Testing for Anticipated Changes in Spot Volatility at Event Times*

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Abstract

We propose a test for anticipated changes in spot volatility, either due to continuous or discontinuous price moves, at the times of realization of event risk in the form of pre-scheduled releases of economic information such as earnings announcements by firms and macroeconomic news announcements. These events can generate nontrivial volatility in asset returns, which does not scale even locally in time. Our test is based on short-dated options written on an underlying asset subject to event risk, which takes place after the options' observation time and prior to or after their expiration. We use options with different tenors to estimate the conditional (risk-neutral) characteristic functions of the underlying asset log-returns over the horizons of the options. Using these estimates and a relationship between the conditional characteristic functions with three different tenors, which holds true if and only if continuous and discontinuous spot volatility does not change at the event time, we design a test for this hypothesis. In an empirical application, we study anticipated individual stocks' volatility changes following earnings announcements for a set of stocks with good option coverage.

Keywords: characteristic function, event risk, functional convergence, jumps, options, volatility, testing.

JEL classification: C51, C52, G12.

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1 Introduction

Inference for time-varying volatility has a long history in financial econometrics. Early work has considered estimation of volatility dynamics using low frequency asset return data. In this case, one cannot estimate consistently the realized volatility path in general but nevertheless studying the dynamics of the latent volatility process is possible using various analytic and simulation-based techniques.

When asset prices can be sampled at high frequencies, then one can estimate consistently either spot or integrated volatility. Following the studies of Andersen et al. (2003) and Barndorff-Nielsen and Shephard (2004a,b, 2006), there is a large body of work that deals with the problem of non-parametric high-frequency volatility estimation in various settings, see e.g., Aït-Sahalia and Jacod (2014) and references therein. With high-frequency data, volatility can be treated as observable and this facilitates inference for the volatility dynamics, see e.g., Barndorff-Nielsen and Shephard (2002), Bollerslev and Zhou (2002), Bandi and Phillips (2003), Corradi and Distaso (2006) and Todorov (2009), among others.

An alternative source of information for volatility is options written on the underlying asset. Indeed, in the classical Black-Scholes model of Black and Scholes (1973), there is a one-to-one map between the option price and the volatility parameter (volatility is constant in this model). In more general settings with stochastic volatility, one cannot estimate consistently volatility using a limited number of noisy option observations at any given point in time. This asymptotic setup is thus akin to the low frequency sampling of asset prices. The fixed cross-section of options can be used, nevertheless, to study the volatility dynamics, see e.g., Pan (2002), Bates (2003) and Eraker (2004), among others. On the other hand, if the cross-section of option observations is large in an asymptotic sense, then one can estimate consistently volatility from options in a parametric setting as shown in Andersen et al. (2015).

For nonparametric identification of spot volatility from options one needs their time-to-maturity (tenor) to be short. This asymptotic setting can be viewed as the natural analogue of the high-frequency sampling of the asset price. In this setting, one can pretend that the semimartingale characteristics (diffusive volatility and jump compensator) are constant over the short interval, i.e., that the process is conditionally Lévy (a process with i.i.d. increments). In the conditional Lévy setting, the conditional return distribution, which can be uniquely identified from options, see e.g., Breeden and Litzenberger (1978), can be used to estimate the spot diffusive volatility in a nonparametric way. One such way is to use the characteristic function as proposed by Todorov (2019).[1]

[1] An alternative is to use the Black-Scholes implied volatility, see e.g., Medvedev and Scaillet (2007) and Durrleman (2008). As shown in Todorov (2019), Black-Scholes implied volatility has a much larger bias in general than a spot volatility estimate based on the characteristic function that we consider here.

More specifically, utilizing the dominant role of the diffusion for high values of the characteristic exponent, see e.g., chapter 2 of Sato (1999), we have for the conditional characteristic function $\mathcal{L}_{t,T}(u) = \mathbb{E}_t(e^{iu(x_{t+T}-x_t)})$ of a log-asset price x_t:

$$-\frac{2}{u^2T}\log|\mathcal{L}_{t-,T}(u)| \to \sigma^2_{t-}, \quad \text{as } u \to \infty, \text{ a.s.,} \tag{1}$$

for some small $T > 0$ and where σ_t is the spot diffusive volatility of the asset. This is illustrated in Figure 1, which plots $-\frac{2}{u^2T}\log|\mathcal{L}_{t-,T}(u)|$ for a parametric model that we will use in our Monte Carlo study. For u approaching zero, $-\frac{2}{u^2T}\log|\mathcal{L}_{t-,T}(u)|$ is an estimate of total spot variance (including the one due to the jumps in x). As u increases, the positive bias of the estimator due to the price jumps gradually disappears. The empirical analysis in Todorov and Zhang (2022) shows that an option-based estimator of volatility based on (1) plays a nontrivial role in optimal measurement of spot volatility that combines options and returns data.

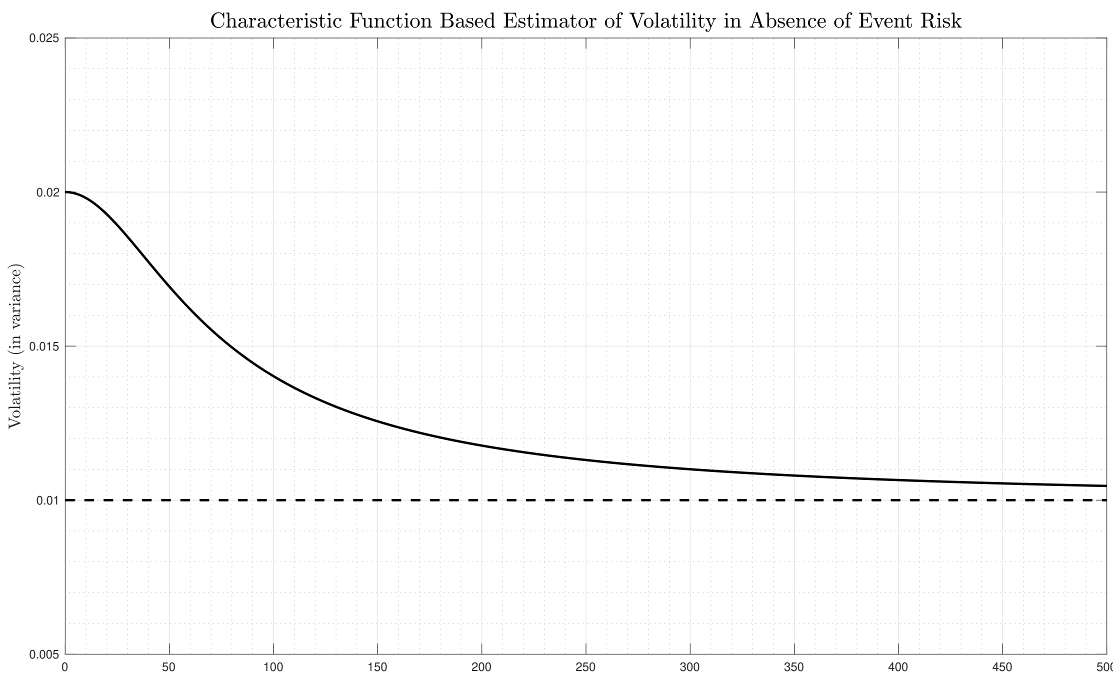

Figure 1: Characteristic Function Estimation of Diffusive Variance in Absence of Event Risk. Dashed line corresponds to true variance and solid line to $-\frac{2}{u^2T}\log|\mathcal{L}_{t-,T}(u)|$. The parametric model for computing the characteristic function is given in (33)-(36), with $\sigma^2_{t^*} = \sigma^2_{t^*-} = 0.01$, $\Delta x_{t^*} = 0$ and $T = 5/252$.

The above approach of identifying spot volatility from options breaks down, however, in the presence of *event risk*. That is, in a situation where prior to the options' expiration, there is a pre-announced event at a fixed and known in advance time in the future that can trigger a jump in the asset price with a positive probability. In this case, as the length of the time window shrinks, then the event risk starts to dominate the price increment. More specifically, suppose that at $t^* \in (t, t+\epsilon)$ for some $\epsilon > 0$, the stock price is exposed to event risk, i.e., that $\Delta x_{t^*} \neq 0$ with positive probability.[2] If this is the case, then it is easy to see that $\mathbb{E}_{t-}(x_{t+\epsilon} - x_t)^2 / \mathbb{E}_{t-}(\Delta x_{t^*})^2 \to 1$ as $t \uparrow t^*$ and $\epsilon \to 0$. In other words, the price increment is dominated by the event risk Δx_{t^*} when the length of the increment shrinks. In this case, the signal about the non-event risk in $x_{t+\epsilon} - x_t$ becomes small in relative terms.

We illustrate this in Figure 2 by plotting the time series of risk-neutral return variance backed out from short-dated options written on the Facebook stock, i.e., we plot in the top panel of Figure 2 the end-of-day estimates from options of $\frac{1}{T}\mathbb{E}_{t-}(x_{t+T} - x_t)^2$, where expectation is under the risk-neutral probability measure and T is the shortest available tenor on that day exceeding 2 business days. The volatility time series exhibits big spikes that occur periodically. They are due to the risk associated with pre-scheduled earnings announcements that happen once every quarter.[3] When the interval $[t, t+T]$ includes such event risk, the continuously compounded return $x_{t+T} - x_t$ is dominated by it, and since this risk is not proportional to time, the annualized variance $\frac{1}{T}\mathbb{E}_{t-}(x_{t+T} - x_t)^2$ explodes when T approaches zero. We can compare the risk-neutral variance estimates with the daily realized volatility, which is displayed in the bottom panel of Figure 2. This series exhibits occasionally very big spikes during the earnings announcement periods but the periodicity pattern in it is much weaker.[4]

How can we measure nonparametrically volatility from options in the case of event risk? Suppose that the occurrence of the event risk does not trigger a jump in the diffusive volatility and/or the jump compensator. In this case, we can recover the characteristic function of the non-event part of the asset return by using the ratio of the characteristic functions of returns over two horizons. More specifically, one can show under some conditions that

$$-\frac{2}{u^2(T_2 - T_1)} \log \left| \frac{\mathcal{L}_{t-,T_2}(u)}{\mathcal{L}_{t-,T_1}(u)} \right| \to \sigma_{t-}^2, \quad \text{as } u \to \infty, \text{ a.s.,} \tag{2}$$

for some $0 < T_1 < T_2 \leq \epsilon$. This strategy will work, however, only if the semimartingale characteristics do not jump at the event time. Indeed, if say the diffusive volatility can jump at time t^* to a level σ_{t^*}, then the diffusive component of the price increment, even after conditioning on $\mathcal{F}_{t-}$, has a mixed Gaussian law and hence it can be fat-tailed. Therefore, one can no longer disentangle this

[2]We note that for the common way of modeling asset prices via Itô semimartingales, the probability of a jump at a fixed point in time is zero, see e.g., Corollary II.1.19 in Jacod and Shiryaev (2003).

