, X_t) = \mathrm{E}_t \int_0^{\infty} \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds. \tag{A8}$$

Choose a time t at which the firm is in the interior of its continuation region $S(X_t)$, specified by (47). Set $T_0 = t$ and define the hitting times T_i for the proposed ISD strategy as in Lemma 7. For any j $(1 \leqslant j \leqslant n)$, rewrite $q_j(K, X)$ in (A8) as

$$q_j(K_t, X_t) = \sum_{i=1}^{j} \mathrm{E}_t \int_{T_{i-1}-t}^{T_i - t} \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds + \mathrm{E}_t \int_{T_j - t}^{\infty} \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds. \quad \text{(A9)}$$

(Notice that T_i may be equal to T_{i+1}: If factor 1 is next adjusted alone, then $T_i < T_{i+1}$ for all $i = 1, ..., j$. If however, factors 1, ..., j are next adjusted simultaneously, then $T_1 = T_2 = \cdots = T_j$ in the summation above.) In order to evaluate (A9), any element in its summation can be written as

$$\begin{aligned}
&\mathrm{E}_t \int_{T_{i-1}-t}^{T_i - t} \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds \\
&= \mathrm{E}_t \int_{T_{i-1}-t}^{T_i - t} \pi_{K_j}([K_{t+s}^{1:i-1}, K_t^{i:n} e^{-\lambda s}], X_{t+s})\, e^{-(\rho+\lambda)s}\, ds \\
&= \mathrm{E}_t e^{-(\rho+\lambda)(T_{i-1}-t)} \int_0^{\infty} \pi_{K_j}([K_{T_{i-1}+s}^{1:i-1}, K_{T_{i-1}}^{i:n} e^{-\lambda s}], X_{T_{i-1}+s})\, e^{-(\rho+\lambda)s}\, ds \\
&\quad - \mathrm{E}_t e^{-(\rho+\lambda)(T_i - t)} \int_0^{\infty} \pi_{K_j}([K_{T_i+s}^{1:i-1}, K_{T_i}^{i:n} e^{-\lambda s}], X_{T_i+s})\, e^{-(\rho+\lambda)s}\, ds.
\end{aligned} \quad \text{(A10)}$$

Recognizing that $i - 1 < j$ in (A9), we use Lemma 3 to obtain

$$\begin{aligned}
&\mathrm{E}_t \int_{T_{i-1}-t}^{T_i - t} \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds \\
&= \mathrm{E}_t e^{-(\rho+\lambda)(T_{i-1}-t)} \frac{\gamma_j(1-\theta)\, Y_{i-1, T_{i-1}}}{K_{j, T_{i-1}}} \\
&\quad - \mathrm{E}_t e^{-(\rho+\lambda)(T_i - t)} \frac{\gamma_j(1-\theta)\, Y_{i-1, T_i}}{K_{j, T_i}}.
\end{aligned} \quad \text{(A11)}$$

We evaluate this expression first in the special case of $i = 1$, and then in the general case of $i = 2, ..., n$. Then we evaluate the last term in equation (A9). First, use Lemmas 3 and 4 to evaluate the first term $(i = 1)$ in (A9):

$$
\begin{aligned}
&\mathrm{E}_t \int_0^{T_1 - t} \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds \\
&\quad = \frac{\gamma_j (1-\theta)\, Y_{0, T_0}}{K_{j, T_0}} - \mathrm{E}_t e^{-(\rho+\lambda)(T_1 - t)} \frac{\gamma_j (1-\theta)\, Y_{0, T_1}}{K_{j, T_1}} \\
&\quad = \frac{\pi_{K_j}(K_t, X_t)}{R} - \mathrm{E}_t e^{-(\rho+\lambda)(T_1 - t)} \frac{\gamma_j (1-\theta)\, \pi(K_{T_1}, X_{T_1})}{R K_{j, T_1}}. \qquad \text{(A12)}
\end{aligned}
$$

