Technical Companion to:

Global Dual Sourcing and Order Smoothing:

The Impact of Capacity and Leadtimes

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This technical companion has three parts: First, it shows the proofs of the propositions in the paper. Then, it presents the exact solutions for special cases if $L_l = \theta_l = \theta_g = 0$ and the Lagrange series for dual sourcing where one or both sources incur capacity costs as Propositions 15 and 16. The last part show the very detailed derivations of the single sourcing analysis and of the exact dual sourcing solutions to the first order conditions: the Cardano solution ($L = 1$ and $L_l = 0$ on Companion Page 17); the Ferrari solution ($L = 3$ and $L_l = 0$ for uncapacitated dual sourcing on Companion Page 20); and the Lagrange series (for the other cases).

1 Proofs of Propositions in the Paper

Proposition 1 was shown in the main text. We proceed with:

Proof of Proposition 2: Given the order process, we can analytically track the net inventory dynamics for $t = 1, ..., T$ and $\alpha \in [0, 1)$:

$$\begin{aligned} q_t &= \sum_{k=L_i}^{t} (1-\alpha)\alpha^{k-L_i} D_{t-k} \\ &= \sum_{k=0}^{t-L_i} (1-\alpha)\alpha^{k} D_{t-L_i-k} \end{aligned}$$

$$
\begin{aligned}
I_t &= I_{t-1} + q_{t-1} - D_t = I_{t-1} + \sum_{k=0}^{t-L_i-1} (1-\alpha)\alpha^k D_{t-L_i-k-1} - D_t \\
&= I_{t-2} + \sum_{k=0}^{t-L_i-2} (1-\alpha)\alpha^k D_{t-L_i-k-2} - D_{t-1} + \sum_{k=0}^{t-L_i-1} (1-\alpha)\alpha^k D_{t-L_i-k-1} - D_t \\
&= I_{-1} + \sum_{j=1}^{t-L_i} \sum_{k=0}^{t-L_i-j} (1-\alpha)\alpha^k D_{t-L_i-k-j} - \sum_{k=0}^{t} D_{t-k} \\
&= I_{-1} + \sum_{j=1}^{t-L_i} \sum_{k=0}^{t-L_i-j} (1-\alpha)\alpha^k D_{t-L_i-k-j} - \sum_{k=0}^{t-L_i} D_{t-L_2-k} - \sum_{k=0}^{L_i-1} D_{t-k} \\
&= I_{-1} - \sum_{k=0}^{t-L_i} \alpha^k D_{t-L_i-k} - \sum_{k=0}^{L_i-1} D_{t-k}
\end{aligned}
$$

Hence, for $t \to \infty$,

$$
\begin{aligned}
Var(q_t) &= \frac{1-\alpha}{1+\alpha}\sigma^2 \\
Var(I_t) &= \frac{1}{1-\alpha^2}\sigma^2 + L_i\sigma^2.
\end{aligned}
$$

Note that this policy encompasses global single sourcing with the standard base-stock policy. Check: if $\alpha = 0$, then we have $Var(I_t) = (L_g + 1)\sigma^2$. If $\alpha = 1$, we have that $q_t = \mu$ and

$$
I_t = I_{-1} + \sum_{i=0}^{t-1} q_i - \sum_{i=0}^{t} D_i = I_s + t\mu - \sum_{i=0}^{t} D_i.
$$

Taking expectations and variances directly yield $\mathbb{E}I_t$ and $Var(I_t)$.$\blacksquare$

Proof of Proposition 3: This is not a standard newsvendor problem because the decision variable I_s is the mean of the distribution but it can be reduced to a newsvendor model. The limiting net inventory process I_∞ is normally distributed with mean $\mathbb{E}I_t = I_s$ and variance $\sigma_I^2 = \frac{1}{1-\alpha^2}\sigma^2 + L_i\sigma^2$. Let $f(x|I_s, \sigma_I)$ denote its density. The associated expected inventory cost rate is

$$
C_I(I_s) = \int_0^\infty hxf(x|I_s, \sigma_I)dx - \int_{-\infty}^0 bxf(x|I_s, \sigma_I)dx.
$$

Substitute to standardized units $z = (x - I_s)/\sigma_I$ and define $z_I = I_s/\sigma_I$. With $x = I_s + z\sigma_I = (z_I + z)\sigma_I$ and $f(x|I_s, \sigma_I) = \phi(z)/\sigma_I$, we get

$$
\begin{aligned}
C_I(I_s) &= \int_{-z_I}^\infty h(z_I + z)\sigma_I\phi(z)dz - \int_{-\infty}^{-z_I} b(z_I + z)\sigma_I\phi(z)dz \\
&= \sigma_I \left[hz_I\left(1 - \Phi(-z_I)\right) + h\phi(-z_I) - bz_I\Phi(-z_I) + b\phi(-z_I)\right], \\
&= \sigma_I \left[hz_I + (h+b)\left(\phi(-z_I) - z_I\Phi(-z_I)\right)\right]
\end{aligned}
$$

where we used the identity $\phi'(x) = -x\phi(x) \Rightarrow -\int_{-\infty}^{x} z\phi(z)dz = \int_{x}^{\infty} z\phi(z)dz = \phi(x)$. Invoking $\phi(-x) = \phi(x)$ and $\Phi(-x) = 1 - \Phi(x)$, it directly follows that

$$\begin{aligned} C_I(I_s) &= \sigma_I \left[hz_I + (h+b)\left(\phi(z_I) - z_I(1-\Phi(z_I))\right)\right] = \sigma_I \left[hz_I + (h+b)I_N(z_I)\right] \\ &= \sigma_I \left[-bz_I + (h+b)\left(\phi(z_I) + z_I\Phi(z_I)\right)\right] \end{aligned}$$

The inventory cost is convex increasing in α and taking the first derivative

$$\frac{d}{dI_s}C_I(I_s) = \frac{d}{\sigma_I dz_I}C_I(I_s) = -b + (h+b)\left(\phi'(z_I) + \Phi(z_I) + z_I\phi(z_I)\right) = -b + (h+b)\Phi(z_I).$$

Clearly $\frac{d^2}{dI_s^2}C_I(I_s) = \sigma_I^{-1}(h+b)\phi(z_I) > 0$ so that C_I is convex with unique minimum given by the familiar critical fractile condition $\Phi(z_I) = b/(h+b)$.■

Proof of Proposition 4: The function $\widehat{C^{ss}}(\alpha)$ is continuous in $[0,1)$ with $\widehat{C}(0) = \theta_i + \sqrt{1+L_i}$ and $\widehat{C}(1) = +\infty$. Its first derivative is:

$$\frac{d}{d\alpha}\widehat{C}(\alpha) = \left(1-\alpha^2\right)^{-\frac{3}{2}}\left\{\alpha\left(1+L_i-L_i\alpha^2\right)^{-\frac{1}{2}} - \theta_i(1-\alpha)\right\}$$

The first term is positive (hence inventory cost is increasing), while the second is negative (hence capacity cost is decreasing). For $0 \le \alpha < 1$, its sign equals the sign of (which also becomes the optimality condition:)

$$F(\alpha) - \theta_i, \text{ where } F(\alpha) = \alpha(1-\alpha)^{-1}\left(1+L_i-L_i\alpha^2\right)^{-\frac{1}{2}}. \tag{30}$$

F is strictly convex and increasing so that F^{-1} is uniquely defined and concave increasing:

$$\begin{aligned} F'(\alpha) &= (1-\alpha)^{-2}\left(1+L_i-L_i\alpha^2\right)^{-\frac{3}{2}}\left\{1+L_i(2\alpha+1)(1-\alpha)\right\} > 0 \\ F''(\alpha) &= (1-\alpha)^{-3}\left(1+L_i-L_i\alpha^2\right)^{-\frac{5}{2}}\left(2+L_i(1-\alpha)\left(3L_i(5\alpha+3)(1-\alpha)+2\alpha+11\right)\right) > 0 \end{aligned}$$

Thus:

1. For any $L_i \ge 0$ and any $\theta_i > 0$, there is a single extremum α^* where $F(\alpha^*) = \theta_i$, so that $\widehat{C^{ss}}$ is first decreasing and then increasing. Thus, $\widehat{C^{ss}}(\alpha^*) < \widehat{C^{ss}}(0)$, which is the cost when using a base-stock policy. If $\theta_i = 0$, $\alpha^* = 0$ and $\widehat{C}$ is increasing.

2. For $\theta_i > 0$, the optimal $\alpha^*(\theta_i)$ is concave increasing in θ_i with

$$\alpha^*|_{\theta_i=0} = 0 < \alpha^* < \alpha^*|_{\theta_i\to\infty} = 1.$$

3. For $\theta_i > 0$, the optimal $\alpha^*(\theta_i)$ is increasing in L_i.We directly have a bound:

$$\alpha^*|_{L_i=0} = \frac{\theta_i}{\theta_i+1} < \alpha^*|_{L_i>0} = \frac{\theta_i}{\theta_i + \left(1+L_i-L_i\alpha^{*2}\right)^{-\frac{1}{2}}}$$

4. Below we show that $\widehat{C}$ is strictly convex in $[0, \frac{1}{2})$ and near $\alpha = 1$. For small values of θ_i, it is convex everywhere. For large values of θ_i, it can be convex-concave-convex. Indeed, consider the second derivative:

$$\frac{d^2}{d\alpha^2}\widehat{C}(\alpha) = \left(1-\alpha^2\right)^{-\frac{5}{2}} \left\{\left(1+L_i\left(1-\alpha^2\right)\right)^{-\frac{3}{2}} \left(1+\alpha^2+L_i\left(1-\alpha^2\right)\left(1+2\alpha^2\right)\right) + \theta_i(1-\alpha)\left(1-2\alpha\right)\right\}$$

where the first term inside the curly brackets is positive (and thus the inventory cost $\sqrt{L_i + \frac{1}{1-\alpha^2}}$ is convex increasing) and the last term is positive for $\alpha < \frac{1}{2}$ and negative for $\frac{1}{2} < \alpha < 1$ (so that the capacity cost is convex-concave with inflection point at $\alpha = \frac{1}{2}$). As θ_i increases, the capacity term becomes more important, but is always dominated for $\alpha \to 1$ by the convex inventory cost.

The value of smoothing for $L_i = 0$: then $\widehat{C}(\alpha_0) = \widehat{C}(\alpha^*) = \sqrt{2\theta_i + 1}$ and $\widehat{C}(0) = 1 + \theta_i$ so that

$$\text{Value} = \frac{1+\theta_i - \sqrt{2\theta_i+1}}{1+\theta_i} = 1 - \frac{\sqrt{2\theta_i+1}}{\theta_i+1},$$

which increases in θ_i towards a maximum of 100%.

Proof of Proposition 5: Extending L to a continuous variable and applying the chain rule on the necessary optimality condition gives:

$$\begin{aligned}
\frac{d}{dL}\widehat{C}(\alpha^*) &= \frac{\partial \widehat{C}}{\partial \alpha}(\alpha^*)\frac{d\alpha^*}{dL} + \frac{\partial \widehat{C}}{\partial L}(\alpha^*) = \frac{\partial \widehat{C}}{\partial L}(\alpha^*) \\
&= \widehat{h}\alpha^{*L} - \theta_c \alpha^{*L} \ln \alpha^* \geq 0.
\end{aligned}$$

To show that the optimal allocation also decreases when L increases from 1 to 2, express the smoothing FOC (16) as a function of the optimal allocation:

$$\theta_c = G(a, L) = \frac{a^{*\frac{2-L}{L}}}{L\left(1-a^{*2/L}\right)^{3/2}\sqrt{1+L_l\left(1-a^{*2/L}\right)}} \tag{31}$$

Extending L to a continuous variable and applying the implicit function theorem on the FOC of the allocation yields:

$$\frac{da^*}{dL} = -\frac{\widehat{h} + \frac{\partial G}{\partial L}(a^*, L)}{\frac{\partial G}{\partial a}(a^*, L)} \tag{32}$$

It can be shown that $\frac{\partial G}{\partial L} > 0$ for all L, and $\frac{\partial G}{\partial a} > 0$ for $L \leq 2$, so that a^* decreases as L increases over $L \in [1, 2]$.

Finally, the comparative statics of θ_c follow from:

$$\begin{aligned}\frac{d}{d\theta_c}\widehat{C}(\alpha^*) &= \frac{\partial \widehat{C}}{\partial \alpha}(\alpha^*)\frac{d\alpha^*}{d\theta_c} + \frac{\partial \widehat{C}}{\partial \theta_c}(\alpha^*) = \frac{\partial \widehat{C}}{\partial \theta_c}(\alpha^*) = -\alpha^{*L} < 0. \\ \frac{d\alpha^*}{d\theta_c} &= -\frac{\frac{\partial^2 C}{\partial \theta_c \partial \alpha}(\alpha^*)}{\frac{\partial^2 C}{\partial \alpha^2}(\alpha^*)} = \frac{L\left(\alpha^*\right)^{L-1}}{\frac{\partial^2 C}{\partial \alpha^2}(\alpha^*)} > 0,\end{aligned}$$

given that $\frac{\partial^2 C}{\partial \alpha^2}(\alpha^*) > 0$ because α^* is a minimizer.■

Proof of Proposition 6: If $L = 1$ and $L_l = 0$, $\widehat{C}(\cdot)$ is strictly convex with $\widehat{C}'(0) = -\theta_c$. If, and only if, $\theta_c > 0$, there is a unique interior minimum $\alpha^* \in (0,1)$ that solves the sufficient first order equation (16), which simplifies to $\theta_c^2 x^3 + x - 1 = 0$ where $x = 1 - \alpha^2$. The third order polynomial is strictly increasing and thus has one real root (34), which is found using Cardano's rule–see Appendix.

For $L = 2$ and $L_l = 0$, $\widehat{C}(\cdot)$ is concave-convex with $\widehat{C}'(\alpha) = -2\theta_c\alpha + \alpha(1-\alpha^2)^{-3/2} = -2\theta_c\alpha + \alpha(1 + \frac{3}{2}\alpha^2 + o(\alpha^2)) = (1 - 2\theta_c)\alpha + o(\alpha)$. If, and only if, $\theta_c > 1/2$, there is a unique interior minimum that solves the sufficient first order equation (16), which simplifies to $(2\theta_c)^2 x^3 - 1 = 0$ where $x = 1 - \alpha^2$. Notice that, if $\theta_c = 1/2$, $\widehat{C}(\alpha^*) = 1$ and that cost decreases as θ_c increases, hence dominating single local sourcing.

For $L = 3$ and $L_l = 0$, $\widehat{C}(\cdot)$ is convex-concave-convex with $\widehat{C}'(\alpha) = -3\theta_c\alpha^2 + \alpha(1-\alpha^2)^{-3/2} = -3\theta_c\alpha^2 + \alpha(1 + \frac{3}{2}\alpha^2 + o(\alpha^2)) = \alpha - 3\theta_c\alpha^2 + \frac{3}{2}\alpha^3 + o(\alpha^3)$. So cost is initially increasing convexly. If θ_c is sufficiently large, cost will go through an inflection point, turn concave and achieve a local maximum, go through another inflection point and turn convex with local minimum. The two extrema in $(0,1)$ solve the first order equation (16), which simplifies to $x^4 - x^3 + (3\theta_c)^{-2} = 0$ where $x = 1 - \alpha^2$. The roots of the quartic can be found using Ferrari's method–see Appendix.

For $L \geq 4$ and $L_l = 0$, the solution to the first order equation are the roots of a polynomial of order $L + 1 > 4$ for which Galois showed there is no general formula, using only a finite number of the usual algebraic operations and radicals. However, Lagrange's series for the inverse of a function Markushevich (1985, II, pp. 88) applies here. Let $x^3\left(1 + L_l x\right)\left(1 - x\right)^{L-2}$, then f^{-1} expanded around $x = 0$ yields:

$$f^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{\left[f(z)\right]_s^{1/3}}\right)^n\right]\Bigg|_{z=z_0} z^{n/3}$$

where the subscript s denotes any fixed single-valued branch of the 3-valued function $\left[f(z)\right]^{1/3}$. Evaluating the derivatives, similar to the Appendix in the paper, yields (19) and it can be shown that the radius of convergence is exactly $\theta_c \geq \underline{\theta}_L$.■

Proof of Proposition 7: For $L = 1$, the approximation is found by intersecting the marginal benefit with a higher marginal cost: α^* solves $\frac{\alpha^*}{\left(1-\alpha^{*2}\right)^{3/2}} = \theta_c$ while α_0 solves $\theta_c = \frac{1}{\left(1-\alpha_0^2\right)^{3/2}}$. Given that $\frac{1}{\left(1-\alpha_0^2\right)^{3/2}} = \frac{\alpha^*}{(1-\alpha^{*2})^{3/2}} \leq \frac{1}{(1-\alpha^{*2})^{3/2}}$ we have that $\alpha_0 \leq \alpha^*$. Clearly, both $\alpha_0 \to 1$ and $\alpha^* \to 1$ as $\theta_c \to \infty$.

For $L \geq 3$, the approximation α_0 corresponds to the first order expansion of the Langrange series (19), but it also can be found directly as follows: α^* solves $\frac{1}{(1-\alpha^{*2})^{3/2}} = L\theta_c\,(\alpha^*)^{L-2} < L\theta_c = \frac{1}{(1-\alpha_0^2)^{3/2}}$, so that $\alpha_0 \geq \alpha^*$. Again, both $\alpha_0 \to 1$ and $\alpha^* \to 1$ as $\theta_c \to \infty$.

The lower bound on the value of dual sourcing follows from $\widehat{C}(\alpha^*) \leq \widehat{C}(\alpha_0)$. The expansion follows from the binomial theorem:

$$
\begin{aligned}
\widehat{C}(\alpha_0) &= -\theta_c\left(1-(L\theta_c)^{-2/3}\right)^{L/2} + \left(L_l + (L\theta_c)^{2/3}\right)^{\frac{1}{2}} \\
&= -\theta_c\left(1 - \frac{L}{2}\,(L\theta_c)^{-2/3} + \frac{L(L-2)}{2^2\cdot 2!}\,(L\theta_c)^{-4/3} - \frac{L(L-2)(L-4)}{2^3\cdot 3!}\,(L\theta_c)^{-6/3} + \cdots\right) \\
&\quad + (L\theta_c)^{1/3}\left(1 + \frac{1}{2}L_l\,(L\theta_c)^{-2/3} - \frac{1}{8}L_l^2\,(L\theta_c)^{-4/3} + O(L_l^3\,(L\theta_c)^{-6/3})\right) \\
&= -\theta_c + \frac{3}{2}\,(L\theta_c)^{\frac{1}{3}} - \frac{(L-2)-2!L_l}{8}\,(L\theta_c)^{-\frac{1}{3}} + \left((L-2)(L-4)-3!L_l^2\right)O\,(L\theta_c)^{-1}
\end{aligned}
$$

■

Proof of Proposition 8: Extending L to a continuous variable and applying the chain rule on the necessary optimality condition gives:

$$
\begin{aligned}
\frac{d}{dL}\widehat{C}(\alpha^*) &= \frac{\partial\widehat{C}}{\partial\alpha}(\alpha^*)\frac{d\alpha^*}{dL} + \frac{\partial\widehat{C}}{\partial L}(\alpha^*) = \frac{\partial\widehat{C}}{\partial L}(\alpha^*) \\
&= \frac{\partial}{\partial L}\left(-\theta_c\alpha^L + \theta_g\alpha^L\sqrt{\frac{1-\alpha}{1+\alpha}} + \theta_l\sqrt{\frac{1-\alpha}{1+\alpha}\,(1-\alpha^{2L})}\right) \\
&= \widehat{h}\alpha^{*L} + \left(-\theta_c\alpha^{*L} + \theta_g\alpha^{*L}\sqrt{\frac{1-\alpha^*}{1+\alpha^*}} + \theta_l\sqrt{\frac{1-\alpha^*}{1+\alpha^*}}\left(1-\alpha^{*2L}\right)^{-1/2}\alpha^{*2L}\right)\ln\alpha^* \\
&= \widehat{h}\alpha^{*L} - \theta_c\alpha^{*L}\ln\alpha^* + \left(\theta_g\alpha^{*L}\sqrt{\frac{1-\alpha^*}{1+\alpha^*}} + \theta_l\sqrt{\frac{1-\alpha^*}{1+\alpha^*}}\left(1-\alpha^{*2L}\right)^{-1/2}\alpha^{*2L}\right)\ln\alpha^*,
\end{aligned}
$$

where the first two terms are positive but the term in parentheses, called B, is positive. We bound B as follows: recall that

$$
\frac{1-\alpha^L}{1-\alpha} = \sum_{k=0}^{L-1}\alpha^k = g(\alpha)
$$

$$\begin{aligned}
\sqrt{\frac{1-\alpha}{1+\alpha}} &\leq 1 \\
\sqrt{\frac{1-\alpha}{(1+\alpha)(1-\alpha^{2L})}} &= \sqrt{\frac{1-\alpha}{(1+\alpha)(1+\alpha^L)(1-\alpha^L)}} \\
&= \sqrt{\frac{1}{(1+\alpha)(1+\alpha^L)\,g(\alpha)}} \leq 1 \\
B &\leq \theta_g \alpha^{*L}\sqrt{\frac{1-\alpha^*}{1+\alpha^*}} + \theta_l \sqrt{\frac{1-\alpha^*}{1+\alpha^*}}\left(1-\alpha^{*2L}\right)^{-1/2}\alpha^{*2L} \\
&\leq \theta_g + \theta_l
\end{aligned}$$

hence:

$$\frac{d}{dL}\widehat{C}(\alpha^*) \geq \left(\widehat{h} - \left(\theta_c - \theta_g - \theta_l \alpha^{*L}\right)\ln \alpha^*\right)\alpha^{*L}$$

So, a sufficient (but not necessary) condition is $\theta_c \geq \theta_g + \theta_l$. Finally, the comparative statics of θ follow from:

$$\begin{aligned}
\frac{d}{d\theta_i}\widehat{C}(\alpha^*) &= \frac{\partial \widehat{C}}{\partial \alpha}(\alpha^*)\frac{d\alpha^*}{d\theta_i} + \frac{\partial \widehat{C}}{\partial \theta_i}(\alpha^*) = \frac{\partial \widehat{C}}{\partial \theta_i}(\alpha^*) \\
\frac{\partial \widehat{C}}{\partial \theta_c}(\alpha^*) &= -\alpha^{*L} < 0. \\
\frac{\partial \widehat{C}}{\partial \theta_g}(\alpha^*) &= \alpha^{*L}\sqrt{\frac{1-\alpha^*}{1+\alpha^*}} < \alpha^{*L} \\
\frac{\partial \widehat{C}}{\partial \theta_l}(\alpha^*) &= \sqrt{\frac{1-\alpha}{1+\alpha}\left(1-\alpha^{2L}\right)} < 1.
\end{aligned}$$

and, given that $\frac{\partial^2 C}{\partial \alpha^2}(\alpha^*) > 0$ because α^* is a minimizer, we have:

$$\begin{aligned}
\frac{d\alpha^*}{d\theta_c} &= -\frac{\frac{\partial^2 C}{\partial \theta_c \partial \alpha}(\alpha^*)}{\frac{\partial^2 C}{\partial \alpha^2}(\alpha^*)} = \frac{L\left(\alpha^*\right)^{L-1}}{\frac{\partial^2 C}{\partial \alpha^2}(\alpha^*)} > 0, \\
sign\frac{d\alpha^*}{d\theta_g} &= -sign\frac{\partial^2 C}{\partial \theta_g \partial \alpha}(\alpha^*) = -sign\frac{L\alpha^{L-1}(1-\alpha)(1+\alpha) - \alpha^L}{(1-\alpha)^{1/2}(1+\alpha)^{3/2}} \\
&= -sign\left(L\left(1-\alpha^*\right)\left(1+\alpha^*\right) - \alpha^*\right) \\
&\leq 0 \text{ if } \alpha^* \leq \frac{\sqrt{1+4L}-1}{2L}, \ > 0 \text{ otherwise.} \\
sign\frac{d\alpha^*}{d\theta_l} &= -sign\frac{\partial^2 C}{\partial \theta_l \partial \alpha}(\alpha^*) = sign\frac{L\alpha^{2L-1}(1-\alpha)(1+\alpha) + \left(1-\alpha^{2L}\right)}{\left(1-\alpha^{2L}\right)^{1/2}(1-\alpha)^{1/2}(1+\alpha)^{3/2}} > 0. \qquad \blacksquare
\end{aligned}$$

Proof of Proposition 9: As $L \to \infty$, the function $\widehat{MB}$ converges uniformly to $\widehat{MB}_\infty$ over $[0,1)$, where

$$\widehat{MB}_\infty(\alpha) = \theta_l\left(1-\alpha\right)^{-\frac{1}{2}}\left(1+\alpha\right)^{-\frac{3}{2}}.$$

Consequently, the optimal smoothing level $\alpha^* \to \alpha^*_\infty$ as $L \to \infty$ where

$$\widehat{MB}_\infty(\alpha^*_\infty) = \frac{\theta_l}{(1-\alpha^*_\infty)^{\frac{1}{2}}(1+\alpha^*_\infty)^{\frac{3}{2}}} = \widehat{MC}(\alpha^*_\infty) = \frac{\alpha^*_\infty}{(1-\alpha^{*2}_\infty)^{\frac{3}{2}}(1+L_l(1-\alpha^{*2}_\infty))^{\frac{1}{2}}}$$

$$\Leftrightarrow \alpha^*_\infty = \frac{\theta_l}{\theta_l + (1+L_l(1-\alpha^{*2}_\infty))^{\frac{1}{2}}}. \qquad \blacksquare$$

Proof of Proposition 10 For $L=1$, the FOC are identical to the uncapacitated case provided we replace θ_c by $\theta_c + \theta_l$. Hence, Cardano (Proposition 14 on Companion Page 11) gives the exact solution.

For $L \geq 2$, the solution uses Lagrange's inversion theorem. In contrast to the uncapacitated case, however, there is no simple general expression for the $n-th$ order term in the Lagrange series. Each term requires substantial work. The derivations are shown in painstaking detail for $L=2$ starting from Companion Page 46 and for $L>2$ from Companion Page 53.$\blacksquare$

Proof of Proposition 11 For $L=1$, the FOC are identical to the uncapacitated case provided we replace replace θ_c by $\theta_c + \theta_l$. The square root remains a lower bound as follows directly from the Proof of Prop. 7. For $L=2$, the proof starts on Companion Page 50. For $L>2$, the proof starts on Companion Page 59.