[3]We refer to Dubinsky et al. (2019) for a parametric analysis of earnings announcement risk. Da and Warachka (2009) and Savor and Wilson (2016) show that earnings announcement risk has a systematic component and is therefore important for the aggregate pricing of risk.

[4]Hence, predicting realized volatility using option-implied volatility (as often done in the volatility forecasting literature) without adjustment for the earnings announcement effect on both series will typically lead to poor forecasting results.

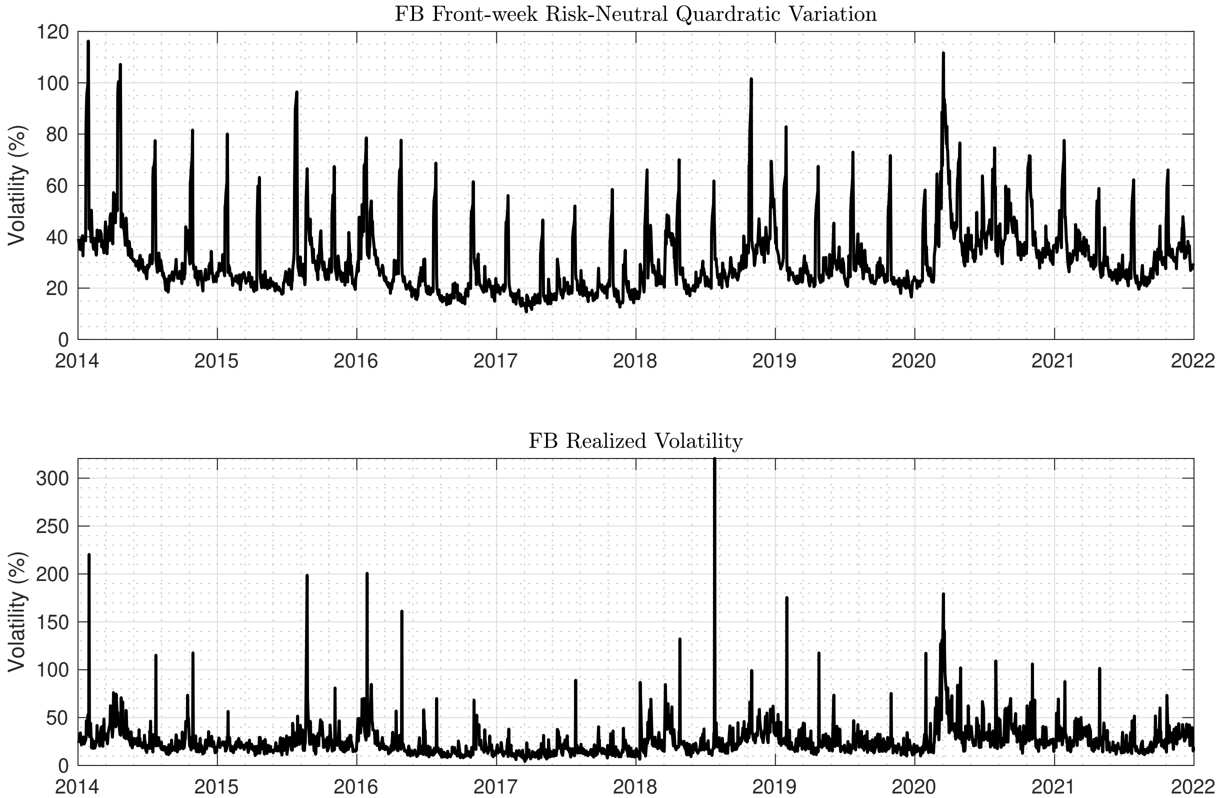

Figure 2: Risk-Neutral Return Variance and Daily Realized Volatility for Facebook. Both quantities are reported in annualized percentage units.

piece in a nonparametric way from the jump part of the asset price like in the standard no-event case.

In this paper we propose a nonparametric test based on options with short tenor observed shortly before the occurrence of an event risk, which allows us to decide whether the semimartingale characteristics (diffusive volatility and jump intensity) are anticipated to jump at the event time (known to the econometrician) with positive probability. As argued above, the test can be used to decide on whether spot volatility can be extracted in a nonparametric way from short-dated options prior to event times. More generally, the developed techniques in the paper can be used to study anticipated changes in volatility at event times. As discussed later in the paper, short-dated options are a unique source of information about this and contain information about volatility jumps at event times which cannot be extracted from return data in general.

The proposed test is based on option-based estimates of the characteristic functions of returns over three different periods recorded prior to the time of the event risk. The first period does not cover the event time while the other two time periods do. The test utilizes the fact that, under the null hypothesis, the characteristic exponent of the non-event risk component of the asset price is proportional to time. This implies a simple functional relation between the three characteristic functions under the null hypothesis, which does not hold under the alternative hypothesis.

For characterizing the asymptotic behavior of the test statistic under the null hypothesis, we derive a functional Central Limit Theorem (CLT) for certain transforms of the three characteristic functions used in the test. The convergence takes place in a weighted L^2 Hilbert space of complex-valued functions. The limit is a mixed Gaussian process with conditional volatility depending on the risks in the underlying asset price as well as on the variance of the option observation error. The limit distribution of the test statistic is non-standard. To determine the critical values of the test, we develop an easy to implement wild bootstrap approach. Using noisy estimates of the option observation error and a sequence of standard normal random variables defined on an extension of the original probability space, we generate new option prices by perturbing by the right amount the observed ones. We then recompute the characteristic functions. The deviation of the newly generated statistic from the one computed from the observed data has the same limit distribution as the one of our test statistic and its quantiles can be easily evaluated via simulation.

We find good finite sample performance of the developed test procedure on simulated data in a Monte Carlo study. In an empirical application we study the behavior of stocks' volatility around quarterly earnings announcements for a cross-section of stocks with liquid option markets over the period 2016-2021. We find evidence for jumps in volatility and/or jump intensity at the announcement times for more than half of the stock and announcement pairs. We document that this shock to volatility is a source of priced risk by investors. More specifically, the expected return variation after the announcement backed out from the options is on average higher than its realization.

The test developed in this paper complements the one in Todorov (2020). Todorov (2020) proposes a test based on short-dated options for deciding whether the underlying asset price is exposed to event risk (which in technical terms means fixed times of discontinuity in the price process). By contrast, in the current paper we assume that the econometrician knows that the stock price is exposed to event risk and we are interested in the spot volatility behavior at the event time. Knowledge of the arrival of the event risk by the econometrician seems proper for certain events such as earnings announcements by firms and macroeconomic news announcements which are pre-scheduled and information for which is easily accessible both to market participants and the econometrician. The test developed here and the one in Todorov (2020) are both based on measuring distances between certain transforms of characteristic functions of asset returns over several short intervals. The different testing goals, however, manifest in very different properties of the test statistics. Mainly, Todorov (2020) derives a CLT for the test statistics of that paper when there is no event risk while here we derive a CLT for our test statistic when event risk is present in the asset price. With no event risk, short-dated out-of-the-money options are asymptotically shrinking which is not the case in presence of event risk. This means faster rate of convergence and in general different asymptotic behavior of the option-based estimate of the characteristic function with and without event risk. This leads to different behavior of the test statistic of this paper and the one in Todorov (2020).

The rest of the paper is organized as follows. We start in Section 2 with introducing our formal setup and assumptions. Section 3 contains the theoretical results of the paper. In Section 4, we present a Monte Carlo study of the finite sample behavior of the test. In Section 5 we implement our specification test to study firms' volatility behavior following earnings announcements. Section 6 concludes. Proofs are given in Section 7.

2 Setting and Assumptions

2.1 Asset Price Dynamics

The asset price process is denoted with X and the logarithm of it with x. The price process is defined on the sample space Ω, with the associated σ-algebra $\mathcal{F}$, and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ being the filtration. We will consider two probability measures, one being the true (statistical) one, denoted with $\mathbb{P}$, and the other one being the risk-neutral one, denoted with $\mathbb{Q}$. The latter, under the weak condition of arbitrage-free asset prices, is locally equivalent to the true one, see e.g., Duffie (2001) and definition III.3.2 in Jacod and Shiryaev (2003). The significance of $\mathbb{Q}$ stems from the fact that the discounted at the risk-free rate payoff process of any asset is a local martingale under $\mathbb{Q}$. We introduce the measure $\mathbb{Q}$ here because our inference will be based on option prices whose theoretical values equal conditional risk-neutral expectations of certain functions of the underlying stock price. Naturally, inference will be conducted under the statistical probability measure.

We proceed with our assumption for the dynamics of the underlying stock price. Our interest in this paper is the behavior of the asset price around a fixed time denoted with t^*. At time t^*, there is an event, such as a pre-scheduled release of economic news, that generates price and potentially volatility jump risk in the asset price. The dynamics of x in a local small window around t^* is given in the following assumption.