From the expression (47) of the continuation region of the proposed ISD policy under analysis, we know that at time T_1, when the first factor is adjusted, $\pi_{K_1}(K, X)$ must equal either the lower bound $\bar{r}_1$ or the upper bound $\bar{c}_1$. Using this information and the law of iterated expectations, the last term in (A12) becomes:

$$
\begin{aligned}
&E_t e^{-(\rho+\lambda)(T_1 - t)} \frac{\gamma_j (1-\theta)\, \pi(K_{T_1}, X_{T_1})}{R K_{j, T_1}} \\
&\quad = E_t e^{-(\rho+\lambda)(T_1 - t)} \mathrm{E}_{T_1} \frac{\gamma_j K_{1, T_1} \pi_{K_1}(K_{T_1}, X_{T_1})}{R \gamma_1 K_{j, T_1}} \\
&\quad = \frac{\gamma_j K_{1, t} \bar{c}_1}{R \gamma_1 K_{j, t}} \mathrm{E}_t [e^{-(\rho+\lambda)(T_1 - t)}; \pi_{K_1}(T_1) = \bar{c}_1] \\
&\qquad + \frac{\gamma_j K_{1, t} \bar{r}_1}{R \gamma_1 K_{j, t}} \mathrm{E}_t [e^{-(\rho+\lambda)(T_1 - t)}; \pi_{K_1}(T_1) = \bar{r}_1],
\end{aligned}
$$

where the last equality uses Lemma 7. Using Lemma 2 the expected hitting times can be evaluated to obtain

$$
\begin{aligned}
&\mathrm{E}_t \int_0^{T_1 - t} \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds \\
&\quad = \frac{\pi_{K_j}(K_t, X_t)}{R} + A_{11} \frac{K_{1, t}}{K_{j, t}} \pi_{K_1}^{\alpha}(K_t, X_t) + B_{11} \frac{K_{1, t}}{K_{j, t}} \pi_{K_1}^{\beta}(K_t, X_t). \qquad \text{(A13)}
\end{aligned}
$$

Using the same method, the remaining terms in the summation form of q_j (for $i = 2, \ldots, n$) in (A9) can be evaluated. Applying Lemma 9, rewrite the first term in (A11) using

$$
\begin{aligned}
\mathrm{E}_{T_{i-1}} \frac{Y_{i-1, T_{i-1}}}{K_{j, T_{i-1}}} &= \mathrm{E}_{T_{i-1}} \left[\bar{\zeta}_{i-1} \frac{K_{i-1, T_{i-1}}}{K_{j, T_{i-1}}}; \pi_{K_{i-1}}(T_{i-1}) = \bar{c}_{i-1} \right] \\
&\quad + \mathrm{E}_{T_{i-1}} \left[\underline{\zeta}_{i-1} \frac{K_{i-1, T_{i-1}}}{K_{j, T_{i-1}}}; \pi_{K_{i-1}}(T_{i-1}) = \bar{r}_{i-1} \right], \qquad \text{(A14)}
\end{aligned}
$$

and the second term using (also making use of Lemma 8)

$$\mathrm{E}_{T_i} \frac{Y_{i-1,\,T_i}}{K_{j,\,T_i}}$$

$$= \mathrm{E}_{T_i}\left[\bar{\zeta}_{i-1}\frac{K_{i-1,\,T_i}}{K_{j,\,T_i}};\ \pi_{K_i}(T_i)=\bar{c}_i\right] + \mathrm{E}_{T_i}\left[\underline{\zeta}_{i-1}\frac{K_{i-1,\,T_i}}{K_{j,\,T_i}};\ \pi_{K_i}(T_i)=\bar{r}_i\right]$$

$$= \mathrm{E}_{T_i}\left[\bar{\zeta}_{i-1}\frac{\bar{c}_i\gamma_{i-1}}{\bar{c}_{i-1}\gamma_i}\frac{K_{i,\,T_i}}{K_{j,\,T_i}};\ \pi_{K_i}(T_i)=\bar{c}_i\right] + \mathrm{E}_{T_i}\left[\underline{\zeta}_{i-1}\frac{\bar{r}_i\gamma_{i-1}}{\bar{r}_{i-1}\gamma_i}\frac{K_{i,\,T_i}}{K_{j,\,T_i}};\ \pi_{K_i}(T_i)=\bar{r}_i\right]. \tag{A15}$$