Proof of Proposition 12 Let $\beta \in [0,1]$ and denote $\overline{\beta} = 1-\beta$. Focus on the parameter line $\theta_c - \theta_g + \theta_l = \sqrt{L+1}-1$ by setting

$$\theta_c = \beta\left(\sqrt{L+1}-1\right) \text{ and } \theta_l = \overline{\beta}\left(\sqrt{L+1}-1\right).$$

For $L \geq 2$, the maximal value of dual sourcing V over LS using Prop. 15 then is:

$$\begin{aligned} V = \frac{\widehat{C^l} - \widehat{C}^*}{\widehat{C^l}} \quad &\simeq \quad \frac{1+\theta_l - \left(\frac{3}{2}\left(L\theta_c + \sqrt{L}\theta_l\right)^{\frac{1}{3}} - \theta_c - \frac{L-2}{8}\left(L\theta_c + \sqrt{L}\theta_l\right)^{-\frac{1}{3}}\right)}{1+\theta_l} \\ &= \quad \frac{\sqrt{L+1} - \frac{3}{2}\left(L\theta_c + \sqrt{L}\theta_l\right)^{\frac{1}{3}} + \frac{L-2}{8}\left(L\theta_c + \sqrt{L}\theta_l\right)^{-\frac{1}{3}}}{1+\overline{\beta}\left(\sqrt{L+1}-1\right)} \end{aligned} \tag{33}$$

where $L\theta_c + \sqrt{L}\theta_l = \left(L\beta + \sqrt{L}\overline{\beta}\right)\left(\sqrt{L+1}-1\right)$. Expression (33) is decreasing in β so that the maximal value is attained at $\theta_c = 0$ and $\theta_l = \sqrt{L+1}-1$ (as is evident in Fig. 20 on Companion Page 31). Hence:

$$V = 1 - \frac{3}{2}\frac{\left(\sqrt{L}\left(\sqrt{L+1}-1\right)\right)^{\frac{1}{3}}}{\sqrt{L+1}} + \frac{\frac{L-2}{8}\left(\sqrt{L}\left(\sqrt{L+1}-1\right)\right)^{-\frac{1}{3}}}{\sqrt{L+1}} + O\left(L^{-2.5}\right).$$

Prop. 15 also shows that the parameter regime where DSS outperforms GS increases as the leadtime increases: Assuming $L\theta_c + \sqrt{L}\theta_l > 1$, this domain has parameters θ_c and θ_l such that

$$\begin{aligned}
\widehat{C}(\alpha^*;\theta_c,0,\theta_l) &= -\theta_c + \frac{3}{2}\left(L\theta_c+\sqrt{L}\theta_l\right)^{1/3} - \frac{L-2}{8}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-\frac{1}{3}} \le \widehat{C^g} = -\theta_c + \sqrt{L+1} \\
&\Leftrightarrow \frac{3}{2}\left(L\theta_c+\sqrt{L}\theta_l\right)^{1/3} \le \sqrt{L+1} + \frac{L-2}{8}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-\frac{1}{3}}. \\
&\Leftrightarrow L\theta_c+\sqrt{L}\theta_l \le \left(\frac{2}{3}\sqrt{L+1} + \frac{L-2}{12}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-\frac{1}{3}}\right)^3 \\
&= \left(\frac{2}{3}\sqrt{L+1}\right)^3 + 3\left(\frac{2}{3}\sqrt{L+1}\right)^2 \frac{L-2}{12}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-\frac{1}{3}} + \ldots \\
&= \left(\frac{2}{3}\sqrt{L+1}\right)^3 + \frac{(L+1)(L-2)}{9}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-\frac{1}{3}} + \ldots
\end{aligned}$$

As L increases, the admissible set of values of θ_c and θ_l that satisfy this condition increases.■

Proof of Proposition 13: To prove (26), first recognize that $\widehat{MB}(\alpha)$ has a lower bound by considering its three terms:

1. $0 \le \theta_c L\alpha^{L-1} \le \theta_c L.$
2. $-\theta_g \frac{e^{-1/2}}{2}\frac{L}{\sqrt{L-1}} \le -\theta_g \frac{L\alpha^{L-1}(1-\alpha)(1+\alpha)-\alpha^L}{(1-\alpha)^{1/2}(1+\alpha)^{3/2}} \le \theta_g \frac{\alpha^L}{(1-\alpha)^{1/2}(1+\alpha)^{3/2}}$
3. $\frac{4}{3\sqrt{3}}\theta_l \le \theta_l \frac{L\alpha^{2L-1}(1-\alpha)(1+\alpha)+\left(1-\alpha^{2L}\right)}{(1-\alpha^{2L})^{1/2}(1-\alpha)^{1/2}(1+\alpha)^{3/2}} \le \theta_l\sqrt{L}$

Hence:

$$\frac{4}{3\sqrt{3}}\theta_l - \theta_g\frac{e^{-1/2}}{2}\frac{L}{\sqrt{L-1}} = \frac{1}{\left(1-\underline{\alpha}_2^2\right)^{3/2}} \le \widehat{MB}(\alpha^*) = \widehat{MC}(\alpha^*) \le \frac{1}{(1-\alpha^{*2})^{3/2}}$$

so that $\alpha^* \ge \underline{\alpha}_1$.

To prove the bound $\underline{\alpha}_1$, follow a similar argument with $L=1$ which allows a tighter lower bound $\widehat{MB}(\alpha)$ because two terms are constants (θ_c and θ_l):

$$\widehat{MB}(0) = \theta_c + \theta_l - \theta_g = \left(1-\underline{\alpha}_1^2\right)^{-3/2} \le \widehat{MB}(\alpha^*) = \widehat{MC}(\alpha^*) \le \left(1-\alpha^{*^2}\right)^{-3/2}.$$

The bound $\underline{\alpha}_1$ requires the intercept of $\widehat{MB}(0) > 1$. Otherwise, the strategic allocation is approximated by considering the first order Taylor approximation, evaluated at $\alpha=0$, of $\widehat{MB}(\alpha) = \theta_{cl} - \theta_g + 2\theta_g\alpha + o(\alpha) = \widehat{MC}(\alpha) = (1+L_l)^{\frac{-1}{2}}\alpha + o(\alpha)$.■

There actually is a tighter bound than $\underline{\alpha}_1$: Notice that the marginal benefit term in θ_g is concave-convex increasing and bounded below by the cord:

$$\theta_g\left(-1+\frac{3}{2}\alpha\right) \le \theta_g\frac{\alpha^2+\alpha-1}{(1-\alpha)^{1/2}(1+\alpha)^{3/2}}$$

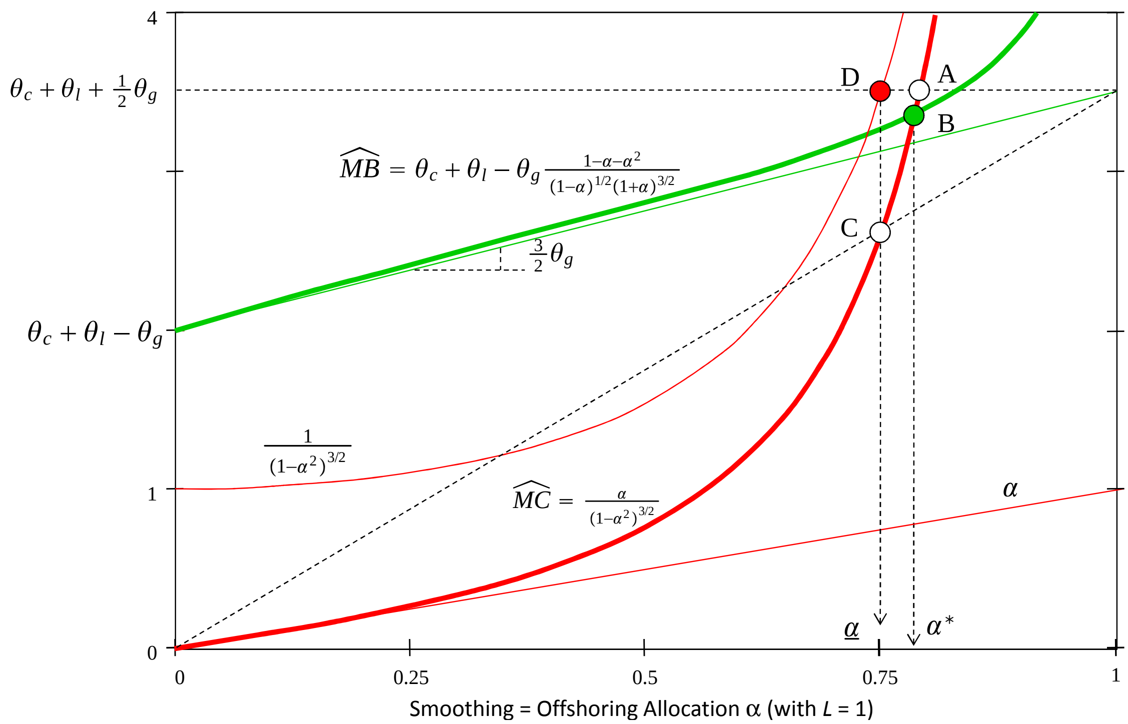

Figure 14: The marginal cost of inventory (solid red) intersects the marginal benefit of sourcing and capacity (solid green) at point B, defining the optimal smoothing level α^*. Bounds on MB and MC intersect at point D, defining the lower bound $\underline{\alpha}$ on α^*.

[Notice that the first order Taylor expansion at $\alpha = 0$, $-1 + 2\alpha$, is not a lower bound because lack of convexity; however, one can verify that $-1 + \frac{3}{2}\alpha$ IS. As is $-1 + \left(\frac{3}{2} + \epsilon\right)\alpha$ where ϵ can be close to 0.1.] Hence, we could do for $L = 1$:

$$\theta_c + \theta_l + \theta_g \left(-1 + \frac{3}{2}\underline{\alpha}\right) = \frac{1}{\left(1 - \underline{\alpha}^2\right)^{3/2}} \leq \widehat{MB}(\alpha^*) = \widehat{MC}(\alpha^*) \leq \frac{1}{\left(1 - \alpha^{*2}\right)^{3/2}}.$$

However, the left equation does not allow a simple expression of $\underline{\alpha}$.]

An explicit bound $\underline{\alpha}$ can be established by considering the maximal value $\theta_c+\theta_l+\theta_g\left(-1 + \frac{3}{2}\underline{\alpha}\right) = \theta_c+\theta_l+\frac{1}{2}\theta_g$ of the lower bound on $\widehat{MB}$ as follows. Notice that $\theta_c+\theta_l+\frac{1}{2}\theta_g = \frac{1}{(1-\underline{\alpha}^2)^{3/2}}$. (Represented by point D in Fig. 14.) To establish that $\underline{\alpha}$ is a lower bound, first note that, keeping $\theta_c+\theta_l+\frac{1}{2}\theta_g = x_0$ constant, the minimal value of θ_g is 0 for which $\widehat{MB}(\alpha) \to x_0$ as $\theta_g \to 0$ and the associated optimal α^* is defined by point A in Fig. 14. As θ_g increases to its maximal value where $\theta_c + \theta_l = \theta_g = \frac{2}{3}x_0$ (recall that $\theta_c + \theta_l - \theta_g \geq 0$), the associated optimal α^* "travels down" the $\widehat{MC}(\alpha)$ curve to point B, near point C and may increase again. But it never goes below point C and it is easily verified that C and D are on the same vertical, so that $\alpha^* \geq \underline{\alpha}$. Clearly, as $\theta_c + \theta_l + \frac{1}{2}\theta_g \to \infty$, then $\underline{\alpha} \to 1$ and thus also $\alpha^* \to 1$.

2 Further Propositions of Special Cases with Exact Results

Proposition 14 (Exact solutions for special cases) *With normal demand, if $L_l = \theta_l = \theta_g = 0$, the optimal smoothing level α^* and allocation $a^* = \alpha^{*L}$ depend on L and θ_c as follows:*

1. *If $L = 1$, then $\alpha^* = 0$ at $\theta_c = 0$ and elsewhere Cardano's rule yields*

$$\alpha^* = \sqrt{1 - \theta_c^{-\frac{2}{3}} \left(\sqrt[3]{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{27\theta_c^2}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4}{27\theta_c^2}}} \right)}. \tag{34}$$

2. *If $L = 2$, then $\alpha^* = 0$ if $\theta_c \leq \underline{\theta}_2^* = \frac{1}{2}$ and elsewhere*

$$\alpha^* = \sqrt{1 - (2\theta_c)^{-\frac{2}{3}}} \text{ and } \widehat{C}(\alpha^*) = \frac{3}{2}(2\theta_c)^{\frac{1}{3}} - \theta_c. \tag{35}$$

3. *If $L = 3$, then $\alpha^* = 0$ if $\theta_c \leq \underline{\theta}_3^* = 1.15$ and elsewhere Ferrari's rule yields*

$$\begin{aligned} \alpha^* &= \sqrt{\frac{3}{4} - \frac{1}{4}\left(\sqrt{1+4u} - \sqrt{2 - 4u + \frac{2}{\sqrt{1+4u}}}\right)} > 0.64, \text{ where} \\ u &= 2R^{\frac{1}{3}} + \frac{2\,(3\theta_c)^{-2}}{3R^{\frac{1}{3}}} \text{ and } R = \frac{(3\theta_c)^{-2}}{2^4}\left(1 - \sqrt{1 - \frac{2^8}{3^3}(3\theta_c)^{-2}}\right). \end{aligned} \tag{36}$$

4. *If $L \geq 4$, then $\alpha^* = 0$ if $\theta_c \leq \underline{\theta}_L^*$ and elsewhere*

$$\alpha^* = \sqrt{1 - \sum_{n=1}^{\infty} \frac{\Gamma\left(n\frac{L+1}{3} - 1\right)}{\Gamma\left(n\frac{L-2}{3}\right)} \frac{(\theta_c L)^{-\frac{2n}{3}}}{n!}}. \tag{37}$$

Proposition 15 (Lagrange's Series for Local Capacity Costs) *With normal demand, and if $L_l = \theta_g = 0$, the optimal smoothing level α^* and offshoring allocation α^{*L} depend on L, θ_c and θ_l:*

1. *If $L = 1$, then $\alpha^* = 0$ at $\theta_c = \theta_l = 0$ and elsewhere:*

$$\alpha^* = \sqrt{1 - (\theta_c + \theta_l)^{-\frac{2}{3}} \left(\sqrt[3]{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{27(\theta_c + \theta_l)^2}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4}{27(\theta_c + \theta_l)^2}}} \right)}.$$

2. *If $L=2$, then $\alpha^* = \frac{\theta_l}{1-2\theta_c+\theta_l} + O\left(\left(\frac{\theta_l}{1-2\theta_c+\theta_l}\right)^3\right)$ if $2\theta_c + \sqrt{2}\theta_l \le 1$ and elsewhere*

$$\begin{aligned} \alpha_4^* &= \sqrt{1-\left(2\theta_c+\sqrt{2}\theta_l\right)^{-\frac{2}{3}} + \theta_l\sqrt{2}\left(\frac{1}{16}\left(2\theta_c+\sqrt{2}\theta_l\right)^{-\frac{9}{3}} + O\left(\left(2\theta_c+\sqrt{2}\theta_l\right)^{-\frac{11}{3}}\right)\right)} \\ \widehat{C}(\alpha^*) &= \frac{3}{2}\left(2\theta_c+\sqrt{2}\theta_l\right)^{\frac{1}{3}} - \theta_c + O\left(\theta_l\left(2\theta_c+\sqrt{2}\theta_l\right)^{-2}\right). \end{aligned}$$

3. *If $L>2$, then $\alpha^* = \frac{\theta_l}{1+\theta_l} + O\left(\left(\frac{\theta_l}{1+\theta_l}\right)^2\right)$ if $L\theta_c + \sqrt{L}\theta_l \le 1$ and elsewhere*

$$\begin{aligned} \alpha^* &= \sqrt{1-\left(L\theta_c+\sqrt{L}\theta_l\right)^{-2/3} + \frac{L}{3}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-7/3}\left(\left(L-2\right)\theta_c+\theta_l\sqrt{L}\right)+\cdots} \\ \widehat{C}(\alpha^*) &= \frac{3}{2}\left(L\theta_c+\sqrt{L}\theta_l\right)^{\frac{1}{3}} - \theta_c - \frac{L-2}{8}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-\frac{1}{3}} + O\left(\left(L\theta_c+\sqrt{L}\theta_l\right)^{-2}\right). \end{aligned}$$

Notice that the expressions for $L=2$ are more accurate (reflecting the fact that they are exact for $\theta_l = 0$) and that, for $L>2$, θ_c is no longer a first-order effect in light offshoring when $\alpha^* = \theta_l/(1+\theta_l)$ (because the marginal sourcing benefit $\theta_c L\alpha^{L-1}$ is of order $L-1>1$). Prop. 15 helps us understand the interaction between the leadtime, standardized cost advantage, and local capacity costs. It demonstrates that local capacity costs have a similar impact as the standardized cost advantage. (Indeed, the key parameters that drives the optimal smoothing level and allocation decision is $L\theta_c + \sqrt{L}\theta_l$, so that θ_c and θ_l are substitutes, up to a factor $\sqrt{L}$.)

Proposition 16 (Lagrange's Series for Local and Global Capacity Costs) *With normal demand, if $L_l = 0$ and $\theta_g > 0$, the optimal smoothing level α^* and offshoring allocation $a^* = \alpha^{*L}$ depend on L, θ_c, θ_g and θ_l as follows:*

1. *If $L=1$, then the Lagrange series around $\alpha_0 = 1$ is:*[8]

$$\alpha^* = \sqrt{1 - \frac{1}{\theta_g+1} - \frac{\theta_g\left(\theta_g+7\right)}{4\left(\theta_g+1\right)^3} + \frac{8\theta_{cl}^2 + \theta_g\left(5\theta_g+1\right)}{5!\left(\theta_g+1\right)^4} - \frac{\left(\theta_g\left(\theta_g+7\right)\right)^2}{5!\left(\theta_g+1\right)^5} + \cdots} \tag{38}$$

[8] We abbreviate $\theta_i + \theta_j$ by θ_{ij}.

Lagrange around $\alpha_0 = 0$ yields: for $\left|\theta_{cl}^2 - \theta_g^2\right| < 1$ and $\theta_g < \frac{1}{2}$:

$$\begin{aligned}
\alpha^* &= \sum_{n=1}^{\infty} \frac{a_n}{n!} (\theta_g^2 - \theta_{cl}^2)^n \\
a_1 &= \frac{1}{2\theta_g (1 - 2\theta_g)} \\
a_2 &= 2\frac{\left(3\theta_{cl}^2 + (1 - 2\theta_g)^2\right)}{(2\theta_g (1 - 2\theta_g))^3} \\
a_3 &= 3 \cdot 4 \frac{\left(3\theta_{cl}^2 + (1 - 2\theta_g)^2\right)^2}{(2\theta_g (1 - 2\theta_g))^5} + 2\frac{4\theta_g^2}{(2\theta_g (1 - 2\theta_g))^4} \\
a_4 &= 4 \cdot 5 \cdot 6 \frac{\left(3\theta_{cl}^2 + (1 - 2\theta_g)^2\right)^3}{(2\theta_g (1 - 2\theta_g))^7} + 3 \cdot 4 \cdot 5 \frac{\left(3\theta_{cl}^2 + (1 - 2\theta_g)^2\right) 4\theta_g^2}{(2\theta_g (1 - 2\theta_g))^6} - 4\frac{6\left(3\theta_{cl}^2 - 2\theta_g (1 - 2\theta_g)\right)}{(2\theta_g (1 - 2\theta_g))^5}
\end{aligned}$$

2. *If $L \geq 2$: then the Lagrange series around $\alpha_0 = 1$ is:*

$$\begin{aligned}
\alpha^* &\simeq \sqrt{1 - \frac{2}{\theta_g}} \\
\alpha^* &= \sqrt{1 - \frac{1}{\theta_g} - \frac{1}{2!}\frac{(\theta_g + 6L - 3)}{2\theta_g^2} - \frac{\theta_g^2 + (6L - 3)\theta_g + 28L^2 - 3L - 20}{5!\theta_g^3} + \frac{\left(\theta_c L + \sqrt{L}\theta_l\right)^2}{15\theta_g^4} + \cdots} \\
&= \sqrt{1 - \frac{1 + \frac{1}{4} + \frac{1}{5!} + \cdots}{\theta_g} - \frac{(6L - 3)\left(\frac{1}{4} + \frac{1}{5!} + \cdots\right)}{\theta_g^2} - \frac{(28L^2 - 3L - 20)\left(\frac{1}{5!} + \cdots\right)}{\theta_g^3} + \frac{\left(\theta_c L + \sqrt{L}\theta_l\right)^2}{15\theta_g^4}}
\end{aligned}$$

Proof: In contrast to the uncapacitated case, there is no simple general expression for the $n-th$ order term in the Lagrange series. Each term requires substantial work. The derivations are shown in painstaking detail for $L = 1$ starting from Companion Page 33 and for $L \geq 2$ from Companion Page 60. ■

3 Exact Solutions for Uncapacitated Dual Sourcing and $L_l = 0$

The FOC become:

$$\begin{aligned}
\theta_c L \alpha^{L-1} &= \frac{\alpha}{(1 - \alpha^2)^{3/2}} \\
\Leftrightarrow (\theta_c L)^2 \alpha^{2(L-1)} \left(1 - \alpha^2\right)^3 &= \alpha^2
\end{aligned}$$

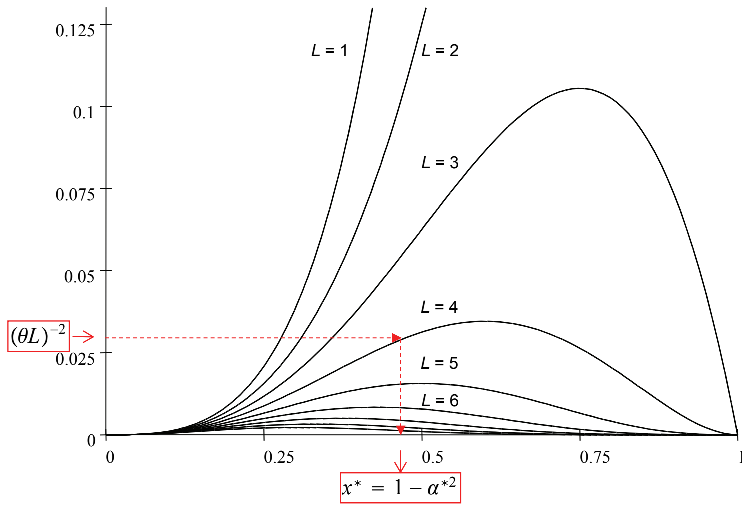

Figure 15: The optimal $x^* = 1 - \alpha^{*2}$ is easily found graphically as a function of $\lambda = (\theta_c L)^{-2}$.

Substituting $x = 1 - \alpha^2$ yields:

$$(\theta_c L)^2 (1-x)^{L-1} x^3 = 1 - x$$

If $\theta_c = 0$, then $x = 1$; otherwise:

$$(1-x)^{L-2} x^3 = \lambda \text{ where } \lambda = (\theta_c L)^{-2}.$$

For $L \geq 3$, the cost function is convex-concave-convex and the FOC has two solutions in $[0, 1]$, represting two local extrema x^*. The local minimum corresponds to the smallest root x^*. While there exist only a simple analytical solution up to $L = 3$, the solution is easily found graphically: draw a horizontal line at $\lambda = (\theta_c L)^{-2}$ and its intersection with the upward curve in Fig gives the optimal $x^* = 1 - \alpha^{*2}$.

We also directly find a necessary condition on λ to have an interior solution:

$$\lambda \leq \overline{\lambda} = \max_{0<x<1} (1-x)^{L-2} x^3. \tag{39}$$

For $L = 2$, $\overline{\lambda} = 1$; For $L > 2$, the maximizer x solves:

$$\begin{aligned} -(L-2)(1-x)^{L-3}x^3 + 3(1-x)^{L-2}x^2 &= 0 \\ \Leftrightarrow -(L-2)x + 3(1-x) &= 0 \\ \Leftrightarrow 3 - (L+1)x &= 0 \\ \Leftrightarrow x &= \frac{3}{L+1} \end{aligned}$$

so that

$$\overline{\lambda} = \left(1 - \frac{3}{L+1}\right)^{L-2} \left(\frac{3}{L+1}\right)^{3}$$

In terms of θ_c:

$$\begin{aligned}
(\theta_c L)^{-2} &\leq \left(1 - \frac{3}{L+1}\right)^{L-2} \left(\frac{3}{L+1}\right)^{3} \\
\theta_c &\geq \frac{1}{L\sqrt{\left(1 - \frac{3}{L+1}\right)^{L-2} \left(\frac{3}{L+1}\right)^{3}}}
\end{aligned}$$

Simplify

$$\frac{1}{L\sqrt{\left(1 - \frac{3}{L+1}\right)^{L-2} \left(\frac{3}{L+1}\right)^{3}}} = \frac{\left(1 - \frac{3}{L+1}\right)}{L\left(\frac{3}{L+1}\right)\sqrt{\left(1 - \frac{3}{L+1}\right)^{L} \left(\frac{3}{L+1}\right)}} = \frac{L-2}{3L\sqrt{\left(\frac{L-2}{L+1}\right)^{L} \left(\frac{3}{L+1}\right)}}$$

The corresponding cost at $x = \frac{3}{L+1} = 1 - \alpha^2$ is $\alpha^2 = 1 - x = \frac{L-2}{L+1}$ so

$$\begin{aligned}
\widehat{C} &= -\theta_c \alpha^L + \frac{1}{\sqrt{1-\alpha^2}} = -\theta_c \left(\frac{L-2}{L+1}\right)^{L/2} + \left(\frac{L+1}{3}\right)^{1/2} \\
&= \frac{-\left(\frac{L-2}{L+1}\right)^{L/2}}{L\sqrt{\left(\frac{L-2}{L+1}\right)^{L-2} \left(\frac{3}{L+1}\right)^{3}}} + \left(\frac{L+1}{3}\right)^{1/2} = \frac{-(L-2)}{3L\sqrt{3\left(\frac{1}{L+1}\right)}} + \left(\frac{L+1}{3}\right)^{1/2} \\
&= \left(\frac{L+1}{3}\right)^{1/2} \left(1 - \frac{L-2}{3L}\right)
\end{aligned}$$

Note that for $L = 2$ the corresponding interior minimal cost $\widehat{C} = 1$. For $L = 3$, we have that $\theta_c = \frac{1}{3\sqrt{\left(1-\frac{3}{3+1}\right)\left(\frac{3}{3+1}\right)^3}} = 1.0264$ and

$$\widehat{C} = \left(\frac{3+1}{3}\right)^{1/2} \left(1 - \frac{1}{3 \cdot 3}\right) = 1.0264$$

We can summarize as:

Proposition 17 *With normal demand, for $L \geq 2$, the optimality of dual sourcing smoothing requires a minimal standardized sourcing cost advantage $\underline{\theta}_L$ and the optimal smoothing level α^* has* *a lower bound* $\underline{\alpha}_L \leq \alpha^*$:[9]

$$\theta_c \quad \geq \quad \underline{\theta}_L = \frac{(L+1)^{\frac{L+1}{2}}}{27^{\frac{1}{2}} L (L-2)^{\frac{L-2}{2}}} = \sqrt{\frac{e^3}{27} L + o\left(\sqrt{L}\right)}, \tag{40}$$

$$\alpha^* \quad \geq \quad \underline{\alpha}_L = \sqrt{1 - \frac{3}{L+1}}. \tag{41}$$

Proof of Proposition 17: With normal demand, the maxima of the functions $(1-x)^{L-2}x^3$ also define a necessary condition on λ for the FOC to have an interior solution:

$$\lambda \leq \overline{\lambda} = \max_{0<x<1} (1-x)^{L-2} x^3. \tag{42}$$

For $L = 2$, $\overline{\lambda} = 1$; For $L > 2$, the maximizer of (42) is $x^* = \frac{3}{L+1}$, which provides an upper bound to any optimal $1 - \alpha^{*2}$, and $\overline{\lambda} = \left(1 - \frac{3}{L+1}\right)^{L-2} \left(\frac{3}{L+1}\right)^3 = e^{-3} \frac{27}{L+1} + o(L)$.■

The proposition quantifies the strategic trade-off between the minimal standardized sourcing cost advantage and the leadtime for dual sourcing smoothing. All our results demonstrate the essential difference between $L = 1$ (which always has an interior solution α^* for which dual sourcing provides lower cost than single local sourcing) and $L > 1$, for which dual sourcing optimality requires that

$$c^l - c^g > hL + \underline{\theta}_L \kappa_I CV = hL + \frac{(L+1)^{\frac{L+1}{2}}}{27^{\frac{1}{2}} L (L-2)^{\frac{L-2}{2}}} \kappa_I CV = hL + \kappa_I CV \sqrt{\frac{e^3}{27} L + o\left(\sqrt{L}\right)}.$$

This expression quantifies the "cost of uncertainty and leadtime:" the global sourcing unit cost advantage must not only outweigh the unit pipeline cost hL but also a "variability penalty" that increases with an increase in the leadtime L, the uncertainty CV, or the financial inventory parameter κ_I. Interestingly, while there is no obvious notion to "leadtime demand" in a smoothing context, the required standardized sourcing cost advantage does (asymptotically) increase with the square root of the leadtime.

Notice that $\theta_c \geq \underline{\theta}_L$ guarantees that the FOC have an interior solution $\alpha^* > 0$. Yet, for dual sourcing smoothing to be optimal, this local cost minimum must have a cost below the single sourcing cost $\widehat{C}(0) = 1$, which requires a more stringent condition $\theta_c \geq \underline{\theta}_L^* = \inf\left\{\theta_c : \widehat{C}(\alpha^*; \theta_c, L) < 1\right\} \geq \underline{\theta}_L$. We can now proceed to specify the optimal smoothing level and offshoring allocation in exact analytic terms.