A. *For $s \in [t^*-\epsilon, t^*+\epsilon]$, where $\epsilon > 0$ is an arbitrary small number, we have under $\mathbb{Q}$:*

$$x_s = x_{t^*-\epsilon} + \int_{t^*-\epsilon}^{s} a_u du + \int_{t^*-\epsilon}^{s} \sigma_{u-} dW_u + \int_{t^*-\epsilon}^{s}\int_{\mathbb{R}} z\mu(du,dz) + \Delta x_{t^*} 1_{\{s\geq t^*\}}, \quad (3)$$

where W is a Brownian motion, μ is a integer-valued random measure on $\mathbb{R}\times\mathbb{R}$ with predictable *compensator* $\phi_{u-}du \otimes \nu(dz)$*,* $a_s = -\frac{1}{2}\sigma_s^2 - \phi_s \int_{\mathbb{R}}(e^z - 1 - z)\nu(dz)$*, and further*

$$\sigma_s = \begin{cases} \sigma_{t^*-}, & s < t^* \\ \sigma_{t^*}, & s \geq t^* \end{cases}, \quad \phi_s = \begin{cases} \phi_{t^*-}, & s < t^* \\ \phi_{t^*}, & s \geq t^* \end{cases}, \quad s \in [t^* - \epsilon, t^* + \epsilon], \tag{4}$$

with $(\sigma_{t^*-}, \phi_{t^*-})$ *being* $\mathcal{F}_{t^*-\epsilon}$*-adapted random variables and* $(\sigma_{t^*}, \phi_{t^*}, \Delta x_{t^*})$ *being* $\mathcal{F}_{t^*}$*-adapted random variables. Furthermore, the jumps* $(\sigma_{t^*}^2 - \sigma_{t^*-}^2, \phi_{t^*} - \phi_{t^*-}, \Delta x_{t^*})$ *are independent from* $\mathcal{F}_{t^*-}$ *and* Δx_{t^*} *is* $\mathcal{F}_{t^*-}$*-conditionally independent from* $\sigma_{t^*}^2$ *and* ϕ_{t^*}*. Finally, for any* $u \in \mathbb{R}$*, we have* $|\mathbb{E}_{t^*-}^{\mathbb{Q}}(e^{iu\Delta x_{t^*}})| > 0$*.*

We make several comments regarding the above assumption. First, assumption A is for the risk-neutral dynamics of x. The local equivalence of $\mathbb{P}$ and $\mathbb{Q}$, however, restricts the diffusive coefficient, σ_t, to be the same under $\mathbb{P}$ and $\mathbb{Q}$. Nevertheless, the risk premium can drive a wedge between the $\mathbb{P}$ and $\mathbb{Q}$ conditional expectations of future σ_t and its jump at time t^*. Related to this, the restriction on the drift coefficient a_t in assumption A is due to the fact that the cum-dividend and discounted at the risk-free stock price is a martingale under $\mathbb{Q}$. To keep notation simple, and since our interest is in asset behavior over short time intervals, we have implicitly set the risk-free rate and the dividend yield to zero. Third, we allow both the diffusive volatility and the jump intensity to jump at time t^* and our interest is to design a test for deciding if this can happen with nontrivial probability. Fourth, the volatility and jump intensity are assumed constant before and after t^*. This is a simplification of the analysis, as the asymptotics here is for shrinking time windows around t^*. This assumption can be further relaxed at the expense of more complicated proofs.

Finally, we note that we need non-vanishing characteristic function for the price jump Δx_{t^*} at arbitrary levels of u. The reason for this is that in our analysis, we use returns over intervals including t^*, and our interest is in the return outside the jump at t^*. The latter, however, is convoluted with that jump in the observed asset return. This situation is analogous to a standard deconvolution problem in statistics. Such a restriction on $\mathbb{E}_{t^*-}^{\mathbb{Q}}\left(e^{iu\Delta x_{t^*}}\right)$ is satisfied for many distributions commonly used in applied work.

2.2 Observation Scheme

We next describe the observation scheme. As already mentioned, we will use options with short time-to-expiration for designing our test. We will use the options to recover the risk-neutral conditional characteristic function of the log-price increment. We denote this quantity with

$$\mathcal{L}_{t,T}(u) = \mathbb{E}_t^{\mathbb{Q}}(e^{iu(x_{t+T} - x_t}), \quad u \in \mathbb{R}, \tag{5}$$

and we note that unlike the introduction, here the expectation has a superscript $\mathbb{Q}$ to signify the fact that it is a risk-neutral expectation.[5] The conditional characteristic function of the log-return can be computed from a portfolio of options over a continuum of strikes. More specifically, following Carr and Madan (2001), we have

$$\mathcal{L}_{t,T}(u) = 1 - \left(u^2 + iu\right) \int_{\mathbb{R}} e^{iu \log(K/X_t)} \frac{O_{t,T}(K)}{K^2} dK, \tag{6}$$

where $O_{t,T}(K)$ denotes the price at time t of a European-style out-of-the-money (OTM) option price, expiring at time $t+T$ with strike K, and whose underlying asset price at time t is X_t. We recall that $O_{t,T}(K)$ is the minimum of the put and call option prices with strike K. In practice, we do not observe options on a continuum of strikes but rather on a discrete grid, which we denote with

$$K_1 < ... < K_{N_{t,T}}, \quad \text{for some integer } N_{t,T}. \tag{7}$$

For simplicity, we will assume that the grid is equidistant and we will denote $\Delta = K_j - K_{j-1}$. In order to keep the notation simple, we suppress dependence of the observed strike grid on (t,T). Option prices are observed with error, i.e., we observe

$$\widehat{O}_{t,T}(K_j) = O_{t,T}(K_j) + \epsilon_{t,T}(K_j), \tag{8}$$

where the errors $\epsilon_{t,T}(K_j)$ are defined on a space $\Omega^{(1)} = \mathbb{R}_+^{\mathbb{R}} \times \mathbb{R}_+^{\mathbb{R}} \times ...$ (each $\mathbb{R}_+^{\mathbb{R}}$ being reserved for the option errors corresponding to a specific pair (t,T)), which is equipped with the product Borel σ-field $\mathcal{F}^{(1)}$, and transition probability $\mathbb{P}^{(1)}(\omega^{(0)}, d\omega^{(1)})$ from the probability space $\Omega^{(0)}$, on which X is defined, to $\Omega^{(1)}$. We further define,

$$\Omega \,=\, \Omega^{(0)} \times \Omega^{(1)}, \quad \mathcal{F} \,=\, \mathcal{F}^{(0)} \times \mathcal{F}^{(1)},$$

and

$$\mathbb{P}(d\omega^{(0)}, d\omega^{(1)}) \,=\, \mathbb{P}^{(0)}(d\omega^{(0)})\, \mathbb{P}^{(1)}(\omega^{(0)}, d\omega^{(1)})\,.$$

Using the observed options, we can construct the feasible counterpart of $\mathcal{L}_{t,T}(u)$ via Riemann sum:

$$\widehat{\mathcal{L}}_{t,T}(u) = 1 - (u^2 + iu) \sum_{j=2}^{N_{t,T}} e^{iu \log(K_{j-1}/X_t)} \frac{\widehat{O}_{t,T}(K_{j-1})}{K_{j-1}^2} (K_j - K_{j-1}). \tag{9}$$

The above option-based estimator of the characteristic function is very similar to that of Todorov (2020) given in equation (3.2) in that paper (see also (3.12) in Todorov (2019)). The slight difference comes from the fact that in Todorov (2019, 2020), the feasible estimator is constructed on the basis of discretizing the equivalent representation of $\mathcal{L}_{t,T}(u)$ in equation (6) above in terms of an integral

[5]For the discussion in the introduction, there was no need to specify the probability measure under which the expectations were taken.

with respect to log-strike. This difference is numerically small and is asymptotically of higher order relative to the rate of convergence in a CLT for $\widehat{\mathcal{L}}_{t,T}(u)$.

For our testing procedures, we will use option-based characteristic functions all computed at a point in time t and which have three different times-to-maturity, T_1, T_2 and T_3. For characterizing the asymptotic behavior of $\widehat{\mathcal{L}}_{t,T}(u)$, for $T = T_1, T_2$ and T_3, we need a set of assumptions on the strike grid and the option observation errors. They are stated below.

B1. *We have $\epsilon_{t,T_l}(K_j) = s_\Delta \bar{\epsilon}_{t,l,j} O_{t,T_l}(K_j)$ for $l = 1,2,3$, where s_Δ is a deterministic sequence satisfying $s_\Delta \to 0$ and $s_\Delta/\sqrt{\Delta} \to \infty$ as $\Delta \downarrow 0$. The three sequences $\{\bar{\epsilon}_{t,1,j}\}_{j=1}^{N_{t,T_1}}$, $\{\bar{\epsilon}_{t,2,j}\}_{j=1}^{N_{t,T_2}}$ and $\{\bar{\epsilon}_{t,3,j}\}_{j=1}^{N_{t,T_3}}$ are defined on $\mathcal{F}^{(1)}$, are i.i.d. and independent of each other and of $\mathcal{F}^{(0)}$, and bounded in absolute value. We further have $\mathbb{E}(\bar{\epsilon}_{t,l,j}|\mathcal{F}^{(0)}) = 0$ and $\mathbb{E}((\bar{\epsilon}_{t,l,j})^2|\mathcal{F}^{(0)}) = \bar{v}_{t,l}$ for $l = 1,2,3$ and some positive $\mathcal{F}_t^{(0)}$-adapted processes $\bar{v}_{t,1}$, $\bar{v}_{t,2}$ and $\bar{v}_{t,3}$ which are left continuous.*

B2. *There exist sufficiently small $\epsilon > 0$ such that for $t \in [t^* - \epsilon, t^*)$ and $s \in [t^*, t^* + \epsilon]$, we have $\mathbb{E}_t^{\mathbb{Q}}\left(e^{(2+\iota)|x_s|}\right) < \infty$, almost surely, for some $\iota > 0$.*

B3. *We have $K_1^2/\Delta \to 0$ and $K_{N_{t,T_l}}^2 \Delta \to \infty$, as $\Delta \downarrow 0$ and for $l = 1,2,3$.*

We make several comments about these assumptions. First, the observation errors are assumed to have $\mathcal{F}^{(0)}$-conditional means of zero and they are assumed proportional to the true option prices they are added to. The size of the error is assumed to be shrinking asymptotically as $\Delta \downarrow 0$ at an arbitrary slow rate. Obviously, in practice Δ is fixed. The current asymptotic setup for the observation error seems proper in light of the fact that observed bid-ask spreads are small relative to the mid option quotes, which we take as noisy proxies for the true option prices in the application of the developed theory, see e.g., Figure 2 in Andersen et al. (2015). The empirical relevance of the above shrinking error assumption will be confirmed later on in our numerical analysis.