Substituting (A14) and (A15) into the expression in (A11) and invoking Lemma 7, we obtain

$$\mathrm{E}_t \int_{T_{i-1}-t}^{T_i-t} \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds$$

$$= \gamma_j(1-\theta)\,\mathrm{E}_t e^{-(\rho+\lambda)(T_{i-1}-t)} \left\{ \begin{aligned} &\mathrm{E}_{T_{i-1}}\left[\bar{\zeta}_{i-1}\frac{K_{i-1,\,t}}{K_{j,\,t}};\ \pi_{K_{i-1}}(T_{i-1})=\bar{c}_{i-1}\right] \\ &+\mathrm{E}_{T_{i-1}}\left[\underline{\zeta}_{i-1}\frac{K_{i-1,\,t}}{K_{j,\,t}};\ \pi_{K_{i-1}}(T_{i-1})=\bar{r}_{i-1}\right] \end{aligned} \right\}$$

$$- \gamma_j(1-\theta)\,\mathrm{E}_t e^{-(\rho+\lambda)(T_i-t)} \left\{ \begin{aligned} &\mathrm{E}_{T_i}\left[\bar{\zeta}_{i-1}\frac{\bar{c}_i\gamma_{i-1}}{\bar{c}_{i-1}\gamma_i}\frac{K_{i,\,t}}{K_{j,\,t}};\ \pi_{K_i}(T_i)=\bar{c}_i\right] \\ &+\mathrm{E}_{T_i}\left[\underline{\zeta}_{i-1}\frac{\bar{r}_i\gamma_{i-1}}{\bar{r}_{i-1}\gamma_i}\frac{K_{i,\,t}}{K_{j,\,t}};\ \pi_{K_i}(T_i)=\bar{r}_i\right] \end{aligned} \right\}. \tag{A16}$$

Using Lemma 2, the hitting times can be evaluated, producing the expression

$$\mathrm{E}_t \int_{T_{i-1}-t}^{T_i-t} \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds$$

$$= A_{i-1,\,i}\frac{K_{i-1,\,t}}{K_{j,\,t}} K_{i-1,\,t}^{\alpha\sum_{k=0}^{i-2}\gamma_k(1-\theta)}\pi_{K_{i-1}}^{\alpha}((\mathrm{e},\, K_t^{i-1:n}),\, X_t)$$

$$+ B_{i-1,\,i}\frac{K_{i-1,\,t}}{K_{j,\,t}} K_{i-1,\,t}^{\beta\sum_{k=0}^{i-2}\gamma_k(1-\theta)}\pi_{K_{i-1}}^{\beta}((\mathrm{e},\, K_t^{i-1:n}),\, X_t)$$

$$+ A_{i,i}\frac{K_{i,\,t}}{K_{j,\,t}} K_{i,\,t}^{\alpha\sum_{k=0}^{i-1}\gamma_k(1-\theta)}\pi_{K_i}^{\alpha}((\mathrm{e},\, K_t^{i:n}),\, X_t)$$

$$+ B_{i,\,i}\frac{K_{i,\,t}}{K_{j,\,t}} K_{i,\,t}^{\beta\sum_{k=0}^{i-1}\gamma_k(1-\theta)}\pi_{K_i}^{\beta}((\mathrm{e},\, K_t^{i:n}),\, X_t). \tag{A17}$$