[9] The Landau notations specify functions $o(f)$ that are of smaller order than f, and $O(g)$ which is of similar order as g. Formally: $\lim_{x\to\infty} o(f)(x)/f(x) = 0$ while $\lim_{x\to\infty} O(g)(x)/g(x)$ is a finite constant.

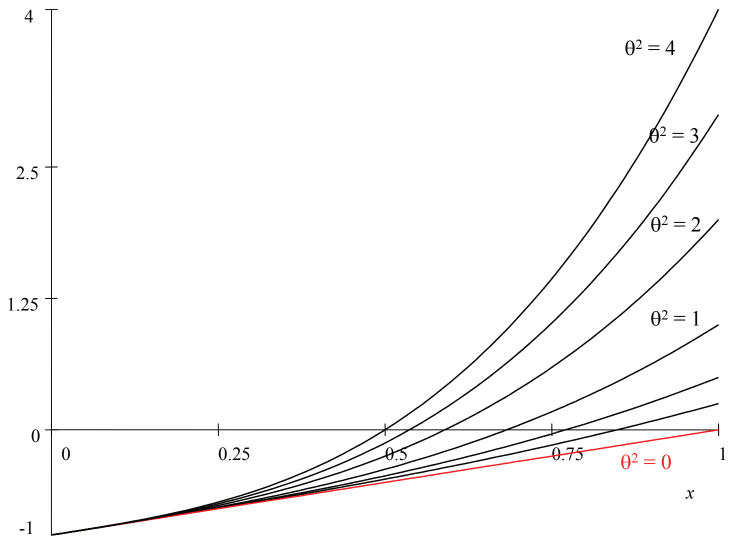

Figure 16: For $L = 1$ we seek the zeros of the functions $f(x) = \theta_c^2 x^3 + x - 1 = 0$.

3.1 $L = 1$ and $L_l = 0$ Uncapacitated: Cardano-Tartaglia

For $L = 1$, this is a cubic in α^2 which can be solved exactly using Cardano's solution: Subsitute $x = 1 - \alpha^2$ in the FOC:

$$\begin{aligned}
\theta_c \left(1 - \alpha^2\right)^{3/2} &= \alpha \\
\theta_c^2 \left(1 - \alpha^2\right)^3 &= \alpha^2 \\
f(y) &= cx^3 + x - 1 = 0
\end{aligned}$$

This cubic has always a unique root in $(0, 1)$ because $f(0) = -1$ and $f(1) = c = \theta_c^2 > 0$ and $f' = 3cx^2 + 1 > 0$ so that f is strictly increasing. Also, $f'' = 6cx$ so strictly convex for $x > 0$ as shown in Figure 16.

Cardano's solution for the real root of:

$$t^3 + pt + q = 0 \Rightarrow t = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

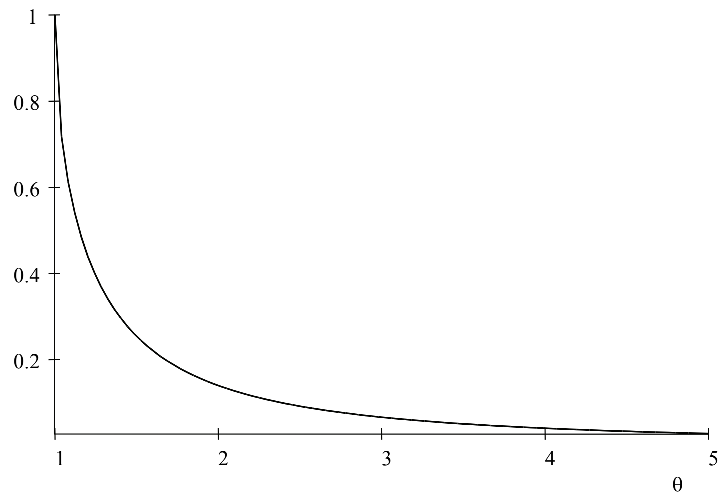

Figure 17: For $L = 1$: The absolute error between α^* and the square root allocation α_0.

so, with $p = -q = c^{-1} = \theta_c^{-2}$ we get:

$$\begin{aligned}
1 - \alpha^{*2} &= \sqrt[3]{\frac{1}{2\theta_c^2} + \sqrt{\frac{1}{4\theta_c^4} + \frac{1}{27\theta_c^6}}} + \sqrt[3]{\frac{1}{2\theta_c^2} - \sqrt{\frac{1}{4\theta_c^4} + \frac{1}{27\theta_c^6}}} \\
&= \sqrt[3]{\frac{1}{2\theta_c^2} + \frac{1}{2\theta_c^2}\sqrt{1 + \frac{4}{27\theta_c^2}}} + \sqrt[3]{\frac{1}{2\theta_c^2} - \frac{1}{2\theta_c^2}\sqrt{1 + \frac{4}{27\theta_c^2}}} \\
&= \theta_c^{-2/3}\left(\sqrt[3]{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{27\theta_c^2}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4}{27\theta_c^2}}}\right)
\end{aligned}$$

Hence:

$$\alpha^* = \sqrt{1 - \theta_c^{-2/3}\left(\sqrt[3]{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{27\theta_c^2}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4}{27\theta_c^2}}}\right)}$$

Clearly, for large θ_c, we recover the square-root bound. The error of the square-root lower bound is function of:

$$\begin{aligned}
\text{Absolute error} &= \sqrt{1 - \theta_c^{-2/3}\left(\sqrt[3]{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{27\theta_c^2}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4}{27\theta_c^2}}}\right)} - \sqrt{1 - \theta_c^{-2/3}} \\
A &= \sqrt[3]{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{27\theta_c^2}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4}{27\theta_c^2}}} - 1,
\end{aligned}$$

which is strictly decreasing in θ_c. (Fig. 17)

And we know that, for the square root formula $\theta_c > 1$, so that for large θ_c:

$$\begin{aligned} A &\simeq \sqrt[3]{\frac{1}{2}+\frac{1}{2}\left(1+\frac{2}{27\theta_c^2}\right)}+\sqrt[3]{\frac{1}{2}-\frac{1}{2}\left(1+\frac{2}{27\theta_c^2}\right)}-1 \\ &= \sqrt[3]{1+\frac{1}{27\theta_c^2}}-\sqrt[3]{\frac{1}{27\theta_c^2}}-1 \\ &\simeq \frac{1}{3\cdot 27\theta_c^2}-\frac{1}{3\theta_c^{2/3}}=\frac{1}{3\theta_c^{2/3}}\left(\left(\frac{1}{3\theta_c}\right)^3-1\right)<0 \end{aligned}$$

Indeed, the square root approximation is a lower bound with maximal error at $\theta_c = 1$ equal to:

$$\alpha^* - \alpha_0 = \sqrt{1-\sqrt[3]{\frac{1}{18}\sqrt{93}+\frac{1}{2}}-\sqrt[3]{\frac{1}{2}-\frac{1}{18}\sqrt{93}}}-0 = 0.5636$$

An approximation of Cardano for small θ_c follows directly from a first order expansion around $\alpha = 0$ of the squared FOC:

$$\begin{aligned} \alpha^2 &= \theta_c^2\left(1-\alpha^2\right)^3=\theta_c^2\left(1-3\alpha^2+O(\alpha^4)\right) \\ &\Rightarrow \ \alpha=\frac{\theta_c}{\sqrt{1+3\theta_c^2}}+O(\theta_c^3) \\ \widehat{C}\left(\frac{\theta_c}{\sqrt{1+3\theta_c^2}};\theta_c\right) &= \frac{-\theta_c^2}{\sqrt{1+3\theta_c^2}}+\left(1-\frac{\theta_c^2}{1+3\theta_c^2}\right)^{-\frac{1}{2}} \\ &= \frac{-\theta_c^2}{\sqrt{1+3\theta_c^2}}+\left(\frac{1+3\theta_c^2}{1+2\theta_c^2}\right)^{\frac{1}{2}}=\frac{1+3\theta_c^2-\theta_c^2\sqrt{1+2\theta_c^2}}{\sqrt{\left(1+2\theta_c^2\right)\left(1+3\theta_c^2\right)}} \\ &\simeq \frac{1+3\theta_c^2-\theta_c^2\left(1+\theta_c^2\right)}{\sqrt{1+5\theta_c^2+6\theta_c^4}}\simeq\left(1+2\theta_c^2\right)\left(1-\frac{5}{2}\theta_c^2\right)\simeq 1-\frac{1}{2}\theta_c^2 \end{aligned}$$

Evaluate value of dual sourcing at $\theta_c = \sqrt{2}-1$:

$$1-\left(\frac{-\theta_c^2}{\sqrt{1+3\theta_c^2}}+\left(\frac{1+3\theta_c^2}{1+2\theta_c^2}\right)^{\frac{1}{2}}\right)=7.75\%$$

Figure 18 compares the exact Cardano (black) with the two approximations: the square root (red) for large θ_c and the almost linear (in green) for small θ_c.

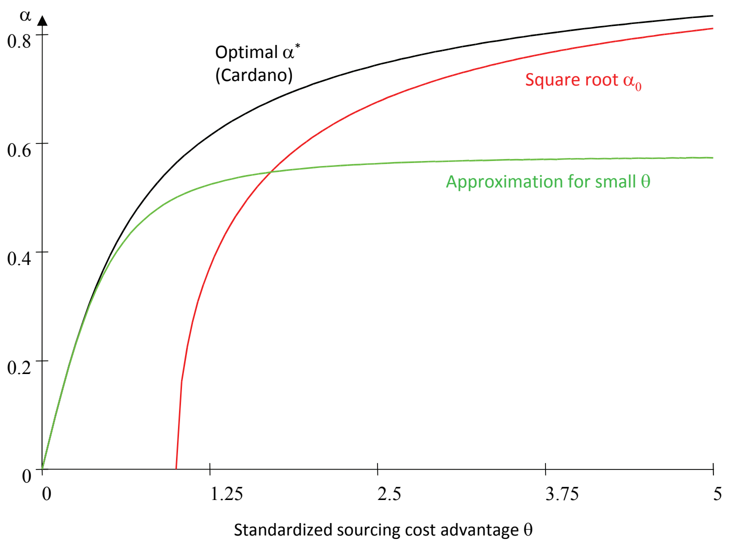

Figure 18: For $L = 1$: The optimal α^* compared to the square root allocation α_0 and the approximation for small θ_c.

3.2 $L = 2$ and $L_l = 0$ Uncapacitated: Simple Square Root

For $L = 2$, it is even simpler and the exact solution is [and equals our "general" approximation!]:

$$\begin{aligned} \theta_c 2\alpha &= \frac{\alpha}{(1-\alpha^2)^{3/2}} && (43) \\ \alpha^{*2} &= 1 - (2\theta_c)^{-2/3} && (44) \end{aligned}$$

This requires $2\theta_c > 1$ and the optimal cost is

$$\widehat{C}(\alpha^*;\theta_c) = \frac{3}{2^{2/3}}\theta_c^{1/3} - \theta_c.$$

Notice that $\widehat{C}(\alpha^*;\theta_c = 1/2) = 1$ and that cost decreases as $\theta_c > 1/2$, hence dominating single sourcing. For $\theta_c \leq 1/2, \alpha^* = 0$.

3.3 $L = 3$ and $L_l = 0$ Uncapacitated: Ferrari

For $L = 3$, we get a quartic, which still can be solved exactly: Subsitute $x = 1 - \alpha^2$

$$\begin{aligned} (\theta_c L)^2 \alpha^{2L-2} \left(1-\alpha^2\right)^3 &= \alpha^2 \\ \Leftrightarrow (3\theta_c)^2 \alpha^2 \left(1-\alpha^2\right)^3 &= 1 \\ \Leftrightarrow (3\theta_c)^2 (1-x)\, x^3 &= 1 \\ \Leftrightarrow f = x^4 - x^3 + (3\theta_c)^{-2} &= 0 \end{aligned}$$

For $\theta_c > 0$, $f(0) = f(1) = (3\theta_c)^{-2}$. Investigate this quartic in $(0, 1)$:

$$
\begin{aligned}
f' &= 4x^3 - 3x^2 = (4x-3)x^2, \\
f'' &= 12x^2 - 6x = 6x(2x-1).
\end{aligned}
$$

Hence, f is decreasing for $x < 3/4$, and increasing elsewhere. It has inflection points at 0 and $\frac{1}{2}$; convex for $x < 0$ and $x > 1/2$ and concave in between.

Thus, one global minimum at $x = \frac{3}{4}$ where $f = \left(\frac{3}{4}\right)^4 - \left(\frac{3}{4}\right)^3 + (3\theta_c)^{-2} = -\frac{27}{256} + (3\theta_c)^{-2}$. Thus, necessary condition for there to be two roots (and hence local maximum and then minimum) is:

$$
\begin{aligned}
-\frac{27}{256} + (3\theta_c)^{-2} &< 0 \Leftrightarrow (3\theta_c)^{-2} < \frac{27}{256} = 0.10547 \\
&\Leftrightarrow \quad \theta_c > \frac{1}{3}\sqrt{\frac{256}{27}} = \frac{16}{27}\sqrt{3} = 1.026
\end{aligned}
$$

Note that we have two real roots in $(0, 1)$. (Indeed, the larger root corresponds to a local maximum in the cost curve. We need the smaller root: recall that higher θ_c, means higher Δ, hence more offshoring, or smaller x. This suggest we should take the smaller root in x.) Solve using Ferrari's method (using his parameters, so do not confuse here with our α):

$$
\begin{aligned}
f &= Ax^4 + Bx^3 + Cx^2 + Dx + E = 0 \\
&= x^4 - x^3 + a = 0 \text{ where } a = (3\theta_c)^{-2} < \frac{27}{256} = 0.10547
\end{aligned}
$$

Thus:

$$
A = 1, B = -1, C = D = 0, E = a
$$

$$
\begin{aligned}
\alpha &= -\frac{3}{8}, \beta = \frac{-1}{8}, \gamma = -\frac{3}{256} + a. \\
P &= -\frac{\alpha^2}{12} - \gamma \\
&= -\frac{1}{12}\left(\frac{3}{8}\right)^2 + \frac{3}{256} - a = -a
\end{aligned}
$$

$$
\begin{aligned}
Q &= -\frac{\alpha^3}{108} + \frac{\alpha\gamma}{3} - \frac{\beta^2}{8} \\
&= \frac{1}{108}\left(\frac{3}{8}\right)^3 + \left(-\frac{1}{8}\right)\left(-\frac{3}{256} + a\right) - \frac{1}{8^3} \\
&= -\frac{1}{8}a
\end{aligned}
$$

$$
\begin{aligned}
R &= -\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}} \\
&= \frac{1}{16} a \pm \sqrt{\frac{1}{4}\frac{1}{8^2} a^2 - \frac{1}{27} a^3} \\
&= \frac{1}{2^4} a \pm \sqrt{\frac{1}{2^8} a^2 - \frac{1}{3^3} a^3} \\
&= \frac{a}{2^4} \left(1 \pm \sqrt{1 - \frac{2^8}{3^3} a} \right)
\end{aligned}
$$

and

$$
1 > 1 - \frac{2^8}{3^3} a > 1 - \frac{2^8}{3^3} \frac{27}{256} = 0
$$

so that both values of R are real and positive:

$$
\begin{aligned}
a &= (3\theta_c)^{-2} = \frac{27}{256} \to R = \frac{a}{2^4} \\
a &= 0 \to R = \frac{a}{2^4} (1 \pm 1) = 0,
\end{aligned}
$$

if from here on we take the minus sign. (Apparently, either sign of the square root will do.) Continuing on:

$$
\begin{aligned}
y &= \frac{-5}{6} \alpha + R^{1/3} - \frac{P}{3R^{1/3}} \\
&= \frac{5}{6}\frac{3}{8} + \frac{a^{1/3}}{2^{4/3}} \left(1 \pm \sqrt{1 - \frac{2^8}{3^3} a} \right)^{1/3} + \frac{a}{3 \frac{a^{1/3}}{2^{4/3}} \left(1 \pm \sqrt{1 - \frac{2^8}{3^3} a} \right)^{1/3}} \\
&= \frac{5}{16} + \frac{a^{1/3}}{2^{4/3}} \left(1 \pm \sqrt{1 - \frac{2^8}{3^3} a} \right)^{1/3} + \frac{2^{4/3} a^{2/3}}{3 \left(1 \pm \sqrt{1 - \frac{2^8}{3^3} a} \right)^{1/3}} \\
&= \frac{5}{16} + R^{1/3} + \frac{a}{3} R^{-1/3}
\end{aligned}
$$

$$
\begin{aligned}
W &= \sqrt{\alpha + 2y} = \\
&= \sqrt{-\frac{3}{8} + \frac{5}{8} + \frac{a^{1/3}}{2^{1/3}} \left(1 \pm \sqrt{1 - \frac{2^8}{3^3} a} \right)^{1/3} + \frac{2^{7/3} a^{2/3}}{3 \left(1 \pm \sqrt{1 - \frac{2^8}{3^3} a} \right)^{1/3}}} \\
&= \sqrt{\frac{1}{4} + \frac{a^{1/3}}{2^{1/3}} \left(1 \pm \sqrt{1 - \frac{2^8}{3^3} a} \right)^{1/3} + \frac{2^{7/3} a^{2/3}}{3 \left(1 \pm \sqrt{1 - \frac{2^8}{3^3} a} \right)^{1/3}}} \\
&= \sqrt{\frac{1}{4} + 2R^{1/3} + \frac{2a}{3} R^{-1/3}}
\end{aligned}
$$

Finally:

$$\begin{aligned} x^* &= -\frac{B}{4A}+\frac{1}{2}\left(\pm_s W \mp \sqrt{-\left(3\alpha+2y\pm_s\frac{2\beta}{W}\right)}\right) \\ &= \frac{1}{4}+\frac{1}{2}\left(\pm_s W \mp \sqrt{\frac{9}{8}-2y\pm_s\frac{\frac{1}{4}}{W}}\right) \end{aligned}$$

and

$$\begin{aligned} &\frac{9}{8}-2y\pm_s\frac{\frac{1}{4}}{W} \\ =\ &\frac{9}{8}-\frac{5}{8}-2R^{1/3}-\frac{2a}{3}R^{-1/3}\pm_s\frac{1}{2}\left(1+8R^{1/3}+\frac{8a}{3}R^{-1/3}\right)^{-1/2} \\ =\ &\frac{1}{2}-2R^{1/3}-\frac{2a}{3}R^{-1/3}\pm_s\frac{1}{2}\left(1+8R^{1/3}+\frac{8a}{3}R^{-1/3}\right)^{-1/2} \end{aligned}$$

So, taking the negative sign in R and denoting

$$\begin{aligned} u &= 2R^{1/3}+\frac{2a}{3}R^{-1/3} \\ &= \frac{a^{1/3}}{2^{1/3}}\left(1-\sqrt{1-\frac{2^8}{3^3}a}\right)^{1/3}+\frac{2^{7/3}a^{2/3}}{3\left(1-\sqrt{1-\frac{2^8}{3^3}a}\right)^{1/3}} \end{aligned}$$

we get

$$\begin{aligned} x^* &= \frac{1}{4}+\frac{1}{2}\left(\pm_s\sqrt{\frac{1}{4}+u}\mp\sqrt{\frac{1}{2}-u\pm_s\frac{1}{2\sqrt{1+4u}}}\right) \\ &= \frac{1}{4}+\frac{1}{4}\left(\pm_s\sqrt{1+4u}\mp\sqrt{2-4u\pm_s\frac{2}{\sqrt{1+4u}}}\right) \end{aligned}$$

There should only be two roots in $(0,1)$ and we should take the smaller one. When a reaches its minimal value 0, we know $R=0$, we must have $x^*=0$ and another root at 1. Hence:

$$\begin{aligned} a &= R=0\to u=0 \\ x^* &= \frac{1}{4}+\frac{1}{4}\left(\pm_s 1\mp\sqrt{2\pm_s 2}\right) \end{aligned}$$

If we take $\pm_s=+$, then

$$x^*=\frac{1}{4}+\frac{1}{4}(1\mp 2)=0 \text{ and } 1. \text{ (OK)}$$

If we take $\pm_s=-$, then

$$x^*=\frac{1}{4}+\frac{1}{4}(-1)=0 \text{ (double root). (not OK)}$$

Hence, we go with $\pm_s = +$ and we need the smaller root, so $\mp = -$. Summary:

$$x^* = \frac{1}{4} + \frac{1}{4}\left(\sqrt{1+4u} - \sqrt{2 - 4u + \frac{2}{\sqrt{1+4u}}}\right) \tag{45}$$

$$\alpha^* = \sqrt{1-x^*} = \sqrt{\frac{3}{4} - \frac{1}{4}\left(\sqrt{1+4u} - \sqrt{2 - 4u + \frac{2}{\sqrt{1+4u}}}\right)} \tag{46}$$

Verifications: When a reaches its maximal value $a = \frac{27}{256}$, we know $R = \frac{a}{2^4} = \frac{27}{2^4 \cdot 2^8}$ and we should have a double root at $x^* = 3/4$ (or $\alpha^* = \frac{1}{2}$). Indeed: $f(\frac{3}{4}) = x^4 - x^3 + a = (\frac{3}{4})^4 - (\frac{3}{4})^3 + \frac{27}{256} = 0$. Also, the term in the parenthesis is equal to 2:

$$\begin{aligned}
a &= \frac{27}{256} = \frac{3^3}{2^8} \to R = \frac{a}{2^4} = \frac{3^3}{2^{12}} \to R^{1/3} = \frac{3}{2^4} \\
u &= 2R^{1/3} + \frac{2a}{3}R^{-1/3} = \frac{3}{2^3} + \frac{2}{3}\frac{3^3}{2^8}\frac{2^4}{3} = \frac{3}{4} \\
\sqrt{1+4u} &= 2 \text{ and } \sqrt{2 - 4u + \frac{2}{\sqrt{1+4u}}} = \sqrt{2 - 3 + \frac{2}{2}} = 0
\end{aligned}$$

Tranforming back to smoothing levels using $\alpha^* = \sqrt{1-x^*}$, we have:

$$\begin{aligned}
\alpha^* &= \sqrt{\frac{3}{4} - \frac{1}{4}\left(\sqrt{1+4u} - \sqrt{2 - 4u + \frac{2}{\sqrt{1+4u}}}\right)}, \text{ where} \\
u &= 2R^{1/3} + \frac{2\,(3\theta_c)^{-2}}{3R^{1/3}} \text{ and } R = \frac{(3\theta_c)^{-2}}{2^4}\left(1 - \sqrt{1 - \frac{2^8}{3^3}\,(3\theta_c)^{-2}}\right)
\end{aligned} \tag{47}$$

or

$$\begin{aligned}
u &= \frac{(3\theta_c)^{-2/3}}{2^{4/3}}\left(1 - \sqrt{1 - \frac{2^8}{3^3}\,(3\theta_c)^{-2}}\right)^{1/3} + \frac{2\,(3\theta_c)^{-2}}{3}\frac{(3\theta_c)^{2/3}}{2^{-4/3}}\left(1 - \sqrt{1 - \frac{2^8}{3^3}\,(3\theta_c)^{-2}}\right)^{-1/3} \\
&= \frac{(3\theta_c)^{-2/3}}{2^{4/3}}\left(1 - \sqrt{1 - \frac{2^8}{3^3}\,(3\theta_c)^{-2}}\right)^{1/3} + \frac{2}{3}\frac{(3\theta_c)^{-4/3}}{2^{-4/3}}\left(1 - \sqrt{1 - \frac{2^8}{3^3}\,(3\theta_c)^{-2}}\right)^{-1/3}
\end{aligned}$$

A first order approximation of (47) around $u = 0$ (which means a and R near 0 too) yields

$$\begin{aligned}
\sqrt{1+4u} &= 1 + 2u + o(u) \\
\sqrt{2 - 4u + \frac{2}{\sqrt{1+4u}}} &= \sqrt{2 - 4u + 2\,(1 - 2u + o(u))} \\
&= \sqrt{4 - 8u + o(u)} \\
&= 2\sqrt{1 - 2u + o(u)} \\
&= 2\,(1-u) + o(u)
\end{aligned}$$

Hence:

$$\alpha^* = \sqrt{\frac{3}{4} - \frac{1}{4}\left(-1+4u\right) + o(u)} = \sqrt{1-u+o(u)}$$

Around $R = 0$ (and thus $a = 0$), we get

$$\begin{aligned}
R &= \frac{a}{2^4}\left(1 - \sqrt{1 - \frac{2^8}{3^3}a}\right) = \frac{a}{2^4}\left(1 - \left(1 - \frac{2^7}{3^3}a + o(a)\right)\right) = \frac{2^3}{3^3}a^2 + o(a^2) \\
u &= 2R^{1/3} + \frac{2\left(3\theta_c\right)^{-2}}{3R^{1/3}} = 2\frac{2a^{2/3}}{3} + o(a^{2/3}) + \frac{2a}{3}\left(\frac{3a^{-2/3}}{2} + o(a^{-2/3})\right) \\
&= a^{1/3} + o(a^{1/3}) \\
&= (3\theta_c)^{-2/3} + o((3\theta_c)^{-2/3})
\end{aligned}$$

Finally, using $a = (3\theta_c)^{-2}$:

$$\alpha^* = \sqrt{1 - (3\theta_c)^{-2/3} + o((3\theta_c)^{-2/3})},$$

which coincides with our square root rule, which thus is a first order approximation for large θ_c!

In addition, there needs to be a stronger condition on θ_c: not only do we need a local minimum, but it needs to be global. (i.e., less than cost at $\alpha = 0$ which is 1). This will set a slightly higher bar on θ_c than $\theta_c > \frac{1}{3}\sqrt{\frac{256}{27}}$. Numerical evaluation of the optimal cost shows that $\widehat{C}(\alpha^*(\theta_c)) = -\theta_c\alpha^{*3} + \frac{1}{\sqrt{1-\alpha^{*2}}} = 1$ for $\theta_c = 1.14993$.

3.4 $L > 3$ and $L_l = 0$ (Lagrange Series)

The derivation of Lagrange's series is in the Appendix of the paper. Here we show the radius of convergence for the function $f\left(z\right) = z^{\alpha}\left(1-z\right)^{\beta}$ which is analytic in z provided α and β are integers. Its Lagrange series around $z = 0$ is

$$f^{-1}(z) = \sum_{n=1}^{\infty} \frac{\Gamma\left(n\left(\frac{\beta}{\alpha}+1\right)-1\right)}{\Gamma\left(\frac{n\beta}{\alpha}\right)} \frac{z^{n/\alpha}}{n!} \tag{48}$$

where $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{C}$. The radius R of convergence of the power series $f(z) = \sum_{k=0}^{\infty} f_k\, z^k$ is given by

$$\frac{1}{R} = \lim_{k\to\infty}\left|\frac{f_{k+1}}{f_k}\right|$$

Let $R=\rho^{1/\alpha}$ where ρ is the radius of convergence of the Lagrange series of $f^{-1}(z)$ above, yields

$$\begin{aligned}
\frac{1}{R} &= \lim_{n\to\infty}\left|\frac{\Gamma\left((n+1)\left(\frac{\beta}{\alpha}+1\right)-1\right)\Gamma\left(\frac{n\beta}{\alpha}\right)n!}{\Gamma\left(n\left(\frac{\beta}{\alpha}+1\right)-1\right)\Gamma\left(\frac{(n+1)\beta}{\alpha}\right)(n+1)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{\Gamma\left(n\left(\frac{\beta}{\alpha}+1\right)+\frac{\beta}{\alpha}\right)\Gamma\left(\frac{n\beta}{\alpha}\right)}{\Gamma\left(n\left(\frac{\beta}{\alpha}+1\right)-1\right)\Gamma\left(\frac{n\beta}{\alpha}+\frac{\beta}{\alpha}\right)(n+1)}\right|
\end{aligned}$$

Using $\frac{\Gamma(n+p)}{\Gamma(n+q)}\sim n^{p-q}$ for large n,

$$\begin{aligned}
\frac{1}{R} &= \lim_{n\to\infty}\left|\left(n\left(\frac{\beta}{\alpha}+1\right)\right)^{\frac{\beta}{\alpha}+1}\left(\frac{n\beta}{\alpha}\right)^{-\frac{\beta}{\alpha}}\frac{1}{n+1}\right| \\
&= \lim_{n\to\infty}\left|n^{\frac{\beta}{\alpha}+1}\left(\frac{\beta}{\alpha}+1\right)^{\frac{\beta}{\alpha}+1}n^{-\frac{\beta}{\alpha}}\left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha}}\frac{1}{n+1}\right| \\
&= \frac{\left(\frac{\beta}{\alpha}+1\right)^{\frac{\beta}{\alpha}+1}}{\left(\frac{\beta}{\alpha}\right)^{\frac{\beta}{\alpha}}}
\end{aligned}$$

Hence, the radius of convergence for the series (29) is $\rho^{1/\alpha}=\frac{\left(\frac{\beta}{\alpha}\right)^{\frac{\beta}{\alpha}}}{\left(\frac{\beta}{\alpha}+1\right)^{\frac{\beta}{\alpha}+1}}$, or, equivalently, $|z|<\frac{\left(\frac{\beta}{\alpha}\right)^{\beta}}{\left(\frac{\beta}{\alpha}+1\right)^{\beta+\alpha}}$. Applied to $\alpha=3$ and $\beta=L-2$, we get:

$$x^{*}=f^{-1}((\theta_cL)^{-2})=\sum_{n=1}^{\infty}\frac{\Gamma\left(n\frac{L+1}{3}-1\right)}{\Gamma\left(\frac{n(L-2)}{3}\right)}\frac{(\theta_cL)^{-\frac{2n}{3}}}{n!}\quad\text{for}\quad(\theta_cL)^{-2}<\frac{3^3(L-2)^{L-2}}{(L+1)^{L+1}},$$

where the radius of convergence is equivalent to $\lambda=(\theta_cL)^{-2}\leq\overline{\lambda}$.