We also note that for deriving infeasible CLT result in Theorem 2 below, we do not need $s_\Delta \to 0$ and s_Δ fixed will work too. The shrinking observation error, i.e., $s_\Delta \to 0$ is needed only for the approach adopted here for feasible estimation of the asymptotic variance of the limiting distribution. Finally, the requirement $s_\Delta/\sqrt{\Delta} \to \infty$ guarantees that bias terms in our test statistic are of higher asymptotic order relative to the term due to the observation error that drives the CLT for the statistic.

Assumption B2 is a conditional moment restriction. The existence of conditional moments of the increments of x_t have implications for how fast the true option price, $O_{t,T}(K)$, decays as $K \downarrow 0$ and $K \uparrow \infty$. We note that for the existence of $O_{t,T}(K)$ for $K > X_t$, we need $\mathbb{E}_t^{\mathbb{Q}}(e^{x_{t+T}}) < \infty$. Finally, assumption B3 imposes conditions on the strike range and the mesh of the strike grid. It is clear that for $\widehat{\mathcal{L}}_{t,T}(u)$ to be a consistent estimate of $\mathcal{L}_{t,T}(u)$, we need $\Delta \downarrow 0$, $K_1 \downarrow 0$ and $K_{N_{t,T}} \uparrow \infty$. Assumption B3 restricts the rate at which this happens and ensures that the error due to the discrete strike grid is larger asymptotically than the error due to the lack of option observations for $K < K_1$ and $K > K_{t,T}$. These conditions are connected with the moment condition in B3, with a stronger moment condition corresponding to a weaker requirement for K_1 and $K_{t,T}$.

3 Tests for Anticipated Event Risk Volatility Jumps

This section contains the main theoretical results of the paper. Formally, our interest is to design a test that can allow us to determine in which of the following two subsets of Ω, the random outcome ω belongs to:

$$\begin{aligned} \Omega_0 &= \{\omega : \mathbb{Q}_{t^*-}(\sigma^2_{t^*} = \sigma^2_{t^*-}) = 1 \text{ and } \mathbb{Q}_{t^*-}(\phi_{t^*} = \phi_{t^*-}) = 1\}, \\ \Omega_A &= \{\omega : \mathbb{E}^{\mathbb{Q}}_{t^*-}(\sigma^2_{t^*} - \sigma^2_{t^*-}) \neq 0 \text{ or } \mathbb{E}^{\mathbb{Q}}_{t^*-}(\phi_{t^*} - \phi_{t^*-}) \neq 0\}. \end{aligned} \tag{10}$$

We note that the union of Ω_0 and Ω_A is not Ω. The remaining part of the sample space consists of outcomes for which $\mathbb{E}^{\mathbb{Q}}_{t^*-}(\sigma^2_{t^*} - \sigma^2_{t^*-}) = \mathbb{E}^{\mathbb{Q}}_{t^*-}(\phi_{t^*} - \phi_{t^*-}) = 0$ but either $\mathbb{E}^{\mathbb{Q}}_{t^*-}(\sigma^2_{t^*} - \sigma^2_{t^*-})^2 \neq 0$ and/or $\mathbb{E}^{\mathbb{Q}}_{t^*-}(\phi_{t^*} - \phi_{t^*-})^2 \neq 0$. We discuss this situation in more detail after Theorem 1 below.

We start in Section 3.1 with designing a statistic that can separate the null from the alternative hypothesis on the basis of characteristic functions of price increments over three distinct short time windows. Following this, we derive a functional Central Limit Theorem (CLT) for a version of the statistic constructed from observable options in Section 3.2. Since the limit of the statistic is non-standard, we develop a bootstrap type procedure for determining the quantiles of the test statistic in Section 3.3, and finally we present the test in Section 3.4.

3.1 Return Characteristic Functions in Presence of Event Risk

We start first with decomposing the characteristic function of the log-price increment that covers a time window including t^* in terms of the event price jump and the non-event risks in the asset price. Towards this end, let us denote

$$\psi_t(u) = iua_t - \frac{u^2}{2}\sigma_t^2 + \phi_t \int_{\mathbb{R}} (e^{iuz} - 1)\nu(dz), \;\; u \in \mathbb{R}, \;\; t > 0. \tag{11}$$

Using successive conditioning, Lévy-Khinchine formula (see e.g., Theorem 8.1 in Sato (1999)), the assumption for the spot characteristics of x and the jump Δx_{t^*} in assumption A, we can write

$$\mathcal{L}_{t,T}(u) = e^{(t^*-t)\psi_{t^*-}(u)} \mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*} + (t+T-t^*)\psi_{t^*}(u)}\right), \tag{12}$$

where $t \in (t^* - \epsilon, t^*)$ and $t + T \in (t^*, t^* + \epsilon)$. If σ_t and ϕ_t do not jump at $t = t^*$, then $\mathcal{L}_{t,T}(u)/\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*}}\right)$ is the conditional characteristic function of the increment of a Lévy process and hence its logarithm scales linearly with T. This suggests a testable implication using conditional characteristic functions $\mathcal{L}_{t,T}(u)$ for three different values of T.

Let us consider $t^* - \epsilon < t < t + T_1 < t^* < t + T_2 < t + T_3 < t^* + \epsilon$. Note that the first time window $[t, t + T_1]$ does not include the event time t^* while the other two, $[t, t + T_2]$ and $[t, t + T_3]$, do. Conditional on Ω_0 and using (12), it is easy to see that we have

$$\mathcal{L}_{t,T_1}(u)^{\frac{T_3 - T_2}{T_1}} \mathcal{L}_{t,T_2}(u) = \mathcal{L}_{t,T_3}(u), \text{ for } \{u \in \mathbb{R} : \max\{T_1, T_3 - T_2\} |\Im(\psi_{t^*-}(u))| < \pi\}, \tag{13}$$

where the power in the above expression is uniquely defined by the principal value of the argument of the complex number. Note that we restrict the values of u in (13) to ensure that both $e^{T_1 \psi_{t^*-}(u)}$ and $e^{(T_3 - T_2)\psi_{t^*-}(u)}$ are away from the negative real axis where the complex power function has a discontinuity. Of course, since $\Im(\psi_{t^*-}(u)) = O_p(u)$ as $u \to \infty$, any value of u will be included for sufficiently small T.

Given the identity in (13), it is natural to consider the following quantity as a way to separate the situation in which σ_t and/or ϕ_t jump at $t = t^*$ with positive probability:

$$\mathcal{W}_{t,\mathbb{T}} = \int_{|u| \leq u_T} \left| \mathcal{L}_{t,T_1}(u)^{\frac{T_3 - T_2}{T_1}} \mathcal{L}_{t,T_2}(u) - \mathcal{L}_{t,T_3}(u) \right|^2 w(u) du, \tag{14}$$

for some positive-valued and continuous weight function w and u_T being a deterministic sequence of positive numbers. We will use weight functions with exponential tail decay, i.e., ones for which the following holds:

$$\int_{|z| > u} w(z) dz = o(e^{-\alpha u^\rho}) \text{ as } u \to \infty, \text{ for some } \alpha > 0 \text{ and } \rho > 0. \tag{15}$$

As an example, in our numerical analysis, we will use the probability density function of a normal random variable as our choice for the weight function w. This choice clearly satisfies the exponential tail decay requirement in (15) above.

We can compare the quantity $\mathcal{W}_{t,\mathbb{T}}$, which we will use to test our null hypothesis, with the one considered by Todorov (2020) for testing against presence of event risk. In the notation here, Todorov (2020) considers the following two alternative quantities

$$\widetilde{\mathcal{W}}^{(1)}_{t,\mathbb{T}} = \int_{\mathbb{R}} \left| \mathcal{L}_{t,T_3}(u) - \mathcal{L}_{t,T_1}(u)^{\frac{T_3}{T_1}} \right|^2 w(u) du, \ \widetilde{\mathcal{W}}^{(2)}_{t,\mathbb{T}} = \int_{\mathbb{R}} \left| \mathcal{L}_{t,T_3}(u) - \mathcal{L}_{t,T_2}(u)^{\frac{T_3}{T_2}} \right|^2 w(u) du. \tag{16}$$

If there is no event risk i.e., t^* is outside the interval $[t, t + T_3]$, then it is easy to see based on the analysis above that $\widetilde{\mathcal{W}}^{(1)}_{t,\mathbb{T}} = \widetilde{\mathcal{W}}^{(2)}_{t,\mathbb{T}} = 0$. On the other hand, if $t^* - \epsilon < t < t + T_1 < t^* < t + T_2 < t + T_3 < t^* + \epsilon$, which is our assumption here, $\widetilde{\mathcal{W}}^{(1)}_{t,\mathbb{T}} \neq 0$ and $\widetilde{\mathcal{W}}^{(2)}_{t,\mathbb{T}} \neq 0$ and this outcome is in the alternative hypothesis of Todorov (2020) therefore.

We have the following result for our distance measure $\mathcal{W}_{t,\mathbb{T}}$:

Theorem 1. *Suppose $t^* - \epsilon < t < t + T_1 < t^* < t + T_2 < t + T_3 < t^* + \epsilon$ and $T_1 \asymp T$, $T_2 \asymp T$ and $T_3 \asymp T$, for some $T \downarrow 0$. Let u_T satisfy*

$$u_T \to \infty \quad \text{and} \quad u_T^2 T \to 0. \tag{17}$$

We have:

(a) $\mathbb{P}(\mathcal{W}_{t,\mathbb{T}} \neq 0|\Omega_0) \to 0$.

(b) $\mathcal{W}_{t,\mathbb{T}} \asymp T$ *conditional on* Ω_A.