Note that for $i=2$ the first two terms in this expression are identical to the last two terms in (A13), up to the constants. More generally, each term in the summation in (A9) adds two such terms to the expression for q_j. The final term in the expression for q_j in (A9) is evaluated using the boundary values of q_j of the proposed ISD investment rule (which satisfies the smooth pasting conditions):

$$\begin{aligned}
&\mathrm{E}_t \int_{T_j - t}^{\infty} \pi_{K_j}(K_{t+s}, X_{t+s})\, e^{-(\rho+\lambda)s}\, ds \\
&\quad = \mathrm{E}_t e^{-(\rho+\lambda)(T_j - t)} \int_0^{\infty} \pi_{K_j}(K_{T_j+s}, X_{T_j+s})\, e^{-(\rho+\lambda)s}\, ds \\
&\quad = \mathrm{E}_t e^{-(\rho+\lambda)(T_j - t)} q_j(T_j) \\
&\quad = \mathrm{E}_t \{ \mathrm{E}_{T_j}[e^{-(\rho+\lambda)(T_j - t)} c_j;\, q_j(T_j) = c_j] \\
&\qquad + \mathrm{E}_{T_j}[e^{-(\rho+\lambda)(T_j - t)} r_j;\, q_j(T_j) = r_j] \} \\
&\quad = A_j K_j^{\alpha \sum_{k=0}^{j-1} \gamma_k (1-\theta)} \pi_{K_j}^{\alpha}((\mathrm{e}, K_t^{j:n}), X_t) \\
&\qquad + B_j K_j^{\beta \sum_{k=0}^{j-1} \gamma_k (1-\theta)} \pi_{K_j}^{\beta}((\mathrm{e}, K_t^{j:n}), X_t).
\end{aligned} \tag{A18}$$

Substituting (A13), (A17), and (A18) into the expression for q_j in (A9) we obtain (46), where $A_{ji} \equiv \sum_l A_{l,i}$ and $B_{ji} = \sum_l B_{l,i}$ for each factor $j = 1, \ldots, n$. ∎

REFERENCES

1. A. B. Abel, A. K. Dixit, J. C. Eberly, and R. S. Pindyck, Options, the value of capital, and investment, *Quart. J. Econ.* **111**, No. 3 (1996), 755–777.
2. A. B. Abel and J. C. Eberly, Optimal investment with costly reversibility, *Rev. Econ. Stud.* **63** (1996), 581–593.
3. R. A. Abrams and U. S. Karmarkar, Optimal multiperiod investment–consumption policies, *Econometrica* **48**, No. 2 (1980), 333–353.
4. Deleted in proof.
5. K. J. Arrow, Optimal capital policy with irreversible investment, *in* "Value, Capital and Growth. Papers in Honour of Sir John Hicks," pp. 1–19, Edinburgh Univ. Press, 1968.
6. J. C. Bean, J. L. Higle, and R. L. Smith, Capacity expansion under stochastic demands, *Operations Res.* **40**, Supp. No. 2 (1992), S210–S216.
7. S. Bentolila and G. Bertola, Firing costs and labour demand: How bad is eurosclerosis? *Rev. Econ. Stud.* **57** (1990), 381–402.
8. B. Bernanke, Irreversibility, uncertainty, and cyclical investment, *Quart. J. Econ.* **98**, No. 1 (1983), 85–106.