4 Lagrange Solutions for uncapacitated and $L_l>0$

For any $0\leq L_1$ and $L>0$, f is analytic around $z_0=0$ with $f(0)=0$ and with $f^{(m)}(z_0)=0$ but $f^{(3)}(z_0)\neq 0$ for all $m<3$, then (27) generalizes to (Markushevich 1985, II, pp. 92)

$$f^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{[f(z)]_s^{1/3}}\right)^{n}\right]\Bigg|_{z=z_0}z^{n/3}=\sum_{n=1}^{\infty}\frac{1}{n!}a_nz^{n/3},\tag{49}$$

where

$$\left(\frac{z}{[f(z)]_s^{1/3}}\right)^{n}=\left(\frac{z}{z(1+L_1z)^{\frac{1}{3}}(1-z)^{\frac{L-2}{3}}}\right)^{n}=(1+L_1z)^{\frac{-n}{3}}(1-z)^{\frac{-n(L-2)}{3}}$$

Now the derivatives become more complex than before. First we calculate a few specific instances:

First order term is simple: $a_1 = 1$ so we get, as before:

$$f^{-1}(z) \simeq z^{\frac{1}{3}} \text{ so that } \alpha^* \simeq \sqrt{1 - (L\theta_c)^{-\frac{2}{3}}} \tag{50}$$

Second order term:

$$\begin{aligned} & \frac{d}{dz}\left(1 + L_1 z\right)^{\frac{-2}{3}} (1-z)^{\frac{-2(L-2)}{3}} \\ = \; & \frac{-2}{3} L_1 \left(1 + L_1 z\right)^{\frac{-5}{3}} (1-z)^{\frac{-2(L-2)}{3}} + \frac{2(L-2)}{3} \left(1 + L_1 z\right)^{\frac{-2}{3}} (1-z)^{\frac{-2(L-2)}{3} - 1} \\ a_2 \; = \; & \frac{-2}{3} L_1 + \frac{2(L-2)}{3} = \frac{2(L - L_1 - 2)}{3} \end{aligned}$$

so that

$$f^{-1}(z) = z^{\frac{1}{3}} + \frac{1}{2!}\frac{2(L-L_1-2)}{3} z^{\frac{2}{3}} \text{ so that } \alpha^* \simeq \sqrt{1 - (L\theta_c)^{-\frac{2}{3}} + \frac{(L-L_1-2)}{3}(L\theta_c)^{-\frac{4}{3}}}$$

The n-th order terms is best calculated via series expansion: Use the general binomial theorem, valid for all real α and $|z| < 1$:

$$(1-z)^\alpha = \sum_{i=0}^{\infty} \binom{\alpha}{i} (-z)^i$$

to get

$$\left(1 + L_1 z\right)^{\frac{-n}{3}} (1-z)^{\frac{-n(L-2)}{3}} = \sum_{i=0}^{\infty} \binom{\frac{-n}{3}}{i} (L_1 z)^i \sum_{j=0}^{\infty} \binom{\frac{-n(L-2)}{3}}{j} (-z)^j .$$

If $L = 2$ only one series applies and a_n is the coefficient of z^{n-1} multiplied by $(n-1)!$

$$a_n = (n-1)! \binom{\frac{-n}{3}}{n-1} L_1^{n-1} = (n-1)! \frac{(-1)^{n-1}}{(n-1)!} \frac{\Gamma\left(\frac{n}{3} + n - 1\right)}{\Gamma\left(\frac{n}{3}\right)} L_1^{n-1} = (-1)^{n-1} \frac{\Gamma\left(\frac{4n}{3} - 1\right)}{\Gamma\left(\frac{n}{3}\right)} L_1^{n-1}$$

If $L \neq 2$, regroup powers by setting $i + j = k$ and invoking the Cauchy product: we have $i = k - j \geq 0$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{\frac{-n}{3}}{k-j} \binom{\frac{-n(L-2)}{3}}{j} (-1)^j L_1^{k-j} z^k .$$

Now, a_n is the coefficient of z^{n-1} multiplied by $(n-1)!$

$$\begin{aligned}
a_n &= (n-1)!\sum_{j=0}^{n-1}\begin{pmatrix}\frac{-n}{3}\\ n-1-j\end{pmatrix}\begin{pmatrix}\frac{-n(L-2)}{3}\\ j\end{pmatrix}(-1)^j L_1^{n-1-j}\\
&= (n-1)!\sum_{i=0}^{n-1}\begin{pmatrix}\frac{-n}{3}\\ i\end{pmatrix}\begin{pmatrix}\frac{-n(L-2)}{3}\\ n-1-i\end{pmatrix}(-1)^{n-1-i}L_1^i\\
&= \Gamma(n)\sum_{i=0}^{k}\frac{\Gamma\left(\frac{-n}{3}+1\right)}{\Gamma(i+1)\,\Gamma\left(\frac{-n}{3}-i+1\right)}\frac{\Gamma\left(\frac{-n(L-2)}{3}+1\right)}{\Gamma(n-i)\,\Gamma\left(\frac{-n(L-2)}{3}-n+i+2\right)}(-1)^{n-1-i}L_1^i
\end{aligned}$$

Getting rid of the negative signs:

$$\begin{aligned}
\begin{pmatrix}\frac{-n(L-2)}{3}\\ n-1-i\end{pmatrix} &= \frac{\left(\frac{-n(L-2)}{3}\right)\left(\frac{-n(L-2)}{3}-1\right)\left(\frac{-n(L-2)}{3}-2\right)\ldots\left(\frac{-n(L-2)}{3}-n-i+2\right)}{(n-1-i)!}\\
&= \frac{(-1)^{n-i-1}}{(n-1)!}\left(\frac{n(L-2)}{3}\right)\left(\frac{n(L-2)}{3}+1\right)\left(\frac{n(L-2)}{3}+2\right)\ldots\left(\frac{n(L-2)}{3}+n+i-2\right)\\
&= \frac{(-1)^{n-i-1}}{(n-1)!}\frac{\Gamma\left(\frac{n(L-2)}{3}+n+i-1\right)}{\Gamma\left(\frac{n(L-2)}{3}\right)}
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix}\frac{-n}{3}\\ i\end{pmatrix} &= \frac{\left(\frac{-n}{3}\right)\left(\frac{-n}{3}-1\right)\left(\frac{-n}{3}-2\right)\ldots\left(\frac{-n}{3}-i+1\right)}{i!}\\
&= \frac{(-1)^i}{i!}\left(\frac{n}{3}\right)\left(\frac{n}{3}+1\right)\left(\frac{n}{3}+2\right)\ldots\left(\frac{n}{3}+i-1\right)\\
&= \frac{(-1)^i}{i!}\frac{\Gamma\left(\frac{n}{3}+i\right)}{\Gamma\left(\frac{n}{3}\right)}
\end{aligned}$$

Hence, if $L \neq 2$:

$$\alpha^* = \sqrt{1-f^{-1}((L\theta_c)^{-2})} = \sqrt{1-\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{\Gamma\left(\frac{n}{3}+i\right)\Gamma\left(\frac{n(L-2)}{3}+n+i-1\right)}{\Gamma\left(\frac{n}{3}\right)\Gamma\left(\frac{n(L-2)}{3}\right)}L_1^i\,(L\theta_c)^{-\frac{2n}{3}}}$$

The radius R of convergence of the power series $f(z)=\sum_{k=0}^{\infty} f_k\, z^k$ is given by

$$\frac{1}{R} = \lim_{k\to\infty}\left|\frac{f_{k+1}}{f_k}\right|$$

Applied to our Lagrange series, where $R=\rho^{1/(-2/3)}$ and ρ is the radius of convergence of (29), yields

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}\frac{\Gamma\left(\frac{n+1}{3}+i\right)\Gamma\left(\frac{(n+1)(L-2)}{3}+n+i\right)}{\Gamma\left(\frac{n+1}{3}\right)\Gamma\left(\frac{(n+1)(L-2)}{3}\right)}L_1^i}{(n+1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{\Gamma\left(\frac{n}{3}+i\right)\Gamma\left(\frac{n(L-2)}{3}+n+i-1\right)}{\Gamma\left(\frac{n}{3}\right)\Gamma\left(\frac{n(L-2)}{3}\right)}L_1^i}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n!\Gamma\left(\frac{n}{3}\right)\Gamma\left(\frac{n(L-2)}{3}\right)\sum_{i=0}^{n}\frac{(-1)^i}{i!}\Gamma\left(\frac{n+1}{3}+i\right)\Gamma\left(\frac{(n+1)(L-2)}{3}+n+i\right)L_1^i}{(n+1)!\Gamma\left(\frac{n+1}{3}\right)\Gamma\left(\frac{(n+1)(L-2)}{3}\right)\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\Gamma\left(\frac{n}{3}+i\right)\Gamma\left(\frac{n(L-2)}{3}+n+i-1\right)L_1^i}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\Gamma\left(\frac{n}{3}\right)\Gamma\left(\frac{n(L-2)}{3}\right)\sum_{i=0}^{n}\frac{(-1)^i}{i!}\Gamma\left(\frac{n+1}{3}+i\right)\Gamma\left(\frac{(n+1)(L-2)}{3}+n+i\right)L_1^i}{(n+1)\Gamma\left(\frac{n}{3}+\frac{1}{3}\right)\Gamma\left(\frac{n(L-2)}{3}+\frac{(L-2)}{3}\right)\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\Gamma\left(\frac{n}{3}+i\right)\Gamma\left(\frac{n(L-2)}{3}+n+i-1\right)L_1^i}\right|$$

Using $\frac{\Gamma(n+p)}{\Gamma(n+q)} \sim n^{p-q}$ for large n,

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{\sum_{i=0}^{n}\frac{(-1)^i}{i!}\Gamma\left(\frac{n+1}{3}+i\right)\Gamma\left(\frac{(n+1)(L-2)}{3}+n+i\right)L_1^i}{(n+1)\,n^{\frac{1}{3}+\frac{L-2}{3}}\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\Gamma\left(\frac{n}{3}+i\right)\Gamma\left(\frac{n(L-2)}{3}+n+i-1\right)L_1^i}\right|.$$

5 Value of Uncapacitated Dual Sourcing and Order Smoothing

In this section we compare the value of incapacitated dual sourcing when $L_l = 0$. The DSS policy encompasses local single sourcing LS ($\alpha = 0$) under the optimal base-stock policy for which $\widehat{C^l} = \widehat{C}(0) = 1$ and dual sourcing smoothing ($\alpha \in (0,1)$). The square root formula directly provides a bound on the value of dual sourcing over single local sourcing:

$$C(0) - C(\alpha^*) \geq \left[1 - \widehat{C}(\alpha_0)\right]\kappa_I\sigma = \left[1 + \theta_c\left(1 - (\theta_c L)^{-\frac{2}{3}}\right)^{\frac{L}{2}} - (\theta_c L)^{\frac{1}{3}}\right]\kappa_I\sigma$$

The DSS policy space, however, does not contain global single sourcing GSS. Global single sourcing with the optimal base-stock policy is a demand-replacement policy where $q_t^g = D_{t-L}$ and $q_t^l = 0$. Then, $\mathbb{E}q_t^g = \mu$ and $Var(q_t^g) = \sigma^2$, and the inventory process is

$$I_t = I_{t-1} + q_{t-1}^g - D_t = I_{-1} + \sum_{i=1}^{L+1} q_{-i}^g - \sum_{i=t-L}^{t} D_i = I_s + (L+1)\mu - \sum_{i=t-L}^{t} D_i,$$

so that $\mathbb{E}I_t = I_s$ and $Var(I_t) = (L+1)\sigma^2$. The corresponding average cost is $C^g = (c^g + hL)\mu + \sqrt{L+1}\kappa_I\sigma$ and normalizing:

$$\widehat{C^g} = \frac{C^g - c^l\mu}{\kappa_I\sigma} = -\theta_c + \sqrt{L+1}.$$

It directly follows that, between the two single sourcing strategies, GS dominates LS if and only if $\theta_c > \theta_g = \sqrt{L+1} - 1$.

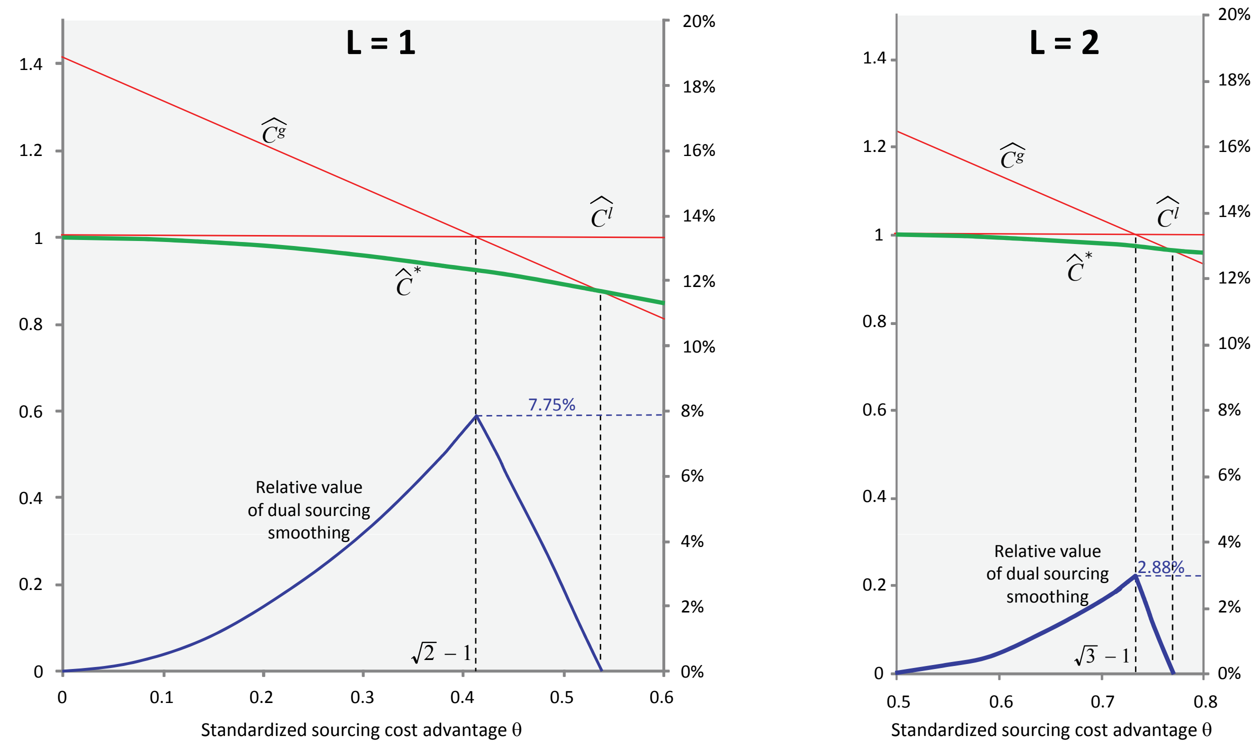

Figure 19: Comparing the cost of dual sourcing smoothing with global and local single sourcing (left axis) and the corresponding relative value of DSS (right axis) for $L = 1$ (left panel) and $L = 2$ (right panel).

Figure 19 compares DSS with GS and LS by considering the optimal cost as a function of θ_c. First consider the left panel where $L = 1$: As θ_c rises, α^* rises and dual sourcing smoothing yields cost $\widehat{C}(\alpha^*) \leq 1$ that falls at rate slower than -1 (Prop. 5). While global single sourcing initially has a higher cost ($\sqrt{2}$ at $\theta_c = 0$), if falls at a faster rate of -1 and thus eventually intersects and then dominates the DSS cost. If $\theta_c < 0.54$, DSS dominates LS and GS and the maximal cost improvement of DSS over single sourcing occurs at $\theta_c = \theta_g$. For $L = 1$, the maximal relative value $1 - \frac{\widehat{C}}{\min(C^l, C^g)}$ is 7.75%. Second, consider the right panel where $L = 2$. A similar situation applies, except that $\alpha^* = 0$ and thus LS is optimal for $\theta_c < 0.5$. Furthermore, the region where DSS dominates single sourcing is smaller ($0.5 < \theta_c < \frac{4}{\sqrt{27}} = 0.77$) and the maximal relative value of DSS reduces to 2.88%.

The cost patterns are similar for $L \geq 3$ with one important exception: DSS is always dominated by single sourcing. Indeed, recall that $\widehat{C}^* = 1$ at $\theta_c = \underline{\theta}_L^* \geq \underline{\theta}_L$ and that $\widehat{C}^*$ falls at rate slower than -1 as θ_c rises beyond $\underline{\theta}_L^*$. It is easily verified that, for $L \geq 3$, $\underline{\theta}_L > \theta_g = \sqrt{L+1} - 1$ so that GS has lower cost than DSS at, and beyond, $\underline{\theta}_L$.

The implication so far is that DSS is attractive for short leadtimes and relatively light offshoring: For $L = 1$, $\alpha = 0$ at $\theta_c = 0$ and maximal $\alpha^* = a^* = .41$ at $\theta_c = 0.54$. For $L = 2$, $a^* = \alpha^{*2} = 0$ at $\theta_c = 0.5$ and $a^* = \alpha^{*2} = 1 - (2\frac{4}{\sqrt{27}})^{-2/3} = \frac{1}{4}$. Yet, dual sourcing with order smoothing becomes significantly more attractive when supply sources incur capacity costs.

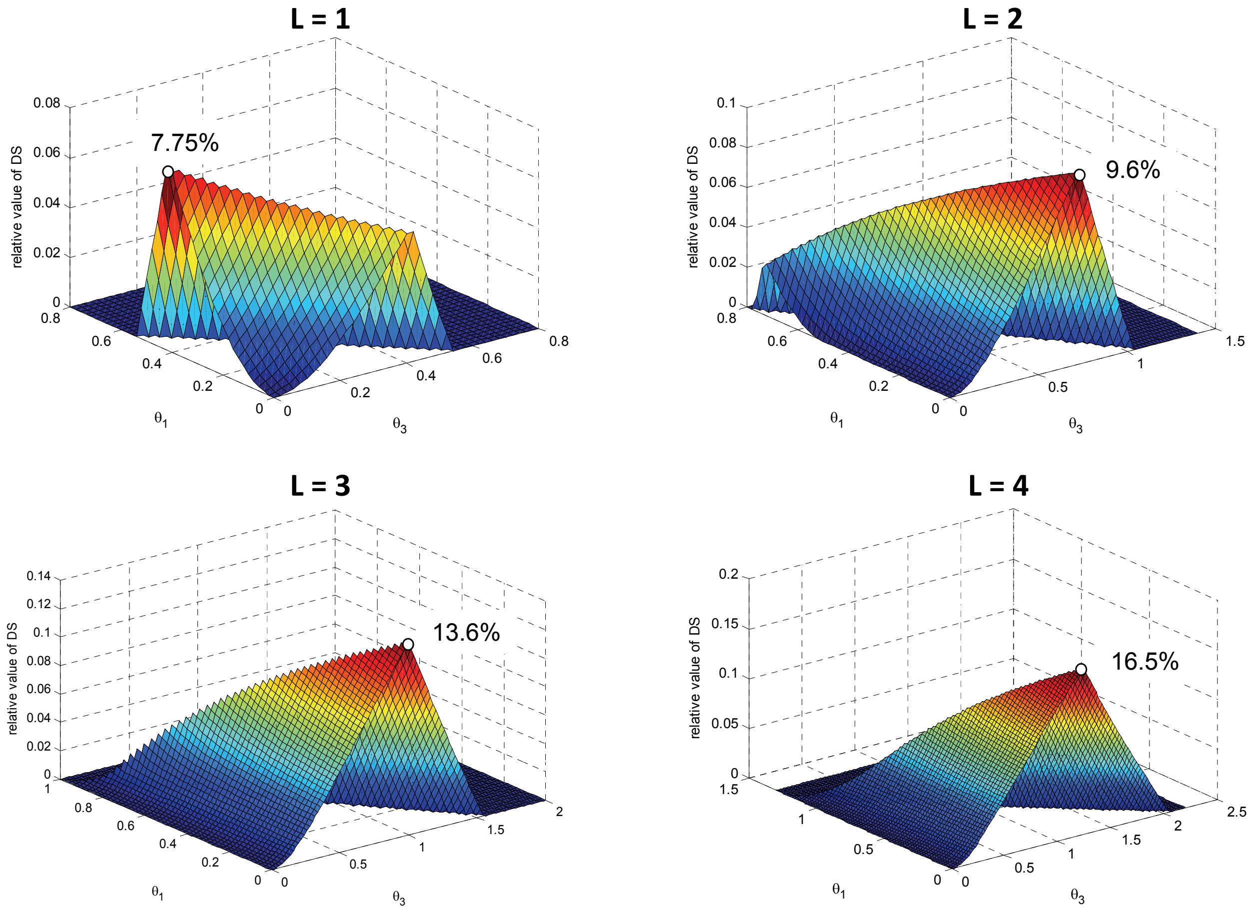

Figure 20: With local capacitated supply, the relative value of DSS over LS and GS increases as the leadtime L increases. The parameter domain (θ_c, θ_3) where DSS outperforms also increases.

6 Value of Local-Capacitated Dual Sourcing and Smoothing

Figure 20 shows the numerical evaluation of the value of dual sourcing with local capacity costs compared to the traditional single sourcing policies LS and GS. Notice that, like in the uncapacitated case, the relative value V of dual sourcing over single sourcing is maximal where the costs of LS and GS, which are linear in θ, intersect:

$$\widehat{C^l} = 1 + \theta_l = \widehat{C^g} = -\theta_c + \sqrt{L+1} + \theta_g \Leftrightarrow \theta_c - \theta_g + \theta_l = \sqrt{L+1} - 1.$$

(Notice that such parameter value can always be attained: when the local capacity cost k^l increases, both θ_c and θ_l increase while θ_g remains unchanged.) With $\theta_g = 0$, the maximal value $\overline{V}$ of dual sourcing thus is attained over the parameter line $\theta_c + \theta_l = \sqrt{L+1} - 1$, as is clearly evident in Figure 20. Comparing these maximal values with the uncapacitated case (Fig. 19) demonstrates our second key message: dual sourcing with the DSS policy is significantly more attractive when local supply is capacitated and leadtimes increase. In contrast to uncapacitated sourcing, DSS then always dominates LS (and increasingly so as local capacity costs increase) and also GS over a parameter domain that enlarges for large leadtimes. This finding can be corroborated and generalized analytically as show in Proposition 12 in the paper.

7 Exact Solutions for Capacitated Dual Sourcing and $L_l = 0$

We need the zero of the FOC and will abbreviate notation: $a = \theta_c$, $b = \theta_g$, and $c = \theta_l$:

$$
\begin{aligned}
\widehat{C}' &= \theta_c L\alpha^{L-1} - \theta_g \frac{L\alpha^{L-1}(1-\alpha)(1+\alpha) - \alpha^L}{(1-\alpha)^{1/2}(1+\alpha)^{3/2}} + \theta_l \frac{L\alpha^{2L-1}(1-\alpha)(1+\alpha) + \left(1-\alpha^{2L}\right)}{\left(1-\alpha^{2L}\right)^{1/2}(1-\alpha)^{1/2}(1+\alpha)^{3/2}} - \frac{\alpha}{(1-\alpha^2)^{3/2}} \\
&= aL\alpha^{L-1} - b\frac{L\alpha^{L-1}(1-\alpha)^{1/2}}{(1+\alpha)^{1/2}} + b\frac{\alpha^L}{(1-\alpha)^{1/2}(1+\alpha)^{3/2}} \\
&\quad + c\frac{L\alpha^{2L-1}(1-\alpha)^{1/2}}{\left(1-\alpha^{2L}\right)^{1/2}(1+\alpha)^{1/2}} + c\frac{\left(1-\alpha^{2L}\right)^{1/2}}{(1-\alpha)^{1/2}(1+\alpha)^{3/2}} - \frac{\alpha}{(1-\alpha^2)^{3/2}}
\end{aligned}
$$

7.1 Capacitated Solutions For $L = 1$

Proposition 18 *With normal demand, the total scaled cost rate has two independent parameters $\theta_{cl} = \theta_c + \theta_l$ and θ_g:*

$$
\widehat{C}(\alpha; \theta_c, \theta_g) = \frac{C(\alpha) - C_0}{\kappa_I \sigma} = -\theta_{cl}\alpha + \theta_g \alpha \sqrt{\frac{1-\alpha}{1+\alpha}} + \frac{1}{\sqrt{1-\alpha^2}} \tag{51}
$$

and is concave-convex in $\alpha \in [0,1]$ with a unique interior minimum α^* satisfying:

$$
\widehat{MB}(\alpha^*) = \theta_{cl} - \theta_g \frac{1-\alpha^* - \alpha^{*2}}{(1-\alpha^*)^{1/2}(1+\alpha^*)^{3/2}} = \widehat{MC}(\alpha^*) = \frac{\alpha^*}{(1-\alpha^{*2})^{3/2}}. \tag{52}
$$

Proof: Both marginal cost and marginal benefits are convex increasing, with MB initially dominating MC and then reversing. Thus, they always have a unique interior intersection, represented by point B in Fig. 14. ■

Proposition 19 *With normal demand and if $\theta_g > 0$, there is no general formula, using only a finite number of the usual algebraic operations and radicals, to express the optimality of capacitated dual sourcing.*

Proof: Rewrite the FOC as polynomial and consider $\alpha \in (0,1)$:

$$
\begin{aligned}
\theta_{cl} - \theta_g \frac{1-\alpha-\alpha^2}{(1-\alpha)^{1/2}(1+\alpha)^{3/2}} - \frac{\alpha}{(1-\alpha^2)^{3/2}} &= 0 \\
\Longleftrightarrow \theta_{cl} \frac{(1-\alpha)^{3/2}(1+\alpha)^{3/2}}{(1-\alpha)^{3/2}(1+\alpha)^{3/2}} - \theta_g \frac{\left(1-\alpha-\alpha^2\right)(1-\alpha)}{(1-\alpha)^{3/2}(1+\alpha)^{3/2}} - \frac{\alpha}{(1-\alpha)^{3/2}(1+\alpha)^{3/2}} &= 0 \\
\Longleftrightarrow \theta_{cl}(1-\alpha)^{3/2}(1+\alpha)^{3/2} = \theta_g \left(1-\alpha-\alpha^2\right)(1-\alpha) + \alpha &
\end{aligned}
$$

Given that $\alpha \in (0,1)$, both sides are positive and squaring yields:

$$\begin{aligned}
\theta_{cl}^2\left(1-\alpha^2\right)^3-\left(\theta_g\left(1-\alpha-\alpha^2\right)(1-\alpha)+\alpha\right)^2 &= 0\\
\Longleftrightarrow \left(\theta_{cl}^2+\theta_g^2\right)\alpha^6-\left(3\theta_{cl}^2-2\theta_g\left(1-2\theta_2\right)\right)\alpha^4+2\theta_g^2\alpha^3 & \\
+\left(3\theta_{cl}^2+\left(1-2\theta_g\right)^2\right)\alpha^2+2\theta_g\left(1-2\theta_g\right)\alpha-\theta_{13}^2+\theta_g^2 &= 0
\end{aligned}$$

If $\theta_g = 0$, we have a cubiq in α^2 whose roots can be solved for with Cardano; if $\theta_g > 0$, however, α^* corresponds to the roots of a general 6-th order polynomial, for which there exists no general formula, using only a finite number of the usual algebraic operations and radicals.■

7.1.1 For $L = 1$ and $\theta_g = 0$: exact solutions and square root is a lower bound

The FOC are identical to the uncapacitated case provided we replace θ by θ_{cl}. Hence, Cardano gives the exact solution and the square root remains a lower bound as follows directly from the Proof of Prop. 7.