In Theorem 1, we restrict the rate of growth of u_T. With this restriction, given the expression for (11), we have that $\mathcal{L}_{t,T_1}(u)$ converges to 1 as $T \to 0$. This restriction on u_T guarantees that, with probability approaching 1, the range requirement for u in (13) is satisfied. Conditional on Ω_A, the distance measure $\mathcal{W}_{t,\mathbb{T}}$ is $O_p(T)$. We note in this regard that in the definition of Ω_A we require either $\mathbb{E}^{\mathbb{Q}}_{t^*-}(\sigma^2_{t^*}) \neq \sigma^2_{t^*-}$ and/or $\mathbb{E}^{\mathbb{Q}}_{t^*-}(\phi_{t^*}) \neq \phi_{t^*-}$. If this is not the case but we still have jump at $t = t^*$ in either σ^2_t or ϕ_t with positive probability, i.e., if $\mathbb{E}^{\mathbb{Q}}_{t^*-}(\sigma^2_{t^*} - \sigma^2_{t^*-})^2 \neq 0$ and/or $\mathbb{E}^{\mathbb{Q}}_{t^*-}(\phi_{t^*} - \phi_{t^*-})^2 \neq 0$, then one can show that $\mathcal{W}_{t,\mathbb{T}} \asymp T^2$.

Since for sufficiently small T, u_T exceeds the absolute value of any u, the measure $\mathcal{W}_{t,\mathbb{T}}$ can detect jumps in either σ^2_t or ϕ_t. For higher values of $|u|$, $\psi_t(u)/u^2$ converges to $-\frac{1}{2}\sigma^2_t$. Therefore, if in defining the statistic $\mathcal{W}_{t,\mathbb{T}}$, we use only large in absolute value u-s, then we can detect expected jumps only in σ^2_t at $t = t^*$.

3.2 Infeasible Limit Theory

We note that $\mathcal{W}_{t,\mathbb{T}}$ is an integral over a function of u, which we can estimate from the data. Therefore, we will need to derive functional convergence results for the characteristic function estimates $\widehat{\mathcal{L}}_{t,T}(u)$. This is what we do in this section. The functions that we consider take value in the complex-valued Hilbert space $\mathcal{L}^2(w)$:

$$\mathcal{L}^2(w) = \left\{ f : \mathbb{R} \to \mathbb{C} \middle| \int_{\mathbb{R}} |f(u)|^2 w(u) du < \infty \right\}, \tag{18}$$

where w is the weight function in (14) and satisfys the exponential tail decay condition in (15). The inner product on $\mathcal{L}^2(w)$ is induced from the inner products of its real and imaginary parts, i.e., for f and g two elements of $\mathcal{L}^2(w)$, we set

$$\langle f, g \rangle = \int_{\mathbb{R}} f(z)\overline{g(z)} w(z) dz. \tag{19}$$

Next, for a random complex function Z taking values in $\mathcal{L}^2(w)$, we introduce the covariance operator $Kh = \mathbb{E}[(Z - \mathbb{E}(Z))\langle h, Z - \mathbb{E}(Z)\rangle]$ and the relation operator $Ch = \mathbb{E}[(Z - \mathbb{E}(Z))\langle h, \overline{Z - \mathbb{E}(Z)}\rangle]$, where $h \in \mathcal{L}^2(w)$. We recall that a Gaussian law on $\mathcal{L}^2(w)$ is uniquely identified by the mean, covariance and relation operators and we denote it with $\mathcal{CN}(\mu, K, C)$, for μ being the mean, K being the covariance and C being the relation operator, see e.g., Section 2 of Cerovecki and Hörmann (2017).

We are now ready to derive our functional convergence results. We denote the counterpart of $\mathcal{W}_{t,\mathbb{T}}$ estimated from the data with $\widehat{\mathcal{W}}_{t,\mathbb{T}}$. We further define

$$\widehat{Z}_{t,\mathbb{T}}(u) = \begin{cases} \widehat{\mathcal{L}}_{t,T_1}(u)^{\frac{T_3-T_2}{T_1}}\widehat{\mathcal{L}}_{t,T_2}(u) - \widehat{\mathcal{L}}_{t,T_3}(u) - (\mathcal{L}_{t,T_1}(u)^{\frac{T_3-T_2}{T_1}}\mathcal{L}_{t,T_2}(u) - \mathcal{L}_{t,T_3}(u)), & \text{if } |u| \leq u_T, \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

where u_T is the sequence used in $\mathcal{W}_{t,\mathbb{T}}$. From Theorem 1, conditional on Ω_0 and with probability approaching one, we have that $||\widehat{\mathcal{W}}_{t,\mathbb{T}}|| = ||\widehat{Z}_{t,\mathbb{T}}||$. For characterizing the behavior of $\widehat{\mathcal{W}}_{t,\mathbb{T}}$ conditional on Ω_0, we need therefore a functional CLT for $\widehat{Z}_{t,\mathbb{T}}$. This is given in the following theorem in the statement of which we use s_Δ defined in assumption B1.

Theorem 2. *Suppose assumptions A and B1-B3 hold and consider $t < t^* < t+T_1 < t^* < t+T_2 < t+T_3$. Let $t \uparrow t^*$, together with $\Delta \to 0$, $T_1 \asymp T$, $T_2 \asymp T$ and $T_3 \asymp T$, for some $T \downarrow 0$, and $T^\alpha/\Delta \to 0$, for some arbitrary big $\alpha > 0$. Let u_T satisfy (17). We then have*

$$\frac{1}{s_\Delta\sqrt{\Delta}}\widehat{Z}_{t,\mathbb{T}} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} Z, \tag{21}$$

with Z defined on an extension of the original probability space and having $\mathcal{F}^{(0)}$-conditional law of $\mathcal{CN}(0, K, C)$, for K and C being covariance and relation operators with integral representations

$$Kh(z) = \int_{\mathbb{R}} k(z,u)h(u)w(u)du, \;\; Ch(z) = \int_{\mathbb{R}} c(z,u)h(u)w(u)du, \quad \forall h \in \mathcal{L}^2(w), \tag{22}$$

for some functions $k(z,u)$ and $c(z,u)$ defined in (61)-(62) the proof.

The rate of convergence of $\widehat{Z}_{t,\mathbb{T}}$ is determined by the mesh of the strike grid and the asymptotic size of the observation error. The above limit result is therefore of joint type, i.e., both $\Delta \downarrow 0$ and $T \downarrow 0$. The requirement that $T^\alpha/\Delta \to 0$, for some arbitrary big $\alpha > 0$, is relatively weak and can be further relaxed. It is needed because $\widehat{Z}_{t,\mathbb{T}}(u)$ takes value of zero for $|u| > u_T$ while the limit $Z(u)$ does not. We do not provide explicit expressions for K and C here as we will not need them in our feasible implementation of the above result. These quantities are governed by the conditional distribution of the jump Δx_{t^*}. If the variance of the latter is zero or of asymptotically shrinking size, then the rate of convergence in (21) will change as well.

We can compare the above limit result with that of Todorov (2020) given in Theorem 1 of that paper. The test statistic of Todorov (2020) is the feasible counterpart of the quantity in (16). The asymptotic result in Theorem 2 and the one in Theorem 1 of Todorov (2020) are very different. We derive the asymptotic result under the assumption that event risk is present while the CLT result in Todorov (2020) holds only under the assumption of no event risk. That results in a very different asymptotic behavior of the cross-sections of options considered in the analysis in our setting and that of Todorov (2020). Indeed, when no event risk is present, then the option prices shrink asymptotically to zero as their tenor shrinks. Moreover, this rate of time decay of the option prices is different depending on the distance of the strikes of the options to the current stock price. This does not happen when event risk is present in the underlying price between the observation time and the expiration of the options. As a result, the rate of convergence of the statistic in Todorov (2020) is faster than the one here. In addition, the limit of our statistic here depends on the observation errors of all options used in the estimation while the limit of the statistic in Todorov (2020) is driven only by the option prices with the strikes in the vicinity of the current stock price given their asymptotically dominant role in the case of no event risk.

3.3 Feasible Limit Theory

For feasible inference, we will need estimates of the $\mathcal{F}^{(0)}$-conditional variance of the option observation errors. They are not directly observed but we can take advantage of the in-fill asymptotic setting here and the fact that the true option prices, with strikes away from X_t, are differentiable as functions of their strikes. A natural choice for an error estimate would be to use $\sqrt{\frac{2}{3}}\left[\widehat{O}_{t,T_i}(K_j) - \frac{1}{2}\left(\widehat{O}_{t,T_i}(K_{j-1}) + \widehat{O}_{t,T_i}(K_{j+1})\right)\right]$ as done in Andersen et al. (2021) and Todorov (2020) for example. Since $\Delta \downarrow 0$, $O_{t,T_i}(K_j) - \frac{1}{2}\left(O_{t,T_i}(K_{j-1}) + O_{t,T_i}(K_{j+1})\right)$ is approximately zero, and the above is an estimate of $\epsilon_{t,T_i}(K_j) - \frac{1}{2}\left(\epsilon_{t,T_i}(K_{j-1}) + \epsilon_{t,T_i}(K_{j+1})\right)$. This approach is not going to work very well, however, in finite samples with relatively coarse strike grid for two reasons. First, it does not utilize the fact that the estimation error is proportional to the option price and the latter varies a lot across strikes. Second, when the strike grid is coarse, the convexity of the true option price can be nontrivial, particularly for strikes close to the current stock price.

For this reason, we propose an alternative estimate for the volatility of the observation error, which utilizes the observation error structure of assumption B1. Our estimate for $\epsilon_{t,T_i}(K_j)$ is given by

$$\widehat{\epsilon}_{t,T_i}(K_j) = \widehat{s}_{i,\Delta}\widehat{O}_{t,T_i}(K_j),\ j = 1, ..., N_{t,T_i},\ i = 1, 2, 3, \tag{23}$$

where the estimates $\widehat{s}_{i,\Delta}$ are given by

$$\widehat{s}_{i,\Delta} = \begin{cases} \sqrt{\frac{2}{3}\frac{1}{|I_t^i|}\sum_{j\in I_t^i}\left(\log(\widehat{O}_{t,T_i}(K_j)) - \frac{1}{2}\log(\widehat{O}_{t,T_i}(K_{j-1})) - \frac{1}{2}\log(\widehat{O}_{t,T_i}(K_{j+1}))\right)^2}, & \text{if } I_t^i \neq \emptyset, \\ 1, \quad \text{if } I_t^i = \emptyset, \end{cases} \tag{24}$$

and the sets I_t^i are defined as:

$$I_t^1 = \left\{j \in \{2, ..., N_{t,T_1} - 1\} \setminus \{j_1^*\} : |K_j - X_t| \leq \sqrt{T_1} \times C_{t,1}\right\}, \tag{25}$$

$$I_t^i = \left\{j \in \{2, ..., N_{t,T_i} - 1\} \setminus \{j_i^*\} : \widehat{O}_{t,T_i}(K_j) > C_{t,i}\right\},\ i = 2, 3, \tag{26}$$

with $\{C_{t,i}\}_{i=1,2,3}$ being $\mathcal{F}_t^{(0)}$-adapted and strictly positive random variables and j_i^* being the smallest element of $\{2, ..., N_{t,T_i} - 1\}$ for which $|K_j - X_t|$ is the smallest.