9. G. Bertola, Irreversible investment, Unpublished manuscript, M.I.T., 1987.
10. D. Blackwell, D. Freedman, and M. Orkin, The optimal reward operator in dynamic programming, *Ann. Prob.* **2**, No. 5 (1974), 926–941.
11. W. A. Brock and L. A. Mirman, Optimal economic growth and uncertainty: The discounted case, *J. Econ. Theory* **3** (1972), 479–513.
12. R. S. Chirinko, Multiple capital inputs, *Q*, and investment spending, *J. Econ. Dynam. Control* **17** (1993), 907–928.
13. M. H. A. Davis, M. A. H. Dempster, and S. P. Vermes, Optimal capacity expansion under uncertainty, *Adv. Appl. Prob.* **19** (1987), 156–176.
14. A. Dixit, Entry and exit decisions under uncertainty, *J. Polit. Econ.* **97**, No. 3 (1989), 620–638.
15. A. Dixit, Investment and employment dynamics in the short run and the long run, *Oxford Econ. Pap.* **49** (1997), 1–20.
16. G. Eppen and E. F. Fama, Cash balance and simple dynamic portfolio problems with proportional costs, *Int. Econ. Rev.* **10** (1969), 119–133.
17. E. F. Fama, Multiperiod consumption–investment decisions, *Amer. Econ. Rev.* **60** (1970), 163–174.
18. C. H. Fine and R. M. Freund, Optimal investment in product-flexible manufacturing capacity, *Manage. Sci.* **36**, No. 4 (1990), 449–466.
19. M. Galeotti and F. Schiantarelli, Generalized *Q* models for investment, *Rev. Econ. Statist.* **73**, No. 3 (1991), 383–392.
20. J. M. Harrison, "Brownian Motion and Stochastic Flow Systems," Wiley, New York, 1985.
21. F. Hayashi and T. Inoue, The relation between firm growth and *Q* with multiple capital goods: Theory and evidence from panel data on Japanese firms, *Econometrica* **59**, No. 3 (1991), 731–754.
22. D. Heyman and M. Sobel, "Stochastic Models in Operations Research," Vol. 2, McGraw–Hill, New York, 1984.
23. D. W. Jorgenson, Capital theory and investment behavior, *Amer. Econ. Rev. Papers and Proceedings* **53** (1963), 247–259.
24. U. S. Karmarkar, Policy structure in multi-state production/inventory problems: An application of convex analysis, *in* "Multilevel Production/Inventory Control Systems: Theory and Practice" (L. B. Schwarz, Ed.), pp. 331–352, TSMS 16, North-Holland, New York, 1981.
25. U. S. Karmarkar, The multilocation multiperiod inventory problem: Bounds and approximations, *Manage. Sci.* **33**, No. 1 (1987), 86–94.
26. H. Luss, Operations and capacity expansion problems: A survey, *Operations Res.* **30** (1982), 907–947.
27. A. Maitra, Discounted dynamic programming on compact metric spaces, *Sankhyā Ser. A* **30** (1968), 211–216.
28. A. S. Manne, Capacity expansion and probabilistic growth, *Econometrica* **29**, No. 4 (1961), 632–649.
29. A. S. Manne, Ed., "Investments for Capacity Expansion," M.I.T. Press, Cambridge, MA, 1967.
30. R. McDonald and D. Siegel, The value of waiting to invest, *Quart. J. Econ.* **101** (1986), 707–728.
31. I. Nadiri and S. Rosen, Interrelated factor demand functions, *Amer. Econ. Rev.* **59**, No. 4 (1969), 457–471.
32. W. Oi, Labor as a quasi-fixed factor, *J. Polit. Econ.* **70**, No. 6 (1962), 538–555.
33. D. Paraskevopoulos, E. Karakitsos, and B. Rustem, Robust capacity planning under uncertainty, *Manage. Sci.* **37**, No. 7 (1991), 787–800.

34. R. S. Pindyck, Irreversible investment, capacity choice, and the value of the firm, *Amer. Econ. Rev.* **78**, No. 5 (1988), 969–985.
35. E. Porteus, On the optimality of structured policies in countable stage decision processes, *Manage. Sci.* **22** (1975), 148–157.
36. E. Porteus, Conditions for characterizing the structure of optimal strategies in infinite-horizon dynamic programs, *J. Optim. Theory Appl.* **36**, No. 3 (1982), 419–432.
37. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.
38. H. L. Royden, "Real Analysis," 3rd ed., Macmillan, New York, 1988.
39. D. M. Topkis, Minimizing a submodular function on a lattice, *Operations Res.* **26**, No. 2 (1978), 305–321.
40. D. E. Wildasin, The q theory of investment with many capital goods, *Amer. Econ. Rev.* **74**, No. 1 (1984), 203–210.