7.1.2 Lagrange series for $L = 1$ ($\theta_g > 0$): expanded around $x = 0$ or $\alpha = 1$

Start from the FOC but make them analytic around $\alpha = 1$ by squaring:

$$\begin{aligned}
\theta_{cl}(1-\alpha)^{3/2}(1+\alpha)^{3/2} &= \theta_g\left(1-\alpha-\alpha^2\right)(1-\alpha)+\alpha\\
\Longleftrightarrow \theta_{cl}^2\left(1-\alpha^2\right)^3 &= \left(\theta_g\left(1-\alpha-\alpha^2\right)(1-\alpha)+\alpha\right)^2
\end{aligned}$$

Denote $a = \theta_{cl}$ and $b = \theta_g$ and use the transformation $x = 1-\alpha^2$ (similar to uncapacitated):

$$\begin{aligned}
\Leftrightarrow a^2\left(1-z^2\right)^3 &= \left(b\left(1-z-z^2\right)(1-z)+z\right)^2 = \left(b+(1-2b)z+bz^3\right)^2\\
\Leftrightarrow a^2x^3 &= \left(b+(1-2b)(1-x)^{1/2}+b(1-x)^{3/2}\right)^2\\
&= \left(b+(1-x)^{1/2}(1-b-bx)\right)^2\\
&= b^2+(1-b-bx)^2(1-x)+2b(1-b-bx)(1-x)^{1/2}
\end{aligned}$$

We seek the root of

$$f(x) = a^2x^3-b^2-(1-b-bx)^2(1-x)-2b(1-b-bx)(1-x)^{1/2}$$

The function is analytic around $x_0 = 0$ with $f'(0) = 1 + b > 0$ always:

$$\begin{aligned}
f(x_0) &= -b^2 - (1-b)^2 - 2b(1-b) = -1 \\
f'(0) &= \frac{d}{dx}\left[a^2x^3 - b^2 - (1-b-bx)^2(1-x) - 2b(1-b-bx)(1-x)^{1/2}\right]_{x=0} \\
&= \left[3a^2x^2 + (1-bx-b)^2 + 2b(1-x)(1-bx-b) + 2b^2\sqrt{1-x} + \frac{b}{\sqrt{1-x}}(1-bx-b)\right]_{x=0} \\
&= (1-b)^2 + 2b(1-b) + 2b^2 + b(1-b) \\
&= b + 1 > 0
\end{aligned}$$

Hence, the Lagrange inversion series applies at $x_0 = 0$ and $f(x_0) = -1$:

$$\begin{aligned}
f^{-1}(z) &= z_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z-z_0}{f(z)-f(z_0)}\right)^n\right]\Bigg|_{z=z_0}(z - f(z_0))^n \\
x^* &= f^{-1}(0) = \sum_{n=1}^{\infty}\frac{1}{n!}\left[\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{f(z)+1}\right)^n\right] = \sum_{n=1}^{\infty}\frac{a_n}{n!}
\end{aligned}$$

We will need:

$$\begin{aligned}
f''(0) &= \frac{d}{dx}\left[3a^2x^2 + (1-bx-b)^2 + 2b(1-x)(1-bx-b) + 2b^2\sqrt{1-x} + \frac{b}{\sqrt{1-x}}(1-bx-b)\right]_{x=0} \\
&= \left[6a^2x + 2b(b+3bx-2) - 2\frac{b^2}{\sqrt{1-x}} + \frac{1}{2}\frac{b}{(1-x)^{\frac{3}{2}}}(1-bx-b)\right]_{x=0} \\
&= 2b(b-2) - 2b^2 + \frac{1}{2}b(1-b) \\
&= -\frac{b(b+7)}{2}
\end{aligned}$$

$$\begin{aligned}
f^{(3)}(0) &= \frac{d}{dx}\left[6a^2x + 2b(b+3bx-2) - 2\frac{b^2}{\sqrt{1-x}} + \frac{1}{2}\frac{b}{(1-x)^{\frac{3}{2}}}(1-bx-b)\right]_{x=0} \\
&= \left[6a^2 + 6b^2 - \frac{3}{2}\frac{b^2}{(1-x)^{\frac{3}{2}}} + \frac{3}{4}\frac{b}{(1-x)^{\frac{5}{2}}}(1-bx-b)\right]_{x=0} \\
&= 6a^2 + 6b^2 - \frac{3}{2}\frac{b^2}{1} + \frac{3}{4}\frac{b}{1}(1-b) \\
&= 6a^2 + \frac{3}{4}b(5b+1)
\end{aligned}$$

Term 1:

$$a_1 = \lim_{z\to 0}\left(\frac{z}{f(z)+1}\right)^1 = [\frac{0}{0}] \stackrel{\text{l'Hospital}}{=} \frac{1}{f'(0)} = \frac{1}{b+1}$$

Term 2:

$$\lim_{x\to0}\frac{d}{dx}\left(\frac{x}{f(x)+1}\right)^2 = \lim_{z\to0}\frac{d}{dx}\left(\frac{z}{a^2x^3-b^2-(1-b-bx)^2(1-x)-2b(1-b-bx)(1-x)^{1/2}+1}\right)^2$$
$$= \lim_{z\to0}2\left(\frac{z}{f(z)+1}\right)\cdot\frac{d}{dx}\left(\frac{x}{f(x)+1}\right)$$
$$= \lim_{x\to0}2\left(\frac{x}{f(x)+1}\right)\cdot\frac{f(x)+1-x\left(f'(x)\right)}{\left(f(x)+1\right)^2}$$
$$= \lim_{x\to0}2\frac{xf(x)+x-x^2\left(f'(x)\right)}{\left(f(x)+1\right)^3} = [\frac{0}{0}]$$

$$\overset{\text{l'Hospital}}{=} \lim_{x\to0}2\frac{f(x)+1-xf'(x)-x^2f''(x)}{3\left(f(x)+1\right)^2\left(f'(x)\right)} = [\frac{0}{0}]$$
$$\overset{\text{H}}{=} \lim_{x\to0}2\frac{f'(x)-f'(x)-xf''(x)-2xf''(x)-x^2f'''(x)}{6\left(f(x)+1\right)(f'(x))^2+3\left(f(x)+1\right)^2\left(f''(x)\right)}$$
$$= \lim_{x\to0}2\frac{-3xf''(x)-x^2f'''(x)}{6\left(f(x)+1\right)(f'(x))^2+3\left(f(x)+1\right)^2\left(f''(x)\right)} = [\frac{0}{0}]$$
$$\overset{\text{H}}{=} \lim_{x\to0}2\frac{-3f''(x)-3xf^{(3)}(x)-2xf^{(3)}(x)-x^2f^{(4)}(x)}{6(f'(x))^3+12\left(f(x)+1\right)2f'(x)f''(x)+6\left(f(x)+1\right)f''(f''(x))+3\left(f(x)+1\right)^2\left(f^{(3)}(x)\right)}$$
$$a_2 = -\frac{f^{(2)}(0)}{(f^{(1)}(0))^3}$$
$$= \frac{b(b+7)}{2(b+1)^3}$$

so far we have the two-term solution:

$$x^* = \frac{1}{b+1}+\frac{b(b+7)}{4(b+1)^3} = \frac{15b+5b^2+4}{4(b+1)^3}$$

Term 3 (so we get a in it):

$$\frac{d^2}{dx^2}\left(\frac{x}{f(x)+1}\right)^3$$
$$= \frac{d}{dx}\left[3\left(\frac{x}{f(x)+1}\right)^2\frac{d}{dx}\left(\frac{x}{f(x)+1}\right)\right]$$
$$= \frac{d}{dx}\left[3\left(\frac{x}{f(x)+1}\right)^2\frac{d}{dx}\left(\frac{x}{f(x)+1}\right)\right]$$
$$= 3\left[\frac{d}{dx}\left(\frac{x}{f(x)+1}\right)^2\right]\frac{d}{dx}\left(\frac{x}{f(x)+1}\right)+3\left(\frac{x}{f(x)+1}\right)^2\frac{d^2}{dx^2}\left(\frac{x}{f(x)+1}\right)$$
$$= 6\frac{xf(x)+x-x^2\left(f'(x)\right)}{\left(f(x)+1\right)^3}\frac{f(x)+1-x\left(f'(x)\right)}{\left(f(x)+1\right)^2}+3\left(\frac{x}{f(x)+1}\right)^2A$$

where

$$\begin{aligned}
A &= \frac{d}{dx}\left[\frac{f(x)+1-x\left(f'(x)\right)}{\left(f(x)+1\right)^2}\right] \\
&= \frac{\left(f(x)+1\right)^2\left(f'(x)-\left(f'(x)\right)-xf''(x)\right)-2\left(f(x)+1\right)f'(x)\left(f(x)+1-x\left(f'(x)\right)\right)}{\left(f(x)+1\right)^4} \\
&= \frac{-\left(f(x)+1\right)xf''(x)-2f'(x)\left(f(x)+1-x\left(f'(x)\right)\right)}{\left(f(x)+1\right)^3}
\end{aligned}$$

so

$$\begin{aligned}
&\frac{d^2}{dx^2}\left(\frac{x}{f(x)+1}\right)^3 \\
&= 6\frac{x}{\left(f(x)+1\right)^3}\frac{\left(f(x)+1-xf'(x)\right)^2}{\left(f(x)+1\right)^2}+3\left(\frac{x}{f(x)+1}\right)^2\frac{-\left(f(x)+1\right)xf''(x)-2f'(x)\left(f(x)+1-x\left(f'(x)\right)\right)}{\left(f(x)+1\right)^3} \\
&= \frac{3x}{\left(f(x)+1\right)^5}\left[2\left(f(x)+1-xf'(x)\right)^2-\left(f(x)+1\right)x^2f''(x)-2xf'(x)\left(f(x)+1-xf'(x)\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&2\left(f(x)+1-xf'(x)\right)^2-\left(f(x)+1\right)x^2f''(x)-2xf'(x)\left(f(x)+1-xf'(x)\right) \\
&= 2\left(f(x)+1\right)^2-4\left(f(x)+1\right)xf'(x)+2x^2f'(x)^2-2xf'(x)\left(f(x)+1\right)+2x^2f'(x)^2-\left(f(x)+1\right)x^2f''(x) \\
&= 2\left(f(x)+1\right)^2-6\left(f(x)+1\right)xf'(x)+4x^2f'(x)^2-\left(f(x)+1\right)x^2f''(x)
\end{aligned}$$

We will have to take l'Hospital 5 times! In the denominator, the surviving term is

$$\begin{aligned}
5(f+1)^4f'+(f+1)^5f'' &= [0] \\
5\cdot 4(f+1)^3f'^2+O(f+1)^5 &= [0] \\
5\cdot 4\cdot 3(f+1)^2f'^3 &= [0] \\
5!(f'(0))^5 &= 5!\,(b+1)^5
\end{aligned}$$

In the numerator:

$$\begin{aligned}
& 4\left(f+1\right)f'-6xf'^2-6(f+1)f'-6(f+1)xf''+8xf'^2+8x^2f'f''-f'x^2f^{''}-(f+1)2xf''-(f+1)x^2f^{(3)} \\
= \;& -2(f+1)f'-8(f+1)xf''+2xf'^2+7x^2f'f''-(f+1)x^2f^{(3)}=[0] \\
& -2(f+1)f^{''}-2f'^2-8f'xf''-8(f+1)f''-8(f+1)xf^{(3)}+2f'^2+4xf'f'' \\
& +14xf'f''+7x^2f''^2+7x^2f'f^{(3)}-f'x^2f^{(3)}-2(f+1)xf^{(3)}-(f+1)x^2f^{(4)} \\
= \;& -10(f+1)f^{''}-10(f+1)xf^{(3)}+10xf'f''+7x^2f''^2+6x^2f'f^{(3)}-(f+1)x^2f^{(4)}=[0] \\
& -10(f+1)f^{^{(3)}}-10f'f^{''}-10f'xf^{(3)}-10(f+1)f^{(3)}-10(f+1)xf^{(4)}+10f'f'' \\
& +10xf''^2+10xf'f^{(3)}+14xf''^2+14x^2f''f^{(3)}+12xf'f^{(3)}+6x^2f''f^{(3)}+6x^2f'f^{(4)} \\
& -f'x^2f^{(4)}-2(f+1)xf^{(4)}-(f+1)x^2f^{(5)} \\
= \;& -(f+1)\left[20f^{^{(3)}}+12xf^{(4)}+x^2f^{(5)}\right]+x\left[24f''^2+20xf''f^{(3)}+12f'f^{(3)}+5xf'f^{(4)}\right]=[0] \\
& -f'\left[20f^{^{(3)}}+12xf^{(4)}+x^2f^{(5)}\right]-(f+1)\left[20f^{^{(3)}}+12xf^{(4)}+x^2f^{(5)}\right]' \\
& +\left[24f''^2+20xf''f^{(3)}+12f'f^{(3)}+5xf'f^{(4)}\right]+x\left[24f''^2+20xf''f^{(3)}+12f'f^{(3)}+5xf'f^{(4)}\right]' \\
= \;& -20f'(0)f^{^{(3)}}(0)+24f''^2(0)+12f'(0)f^{(3)}(0)=24f''^2(0)-8f'(0)f^{^{(3)}}(0) \\
= \;& 24\left(-\frac{b\left(b+7\right)}{2}\right)^2-8\left(b+1\right)\left(6a^2+\frac{3}{4}b\left(5b+1\right)\right)
\end{aligned}$$

hence

$$\begin{aligned}
a_3 &= \frac{24f''^2(0)-8f'(0)f^{^{(3)}}(0)}{5!(f'(0))^5} \\
&= \frac{24\left(-\frac{b(b+7)}{2}\right)^2-8\left(b+1\right)\left(6a^2+\frac{3}{4}b\left(5b+1\right)\right)}{5!\left(b+1\right)^5}
\end{aligned}$$

The three terms in the series yield where $a=\theta_{cl}$ and $b=\theta_g$:

$$\begin{aligned}
x^* &= \frac{1}{b+1}+\frac{b\left(b+7\right)}{4\left(b+1\right)^3}+\frac{1}{3!}\frac{6\left(b\left(b+7\right)\right)^2-\left(b+1\right)\left(48a^2+6b\left(5b+1\right)\right)}{5!\left(b+1\right)^5} \\
&= \frac{1}{b+1}+\frac{b\left(b+7\right)}{4\left(b+1\right)^3}-\frac{8a^2+b\left(5b+1\right)}{5!\left(b+1\right)^4}+\frac{\left(b\left(b+7\right)\right)^2}{5!\left(b+1\right)^5}
\end{aligned}$$

Written differently:

$$\begin{aligned}
x^* &= \frac{1}{b+1}+\frac{b\left(b+7\right)}{4\left(b+1\right)^3}+\frac{b\left(b\left(b+7\right)^2-\left(5b+1\right)\left(b+1\right)\right)}{5!\left(b+1\right)^5}-\frac{a^2}{15\left(b+1\right)^4} \\
&= \frac{1}{b+1}+\frac{b\left(b+7\right)}{4\left(b+1\right)^3}+\frac{b\left(b^3+9b^2+43b-1\right)}{5!\left(b+1\right)^5}-\frac{a^2}{15\left(b+1\right)^4} \\
&= \frac{5!+689b+793b^2+489b^3+121b^4}{5!\left(b+1\right)^5}-\frac{a^2}{15\left(b+1\right)^4}
\end{aligned}$$

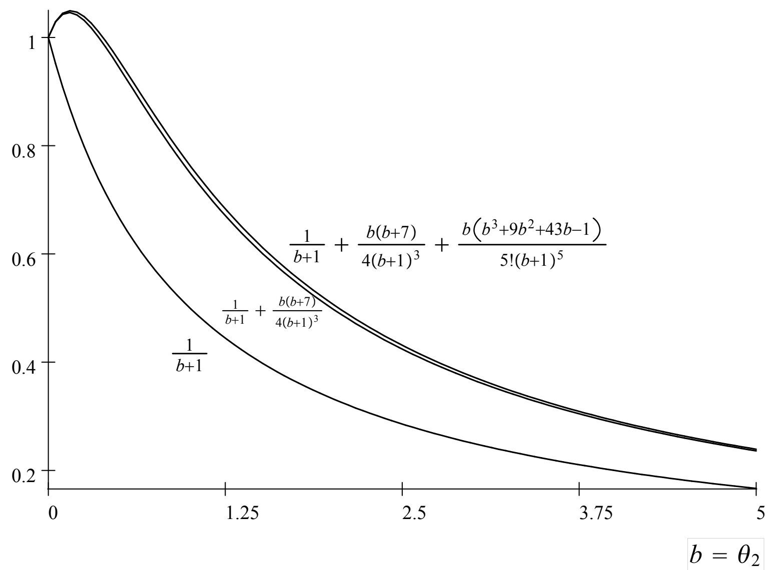

Figure 21: The first, second, and third order Lagrange's series for the general capacitated dual sourcing with $L = 1$.

Summary:

$$\begin{aligned}
\text{For } L=1:\alpha^* &= \sqrt{1-\frac{1}{b+1}-\frac{b\,(b+7)}{4\,(b+1)^3}+\frac{8a^2+b\,(5b+1)}{5!\,(b+1)^4}-\frac{(b\,(b+7))^2}{5!\,(b+1)^5}+\cdots} \\
&= \sqrt{1-\left(\frac{1}{b+1}+\frac{b\,(b+7)}{4\,(b+1)^3}+\frac{b\,(b^3+9b^2+43b-1)}{5!\,(b+1)^5}-\frac{a^2}{15\,(b+1)^4}\right)+\cdots} \\
&= \sqrt{1-\frac{15\,(\theta_g+1)^3-\theta_{cl}^2}{15\,(\theta_g+1)^4}+\cdots}
\end{aligned}$$

As function of b : $\frac{1}{b+1}+\frac{b(b+7)}{4(b+1)^3}+\frac{b\left(b^3+9b^2+43b-1\right)}{5!(b+1)^5}$. Figure 21 shows that the addition of the third term makes little difference–we may be close to convergence? Also, for $x^* < 1$, this requires $b > 0.4$ roughly.

7.1.3 Lagrange series for $L = 1$ ($\theta_g > 0$): expanded around $x = 1$ or $\alpha = 0$

Start from the FOC but make them analytic by squaring:

$$\begin{aligned}
\theta_{cl}\,(1-\alpha)^{3/2}\,(1+\alpha)^{3/2} &= \theta_g\left(1-\alpha-\alpha^2\right)(1-\alpha)+\alpha \\
\Longleftrightarrow \theta_{cl}^2\left(1-\alpha^2\right)^3 &= \left(\theta_g\left(1-\alpha-\alpha^2\right)(1-\alpha)+\alpha\right)^2
\end{aligned}$$

Denote $a = \theta_{cl}$ and $b = \theta_g$ (and it is natural to have $b < a$) and we seek the root of

$$
\begin{aligned}
f(\alpha) &= a^2\left(1-\alpha^2\right)^3 - \left(b\left(1-\alpha-\alpha^2\right)(1-\alpha)+\alpha\right)^2 \\
&= a^2 - b^2 + 2b\,(2b-1)\,\alpha - \left(3a^2 + (1-2b)^2\right)\alpha^2 - 2b^2\alpha^3 + \left(3a^2 - 2b\,(1-2b)\right)\alpha^4 - \left(a^2+b^2\right)\alpha^6
\end{aligned}
$$

Notice that f, as a polynomial, is analytic around any α. Use Lagrange around $\alpha_0 = 0$:

$$
\begin{aligned}
f(0) &= a^2 - b^2 \\
f'(\alpha) &= \frac{d}{d\alpha}\left[a^2\left(1-\alpha^2\right)^3 - \left(b\left(1-\alpha-\alpha^2\right)(1-\alpha)+\alpha\right)^2\right] \\
&= -6a^2\alpha\left(1-\alpha^2\right)^2 - 2\left(\alpha + b\,(1-\alpha)\left(1-\alpha^2-\alpha\right)\right)\left(b\,(1-\alpha)\,(-2\alpha-1) - b\left(1-\alpha^2-\alpha\right)+1\right) \\
&= -6a^2\alpha\left(1-\alpha^2\right)^2 - 2\left(\alpha + b\,(1-\alpha)\left(1-\alpha^2-\alpha\right)\right)\left(b\,(1-\alpha)\,(-2\alpha-1) - b\left(1-\alpha^2-\alpha\right)+1\right) \\
&= -2\left[b - 2b^2 + \alpha\left(3a^2 - 4b + 4b^2 + 1\right) + 3b^2\alpha^2 + \alpha^3\left(4b - 6a^2 - 8b^2\right) + \alpha^5\left(3a^2 + 3b^2\right)\right] \\
f'(0) &= -2b\,(1-2b) > 0
\end{aligned}
$$

If $b > 0$ and $b \neq \frac{1}{2}$, the Lagrange inversion series applies at $x_0 = 0$ and $f(x_0) = a^2 - b^2$:

$$
\begin{aligned}
f^{-1}(z) &= z_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z - z_0}{f(z) - f(z_0)}\right)^n\right]\Bigg|_{z=z_0} (z - f(z_0))^n \\
\alpha^* &= f^{-1}(0) = \sum_{n=1}^{\infty}\left[\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{f(z)+b^2-a^2}\right)^n\right]\frac{(b^2-a^2)^n}{n!} = \sum_{n=1}^{\infty}\frac{a_n}{n!}(b^2-a^2)^n
\end{aligned}
$$

Convergence will probably require $\left|a^2 - b^2\right| < 1$

Consider:

$$
\begin{aligned}
F(z) &= \frac{z}{f(z)+b^2-a^2} \\
&= \left(2b\,(2b-1) - \left(3a^2 + (1-2b)^2\right)z - 2b^2z^2 + \left(3a^2 - 2b\,(1-2b)\right)z^3 - \left(a^2+b^2\right)z^5\right)^{-1} \\
&= g(z)^{-1}
\end{aligned}
$$

We will need:

$$
\begin{aligned}
g &= 2b\,(2b-1) - \left(3a^2 + (1-2b)^2\right)z - 2b^2z^2 + \left(3a^2 - 2b\,(1-2b)\right)z^3 - \left(a^2+b^2\right)z^5 \\
g^{(1)} &= -\left(3a^2 + (1-2b)^2\right) - 4b^2z + 3\left(3a^2 - 2b\,(1-2b)\right)z^2 - 5\left(a^2+b^2\right)z^4 \\
g^{(2)} &= -4b^2 + 6\left(3a^2 - 2b\,(1-2b)\right)z - 20\left(a^2+b^2\right)z^3 \\
g^{(3)} &= 6\left(3a^2 - 2b\,(1-2b)\right) - 60\left(a^2+b^2\right)z^2 \\
g^{(4)} &= -120\left(a^2+b^2\right)z \\
g^{(5)} &= -120\left(a^2+b^2\right) \\
g^{(k)} &= 0 \text{ for } k > 5
\end{aligned}
$$

Term $n=1$:

$$\begin{aligned} a_1 &= F(0) = \frac{1}{g(0)} = \frac{1}{2b\,(1-2b)} \\ \alpha_1 &= \frac{a_1}{1!}(b^2-a^2) = \frac{a^2-b^2}{2b\,(1-2b)} \end{aligned}$$

The natural condition $b \le a$ then requires $b < \frac{1}{2}$:

$$\begin{aligned} 0 &\le a_1 \Leftrightarrow \left(b \ge a \text{ and } b > \frac{1}{2} \text{ (no)}\right) \text{ or } \left(b \le a \text{ and } b < \frac{1}{2} \text{ (yes)}\right) \\ a_1 &\le 1 \Leftrightarrow \left(b \le a \text{ and } b < \frac{1}{2} \text{ (yes)}\right) \text{ and } a^2-b^2 \le 2b\,(1-2b) \\ &\Leftrightarrow b \le a \text{ and } b < \frac{1}{2} \text{ and } a^2 \le b\,(2-3b) \end{aligned}$$

(The feasible region is below an arc but above a straight line.)