We make several observations regarding $\widehat{\epsilon}_{t,T_i}(K_j)$. First, due to the fact that the observation error is asymptotically shrinking, nonlinear transformations of the option price can be made in the estimation of the $\mathcal{F}^{(0)}$-conditional variance of $\epsilon_{t,T_i}(K_j)$. Second, the use of the log transformation in $\widehat{s}_{i,\Delta}$ allows us to estimate the variance of $\epsilon_{t,T_i}(K_j)/O_{t,T_i}(K_j)$ which does not depend on the strike. This way we can pool information across strikes and significantly reduce the error in estimating the option error variance. Third, by taking logarithm of the option prices in $\widehat{s}_{i,\Delta}$, we reduce the positive bias in the estimation that is due to the convexity of the true option price because for a convex, positive and twice-differentiable function $f(k)$, we have $f(k)(\log(f(k))'' \leq f''(k)$. Finally, the sets of strikes, $\{I_t^i\}_{i=1,2,3}$, used in the estimation of $\widehat{s}_{i,\Delta}$, are determined so that the true option prices are strictly above zero, which carries over to the ratio $\frac{O_{t,T_i}(K_j)}{\sqrt{O_{t,T_i}(K_{j-1})O_{t,T_i}(K_{j+1})}}$ as well. Note that this leads to slightly different sets I_t^1 and $\{I_t^i\}_{i=2,3}$. The reason for this is that the event risk takes place after the expiration of the shortest tenor options and before that of the rest of the options used in forming the test. As a result, $\{O_{t,T_1}(K_j)\}_{j\geq 1}$ shrink asymptotically while $\{O_{t,T_i}(K_j)\}_{j\geq 1}$, for $i = 1, 2$, do not.

We continue next with describing a feasible CLT using the estimates $\widehat{\epsilon}_{t,T_i}(K_j)$. As discussed in the previous section, the limit distribution in (21) is non-standard and we will therefore develop a wild bootstrap approach for evaluating its quantiles. With this in mind, using the noisy proxies for the observation error, we generate new option prices as follows:

$$\widehat{O}^*_{t,T_i}(K_j) = \widehat{O}_{t,T_i}(K_j) + \widehat{\epsilon}_{t,T_i}(K_j) z_{i,j}, \quad j = 1, ..., N_{t,T_i}, \quad i = 1, 2, \tag{27}$$

where $\{z_{1,j}\}_{j=1}^{N_{t,T_1}}$, $\{z_{2,j}\}_{j=1}^{N_{t,T_2}}$ and $\{z_{3,j}\}_{j=1}^{N_{t,T_3}}$ are three i.i.d. sequences of standard normal variables defined on an extension of the original probability space and independent from $\mathcal{F}$ and from each other. We then define $\widehat{\mathcal{L}}^*_{t,T_i}$ from $\widehat{\mathcal{L}}_{t,T_i}$ by replacing $\widehat{O}_{t,T_i}(K_{i,j})$ with $\widehat{O}^*_{t,T_i}(K_{i,j})$. With this notation, we set

$$\widehat{Z}^*_{t,\mathbb{T}}(u) = \begin{cases} \widehat{\mathcal{L}}^*_{t,T_1}(u)^{\frac{T_3-T_2}{T_1}} \widehat{\mathcal{L}}^*_{t,T_2}(u) - \widehat{\mathcal{L}}^*_{t,T_3}(u) - (\widehat{\mathcal{L}}_{t,T_1}(u)^{\frac{T_3-T_2}{T_1}} \widehat{\mathcal{L}}_{t,T_2}(u) - \widehat{\mathcal{L}}_{t,T_3}(u)), & \text{if } |u| \leq u_T, \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

The following theorem shows that the $\mathcal{F}$-conditional limit distribution of $\widehat{Z}^*_{t,\mathbb{T}}$ is the same as that of $\widehat{Z}_{t,\mathbb{T}}$.

Theorem 3. *In the setting of Theorem 2, we have*

$$\frac{1}{s_\Delta \sqrt{\Delta}} \widehat{Z}^*_{t,\mathbb{T}} \stackrel{\mathcal{L}|\mathcal{F}}{\longrightarrow} Z, \tag{29}$$

with Z defined on an extension of the original probability space and having the same $\mathcal{F}$-conditional law as that of the limit in Theorem 2.

3.4 The Test

We are now ready to state formally our test, which will be based on $\widehat{\mathcal{W}}_{t,\mathbb{T}}$. Under the null hypothesis, i.e., conditional on Ω_0, we have $\widehat{\mathcal{W}}_{t,\mathbb{T}} = ||\widehat{Z}_{t,\mathbb{T}}||^2$ with probability approaching one and the $\mathcal{F}^{(0)}$-conditional quantiles of $||\widehat{Z}_{t,\mathbb{T}}||$ can be estimated by those of $||\widehat{Z}^*_{t,\mathbb{T}}||$ using simulation. More specifically, we denote

$$\widehat{cv}_\alpha = Q_{1-\alpha}(||\widehat{Z}^*_{t,\mathbb{T}}||^2\,|\mathcal{F}), \quad \alpha \in (0,1), \tag{30}$$

where $Q_\alpha(Z)$ is the α-quantile of the random variable Z. We can evaluate $\widehat{cv}_\alpha$ easily via simulation. The following corollary follows from Theorems 1, 2 and 3:

Corollary 1. *In the setting of Theorem 2 and for $\alpha \in (0,1)$, we have*

(a)

$$\mathbb{P}\left(\widehat{\mathcal{W}}_{t,\mathbb{T}} > \widehat{cv}_\alpha|\Omega_0\right) \;\longrightarrow\; \alpha. \tag{31}$$

(b)

$$\mathbb{P}\left(\widehat{\mathcal{W}}_{t,\mathbb{T}} > \widehat{cv}_\alpha|\Omega_A\right) \;\longrightarrow\; 1, \quad \textit{provided } s_\Delta\sqrt{\Delta}/T \to 0. \tag{32}$$

The requirement $s_\Delta\sqrt{\Delta}/T \to 0$ in part (b) of the above corollary is natural given the results of Theorems 1 and 2. If this condition does not hold, then the option observation error is too big relative to the changes caused by the jumps in volatility and/or jump intensity to the return distribution.

We finish this section with a brief discussion about a natural potential alternative to the above testing procedure that is based on high-frequency returns of the underlying asset. Mainly, one can form estimates of volatility in local blocks before and after the event time and using these two estimates test for volatility jump at the event time, see e.g., Jacod and Todorov (2010). There are several important differences between the two approaches. The first is that here we use data strictly before the occurrence of the event for conducting the test while with the high-frequency return-based approach we have to use data after the event too. Second, with high-frequency return data, in general, we can only recover jump in the diffusive volatility but we cannot say anything about the intensity of the jumps. This is due to the rare nature of jumps: over a short interval of time jumps might not occur while their intensity is nevertheless strictly positive. Third, with the high-frequency return data one has access only to the realized volatility jump. By contrast with the option data, we can infer risk-neutral moments of the jump in the diffusive volatility and the jump intensity. As a result, we can have a situation in which $\omega \in \Omega_A$ but nevertheless no volatility jump takes place at the event time. Fourth, the estimators of spot diffusive volatility from option data can in some cases have higher precision than the return-based ones, see e.g., Todorov and Zhang (2022). Finally, with the option data one can study the pricing of the volatility jump risk at the event time as we will show later on in the empirical section.

4 Monte Carlo Study

4.1 Setup and Choice of Tuning Parameters

The model for the underlying asset price in the Monte Carlo is specified by the following choices:

- The jumps outside t^* are specified by

$$\nu(dz) = \frac{\lambda^2}{2}\frac{e^{-\lambda|z|}}{|z|}, \text{ and } \phi_s = \sigma^2_{t^*-}. \tag{33}$$

  The Lévy measure ν corresponds to Variance Gamma Lévy process, and we set its scale parameter so that $\int_{\mathbb{R}} z^2\nu(dz) = 1$. We choose $\lambda = 50$, which implies Lévy tail decay that is roughly consistent with that found in previous studies.

- The instantaneous drift term is given by

$$a_s = -\frac{1}{2}\sigma_s^2 + \sigma^2_{t^*-}\frac{\lambda^2}{2}\log\left(1 - \frac{1}{\lambda^2}\right). \tag{34}$$

- For the event jump, we set

$$\Delta x_{t^*} \sim N(-\sigma_*^2/2, \sigma_*^2), \tag{35}$$

  for some parameter σ_*. With this specification, we have $\mathbb{E}_{t^*-}(e^{\Delta x_{t^*}}) = 1$.

- The diffusive volatility after the event is specified as

$$\sigma^2_{t^*}|\mathcal{F}_{t^*-} \text{ is Inverse Gaussian with mean } \mu^* \text{ and standard deviation } v^*. \tag{36}$$

We consider different values for the initial diffusive volatility σ_{t^*-}, the variance of the event jump σ_*^2, and the parameters μ_* and v_* of the conditional distribution of $\sigma^2_{t^*}$. The different parameter configurations are given in Table 1. The first five columns correspond to different configurations in which the diffusive volatility does not jump at t^*, which is the null hypothesis we test. We consider three different levels of the diffusive volatility: low, medium and high. For each of these three levels of volatility, we consider a scenario with small and big event risk. The small even risk scenario is one in which the variance of the event risk is half of the diffusive variance over an interval of five days and in the big event risk scenario we double that number. In the specifications under the alternative, reported in the last five columns of Table 1, we add a volatility jump at t^* to their counterparts under the null hypothesis. In all considered scenarios, the mean of $\sigma^2_{t^*}$ is double the value of $\sigma^2_{t^*-}$ and the standard deviation of $\sigma^2_{t^*}$ is 20% of its mean.