Term $n=2$:

$$\begin{aligned} \frac{d}{dz}g(z)^{-2} &= -2g^{-3}g' \\ a_2 &= \lim_{z\to 0}\frac{d}{dz}g(z)^{-2} = 2\frac{\left(3a^2+(1-2b)^2\right)}{(2b\,(2b-1))^3} \end{aligned}$$

Term $n=3$:

$$\begin{aligned} \frac{d^2}{dz^2}g(z)^{-3} &= \frac{d}{dz}\left[-3g^{-4}g'\right] = (-3)\,(-4)\,g^{-5}g'^2 - 2g^{-4}g^{(2)} \\ a_3 &= 3\cdot 4\frac{\left(3a^2+(1-2b)^2\right)^2}{(2b\,(2b-1))^5} + 2\frac{4b^2}{(2b\,(2b-1))^4} \end{aligned}$$

Term $n=4$:

$$\begin{aligned} \frac{d^3}{dz^3}g(z)^{-4} &= \frac{d^2}{dz^2}\left[-4g^{-5}g'\right] \\ &= \frac{d}{dz}\left[4\cdot 5g^{-6}g'^2 - 4g^{-5}g^{(2)}\right] \\ &= -4\cdot 5\cdot 6g^{-7}g'^3 + 2\cdot 4\cdot 5g^{-6}g'g^{(2)} + 4\cdot 5g^{-6}g'g^{(2)} - 4g^{-5}g^{(3)} \\ &= -4\cdot 5\cdot 6g^{-7}g'^3 + 3\cdot 4\cdot 5g^{-6}g'g^{(2)} - 4g^{-5}g^{(3)} \\ a_4 &= 4\cdot 5\cdot 6\frac{\left(3a^2+(1-2b)^2\right)^3}{(2b\,(2b-1))^7} + 3\cdot 4\cdot 5\frac{\left(3a^2+(1-2b)^2\right)4b^2}{(2b\,(2b-1))^6} - 4\frac{6\left(3a^2-2b\,(1-2b)\right)}{(2b\,(2b-1))^5} \end{aligned}$$

So for we have for $\left|\theta_{cg}^2 - \theta_g^2\right| < 1$ and $\theta_g < \frac{1}{2}$

$$\begin{aligned}
\alpha^* &= \sum_{n=1}^{\infty} \frac{a_n}{n!} (\theta_g^2 - \theta_{cg}^2)^n \\
a_1 &= \frac{1}{2\theta_g (1-2\theta_g)} \\
a_2 &= 2\frac{\left(3\theta_{cg}^2 + (1-2\theta_g)^2\right)}{(2\theta_g (1-2\theta_g))^3} \\
a_3 &= 3\cdot 4\frac{\left(3\theta_{cg}^2 + (1-2\theta_g)^2\right)^2}{(2\theta_g (1-2\theta_g))^5} + 2\frac{4\theta_g^2}{(2\theta_g (1-2\theta_g))^4} \\
a_4 &= 4\cdot 5\cdot 6\frac{\left(3\theta_{cg}^2 + (1-2\theta_g)^2\right)^3}{(2\theta_g (1-2\theta_g))^7} + 3\cdot 4\cdot 5\frac{\left(3\theta_{cg}^2 + (1-2\theta_g)^2\right)4\theta_g^2}{(2\theta_g (1-2\theta_g))^6} - 4\frac{6\left(3\theta_{cg}^2 - 2\theta_g (1-2\theta_g)\right)}{(2\theta_g (1-2\theta_g))^5}
\end{aligned}$$

7.2 Capacitated Solutions for $L = 2$

7.2.1 Lagrange Series for $L = 2$: General Case: $\theta_g = b > 0$

Lagrange series for $L = 2$ with $b > 0$ around $\alpha = 1$:

$$\begin{aligned}
a2\alpha - b\frac{2\alpha (1-\alpha)^{1/2}}{(1+\alpha)^{1/2}} + b\frac{\alpha^2}{(1-\alpha^2)^{1/2}(1+\alpha)} & \\
+c\frac{2\alpha^3 (1-\alpha)^{1/2}}{(1-\alpha^2)^{1/2}(1+\alpha^2)^{1/2}(1+\alpha)^{1/2}} + c\frac{(1+\alpha^2)^{1/2}}{(1+\alpha)} &= \frac{\alpha}{(1-\alpha^2)^{3/2}} \\
\Leftrightarrow a2\alpha\left(1-\alpha^2\right)^{3/2} - b\frac{2\alpha (1-\alpha)^{1/2}}{(1+\alpha)^{1/2}}\left(1-\alpha^2\right)^{3/2} + b\frac{\alpha^2\left(1-\alpha^2\right)}{(1+\alpha)} & \\
+c\frac{2\alpha^3 (1-\alpha)^{1/2}\left(1-\alpha^2\right)}{(1+\alpha^2)^{1/2}(1+\alpha)^{1/2}} + c\frac{\left(1+\alpha^2\right)^{1/2}\left(1-\alpha^2\right)^{3/2}}{(1+\alpha)} &= \alpha
\end{aligned}$$

Set $x = 1-\alpha^2$, then $\alpha^2 = (1-x)$ and $\alpha = (1-x)^{1/2}$

$$\begin{aligned}
a2(1-x)^{1/2}x^{3/2} - b\frac{2(1-x)^{1/2}\left(1-(1-x)^{1/2}\right)^{1/2}}{\left(1+(1-x)^{1/2}\right)^{1/2}}x^{3/2} + b\frac{(1-x)x}{\left(1+(1-x)^{1/2}\right)} & \\
+c\frac{2(1-x)^{3/2}\left(1-(1-x)^{1/2}\right)^{1/2}x}{(1+(1-x))^{1/2}\left(1+(1-x)^{1/2}\right)^{1/2}} + c\frac{(1+(1-x))^{1/2}x^{3/2}}{\left(1+(1-x)^{1/2}\right)} &= (1-x)^{1/2}
\end{aligned}$$

and dividing by $(1-x)^{1/2}$ yields:

$$
\begin{aligned}
f(x) = a2x^{3/2} - b\frac{2\left(1-(1-x)^{1/2}\right)^{1/2}}{\left(1+(1-x)^{1/2}\right)^{1/2}}x^{3/2} + b\frac{(1-x)^{1/2}\,x}{\left(1+(1-x)^{1/2}\right)}& \\
+c\frac{2\,(1-x)\left(1-(1-x)^{1/2}\right)^{1/2}x}{(1+(1-x))^{1/2}\left(1+(1-x)^{1/2}\right)^{1/2}} + c\frac{(1+(1-x))^{1/2}\,x^{3/2}}{(1-x)^{1/2}\left(1+(1-x)^{1/2}\right)} &= 1
\end{aligned}
$$

To go further with $z_0 = 0$ and $f(z_0) = 0$, the function must be analytic around z_0 and the above is not due to $x^{3/2}$. To make it analytic, we need to square terms, but smartly by collecting all terms in $x^{3/2}$: Note that

$$
\begin{aligned}
(1+x)^{1/2} &= \sum_{k=0}^{\infty}\begin{pmatrix}1/2\\ k\end{pmatrix}x^k \text{ where } \begin{pmatrix}r\\ k\end{pmatrix} = \frac{r\,(r-1)\cdots(r-k+1)}{k!} \\
&= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\left(1-(1-x)^{1/2}\right)^{1/2} &= \left(1-\left(1-\frac{1}{2}x-\frac{1}{2!2^2}x^2-\frac{3}{3!2^3}x^3-\cdots\right)\right)^{1/2} \\
&= \left(\frac{1}{2}x+\frac{1}{2!2^2}x^2+\frac{3}{3!2^3}x^3+\cdots\right)^{1/2} \\
&= \left(\frac{1}{2}+\frac{1}{2!2^2}x+\frac{3}{3!2^3}x^2+\cdots\right)^{1/2}x^{1/2} \\
&= h(x)x^{1/2}
\end{aligned}
$$

where $h(x)$ is analytic around $x = 0$ with $h(0) = \frac{1}{2^{1/2}}$

$$
\begin{aligned}
&\left[2a + c\frac{(1+(1-x))^{1/2}}{(1-x)^{1/2}\left(1+(1-x)^{1/2}\right)} + c\frac{2\,(1-x)\,h(x)}{(1+(1-x))^{1/2}\left(1+(1-x)^{1/2}\right)^{1/2}}\right]x^{3/2} \\
&= 1 - b\frac{(1-x)^{1/2}}{\left(1+(1-x)^{1/2}\right)}x + b\frac{2h(x)}{\left(1+(1-x)^{1/2}\right)^{1/2}}x^2
\end{aligned}
$$

Squaring both sides yields

$$
\begin{aligned}
f(x) &= \left[2a + c\frac{(2-x)^{1/2}}{(1-x)^{1/2}\left(1+(1-x)^{1/2}\right)} + c\frac{2(1-x)h(x)}{(2-x)^{1/2}\left(1+(1-x)^{1/2}\right)^{1/2}}\right]^2 x^3 && (53)\\
&\quad - \left[1 - b\frac{(1-x)^{1/2}}{\left(1+(1-x)^{1/2}\right)}x + b\frac{2h(x)}{\left(1+(1-x)^{1/2}\right)^{1/2}}x^2\right]^2 \\
f(x) &= Ax^3 - \left[1 - bB_1x + bB_2x^2\right]^2 && (54)
\end{aligned}
$$

which is analytic at $x = 0$ and we seek $f^{-1}(0)$ by expanding around $z_0 = 0$ where $f(0) = -1$ and

$$
\begin{aligned}
f'(x) &= 3x^2A + x^3A' - 2\left[1 - bB_1x + bB_2x^2\right]\left[-bB_1 - bB_1'x + 2bB_2x + bB_2'x^2\right]\\
f'(0) &= -2\,[1]\left[-b\frac{1}{2}\right] = b > 0
\end{aligned}
$$

$$
\begin{aligned}
f^{(2)}(x) &= 6xA + 6x^2A' + x^3A^{''} - 2\left[-bB_1 - bB_1^{'}x + 2bB_2x + bB_2^{'}x^2\right]\left[-bB_1 - bB_1'x + 2bB_2x + bB_2'x^2\right]\\
&\quad -2\left[1 - bB_1x + bB_2x^2\right]\left[-bB_1^{'} - bB_1^{'} - bB_1^{''}x + 2bB_2 + 2bB_2^{'}x + 2bB_2'x + bB_2^{''}x^2\right]\\
&= 6xA + 6x^2A' + x^3A^{''} - 2b^2\left[-B_1 - B_1^{'}x + 2B_2x + B_2^{'}x^2\right]\left[-B_1 - B_1'x + 2B_2x + B_2'x^2\right]\\
&\quad -2b\left[1 - bB_1x + bB_2x^2\right]\left[-2B_1^{'} - B_1^{''}x + 2B_2 + 4B_2^{'}x + B_2^{''}x^2\right]\\
f^{(2)}(0) &= -2b^2\left[-B_1(0)\right]\left[-B_1(0)\right] - 2b\,[1]\left[-2B_1^{'} + 2B_2\right]\\
&= -\frac{1}{2}b^2 - 4b\left[-(-\frac{1}{8}) + \frac{2}{2}\right] = -\frac{1}{2}b^2 - b\frac{9}{2} = -\frac{b\,(b+9)}{2}
\end{aligned}
$$

$$
\begin{aligned}
f^{(3)}(x) &= 6A + 6xA' + 12xA' + 6x^2A'' + 3x^2A^{''} + x^3A^{(3)}\\
&\quad -2b^2\left[-B_1^{'} - B_1^{'} - B_1^{''}x + 2B_2 + 2B_2^{'}x + 2B_2^{'}x + B_2^{''}x^2\right]\left[-B_1 - B_1'x + 2B_2x + B_2'x^2\right]\\
&\quad -2b^2\left[-B_1 - B_1^{'}x + 2B_2x + B_2^{'}x^2\right]\left[-B_1^{'} - B_1^{'} - B_1^{'}{}'x + 2B_2 + 2B_2'x + 2B_2'x + B_2^{'}{}'x^2\right]\\
&\quad -2b\left[-bB_1 - bB_1'x + 2bB_2x + bB_2'x^2\right]\left[-2B_1^{'} - B_1^{''}x + 2B_2 + 4B_2^{'}x + B_2^{''}x^2\right]\\
&\quad -2b\left[1 - bB_1x + bB_2x^2\right]\left[-2B_1^{''} - B_1^{''} - B_1^{(3)}x + 2B_2' + 4B_2^{'} + 4B_2^{''}x + 2B_2^{''}x + B_2^{(3)}x^2\right]\\
&= 6A + 20xA' + 9x^2A'' + x^3A^{(3)}\\
&\quad -2b^2\left[-2B_1' - B_1^{''}x + 2B_2 + 4B_2^{'}x + B_2^{''}x^2\right]\left[-B_1 - B_1'x + 2B_2x + B_2'x^2\right]\\
&\quad -2b^2\left[-B_1 - B_1^{'}x + 2B_2x + B_2^{'}x^2\right]\left[-2B_1' - B_1^{'}{}'x + 2B_2 + 4B_2'x + B_2^{'}{}'x^2\right]\\
&\quad -2b\left[-bB_1 - bB_1'x + 2bB_2x + bB_2'x^2\right]\left[-2B_1^{'} - B_1^{''}x + 2B_2 + 4B_2^{'}x + B_2^{''}x^2\right]\\
&\quad -2b\left[1 - bB_1x + bB_2x^2\right]\left[-3B_1^{''} - B_1^{(3)}x + 6B_2' + 6B_2^{''}x + B_2^{(3)}x^2\right]
\end{aligned}
$$

$$
\begin{aligned}
f^{(3)}(0) &= 6A(0) - 2b^2\left[-2B_1' + 2B_2\right]\left[-B_1\right] - 2b^2\left[-B_1\right]\left[-2B_1' + 2B_2\right] \\
&\quad -2b\left[-bB_1\right]\left[-2B_1^{'} + 2B_2\right] - 2b\left[1\right]\left[-3B_1^{''} + 6B_2'\right] \\
&= 6A(0) + 12b^2\left[-B_1' + B_2\right]\left[B_1\right] - 2b\left[-3B_1^{''} + 6B_2'\right] \\
&= 6\left[2a + c\frac{(2)^{1/2}}{(2)} + c\frac{2\frac{1}{2^{1/2}}}{2}\right]^2 + 12\left(\frac{1}{8} + 1\right)\frac{1}{2}b^2 - 2\left[\frac{3}{8} + \frac{6}{4}\right]b \\
&= 6\left(2a + c\sqrt{2}\right)^2 + \frac{27}{4}b^2 - \frac{15}{4}b
\end{aligned}
$$

And

$$
\begin{aligned}
B_1(x) &= \frac{(1-x)^{1/2}}{1 + (1-x)^{1/2}} = 1 - \frac{1}{1 + (1-x)^{1/2}} \to B_1(0) = \frac{1}{2} \\
B_1'(x) &= -\frac{d}{dx}\frac{1}{1 + (1-x)^{1/2}} = -\frac{1}{2\sqrt{1-x}\left(\sqrt{1-x} + 1\right)^2} \to B_1'(0) = -\frac{1}{2 \cdot 2^2} = -\frac{1}{8}
\end{aligned}
$$

$$
\begin{aligned}
B_1^{''}(x) &= -\frac{d}{dx}\frac{1}{2\sqrt{1-x}\left(\sqrt{1-x} + 1\right)^2} \\
&= \frac{1}{2\left(x-1\right)\left(\sqrt{1-x} + 1\right)^3} - \frac{1}{4\left(\sqrt{1-x} + 1\right)^2\left(\sqrt{1-x} - x\sqrt{1-x}\right)} \\
&\to \quad B_1^{''}(0) = \frac{1}{2\left(-1\right)\left(2\right)^3} - \frac{1}{4\left(2\right)^2} = -\frac{1}{8}
\end{aligned}
$$

$$
\begin{aligned}
B_2(x) &= \frac{2h(x)}{\left(1 + (1-x)^{1/2}\right)^{1/2}} \to B_2(0) = \frac{2\frac{1}{2^{1/2}}}{2^{1/2}} = 1 \\
&= \frac{2\left(1 - (1-x)^{1/2}\right)^{1/2}}{x^{1/2}\left(1 + (1-x)^{1/2}\right)^{1/2}} = \frac{2\left(1 - (1-x)^{1/2}\right)}{x^{1/2}\left(1 + (1-x)^{1/2}\right)^{1/2}\left(1 - (1-x)^{1/2}\right)^{1/2}} \\
&= \frac{2\left(1 - (1-x)^{1/2}\right)}{x^{1/2}\left(1 - (1-x)\right)^{1/2}} = \frac{2\left(1 - (1-x)^{1/2}\right)}{x} = \frac{2\left(\frac{1}{2}x + \frac{1}{2!2^2}x^2 + \frac{3}{3!2^3}x^3 + \cdots\right)}{x} \\
&= 1 + \frac{1}{2!2}x + \frac{3}{3!2^2}x^2 + \cdots \\
B_2^{'}(x) &= \frac{1}{2!2} + \frac{3}{3!2}x + \cdots \to B_2'(0) = \frac{1}{4}
\end{aligned}
$$

Given that f is analytic at $x = 0$ with $f'(0) > 0$ Lagrange applies as:

$$
\begin{aligned}
f^{-1}(z) &= z_0 + \sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z - z_0}{f(z) - f(z_0)}\right)^n\right]\Bigg|_{z=z_0}\left(z - f(z_0)\right)^n \\
x^* &= f^{-1}(0) = \sum_{n=1}^{\infty}\frac{1}{n!}\left[\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{f(z) + 1}\right)^n\right]
\end{aligned}
$$

Term $n = 1$:

$$\lim_{z\to 0} \frac{z}{f(z)+1} = [\frac{0}{0}] \overset{\mathrm{H}}{=} \frac{1}{f'(0)} = \frac{1}{b}$$

Term $n = 2$:

$$\frac{d}{dz}\left(\frac{z}{f+1}\right)^2 = 2\frac{z}{(f+1)}\frac{d}{dz}\frac{z}{(f+1)} = 2\frac{z}{(f+1)}\left(\frac{1}{(f+1)} - \frac{zf'}{(f+1)^2}\right) = \frac{2z\left(f+1-zf'\right)}{(f+1)^3}$$

$$\begin{aligned}
&\lim_{z\to 0}\frac{2z\left(f+1-zf'\right)}{(f+1)^3} = [\frac{0}{0}] \\
&\overset{\mathrm{H}}{=} \frac{2\left(f+1-zf'\right)+2z\left(f'-f'-zf''\right)}{3\left(f+1\right)^2 f'} = \frac{2\left(f+1\right)-2z\left(f'+zf''\right)}{3\left(f+1\right)^2 f'} = [\frac{0}{0}] \\
&\overset{\mathrm{H}}{=} \frac{2f'-2\left(f'+zf''\right)-2z\left(f''+f''+zf^{(3)}\right)}{6\left(f+1\right)f'^2+3\left(f+1\right)^2 f''} = \frac{-2z\left(3f''+zf^{(3)}\right)}{3\left(f+1\right)\left[2f'^2+\left(f+1\right)f''\right]} \\
= \quad & -\frac{2}{3}\frac{z\left[3f''+zf^{(3)}\right]}{\left(f+1\right)\left[2f'^2+\left(f+1\right)f''\right]} = [\frac{0}{0}] \\
&\overset{\mathrm{H}}{=} -\frac{2}{3}\frac{\left[3f''+zf^{(3)}\right]+z\left[3f''+zf^{(3)}\right]'}{f'\left[2f'^2+\left(f+1\right)f''\right]+\left(f+1\right)\left[2f'^2+\left(f+1\right)f''\right]'} \\
= \quad & -\frac{f^{(2)}(0)}{(f^{(1)}(0))^3} = -\frac{-\frac{b(b+9)}{2}}{b^3} = \frac{(b+9)}{2b^2}
\end{aligned}$$

Term $n = 3$: following the above:

$$\begin{aligned}
a_3(0) &= \frac{24f''^2(0)-8f'(0)f^{(3)}(0)}{5!(f'(0))^5} \\
&= \frac{24\left(-\frac{b(b+9)}{2}\right)^2-8b\left(6\left(aL+c\sqrt{2}\right)^2+\frac{27}{4}b^2-\frac{15}{4}b\right)}{5!b^5} \\
&= \frac{b\left(b+9\right)^2-\left(8\left(2a+c\sqrt{2}\right)^2+9b^2-5b\right)}{5\cdot 4b^4} \\
&= \frac{-2\left(2a+c\sqrt{2}\right)^2}{5b^4}+\frac{b^2+9b+86}{5\cdot 4b^3}
\end{aligned}$$

Thus, up to three terms for $L = 2$ where $a = \theta_c$ (cost delta), $b = \theta_g$ (global capacity), and $c = \theta_l$ (local capacity):

$$\begin{aligned}
x_1^* &= \frac{1}{\frac{1}{2}b} \Rightarrow \alpha^* = \sqrt{1 - \left(\frac{1}{2}\theta_g\right)^{-1}} \text{ [cannot be further expanded b/c not analytic]} \\
x_1^* &= \frac{1}{b} \Rightarrow \alpha^* = \sqrt{1 - \theta_g^{-1}} \\
x_2^* &= \frac{1}{b} + \frac{1}{2}\frac{(b+9)}{2b^2} = \frac{1}{b}\left(1 + \frac{b+9}{4b}\right) \\
x_3^* &= \frac{1}{b} + \frac{1}{2!}\frac{(b+9)}{2b^2} + \frac{1}{3!}\frac{b^2+9b+86}{5!b^3} - \frac{\left(a2 + c\sqrt{2}\right)^2}{3!15b^4}
\end{aligned}$$

Note that global capacity $b = \theta_g = 0$ is fundamentally different from $b > 0$ (meaning that it requires a different Lagrange series and the results do not apply for $b \to 0$, and hence we cannot recover our previous uncapacitated result from this). It seems that b is a dominant factor and substitutes for $\left(2a + c\sqrt{2}\right)^{1/2}$. The formulae above require large b and small a and c.

7.2.2 Lagrange Series for $L = 2$: Local Capacity Only: $\theta_g = b = 0$

Start from the optimality equation (53) with $\theta_g = b = 0$ and $L = 2$:

$$\begin{aligned}
f(x) &= \left[2a + c\frac{(2-x)^{1/2}}{(1-x)^{1/2}\left(1 + (1-x)^{1/2}\right)} + c\frac{2(1-x)\,h(x)}{(2-x)^{1/2}\left(1+(1-x)^{1/2}\right)^{1/2}}\right]^2 x^3 - 1 \\
&= A(x)x^3 - 1
\end{aligned}$$

This can be simplified as:

$$A(x) = \left[2a + c\frac{(2-x)^{1/2}}{1 - x + (1-x)^{1/2}} + c\frac{2(1-x)\,h(x)\left(1+(1-x)^{1/2}\right)^{1/2}}{(2-x)^{1/2}\left(1+(1-x)^{1/2}\right)}\right]^2$$

and

$$h(x) = \frac{\left(1 - (1-x)^{1/2}\right)^{1/2}}{x^{1/2}} = \left(\frac{1}{2} + \frac{1}{2!2^2}x + \frac{3}{3!2^3}x^2 + \cdots\right)^{1/2}$$

so

$$A(x) = \left[2a + c\frac{(2-x)^{1/2}}{1 - x + (1-x)^{1/2}} + c\frac{2(1-x)}{(2-x)^{1/2}\left(1+(1-x)^{1/2}\right)}\right]^2$$

which is analytic at $x = 0$ and we seek $f^{-1}(0)$ by expanding around $z_0 = 0$ where $f(0) = -1$ and

$$
\begin{aligned}
f'(x) &= 3x^2 A + x^3 A' \Rightarrow f'(0) = b = 0 \\
f^{(2)}(x) &= 6xA + 6x^2 A' + x^3 A'' \Rightarrow f^{(2)}(0) = 0 \\
f^{(3)}(x) &= 6A + x[] \Rightarrow f^{(3)}(0) = 6A(0) = 6\left(2a + c\sqrt{2}\right)^2 > 0
\end{aligned}
$$

Given that f is analytic at $x = 0$ with $f(0) = -1$ and $f'(0) = f^{(2)}(0) = 0$ and $f^{(3)}(0) > 0$ Lagrange applies in modified form as:

$$
\begin{aligned}
f^{-1}(z) &= z_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} \left(\frac{z - z_0}{[f(z) - f(z_0)]_s^{1/3}} \right)^n \right] \Bigg|_{z=z_0} (z - f(z_0))^n \\
x^* &= f^{-1}(0) = \sum_{n=1}^{\infty} \frac{1}{n!} \left[\lim_{z\to 0} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z}{(f(z)+1)^{1/3}} \right)^n \right]
\end{aligned}
$$

Consider

$$
\begin{aligned}
F(x) &= \frac{x}{(f(x)+1)^{1/3}} = \frac{x}{A^{1/3} x} \\
&= A^{-1/3} = \left[2a + c\frac{(2-x)^{1/2}}{1 - x + (1-x)^{1/2}} + c\frac{2(1-x)}{(2-x)^{1/2}\left(1 + (1-x)^{1/2}\right)} \right]^{-2/3} = B^{-2/3}
\end{aligned}
$$

Term $n = 1$:

$$
\begin{aligned}
a_1 &= \lim_{z\to 0} F(x) = A(0)^{-1/3} = B(0)^{-2/3} = \left(2a + c\sqrt{2}\right)^{-2/3} \\
\alpha_1^* &= \sqrt{1 - \left(2\theta_c + \sqrt{2}\theta_l\right)^{-2/3}},
\end{aligned}
$$

which requires $2\theta_c + \sqrt{2}\theta_l > 1$.

Term $n = 2$:

$$
\frac{d}{dx} B^{-4/3} = -\frac{4}{3} B^{-7/3} B'
$$

$$
\begin{aligned}
B' &= c\left(\begin{array}{c} \frac{\sqrt{2-x}}{\left(\sqrt{1-x}-x+1\right)^2}\left(\frac{1}{2\sqrt{1-x}} + 1\right) - \frac{1}{2\sqrt{2-x}\left(\sqrt{1-x}-x+1\right)} \\ + \frac{\sqrt{1-x}}{\sqrt{2-x}\left(\sqrt{1-x}+1\right)^2} - \frac{2}{\sqrt{2-x}\left(\sqrt{1-x}+1\right)} + \frac{1-x}{\left(\sqrt{1-x}+1\right)\left(2\sqrt{2-x} - x\sqrt{2-x}\right)} \end{array} \right) \\
&= c\left(\begin{array}{c} \frac{\sqrt{2-x}}{\left(\sqrt{1-x}-x+1\right)^2}\left(\frac{1}{2\sqrt{1-x}} + 1\right) - \frac{1}{2\sqrt{2-x}\left(\sqrt{1-x}-x+1\right)} \\ - \frac{\sqrt{1-x}+2}{\sqrt{2-x}\left(\sqrt{1-x}+1\right)^2} + \frac{1-x}{\left(\sqrt{1-x}+1\right)(2-x)^{3/2}} \end{array} \right) \\
&= c\left(\frac{\sqrt{2-x}}{\left(x - \sqrt{1-x} - 1\right)^2}\left(\frac{1}{2\sqrt{1-x}} + 1\right) + \frac{1}{2\sqrt{2-x}\left(x - \sqrt{1-x} - 1\right)} + \frac{x - \sqrt{1-x} - 3}{\left(\sqrt{1-x}+1\right)^2 (2-x)^{3/2}} \right)
\end{aligned}
$$

$$
\begin{aligned}
B'(0) &= c\left(\frac{\sqrt{2}}{2^2}\left(\frac{1}{2}+1\right)-\frac{1}{2\sqrt{2}\,(1+1)}+\frac{1}{\sqrt{2}2^2}-\frac{2}{\sqrt{2}\,(2)}+\frac{1}{(1+1)\left(2\sqrt{2}\right)}\right)=0 \\
a_2 &= \frac{1}{2}\left(-\frac{4}{3}B^{-7/3}(0)B'(0)\right)=-\frac{2}{3}\left(2a+c\sqrt{2}\right)^{-7/3}c\cdot 0=0
\end{aligned}
$$

Thus:

$$
\alpha_1^*=\alpha_2^*=\sqrt{1-\left(2\theta_c+\sqrt{2}\theta_l\right)^{-2/3}},
$$

in other words, the square root formula is also a second order approximation for $L=2$.