Turning next, to the specification of the observation setup, we set $t - t^* = 7/252$, $T_1 = 5/252$, $T_2 = 10/252$, $T_3 = 15/252$. This corresponds to options which expire in one, two and three weeks from now, respectively and the event is approximately in the middle of the second week. This setup

Table 1: Monte Carlo Parameter Settings

Case	**Parameters**				**Case**	**Parameters**			
	$\sigma^2_{t^*-}$	σ^2_*	μ_*	v_*		$\sigma^2_{t^*-}$	σ^2_*	μ_*	v_*
N-L-S	0.005	$0.0025 \times 5/252$	0.005	0	A-L-S	0.005	$0.0025 \times 5/252$	0.0075	0.0075×0.2
N-M-S	0.010	$0.0050 \times 5/252$	0.010	0	A-M-S	0.010	$0.0050 \times 5/252$	0.0150	0.0150×0.2
N-H-S	0.020	$0.0100 \times 5/252$	0.020	0	A-H-S	0.020	$0.0100 \times 5/252$	0.0300	0.0300×0.2
N-L-B	0.005	$0.0050 \times 5/252$	0.005	0	A-L-B	0.005	$0.0050 \times 5/252$	0.0075	0.0075×0.2
N-M-B	0.010	$0.0100 \times 5/252$	0.010	0	A-M-B	0.010	$0.0100 \times 5/252$	0.0150	0.0150×0.2
N-H-B	0.020	$0.0200 \times 5/252$	0.020	0	A-H-B	0.020	$0.0200 \times 5/252$	0.0300	0.0300×0.2

mimics roughly the one in the empirical study. The current value of the underlying asset price is set to $X_0 = 250$. The mesh of the strike grid Δ is set to 1. For each maturity, starting from a strike equal to X_0, we keep adding strikes above and below X_0 at increments of Δ until the true option price falls below 0.01. This strike grid mimics roughly the one of observable individual equity options that we will use in our empirical analysis. Finally, observed options are contaminated with error, i.e., we observe

$$\widehat{O}_{t,T_i}(K_j) = (1 + 0.1 \times z_{t,i,j})O_{t,T_i}(K_j), \tag{37}$$

where $z_{t,i,j}$ are i.i.d. across both i and j, and each $z_{t,i,j}$ has a truncated standard normal distribution with truncation interval $(-1.96, 1.96)$.

We finish this section with describing our choice of tuning parameters. First, we set u_T to

$$\widehat{u}_T = \inf\left\{u \geq 0 : |\widehat{\mathcal{L}}_{t,T_1}(u)| \leq 0.5\right\}. \tag{38}$$

Second, we set the weight function to

$$w(u) = \exp\left(-\frac{u^2}{\overline{u}_T^2}\right), \ \overline{u}_T = \inf\left\{u \geq 0 : |\widehat{\mathcal{L}}_{t,T_3}(u)| \leq 0.5\right\}. \tag{39}$$

Finally, the random variables $C_{t,i}$ in the definition of the sets I_t^i are set to

$$C_{t,1} = 4 \times \sigma_{t,1}^{BSIV} \quad \text{and} \quad C_{t,2} = 0.01, \tag{40}$$

where $\sigma_{t,1}^{BSIV}$ denotes the at-the-money Black-Scholes implied volatility extracted from the options with tenor T_1.

4.2 Results

The results from the Monte Carlo are reported in Table 2. They indicate good finite sample performance of the test under the null hypothesis, with empirical rejection rates close to the significance level of the test in the various configurations under the null hypothesis. The test has good power also under the considered alternatives. The power is higher for higher initial volatility σ_{t^*-}. The reason for this is that when volatility is high, the available number of options is higher as more strikes are needed to cover the effective support of the return distribution. As a result, the effect from the observation error on the estimation gets reduced, which in turn allows for a higher power of the test. We further note that the power of the test decreases when the variance of the jump Δx_{t^*} increases. The reason for this is that the information about the jump in σ_t and ϕ_t in the returns over time increments including t^* gets reduced, in relative terms, when the event risk is larger. This effect is similar to the standard deconvolution problem in which higher noise makes it more difficult to learn about the signal.

Table 2: Monte Carlo Results

Case	**Significance Level**			**Case**	**Significance Level**		
	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 1\%$		$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 1\%$
N-L-S	10.06%	5.56%	1.36%	A-L-S	56.08%	43.20%	20.78%
N-M-S	10.50%	5.34%	1.02%	A-M-S	72.64%	60.48%	35.18%
N-H-S	10.60%	5.56%	1.28%	A-H-S	87.28%	78.28%	53.08%
N-L-B	10.00%	5.42%	1.62%	A-L-B	51.36%	38.28%	18.62%
N-M-B	10.16%	5.48%	1.60%	A-M-B	64.84%	51.06%	26.94%
N-H-B	10.10%	5.06%	1.24%	A-H-B	82.32%	70.78%	46.04%

Note: The entries in the table correspond to the empirical rejection rates of the volatility jump test (in percentage) based on $5,000$ Monte Carlo draws.

5 Empirical Analysis of Earnings Announcement Volatility

In this section, we perform the proposed test for jumps in spot volatility at event times for a sample of individual stocks.[6] We focus on the event risk generated by the pre-scheduled quarterly earnings

[6]As recently documented in Jeon et al. (2022), jumps in stock prices are typically associated with important news releases. Our focus here is on a specific type of firm-related news which are pre-scheduled (and hence there is no uncertainty about their time arrival).

announcements.[7] We select a sub-sample of 38 individual stocks from the S&P 100 Index across different sectors with liquid options. Our sample period is from January 2, 2014 until December 31, 2021. For each stock, we obtain the earnings announcement history from the Zacks earnings calendar, where both earnings dates and times (e.g., after close of before open) are provided.[8] We align the "after close" with the "before open" announcements by adding one business day to the original "after close" announcement dates. Over the period from January 2, 2014 until December 31, 2021, each stock in the sample has 32 earnings announcement events.

The option data is obtained from OptionMetrics. It consists of closing best bid and ask quotes. For each day in the sample, we keep the three shortest available tenors with time-to-maturity of at least 2 business days. We consider only tenors for which there are at least 10 out-of-the-money options with different strikes and non-zero bid quotes. The moneyness is determined by the implied forward rate, which in turn is computed by using put-call parity for three distinct strikes with the smallest gap between call and put mid-quotes. To perform the specification test, three distinct tenors are needed, among which only the last two include the earnings event. Since weekly options for individual stocks always expire on Fridays, we first locate the Friday prior to the week of the earnings announcement, which we refer to as the first prior-earning Friday.[9] We then find the Friday in the week before the first prior-earning Friday and perform the specification test on this date, i.e., on the so-called second prior-earning Friday. If the second prior-earning Friday is a holiday, we then use the following trading day.

Summary statistics for the data are reported in Table 3. On average, the number of strikes for the different stocks is high, which is important for the good finite sample performance of the test. The stocks with the highest number of strikes under consideration are Amazon (AMZN), Google (GOOG), Netflix (NFLX), and Tesla (TSLA). For most stocks, the strike range is also wide. Of course, the length of the strike range naturally depends on the volatility of the underlying stock, with more volatile stocks such as Tesla having wider strike range. We also note that the strike range for essentially all stocks is a bit skewed to the left, relative to the current stock price, which is manifestation of the negative skew of the risk-neutral return distribution. This effect is quite mild though compared to the one observed for market index options.

We implement the test for every stock-announcement pair in the sample. The results from the

[7]The results of the current paper can be also used to study anticipated changes in spot volatility due to macroeconomic announcements. There is a large literature in finance that studies various effects of macroeconomic announcement on asset prices, see e.g., Cochrane and Piazzesi (2002), Rigobon and Sack (2004), Bernanke and Kuttner (2005), Savor and Wilson (2014), Lucca and Moench (2015), Ai and Bansal (2018) and Nakamura and Steinsson (2018), among many others.

[8]The Zacks earnings calendar is publicly accessible via https://www.zacks.com/stock/research/"XXX"/earnings-calendar, where "XXX" represents the ticker symbol of the stock.

[9]If Friday is a holiday, then preceding Thursday is used instead.

Table 3: Summary Statistics for Short-dated Options across Earnings Announcements

Ticker	# of Strikes	$\log(K/S)_{min}$	$\log(K/S)_{max}$	Ticker	# of Strikes	$\log(K/S)_{min}$	$\log(K/S)_{max}$
AAPL	43	0.76	1.22	JNJ	23	0.85	1.09
AMZN	139	0.71	1.30	JPM	30	0.81	1.14
AXP	29	0.83	1.12	KO	21	0.86	1.12
BA	40	0.76	1.23	MCD	22	0.86	1.10
BAC	21	0.79	1.16	MMM	22	0.85	1.11
BLK	35	0.83	1.10	MRK	23	0.88	1.09
C	28	0.82	1.15	MS	21	0.82	1.12
CAT	34	0.80	1.15	MSFT	31	0.81	1.13
COF	31	0.83	1.12	NFLX	63	0.67	1.36
CSCO	23	0.83	1.14	NKE	30	0.84	1.14
CVX	26	0.87	1.10	NVDA	50	0.69	1.30
DIS	32	0.81	1.18	PFE	20	0.87	1.16
FB	41	0.75	1.23	PG	25	0.88	1.08
GE	18	0.77	1.23	T	17	0.85	1.11
GOOG	86	0.82	1.16	TSLA	76	0.56	1.46
GS	28	0.78	1.15	V	27	0.83	1.11
HD	26	0.82	1.11	VZ	17	0.87	1.08
IBM	28	0.82	1.14	WMT	30	0.86	1.14
INTC	26	0.81	1.16	XOM	26	0.87	1.10

Note: The table reports the average number of strikes per tenor, the minimum and maximum log-moneyness of the available options for each stock across earnings announcement events with valid option data (i.e., there are three consecutive weekly expiration dates available on the Friday two weeks prior to the scheduled earnings announcement date and the minimum number of strikes across all three tenors is higher than 10).

tests are summarized in Table 4. Overall, these results provide nontrivial evidence for the anticipated jump at the earnings announcement time t^* in the stocks' volatility and/or jump intensity. Indeed, at 10% significance level, the test rejects more than half of the times for the stock/announcement pairs in our sample. Table 4 also reveals some variation in the test results across the stocks in the sample. For example, for stocks such as Amazon, Google and Netflix, the rejection rates are very high. On the other end of the spectrum are stocks such as Facebook, Home Depot and IBM.