Term $n=3$:

$$
\frac{d^2}{dx^2}B^{-6/3}=-2\frac{d}{dx}B^{-9/3}B'=-2\left(-3B^{-4}B'^2+B^{-3}B''\right)
$$

$$
\begin{aligned}
B'' &= c\frac{d}{dx}\left(\frac{\sqrt{2-x}}{\left(x-\sqrt{1-x}-1\right)^2}\left(\frac{1}{2\sqrt{1-x}}+1\right)+\frac{1}{2\sqrt{2-x}\left(x-\sqrt{1-x}-1\right)}+\frac{x-\sqrt{1-x}-3}{\left(\sqrt{1-x}+1\right)^2(2-x)^{3/2}}\right) \\
&= c\left(\begin{array}{c}
\frac{1}{4\left(x-\sqrt{1-x}-1\right)\left(2\sqrt{2-x}-x\sqrt{2-x}\right)}+\frac{1}{\sqrt{2-x}\left(\sqrt{1-x}-x+1\right)^2}\left(-\frac{1}{2\sqrt{1-x}}-1\right) \\
-2\frac{\sqrt{2-x}}{\left(x-\sqrt{1-x}-1\right)^3}\left(\frac{1}{2\sqrt{1-x}}+1\right)^2+\frac{1}{4}\frac{\sqrt{2-x}}{\left(\sqrt{1-x}-x+1\right)^2\left(\sqrt{1-x}-x\sqrt{1-x}\right)} \\
+\frac{1}{\left(\sqrt{1-x}+1\right)^2}\frac{\frac{1}{2\sqrt{1-x}}+1}{2\sqrt{2-x}-x\sqrt{2-x}}+\frac{1}{\sqrt{1-x}\left(\sqrt{1-x}+1\right)^3}\frac{x-\sqrt{1-x}-3}{2\sqrt{2-x}-x\sqrt{2-x}}+\frac{3}{2\left(\sqrt{1-x}+1\right)^2}\frac{x-\sqrt{1-x}-3}{4\sqrt{2-x}-4x\sqrt{2-x}+x^2\sqrt{2-x}}
\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
B''(0) &= c\sqrt{2}\frac{3}{16} \\
a_3 &= \frac{1}{3!}\left(-2\left(-3B^{-4}B'^2+B^{-3}B''\right)|_0\right)=\frac{-2}{3!}\left(2a+c\sqrt{2}\right)^{-3}c\sqrt{2}\frac{3}{16}=-\left(2a+c\sqrt{2}\right)^{-3}c\sqrt{2}\frac{1}{16}
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\alpha_3^* &= \sqrt{1-\left(2\theta_c+\sqrt{2}\theta_l\right)^{-2/3}\left(1-\left(2\theta_c+\sqrt{2}\theta_l\right)^{-7/3}\theta_l\sqrt{2}\frac{1}{16}\right)} \\
&= \sqrt{1-\left(2\theta_c+\sqrt{2}\theta_l\right)^{-2/3}+\theta_l\sqrt{2}\frac{1}{16}\left(2\theta_c+\sqrt{2}\theta_l\right)^{-9/3}}
\end{aligned}
$$

Term $n=4$:

$$
\begin{aligned}
\frac{d^3}{dx^3}B^{-8/3} &= -\frac{8}{3}\frac{d^2}{dx^2}B^{-11/3}B'=-\frac{8}{3}\frac{d}{dx}\left(\frac{-11}{3}B^{-14/3}B'+B^{-11/3}B''\right) \\
&= \frac{-8}{3}\left(\frac{11}{3}\frac{14}{3}B^{-17/3}B'+\frac{-11}{3}2B^{-14/3}B''+B^{-11/3}B^{(3)}\right)
\end{aligned}
$$

$$
\begin{aligned}
B^{(3)} &= c\frac{d}{dx}\left(\begin{array}{c}
\frac{1}{4\left(x-\sqrt{1-x}-1\right)\left(2\sqrt{2-x}-x\sqrt{2-x}\right)}+\frac{1}{\sqrt{2-x}\left(\sqrt{1-x}-x+1\right)^2}\left(-\frac{1}{2\sqrt{1-x}}-1\right)\\
-2\frac{\sqrt{2-x}}{\left(x-\sqrt{1-x}-1\right)^3}\left(\frac{1}{2\sqrt{1-x}}+1\right)^2+\frac{1}{4}\frac{\sqrt{2-x}}{\left(\sqrt{1-x}-x+1\right)^2\left(\sqrt{1-x}-x\sqrt{1-x}\right)}+\frac{1}{\left(\sqrt{1-x}+1\right)^2}\frac{\frac{1}{2\sqrt{1-x}}+1}{2\sqrt{2-x}-x\sqrt{2-x}}\\
+\frac{1}{\sqrt{1-x}\left(\sqrt{1-x}+1\right)^3}\frac{x-\sqrt{1-x}-3}{2\sqrt{2-x}-x\sqrt{2-x}}+\frac{3}{2\left(\sqrt{1-x}+1\right)^2}\frac{x-\sqrt{1-x}-3}{4\sqrt{2-x}-4x\sqrt{2-x}+x^2\sqrt{2-x}}
\end{array}\right)\\
B^{(3)}(0) &= c\sqrt{2}\frac{24}{32}=c\sqrt{2}\frac{3}{4}\\
a_4 &= \frac{1}{4!}\frac{-8}{3}\left(\frac{11}{3}\frac{14}{3}B^{-17/3}B'+\frac{-11}{3}2B^{-14/3}B''+B^{-11/3}B^{(3)}\right)_{x=0}\\
&= \frac{1}{3}\frac{-1}{3}\left(\frac{-11}{3}2B^{-1}\frac{3}{16}+\frac{3}{4}\right)B^{-11/3}c\sqrt{2}\\
&= \frac{-1}{3}\left(\frac{-11}{3}B^{-1}\frac{1}{8}+\frac{1}{4}\right)B^{-11/3}c\sqrt{2}
\end{aligned}
$$

Finally:

$$
\alpha_4^* = \sqrt{1-\left(2\theta_c+\sqrt{2}\theta_l\right)^{-2/3}+\theta_l\sqrt{2}\left(\begin{array}{c}\frac{1}{16}\left(2\theta_c+\sqrt{2}\theta_l\right)^{-9/3}-\frac{1}{12}\left(2\theta_c+\sqrt{2}\theta_l\right)^{-11/3}\\ +\frac{11}{72}\left(2\theta_c+\sqrt{2}\theta_l\right)^{-14/3}+O\left(2\theta_c+\sqrt{2}\theta_l\right)^{-17/3}\end{array}\right)}
$$

Notice that, as it should be, the formula is exact for $\theta_l = 0$. The associated scaled cost is, setting ${\alpha^*}^2 = 1-x^*$

$$
{\alpha^*}^2 = 1-x^* \simeq 1-\left(2\theta_c+\sqrt{2}\theta_l\right)^{-2/3}+O(\theta_l\left(2\theta_c+\sqrt{2}\theta_l\right)^{-2/3})
$$

$$
\begin{aligned}
\widehat{C}(\alpha;\theta_c,0,\theta_l) &= -\theta_c\alpha^2+\theta_l\sqrt{\frac{1-\alpha}{1+\alpha}\left(1-\alpha^4\right)}+\frac{1}{\sqrt{1-\alpha^2}}\\
\sqrt{\frac{1-\alpha}{1+\alpha}\left(1-\alpha^4\right)} &= \sqrt{\frac{1-\alpha^2}{\left(1+\alpha\right)^2}\left(1-\alpha^2\right)\left(1+\alpha^2\right)}=\frac{1-\alpha^2}{1+\alpha}\sqrt{1+\alpha^2}=\left(1-\alpha\right)\sqrt{1+\alpha^2}\\
\widehat{C}(\alpha;\theta_c,0,\theta_l) &= -\theta_c\alpha^2+\theta_l\left(1-\alpha\right)\sqrt{1+\alpha^2}+\frac{1}{\sqrt{1-\alpha^2}}\\
\widehat{C}(\alpha^*;\theta_c,0,\theta_l) &= -\theta_c\left(1-x^*\right)+\theta_l\left(1-\sqrt{1-x^*}\right)\sqrt{2-x^*}+\frac{1}{\sqrt{x^*}}.
\end{aligned}
$$

Recall that, for any real β and $-1<x<1$:

$$
\begin{aligned}
(1+x)^\beta &= \sum_{k=0}^{\infty}\left(\begin{array}{c}\beta\\ k\end{array}\right)x^k = 1+\beta x+\frac{\beta(\beta-1)}{2!}x^2+\cdots\\
(1-x)^{\frac{1}{2}} &= 1-\frac{1}{2}x-\frac{1}{8}x^2+O(x^3)
\end{aligned}
$$

so that

$$\begin{aligned}
& \widehat{C}(\alpha^*;\theta_c,0,\theta_l) \\
= \quad & -\theta_c+\theta_c x^*+\theta_l\left(1-\left(1-\frac{1}{2}x^*-\frac{1}{8}x^{*2}+O(x^{*3})\right)\right)\sqrt{2}\left(1-\frac{1}{4}x^*-\frac{1}{32}x^{*2}+O(x^{*3})\right)+\frac{1}{\sqrt{x^*}} \\
= \quad & -\theta_c+\theta_c x^*+\theta_l\left(\frac{1}{2}x^*+\frac{1}{8}x^{*2}+O(x^{*3})\right)\sqrt{2}\left(1-\frac{1}{4}x^*-\frac{1}{32}x^{*2}+O(x^{*3})\right)+\frac{1}{\sqrt{x^*}} \\
= \quad & -\theta_c+\frac{1}{2}\left(2\theta_c+\sqrt{2}\theta_l\right)x^*+\frac{1}{\sqrt{x^*}}-\theta_l O(x^{*3})
\end{aligned}$$

Substituting

$$x^*=\left(2\theta_c+\sqrt{2}\theta_l\right)^{-2/3}+O\left(\theta_l\left(2\theta_c+\sqrt{2}\theta_l\right)^{-3}\right)$$

we get

$$\begin{aligned}
\widehat{C}(\alpha^*;\theta_c,0,\theta_l) &= -\theta_c+\frac{1}{2}\left(2\theta_c+\sqrt{2}\theta_l\right)^{1/3}+\left(2\theta_1+\sqrt{2}\theta_l\right)^{1/3}+O\left(\theta_l\left(2\theta_1+\sqrt{2}\theta_l\right)^{-2}\right) \\
&= -\theta_c+\frac{3}{2}\left(2\theta_c+\sqrt{2}\theta_l\right)^{1/3}+O\left(\theta_l\left(2\theta_c+\sqrt{2}\theta_l\right)^{-2}\right)
\end{aligned}$$

Notice that, for $L=2$, the formula is exact for $\theta_l=0$. It also has higher accuracy because the term in x^{*2} falls out and thus this expression has smaller error!

7.2.3 For $L=2$ and $\theta_g=0$: square root is a lower bound

Simplify the FOC:

$$\widehat{MB}=\theta_c L\alpha^{L-1}-\theta_g\frac{\alpha^{L-1}\left(L\left(1-\alpha^2\right)-\alpha\right)}{(1-\alpha)^{1/2}(1+\alpha)^{3/2}}+\theta_l\frac{L\alpha^{2L-1}(1-\alpha)(1+\alpha)+\left(1-\alpha^{2L}\right)}{(1-\alpha^{2L})^{1/2}(1-\alpha)^{1/2}(1+\alpha)^{3/2}}$$

and define

$$g(\alpha;L)=\frac{1-\alpha^L}{1-\alpha}=\sum_{k=0}^{L-1}\alpha^k \tag{55}$$

then

$$\begin{aligned}
\widehat{MB}_3 &= \theta_l\frac{L\alpha^{2L-1}(1-\alpha)(1+\alpha)+\left(1-\alpha^L\right)\left(1+\alpha^L\right)}{\left(1-\alpha^L\right)^{1/2}\left(1+\alpha^L\right)^{1/2}(1-\alpha)^{1/2}(1+\alpha)^{3/2}} \\
&= \theta_l\frac{L\alpha^{2L-1}(1-\alpha)(1+\alpha)+(1-\alpha)g(\alpha)\left(1+\alpha^L\right)}{((1-\alpha)g(\alpha))^{1/2}\left(1+\alpha^L\right)^{1/2}(1-\alpha)^{1/2}(1+\alpha)^{3/2}} \\
&= \theta_l\frac{L\alpha^{2L-1}(1+\alpha)+g(\alpha)\left(1+\alpha^L\right)}{g(\alpha)^{1/2}\left(1+\alpha^L\right)^{1/2}(1+\alpha)^{3/2}}
\end{aligned}$$

For $L = 1 : g(\alpha) = 1$

$$\widehat{MB}_3 = \theta_l$$

For $L = 2 : g(\alpha) = 1 + \alpha$

$$\begin{aligned}
\widehat{MB}_3 &= \theta_l \frac{2\alpha^3 (1+\alpha) + (1+\alpha)\left(1+\alpha^2\right)}{(1+\alpha)^{1/2}(1+\alpha^2)^{1/2}(1+\alpha)^{3/2}} \\
&= \theta_l \frac{2\alpha^3 + \left(1+\alpha^2\right)}{(1+\alpha^2)^{1/2}(1+\alpha)} = \theta_l \frac{(\alpha+1)\left(2\alpha^2-\alpha+1\right)}{(1+\alpha^2)^{1/2}(1+\alpha)} = \theta_l \frac{\left(2\alpha^2-\alpha+1\right)}{(1+\alpha^2)^{1/2}} \\
f_3 &= \frac{\left(2\alpha^2-\alpha+1\right)}{(1+\alpha^2)^{1/2}} \\
f_3' &= \frac{\left(1+\alpha^2\right)^{1/2}(4\alpha-1) - \left(2\alpha^2-\alpha+1\right)\frac{1}{2}2\alpha\left(1+\alpha^2\right)^{-1/2}}{(1+\alpha^2)} \\
&= \frac{\left(1+\alpha^2\right)(4\alpha-1) - \left(2\alpha^2-\alpha+1\right)\alpha}{(1+\alpha^2)^{3/2}} = \frac{\left(2\alpha^3+3\alpha-1\right)}{(1+\alpha^2)^{3/2}} \\
f_3'' &= \frac{\left(1+\alpha^2\right)^{3/2}\left(6\alpha^2+3\right) - \left(2\alpha^3+3\alpha-1\right)\frac{3}{2}2\alpha\left(1+\alpha^2\right)^{1/2}}{(1+\alpha^2)^3} \\
&= \frac{\left(1+\alpha^2\right)\left(6\alpha^2+3\right) - \left(2\alpha^3+3\alpha-1\right)3\alpha}{(1+\alpha^2)^{5/2}} = \frac{3(\alpha+1)}{(1+\alpha^2)^{5/2}} > 0
\end{aligned}$$

Two Taylor approximations for $\widehat{MB}_3 = f_3$, one around $\alpha = 0$, the other around $\alpha = 1$:

$$\begin{aligned}
f_3 &= 1 - \alpha + \frac{3}{2}\alpha^2 + o(\alpha^2) \\
f_3 &= \sqrt{2} + \sqrt{2}(\alpha-1) + \frac{3\sqrt{2}}{2\cdot 4}(\alpha-1)^2 + o((\alpha-1)^2) \\
&= \sqrt{2}\alpha + o(\alpha-1) \\
&= \frac{\sqrt{2}}{8}\left(3+2\alpha+3\alpha^2\right) + o((\alpha-1)^2)
\end{aligned}$$

Given that f_3 is convex, the first order approximation is a lower bound (and so is the second order over the interval $[0,1]$):

$$\sqrt{2}\alpha < \frac{\sqrt{2}}{8}\left(3+2\alpha+3\alpha^2\right) < f_3,$$

as shown in green in Figure 22. In addition, we have the following bounds (in red in Figure 22):

$$\frac{1}{\sqrt{2}}\left(2\alpha^2-\alpha+1\right) < f_3 = \frac{\left(2\alpha^2-\alpha+1\right)}{(1+\alpha^2)^{1/2}} < \left(2\alpha^2-\alpha+1\right)$$

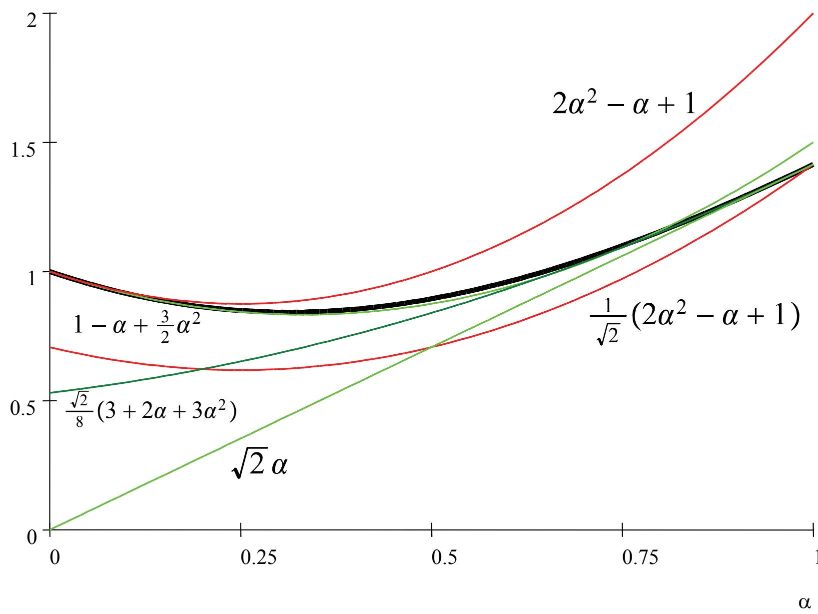

Figure 22: Bounds on the marginal local capacity benefit for $L = 2$.

Hence, for $L = 2$ with $\theta_g = 0$ we have

$$2\theta_c\alpha + \sqrt{2}\theta_l\alpha < \widehat{MB} = \left(2\theta_1 + \sqrt{2}\theta_l\right)\alpha + O((\alpha-1)^2) = \frac{\alpha}{(1-\alpha^2)^{3/2}}$$

So that

$$2\theta_c + \sqrt{2}\theta_l = \frac{1}{(1-\underline{\alpha}^2)^{3/2}} < 2\theta_c + \sqrt{2}\theta_l + O\left(\frac{(\alpha^*-1)^2}{\alpha^*}\right) = \frac{1}{(1-\alpha^{*2})^{3/2}}$$

or

$$\underline{\alpha} = \sqrt{1 - \left(2\theta_c + \sqrt{2}\theta_l\right)^{-2/3}} \leq \alpha^*$$

We also have (accurate for small α) and correct for $\alpha < 0.75$ (and even a little higher):

$$f_3 > 1 - \alpha$$

so

$$2\theta_c\alpha + \theta_l\left(1 - \alpha + \frac{3}{2}\alpha^2\right) < \widehat{MB} + o(\theta_l\alpha^3) < \widehat{MB} = \frac{\alpha}{(1-\alpha^2)^{3/2}}$$

Expand $\widehat{MC}$:

$$\begin{aligned}
\widehat{MC} &= f = \frac{\alpha}{(1-\alpha^2)^{3/2}} \\
f' &= \frac{\left(1-\alpha^2\right)^{3/2} - \alpha\frac{3}{2}(-2\alpha)\left(1-\alpha^2\right)^{1/2}}{(1-\alpha^2)^3} = \frac{\left(2\alpha^2+1\right)}{(1-\alpha^2)^{5/2}} \\
f'' &= \frac{\left(1-\alpha^2\right)^{5/2}(4\alpha) - \left(2\alpha^2+1\right)\frac{5}{2}(-2\alpha)\left(1-\alpha^2\right)^{3/2}}{(1-\alpha^2)^5} = \frac{3\alpha\left(2\alpha^2+3\right)}{(1-\alpha^2)^{7/2}} \\
f^{(3)} &= \frac{\left(1-\alpha^2\right)^{7/2} 3\left(6\alpha^2+3\right) - 3\alpha\left(2\alpha^2+3\right)\frac{7}{2}(-2\alpha)\left(1-\alpha^2\right)^{5/2}}{(1-\alpha^2)^7} = \frac{3\left(24\alpha^2+8\alpha^4+3\right)}{(1-\alpha^2)^{9/2}}
\end{aligned}$$

Hence linear Taylor is accurate up to second degree:

$$\widehat{MC} = \alpha + O(\alpha^3)$$

Solve first order, and we must thus have: $1 > 2\theta_c - \theta_l$ and $\theta_l < 1 - 2\theta_c + \theta_l$ or: $0 < 1 - 2\theta_c$ for any $\theta_l > 0$.

$$\begin{aligned}
& \widehat{MB} = 2\theta_c\alpha + \theta_l(1-\alpha) + O(\alpha^3) = \widehat{MC} = \alpha + O(\alpha^3) \\
\Leftrightarrow\quad & \alpha^* = \frac{\theta_l}{1-2\theta_c+\theta_l} + O\left(\left(\frac{\theta_l}{1-2\theta_c+\theta_l}\right)^3\right)
\end{aligned}$$

Second order solution would give:

$$\begin{aligned}
& 2\theta_c\alpha + \theta_l\left(1-\alpha+\frac{3}{2}\alpha^2\right) = \alpha \\
\Leftrightarrow\quad & 3\theta_3\alpha^2 - 2\left(1-2\theta_c+\theta_l\right)\alpha + 2\theta_3 = 0 \\
\Leftrightarrow\quad & \alpha = \frac{(1-2\theta_c+\theta_l) \pm \sqrt{(1-2\theta_c+\theta_l)^2 - 3\cdot 2\theta_l}}{3\theta_l}
\end{aligned}$$

7.3 Capacitated Solutions for $L > 2$

7.3.1 Lagrange Series for $L > 2$ for $\theta_g = 0$

$$\begin{aligned}
aL\alpha^{L-1} + c\frac{L\alpha^{2L-1}(1-\alpha)^{1/2}}{(1-\alpha^{2L})^{1/2}(1+\alpha)^{1/2}} + c\frac{\left(1-\alpha^{2L}\right)^{1/2}}{(1-\alpha)^{1/2}(1+\alpha)^{3/2}} &= \frac{\alpha}{(1-\alpha^2)^{3/2}} \\
aL\alpha^{L-1} + c\frac{L\alpha^{2L-1}\left(1-\alpha^2\right)^{1/2}}{(1-\alpha^{2L})^{1/2}(1+\alpha)} + c\frac{(1-\alpha)\left(1-\alpha^{2L}\right)^{1/2}}{(1-\alpha^2)^{3/2}} &= \frac{\alpha}{(1-\alpha^2)^{3/2}}
\end{aligned}$$

Set $x = 1 - \alpha^2$, then $\alpha^2 = (1-x)$ and $\alpha = (1-x)^{1/2}$ and multiplying by $x^{3/2}$

$$\begin{aligned} aL(1-x)^{(L-1)/2} + c\frac{L(1-x)^{(2L-1)/2}\left(1-(1-x)^{1/2}\right)^{1/2}}{\left(1-(1-x)^L\right)^{1/2}\left(1+(1-x)^{1/2}\right)^{1/2}} & \\ +c\frac{\left(1-(1-x)^{1/2}\right)\left(1-(1-x)^L\right)^{1/2}}{x^{3/2}} &= \frac{(1-x)^{1/2}}{x^{3/2}} \\ aLx^{3/2}(1-x)^{(L-1)/2} + cx^{3/2}\frac{L(1-x)^{(2L-1)/2}\left(1-(1-x)^{1/2}\right)^{1/2}}{\left(1-(1-x)^L\right)^{1/2}\left(1+(1-x)^{1/2}\right)^{1/2}} & \\ +c\left(1-(1-x)^{1/2}\right)\left(1-(1-x)^L\right)^{1/2} &= (1-x)^{1/2} \end{aligned}$$

Dividing by $(1-x)^{1/2}$

$$\begin{aligned} aLx^{3/2}(1-x)^{(L-2)/2} + cx^{3/2}\frac{L(1-x)^{(2L-1)/2}\left(1-(1-x)^{1/2}\right)^{1/2}}{(1-x)^{1/2}\left(1-(1-x)^L\right)^{1/2}\left(1+(1-x)^{1/2}\right)^{1/2}} & \\ +\frac{c\left(1-(1-x)^{1/2}\right)\left(1-(1-x)^L\right)^{1/2}}{(1-x)^{1/2}} &= 1 \end{aligned}$$

and multiply second term's numerator and denominator by $\left(1+(1-x)^{1/2}\right)^{1/2}$

$$\frac{L(1-x)^{(2L-1)/2}\left(1-(1-x)^{1/2}\right)^{1/2}}{(1-x)^{1/2}\left(1-(1-x)^L\right)^{1/2}\left(1+(1-x)^{1/2}\right)^{1/2}} = \frac{L(1-x)^{(2L-1)/2}x^{1/2}}{(1-x)^{1/2}\left(1-(1-x)^L\right)^{1/2}\left(1+(1-x)^{1/2}\right)}$$

where

$$\left(1-(1-x)^L\right)^{1/2} = \left(\sum_{k=1}^{L}\binom{L}{k}(-1)^{k+1}x^k\right)^{1/2} = x^{1/2}\left(\sum_{k=1}^{L}\binom{L}{k}(-x)^{k-1}\right)^{1/2}$$

and

$$\begin{aligned} \left(1-(1-x)^{1/2}\right)\left(1-(1-x)^L\right)^{1/2} &= \left(\frac{1}{2}x + \frac{1}{2!2^2}x^2 + \frac{3}{3!2^3}x^3 + \cdots\right)\left(\sum_{k=1}^{L}\binom{L}{k}(-1)^{k+1}x^k\right)^{1/2} \\ &= x\left(\frac{1}{2} + \frac{1}{2!2^2}x + \frac{3}{3!2^3}x^2 + \cdots\right)\left(Lx + ... + x^L\right)^{1/2} \\ &= x^{3/2}\left(\frac{1}{2} + \frac{1}{2!2^2}x + \frac{3}{3!2^3}x^2 + \cdots\right)\left(L + ... + x^{L-1}\right)^{1/2} \end{aligned}$$

are both analytic at $x=0$. Denote:

$$h_1(x) = \left(\frac{1}{2} + \frac{1}{2!2^2}x + \frac{3}{3!2^3}x^2 + \cdots\right)$$

Thus, squaring both sides we arrive at

$$\begin{aligned} f(x) &= \left[\begin{array}{c} aL\,(1-x)^{(L-2)/2} + c\frac{L(1-x)^{(2L-1)/2}x^{1/2}}{(1-x)^{1/2}\left(1-(1-x)^L\right)^{1/2}\left(1+(1-x)^{1/2}\right)} \\ +\frac{ch_1(x)\left(L+...+x^{L-1}\right)^{1/2}}{(1-x)^{1/2}} \end{array}\right]^2 x^3 - 1 \\ &= A(x)x^3 - 1 \end{aligned}$$

which is analytic at $x=0$ and we seek $f^{-1}(0)$ by expanding around $z_0 = 0$ where $f(0) = -1$ and

$$\begin{aligned} f'(x) &= 3x^2A + x^3A' \\ f'(0) &= 0 \\ f^{(2)}(x) &= 6xA + 6x^2A' + x^3A'' \\ f^{(2)}(0) &= 0 \\ f^{(3)}(x) &= 6A + x[] \end{aligned}$$

$$\begin{aligned} f^{(3)}(0) &= 6A(0) = 6\left(aL + c\left[\frac{L}{L^{1/2}2} + \frac{\frac{1}{2}L^{1/2}}{1}\right]\right)^2 \\ &= 6\left(aL + cL^{1/2}\right)^2 = 6\left(aL + cL^{1/2}\right)^2 > 0 \end{aligned}$$

Given that f is analytic at $x=0$ with $f(0) = -1$ and $f'(0) = f^{(2)}(0) = 0$ and $f^{(3)}(0) > 0$ Lagrange applies in modified form as:

$$\begin{aligned} f^{-1}(z) &= z_0 + \sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z-z_0}{[f(z)-f(z_0)]_s^{1/3}}\right)^n\right]\Bigg|_{z=z_0}(z-f(z_0))^n \\ x^* &= f^{-1}(0) = \sum_{n=1}^{\infty}\frac{1}{n!}\left[\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{(f(z)+1)^{1/3}}\right)^n\right] \end{aligned}$$

Consider

$$\begin{aligned} F(x) &= \frac{x}{(f(x)+1)^{1/3}} = \frac{x}{A^{1/3}x} = A^{-1/3} \\ &= \left[\begin{array}{c} aL\,(1-x)^{(L-2)/2} + c\frac{L(1-x)^{(2L-1)/2}}{(1-x)^{1/2}(L+...+x^{L-1})^{1/2}\left(1+(1-x)^{1/2}\right)} \\ +c\frac{\left(\frac{1}{2}+\frac{1}{2!2^2}x+\frac{3}{3!2^3}x^2+\cdots\right)\left(L+...+x^{L-1}\right)^{1/2}}{(1-x)^{1/2}} \end{array}\right]^{-2/3} \end{aligned}$$

Term $n = 1$:

$$\begin{aligned} a_1 &= \lim_{z\to 0} F(x) = A(0)^{-1/3} = \left(aL + cL^{1/2}\right)^{-2/3} \\ \alpha_1^* &= \sqrt{1 - \left(L\theta_c + \sqrt{L}\theta_l\right)^{-2/3}} \end{aligned}$$

Term $n = 2$:

$$a_2 = \frac{1}{2!}\frac{d}{dz}A^{-2/3} = \frac{1}{2}\frac{d}{dz}[]^{-4/3} = \frac{-2}{3}[]^{-7/3}[]'$$

where $[]'$ is the sum of the following four terms, each evaluated at $x = 0$:

First: $aL\frac{L-2}{2}(1-x)^{\left(\frac{L}{2}-2\right)} \to aL\frac{L-2}{2}$

Second: $c\frac{L\frac{2L-1}{2}(1-x)^{\left(L-\frac{3}{2}\right)}}{(1-x)^{1/2}\left(L+...+x^{L-1}\right)^{1/2}\left(1+(1-x)^{1/2}\right)} \to c\frac{L\frac{2L-1}{2}}{L^{1/2}2} = c\sqrt{L}\frac{2L-1}{4}$

Third:

$$\begin{aligned} & c\frac{L(1-x)^{(2L-1)/2}\left[\begin{array}{c} -\frac{1}{2}(1-x)^{-\frac{1}{2}}\left(L+...+x^{L-1}\right)^{1/2}\left(1+(1-x)^{1/2}\right) \\ +(1-x)^{1/2}\frac{1}{2}\left(L+...+x^{L-1}\right)^{\frac{-1}{2}}\left(\sum_{k=2}^{L}-\begin{pmatrix} L \\ k \end{pmatrix}(-x)^{k-2}\right)\left(1+(1-x)^{1/2}\right) \\ +(1-x)^{1/2}\left(L+...+x^{L-1}\right)^{1/2}\left(\frac{-1}{2}(1-x)^{-\frac{1}{2}}\right) \end{array}\right]}{(1-x)\left(L+...+x^{L-1}\right)\left(1+(1-x)^{1/2}\right)^2} \\ \to\ & c\frac{L\left[-\frac{1}{2}L^{1/2}(2)+\frac{1}{2}(L)^{\frac{-1}{2}}\left(-\begin{pmatrix} L \\ 2 \end{pmatrix}\right)(2)+(L)^{1/2}\left(\frac{-1}{2}\right)\right]}{(1)(L)(2)^2} \\ =\ & c\frac{1}{4}\left[-L^{1/2}-L^{\frac{1}{2}}\frac{(L-1)}{2}-L^{1/2}\frac{1}{2}\right] = -c\frac{L^{1/2}}{4}\left[1+\frac{L}{2}\right] \\ =\ & -c\sqrt{L}\frac{L+2}{8} \end{aligned}$$

And Fourth: $\frac{d}{dx} c \frac{\left(\frac{1}{2}+\frac{1}{2!2^2}x+\frac{3}{3!2^3}x^2+\cdots\right)\left(L+...+x^{L-1}\right)^{1/2}}{(1-x)^{1/2}} =$

$$c\frac{\begin{array}{c}\left(\frac{1}{2!2^2}+\frac{1}{2^3}x+\cdots\right)\left(L+...+x^{L-1}\right)^{1/2}\\ +\left(\frac{1}{2}+\frac{1}{2!2^2}x+\frac{3}{3!2^3}x^2+\cdots\right)\frac{1}{2}\left(L+...+x^{L-1}\right)^{-1/2}\left(\sum_{k=2}^{L}-\left(\begin{array}{c}L\\k\end{array}\right)(-x)^{k-2}\right)\end{array}}{(1-x)^{1/2}}$$

$$\begin{array}{ll}\rightarrow & c\left[\frac{1}{2!2^2}L^{1/2}+\frac{1}{2}\frac{1}{2}L^{-1/2}\left(-\frac{L(L-1)}{2}\right)\right] = cL^{1/2}\frac{1}{8}\left[2-L\right]\\ & +c\left(\frac{1}{2}+\frac{1}{2!2^2}x+\frac{3}{3!2^3}x^2+\cdots\right)\left(L+...+x^{L-1}\right)^{1/2}\frac{1}{2}\left(1-x\right)^{-3/2}\\ \rightarrow & cL^{1/2}\frac{1}{4}\end{array}$$

So:

$$\begin{aligned} a_2 &= \frac{-2}{3}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-7/3}\left(aL\frac{L-2}{2}+c\sqrt{L}\left(\frac{2L-1}{4}-\frac{L+2}{8}+\frac{2-L}{8}+\frac{1}{4}\right)\right)\\ &= \frac{-2}{3}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-7/3}\left(aL\frac{L-2}{2}+c\sqrt{L}\frac{L}{2}\right)\\ &= \frac{-1}{3}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-7/3}\left(aL\left(L-2\right)+c\sqrt{L}L\right)\end{aligned}$$

Hence:

$$\begin{aligned}\alpha^* &= \sqrt{1-\left(L\theta_c+\sqrt{L}\theta_l\right)^{-2/3}+\frac{L}{3}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-7/3}\left(\left(L-2\right)\theta_c+\theta_l\sqrt{L}\right)+\cdots}\\ \alpha^* &= \sqrt{1-\left(L\theta_c+\sqrt{L}\theta_l\right)^{-2/3}+\frac{L}{3}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-5/3}+\frac{L}{3}\left(L\theta_c+\sqrt{L}\theta_l\right)^{-7/3}\left(-2\theta_c\right)+\cdots}\end{aligned}$$

which is exact when $L=2$ and $\theta_l=0$. (For it to coincide with the formula for $L=2$ and $\theta_l>0$, we would need to calculate one more term (which probably would then offset the term in the power $-7/3$).

The associated scaled cost is, setting ${\alpha^*}^2 = 1-x^* = 1-\left(L\theta_c+\sqrt{L}\theta_l\right)^{-2/3}$

$$\begin{aligned}\widehat{C}(\alpha^*;\theta_c,0,\theta_l) &= -\theta_c\alpha^{*L}+\theta_l\sqrt{\frac{1-\alpha}{1+\alpha}\left(1-\alpha^{2L}\right)}+\frac{1}{\sqrt{1-\alpha^2}}\\ &= -\theta_c\alpha^{*L}+\theta_l\frac{\sqrt{\left(1-\alpha^2\right)\left(1-\alpha^{2L}\right)}}{1+\alpha}+\frac{1}{\sqrt{1-\alpha^2}}\\ &= -\theta_c\left(1-x^*\right)^{L/2}+\theta_l\frac{\sqrt{x^*\left(1-\left(1-x^*\right)^L\right)}}{1+\sqrt{1-x^*}}+\frac{1}{\sqrt{x^*}}\end{aligned}$$

Recall that for $-1 < x < 1$:

$$\begin{aligned}
(1-x)^{\frac{L}{2}} &= 1 - \frac{L}{2}x + \frac{L\,(L-2)}{8}x^2 + O(x^3) \\
(1-x)^{L} &= 1 - Lx + \frac{L\,(L-1)}{2}x^2 + O(x^3) \\
(1-x)^{\frac{1}{2}} &= 1 - \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)
\end{aligned}$$

we have

$$\begin{aligned}
&\widehat{C}(\alpha^*;\theta_c,0,\theta_l) \\
&= -\theta_c\left(1-\frac{L}{2}x^* + \frac{L\,(L-2)}{8}x^{*2} + O(x^{*3})\right) + \theta_l\frac{\sqrt{x^*\left(Lx^* - \frac{L(L-1)}{2}x^{*2} + O(x^{*3})\right)}}{1+1-\frac{1}{2}x^* - \frac{1}{8}x^2 + O(x^{*3})} + \frac{1}{\sqrt{x^*}} \\
&= -\theta_c + \frac{L\theta_c}{2}x^* + \frac{1}{\sqrt{x^*}} - \frac{\theta_c L\,(L-2)}{8}x^{*2} + O(\theta_c x^{*3}) + \theta_l\frac{x^*\sqrt{L\left(1-\frac{(L-1)}{2}x^* + O(x^{*2})\right)}}{2\left(1-\frac{1}{4}x^* - \frac{1}{16}x^2 + O(x^{*3})\right)} \\
&= -\theta_c + \frac{L\theta_c}{2}x^* + \frac{1}{\sqrt{x^*}} - \frac{\theta_c L\,(L-2)}{8}x^{*2} + O(\theta_c x^{*3}) + \theta_l\frac{\sqrt{L}x^*\left(1-\frac{(L-1)}{4}x^* + O(x^{*2})\right)}{2}\left(1+\frac{1}{4}x^* + O(x^{*2})\right)
\end{aligned}$$

and

$$\begin{aligned}
\theta_l\frac{x^*\sqrt{L\left(1-\frac{(L-1)}{2}x^* + O(x^{*2})\right)}}{2\left(1-\frac{1}{4}x^* - \frac{1}{16}x^2 + O(x^{*3})\right)} &= \theta_l\frac{\sqrt{L}x^*\left(1-\frac{(L-1)}{4}x^* + O(x^{*2})\right)}{2}\left(1+\frac{1}{4}x^* + O(x^{*2})\right) \\
&= \frac{\sqrt{L}\theta_l}{2}x^*\left(1-\frac{(L-2)}{4}x^* + O(x^{*2})\right)
\end{aligned}$$

Hence, setting $x^* = \left(L\theta_c + \sqrt{L}\theta_l\right)^{-\frac{2}{3}} + \frac{1}{3}\left(L\theta_c + \sqrt{L}\theta_l\right)^{-7/3}\left(2L\,(L-2)\,\theta_c + \theta_l\sqrt{L}\left(1-\frac{1}{4}\,(2L+1)\,(L-2)\right)\right)$

$$\begin{aligned}
\widehat{C}(\alpha^*;\theta_c,0,\theta_l) &= -\theta_c + \frac{L\theta_c + \sqrt{L}\theta_l}{2}x^* + \frac{1}{\sqrt{x^*}} - \frac{\left(\theta_c L + \sqrt{L}\theta_l\right)(L-2)}{8}x^{*2} + O(x^{*3}) \\
&= -\theta_c + \frac{3}{2}\left(L\theta_c + \sqrt{L}\theta_l\right)^{\frac{1}{3}} - \frac{L-2}{8}\left(L\theta_c + \sqrt{L}\theta_l\right)^{-\frac{1}{3}} + O\left(\left(L\theta_c + \sqrt{L}\theta_l\right)^{-2}\right)
\end{aligned}$$

which is exact when $L = 2$ and $\theta_l = 0$.

First-order expression for α^* when $\alpha^* \to 0$: Below we show that $\widehat{MB}_3'(0) = -1$ for all $L > 1$, hence a first order approximation around $\alpha = 0$ of the optimality equations yields: , and we must thus have: $1 > 2\theta_c - \theta_l$ and $\theta_l < 1 - 2\theta_c + \theta_l$ or: $0 < 1 - 2\theta_c$ for any $\theta_l > 0$.

$$\widehat{MB} = L\theta_c\alpha^{L-1} + \theta_l\,(1-\alpha) + O(\alpha^2) = \widehat{MC} = \alpha + O(\alpha^3)$$

For $L > 2$, the term in θ_c is lower order and we get:

$$\alpha^* = \frac{\theta_l}{1+\theta_l} + O\left(\left(\frac{\theta_l}{1+\theta_l}\right)^2\right)$$

7.3.2 General $L > 2$ and $\theta_g = 0$: square root is an asymptotic upper bound

Does the lower bounding argument for $L = 1, 2$ extend to $L > 2$? No. For that to work we would need that $f_3 > \sqrt{L}\alpha$, which is not the case. To see why: We do know that f_3 is convex (so any cord is a lower bound) with $f_3(1) = \sqrt{L}$. But $f_3'(1) \neq \sqrt{L}$ (we will also compute $f_3'(0)$ because we need that for a first-order expansion of small α^*)

$$\begin{aligned}
f_3' &= \frac{d}{d\alpha}\frac{L\alpha^{2L-1}(1+\alpha) + g(\alpha)\left(1+\alpha^L\right)}{g(\alpha)^{1/2}\left(1+\alpha^L\right)^{1/2}(1+\alpha)^{3/2}} = \frac{d}{d\alpha}\frac{N}{D} \\
N' &= L(2L-1)\alpha^{2L-2}(1+\alpha) + L\alpha^{2L-1} + g(\alpha)L\alpha^{L-1} + g'(\alpha)\left(1+\alpha^L\right) \\
&= L\alpha^{2L-2}(2L-1+2L\alpha) + g(\alpha)L\alpha^{L-1} + g'(\alpha)\left(1+\alpha^L\right) \\
D' &= \frac{d}{d\alpha}\left(g(\alpha)\left(1+\alpha^L\right)(1+\alpha)^3\right)^{1/2} \\
&= \frac{1}{2}\left(g(\alpha)\left(1+\alpha^L\right)(1+\alpha)^3\right)^{-1/2} \\
&\quad \times\left(g'(\alpha)\left(1+\alpha^L\right)(1+\alpha)^3 + g(\alpha)\left(L\alpha^{L-1}\right)(1+\alpha)^3 + g(\alpha)\left(1+\alpha^L\right)3(1+\alpha)^2\right)
\end{aligned}$$

and

$$\begin{aligned}
g(\alpha) &= \sum_{k=0}^{L-1}\alpha^k \to g(0) = 1 \text{ and } g(1) = L \\
g'(\alpha) &= \sum_{k=1}^{L-1}k\alpha^{k-1} \to g'(0) = 1 \text{ and } g'(1) = \frac{(L-1)L}{2}
\end{aligned}$$

so that

$$\begin{aligned}
N(0) &= 1 \text{ and } N(1) = L2 + L(2) = 4L \\
N'(0) &= 1 \text{ and } N'(1) = L(2L-1+2L) + LL + (L-1)L = 6L^2 - 2L \\
D(0) &= 1 \text{ and } D(1) = \left(L2^4\right)^{1/2} \\
D'(0) &= \frac{1}{2}(1+3) = 2 \\
D'(1) &= \frac{1}{2}\left(L2^4\right)^{-1/2}\left(\frac{(L-1)L}{2}2^4 + L(L)2^3 + L(2)3(2)^2\right) \\
&= \frac{(L-1)L8 + L(L)8 + 8L3}{8L^{1/2}} = \frac{2L+2L^2}{L^{1/2}}
\end{aligned}$$

and

$$
\begin{aligned}
f_3'(0) &= \frac{D(0)N'(0) - D'(0)N(0)}{D^2(0)} = \frac{1-2}{1} = -1 \\
f_3^{'}(1) &= \frac{D(1)N'(1) - D'(1)N(1)}{D^2(1)} = \frac{\left(L2^4\right)^{1/2}\left(6L^2 - 2L\right) - \frac{2L+2L^2}{L^{1/2}}4L}{L2^4} \\
&= \frac{\left(L2^4\right)^{1/2}\left(3L^2 - L\right) - \left(2L + 2L^2\right)2L^{1/2}}{L2^3} \\
&= L^{1/2}\frac{2^2\left(3L-1\right) - \left(2+2L\right)2}{2^3} \\
&= L^{1/2}\frac{\left(3L-1\right) - \left(1+L\right)}{2} \\
&= L^{1/2}\left(L-1\right)
\end{aligned}
$$

Thus, we have that

$$
\begin{aligned}
f_3 &= \sqrt{L} + \sqrt{L}\left(L-1\right)\left(\alpha - 1\right) + o(\alpha - 1) \\
&> \sqrt{L} + \sqrt{L}\left(L-1\right)\left(\alpha - 1\right)
\end{aligned}
$$

which is affine, but not proportional in α. Hence the lower bounding argument above does not extend to $L > 2$ (we knew it doesn't for $\theta_l = 0$).

But it is easily numerically verified that for $L > 2$ and $\alpha \geq \frac{1}{2}$, we have that

$$
\frac{f_3(\alpha)}{\alpha} < \frac{f_3(1)}{1} = \sqrt{L}
$$

Hence, if $\theta_g = 0$ and $L > 2$ and if we knew that $\alpha^* > \frac{1}{2}$, then α^* solves $\frac{1}{(1-\alpha^{*2})^{3/2}} = L\theta_c\left(\alpha^*\right)^{L-2} + \theta_l\frac{f_3(\alpha^*)}{\alpha^*} < L\theta_c + \sqrt{L}\theta_l = \frac{1}{(1-\alpha_0^2)^{3/2}}$, so that $\alpha_0 \geq \alpha^*$. Given that α_0 is asymptotically correct, we know that the square root is a tight lower bound asymptotically; we cannot establish a clear zone on α_0 above which it is a lower bound (for that we would need to know α^*).

7.3.3 Lagrange Series for $L > 2$: General case with $\theta_g > 0$

Lagrange for $L > 2$ with $b > 0$ around $\alpha = 1$. The optimality equations are:

$$
\begin{aligned}
&aL\alpha^{L-1} - b\frac{L\alpha^{L-1}\left(1-\alpha\right)\left(1+\alpha\right) - \alpha^L}{\left(1-\alpha\right)^{1/2}\left(1+\alpha\right)^{3/2}} \\
&+c\frac{L\alpha^{2L-1}\left(1-\alpha\right)^{1/2}}{\left(1-\alpha^{2L}\right)^{1/2}\left(1+\alpha\right)^{1/2}} + c\frac{\left(1-\alpha^{2L}\right)^{1/2}}{\left(1-\alpha\right)^{1/2}\left(1+\alpha\right)^{3/2}} = \frac{\alpha}{\left(1-\alpha^2\right)^{3/2}}
\end{aligned}
$$

Rework:

$$aL\alpha^{L-1}-b\frac{\alpha^{L-1}\left(L\left(1-\alpha^2\right)-\alpha\right)(1-\alpha)}{\left(1-\alpha^2\right)^{3/2}}$$
$$+c\frac{L\alpha^{2L-1}\left(1-\alpha^2\right)^{1/2}}{\left(1-\alpha^{2L}\right)^{1/2}(1+\alpha)}+c\frac{(1-\alpha)\left(1-\alpha^{2L}\right)^{1/2}}{\left(1-\alpha^2\right)^{3/2}}=\frac{\alpha}{\left(1-\alpha^2\right)^{3/2}}$$

Multiply both sides by $\alpha^{-1}\left(1-\alpha^2\right)^{3/2}$:

$$aL\alpha^{L-2}\left(1-\alpha^2\right)^{\frac{3}{2}}-b\alpha^{L-2}\left(L\left(1-\alpha^2\right)-\alpha\right)(1-\alpha)$$
$$+c\frac{L\alpha^{2L-2}\left(1-\alpha^2\right)^2}{\left(1-\alpha^{2L}\right)^{1/2}(1+\alpha)}+c\frac{(1-\alpha)\left(1-\alpha^{2L}\right)^{1/2}}{\alpha}=1$$

Set $x=1-\alpha^2$, then $\alpha^2=(1-x)$ and $\alpha=(1-x)^{\frac{1}{2}}$

$$aL(1-x)^{\frac{L-2}{2}}x^{\frac{3}{2}}-b(1-x)^{\frac{L-2}{2}}\left(Lx-(1-x)^{\frac{1}{2}}\right)\left(1-(1-x)^{\frac{1}{2}}\right)$$
$$+c\frac{L(1-x)^{L-1}x^2}{\left(1-(1-x)^L\right)^{\frac{1}{2}}\left(1+(1-x)^{\frac{1}{2}}\right)}+c\frac{\left(1-(1-x)^{\frac{1}{2}}\right)\left(1-(1-x)^L\right)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}=1$$

We showed in the case of $L>2$ and $\theta_g=0$, that the terms in c have a factor into $x^{3/2}$. To go further with $z_0=0$ and $f(z_0)=0$, the function must be analytic around z_0 and the above is not due to $x^{3/2}$. To make it analytic, we need to square terms, but smartly by collecting all terms in $x^{3/2}$ as before:

$$\left(1-(1-x)^L\right)^{1/2}=\left(\sum_{k=1}^{L}\begin{pmatrix}L\\k\end{pmatrix}(-1)^{k+1}x^k\right)^{1/2}=x^{1/2}\left(\sum_{k=1}^{L}\begin{pmatrix}L\\k\end{pmatrix}(-x)^{k-1}\right)^{1/2}=x^{\frac{1}{2}}\left(L+...+x^{L-1}\right)^{1/2}$$

$$\left(1-(1-x)^{1/2}\right)=\left(\frac{1}{2}x+\frac{1}{2!2^2}x^2+\frac{3}{3!2^3}x^3+\cdots\right)=x\left(\frac{1}{2}+\frac{1}{2!2^2}x+\frac{3}{3!2^3}x^2+\cdots\right)=xh_1(x)$$

$$\left[aL(1-x)^{\frac{L-2}{2}}+c\frac{L(1-x)^{(L-1)}}{\left(L+...+x^{L-1}\right)^{1/2}\left(1+(1-x)^{\frac{1}{2}}\right)}+\frac{ch_1(x)\left(L+...+x^{L-1}\right)^{1/2}}{(1-x)^{1/2}}\right]x^{3/2}$$
$$\begin{aligned}&= 1+b(1-x)^{\frac{L-2}{2}}\left(Lx+xh_1(x)-1\right)xh_1(x)\\&= 1-b(1-x)^{\frac{L-2}{2}}h_1(x)x+b(1-x)^{\frac{L-2}{2}}\left(L+h_1(x)\right)h_1(x)x^2\end{aligned}$$

Squaring both sides yields

$$\begin{aligned}
f(x) &= \left[aL\left(1-x\right)^{\frac{L-2}{2}} + c\frac{L\left(1-x\right)^{(L-1)}}{\left(L+...+x^{L-1}\right)^{1/2}\left(1+\left(1-x\right)^{\frac{1}{2}}\right)} + \frac{ch_1(x)\left(L+...+x^{L-1}\right)^{1/2}}{\left(1-x\right)^{1/2}}\right]^2 x^3 \\
&\quad -\left[1-b\left(1-x\right)^{\frac{L-2}{2}}h_1(x)x + b\left(1-x\right)^{\frac{L-2}{2}}\left(L+h_1(x)\right)h_1(x)x^2\right]^2 \\
f(x) &= Ax^3 - \left[1-bB_1x+bB_2x^2\right]^2
\end{aligned}$$

which is analytic at $x=0$ and we seek $f^{-1}(0)$ by expanding around $z_0=0$ where $f(0)=-1$.

Notice that:

$$\begin{aligned}
B_1(x) &= \left(1-x\right)^{\frac{L-2}{2}}h_1(x) \rightarrow B_1(0) = \frac{1}{2} \\
B_1'(x) &= -\frac{L-2}{2}\left(1-x\right)^{\frac{L-4}{2}}h_1(x) + \left(1-x\right)^{\frac{L-2}{2}}\left(\frac{1}{2!2^2}+\frac{3}{3!2^2}x+\cdots\right) \\
B_1'(0) &= \frac{1}{8}-\frac{L-2}{4} = \frac{5-2L}{8}
\end{aligned}$$

$$\begin{aligned}
B_1^{"}(x) &= \frac{L-2}{2}\frac{L-4}{2}\left(1-x\right)^{\frac{L-6}{2}}h_1(x) - 2\frac{L-2}{2}\left(1-x\right)^{\frac{L-4}{2}}\left(\frac{1}{2!2^2}+\frac{3}{3!2^2}x+\cdots\right) \\
&\quad +\left(1-x\right)^{\frac{L-2}{2}}\left(\frac{3}{3!2^2}+\cdots\right) \\
B_1^{"}(0) &= \frac{1}{8}-\frac{L-2}{8}+\frac{\left(L-2\right)\left(L-4\right)}{2} = \frac{4L^2-25L+35}{8}
\end{aligned}$$

$$\begin{aligned}
B_2(x) &= \left(1-x\right)^{\frac{L-2}{2}}\left(L+h_1(x)\right)h_1(x) \rightarrow B_2(0) = \left(L+\frac{1}{2}\right)\frac{1}{2} = \frac{2L+1}{4} \\
B_2'(x) &= -\frac{L-2}{2}\left(1-x\right)^{\frac{L-4}{2}}\left(L+h_1(x)\right)h_1(x) + \left(1-x\right)^{\frac{L-2}{2}}\left(\frac{1}{2!2^2}+\frac{3}{3!2^2}x+\cdots\right)h_1(x) \\
&\quad +\left(1-x\right)^{\frac{L-2}{2}}\left(L+h_1(x)\right)\left(\frac{1}{2!2^2}+\frac{3}{3!2^2}x+\cdots\right)
\end{aligned}$$

$$\begin{aligned}
B_2'(0) &= -\frac{L-2}{2}\left(L+\frac{1}{2}\right)\frac{1}{2}+\frac{1}{2!2^2}\frac{1}{2}+\left(L+\frac{1}{2}\right)\left(\frac{1}{2!2^2}\right) \\
&= -\frac{L-2}{2}\left(L+\frac{1}{2}\right)\frac{1}{2}+\frac{1}{2!2^2}+L\left(\frac{1}{2!2^2}\right) \\
&= \frac{1}{2}L-\frac{1}{4}L^2+\frac{3}{8} = \frac{3+4L-2L^2}{8}
\end{aligned}$$

We can re-use some results from the case of $L>2$ and $\theta_g=0$ above

$$f'(0) = -2\left[1\right]\left[-bB_1(0)\right] = -2\left[1\right]\left[-b\frac{1}{2}\right] = b > 0$$

as before for $L = 2$.

$$
\begin{aligned}
f^{(2)}(0) &= -2b^2\left[-B_1(0)\right]\left[-B_1(0)\right] - 2b\left[1\right]\left[-2B_1^{'}(0) + 2B_2(0)\right] \\
&= -2b^2\left[\frac{-1}{2}\right]\left[\frac{-1}{2}\right] - 2b\left[1\right]\left[-2\frac{5-2L}{8} + 2\frac{2L+1}{4}\right] \\
&= -b^2\frac{1}{2} - 2b\left[\frac{3}{2}L - \frac{3}{4}\right] \\
&= -\frac{1}{2}b^2 - b\frac{6L-3}{2} = -\frac{b\left(b+6L-3\right)}{2},
\end{aligned}
$$

as before for $L = 2$.

$$
\begin{aligned}
f^{(3)}(0) &= 6A(0) + 12b^2\left[-B_1'(0) + B_2(0)\right]\left[B_1(0)\right] - 2b\left[-3B_1^{''}(0) + 6B_2'(0)\right] \\
&= 6\left[aL + c\frac{L}{L^{1/2}2} + \frac{cL^{1/2}}{2}\right]^2 + 12b^2\left[-\frac{5-2L}{8} + \frac{2L+1}{4}\right]\frac{1}{2} \\
&\quad -2b\left[-3\frac{4L^2-25L+35}{8} + 6\frac{3+4L-2L^2}{8}\right] \\
&= 6\left(aL + c\sqrt{L}\right)^2 + b^2\frac{3}{4}\left(6L-3\right) + b\frac{3\left(8L^2-33L+29\right)}{4}
\end{aligned}
$$

as before for $L = 2$.

Given that f is analytic at $x = 0$ with $f'(0) > 0$ Lagrange applies as:

$$
\begin{aligned}
f^{-1}(z) &= z_0 + \sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z-z_0}{f(z)-f(z_0)}\right)^n\right]\Bigg|_{z=z_0}\left(z-f(z_0)\right)^n \\
x^* &= f^{-1}(0) = \sum_{n=1}^{\infty}\frac{1}{n!}\left[\lim_{z\to 0}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{z}{f(z)+1}\right)^n\right]
\end{aligned}
$$

Term $n = 1$:

$$
a_1(0) = \frac{1}{f'(0)} = \frac{1}{b}
$$

Term $n = 2$:

$$
a_2(0) = -\frac{f^{(2)}(0)}{(f^{(1)}(0))^3} = -\frac{-\frac{b(b+6L-3)}{2}}{b^3} = \frac{(b+6L-3)}{2b^2}
$$

Term $n = 3$:

$$
\begin{aligned}
a_3(0) &= \frac{24f''^2(0) - 8f'(0)f^{(3)}(0)}{5!(f'(0))^5} \\
&= \frac{24\left(-\frac{b(b+6L-3)}{2}\right)^2 - 8b\left(6\left(aL + \frac{3}{2}c\sqrt{L}\right)^2 + b^2\frac{3}{4}(6L-3) + b\frac{3\left(8L^2-33L+29\right)}{4}\right)}{5!b^5} \\
&= \frac{6b(b+6L-3)^2 - 8\left(6\left(aL + \frac{3}{2}c\sqrt{L}\right)^2 + b^2\frac{3}{4}(6L-3) + b\frac{3\left(8L^2-33L+29\right)}{4}\right)}{5!b^4} \\
&= \frac{-2\left(aL + \frac{3}{2}c\sqrt{L}\right)^2}{5b^4} + \frac{6(b+6L-3)^2 - 2\left(3b(6L-3) + 3\left(8L^2 - 33L + 29\right)\right)}{5!b^3} \\
&= \frac{-2\left(aL + c\sqrt{L}\right)^2}{5b^4} + \frac{b^2 + (6L-3)b + 28L^2 - 3L - 20}{5 \cdot 4b^3}
\end{aligned}
$$

Thus, up to three terms where $a = \theta_c$ (cost delta), $b = \theta_g$ (global capacity), and $c = \theta_l$ (local capacity):

$$
\begin{aligned}
x_1^* &= \frac{1}{\frac{1}{2}b} \Rightarrow \alpha^* = \sqrt{1 - \left(\frac{1}{2}\theta_g\right)^{-1}} \text{ [cannot be further expanded b/c not analytic]} \\
x_1^* &= \frac{1}{b} \Rightarrow \alpha^* = \sqrt{1 - \theta_g^{-1}} \\
x_2^* &= \frac{1}{b} + \frac{1}{2!}\frac{(b+6L-3)}{2b^2} \\
x_3^* &= \frac{1}{b} + \frac{1}{2!}\frac{(b+6L-3)}{2b^2} + \frac{b^2 + (6L-3)b + 28L^2 - 3L - 20}{5!b^3} - \frac{\left(aL + c\sqrt{L}\right)^2}{15b^4}
\end{aligned}
$$