Table 4: Earnings Announcement Volatility Jump Test Results

Ticker	# of Obs.	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 1\%$	Ticker	# of Obs.	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 1\%$
AAPL	32	22	21	15	JNJ	30	21	15	9
AMZN	32	32	30	29	JPM	30	16	14	10
AXP	30	20	17	14	KO	12	8	5	4
BA	31	17	16	12	MCD	29	13	9	9
BAC	12	6	4	4	MMM	27	14	12	7
BLK	15	13	10	8	MRK	31	22	20	15
C	31	19	15	11	MS	28	19	15	14
CAT	32	19	16	13	MSFT	30	19	16	12
COF	27	21	20	16	NFLX	32	26	23	17
CSCO	16	11	11	9	NKE	28	15	10	6
CVX	32	23	21	18	NVDA	27	17	14	9
DIS	31	18	14	9	PFE	10	6	6	6
FB	32	15	14	11	PG	30	22	19	14
GE	6	3	2	2	T	21	7	4	2
GOOG	30	29	29	25	TSLA	32	24	22	20
GS	31	18	15	8	V	32	16	16	9
HD	31	8	6	5	VZ	26	8	7	5
IBM	31	14	8	5	WMT	30	19	12	8
INTC	25	12	9	6	XOM	32	23	22	16

Note: The table reports the total number of earnings announcement events with valid option data (i.e., there are three consecutive weekly expiration dates available on the Friday two weeks prior to the scheduled earnings announcement date and the minimum number of strikes across all three tenors is higher than 10), and the number of times the volatility jump test rejects for different significance levels.

We note that we conduct the test for every single earnings announcement in the sample. This is similar to the way tests for presence of jumps in high-frequency data are typically implemented. Of course, given the derived limit results in the paper, it is easy to design a test for the null hypothesis that there were no anticipated changes in volatility and jump intensity for a fixed number of earning announcements for a given stock. This can be done either using the supremum or the sum of the test statistics for all earning announcement events.

In Figures 3-4, we illustrate the data used in the construction of the test as well as the two transformations of the characteristic functions that are contrasted in $\widehat{\mathcal{W}}_{t,\mathcal{T}}$ for two earnings announcements of Facebook (FB) in 2021. As seen from the left panels of the two figures, the longer-dated options are considerably more expensive. This is to be expected as the longer dated options contain more risk than their short-dated counterparts. What is unique for the option prices prior to the earnings announcements displayed on the left panels of the figures is that the gaps between the option prices (with the same strike) with tenors T_2 and T_3 is much smaller than the one between those with tenors T_1 and T_2. This is due to the announcement risk anticipated by investors. Recall that the announcement takes place somewhere in the interval $(t+T_1, t+T_2)$ and the risk it generates is, therefore, not priced in the shortest tenor options.

The right panels of the two figures display $|\widehat{\mathcal{L}}_{t,T_1}(u)|^{\frac{T_3-T_2}{T_1}}|\widehat{\mathcal{L}}_{t,T_2}(u)|$ and $|\widehat{\mathcal{L}}_{t,T_3}(u)|$. Under the null hypothesis, these two quantities should be the same up to measurement error. This seems to be the case for the second of the two events with the two lines being almost on top of each other. For the announcement in Figure 3, however, there seems to be considerable and persistent difference between the two lines.

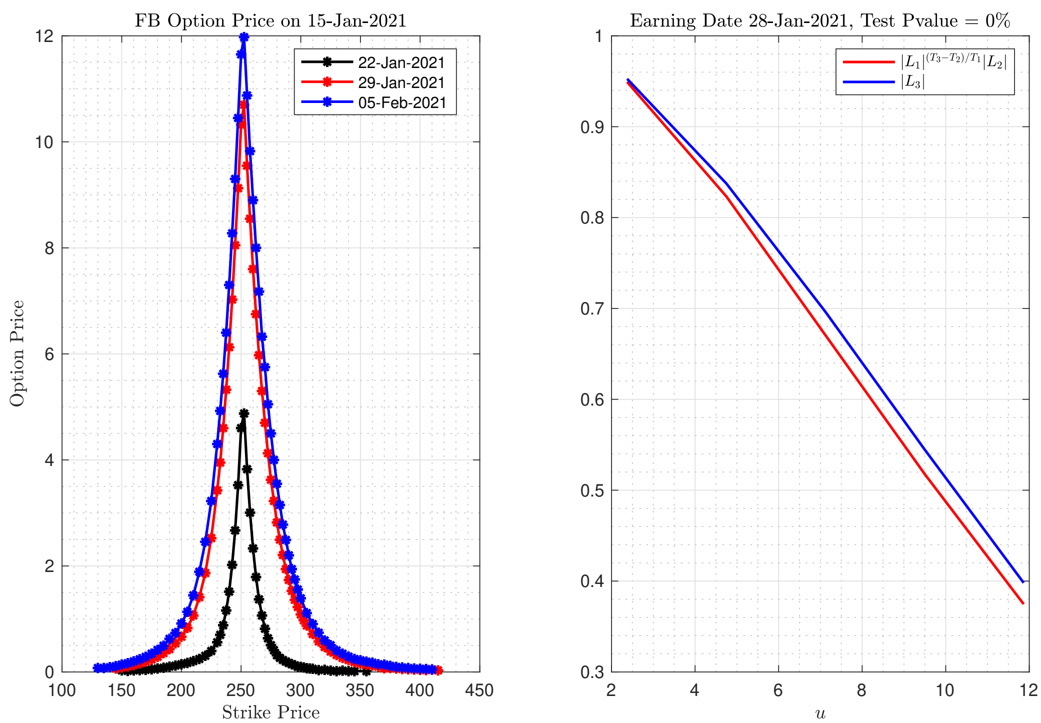

Figure 3: OTM Option Prices and Characteristic Function Estimates for Facebook on 15-Jan-2021.

Given the above evidence for jumps in σ_t and/or ϕ_t, it seems difficult in general to separate nonparametrically the expected volatility jump at t^* from the one in the jump intensity for many

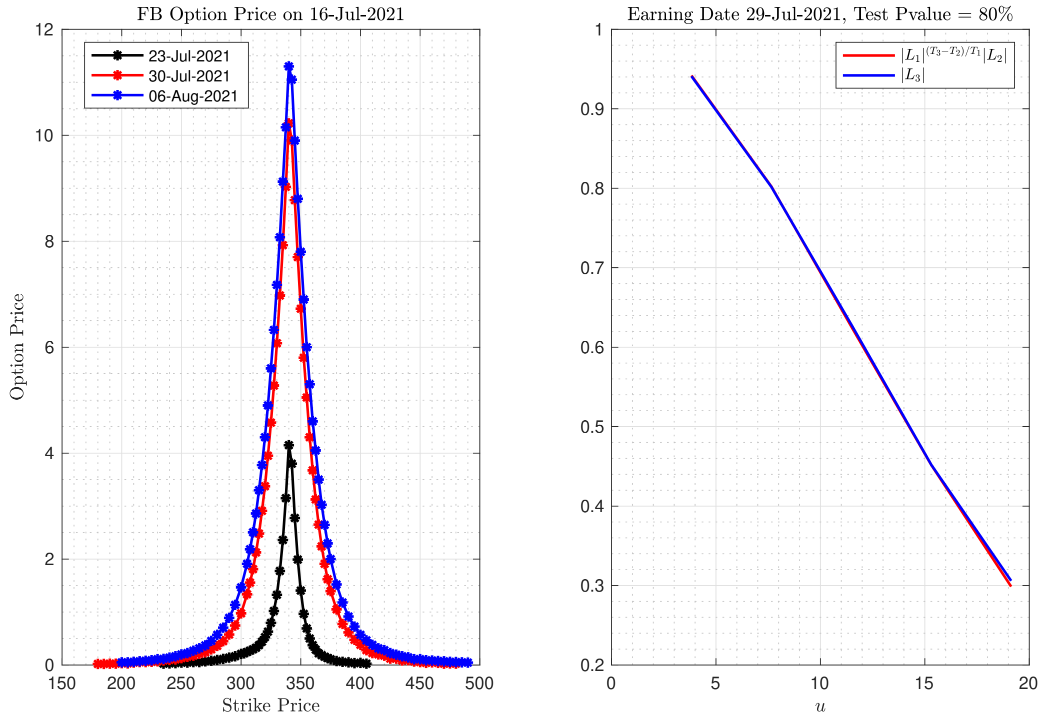

Figure 4: OTM Option Prices and Characteristic Function Estimates for Facebook on 29-July-2021.

of the earnings announcement events in the sample. The risk-neutral expectation of their sum is, however, easy to estimate using estimates of the risk neutral quadratic variation over the different tenors. The latter are given by

$$\widehat{QV}_{t,T_l} = \sum_{j=2}^{N_{t,T}} \left(1 - \log\left(\frac{K_{j-1}}{X_t}\right)\right) \frac{\widehat{O}_{t,T_l}(K_{j-1})}{K_{j-1}^2}(K_j - K_{j-1}), \quad l = 1,2,3. \tag{41}$$

Under similar conditions to the ones for proving the limit behavior of $\widehat{\mathcal{L}}_{t,T_l}(u)$ in our theoretical analysis, one can show that $\widehat{QV}_{t,T_l}$ is a consistent estimator for the expected risk-neutral quadratic variation $\mathbb{E}_t^{\mathbb{Q}}(QV_{t,T_l})$, where we denote

$$QV_{t,T} = \int_t^{t+T} \sigma_s^2 ds + \sum_{s\in[t,t+T]} (\Delta x_s)^2. \tag{42}$$

Now, for our setting with event risk (under assumption A), we have

$$\begin{aligned} &\mathbb{E}_t^{\mathbb{Q}}(QV_{t,T_1}) = T_1(\sigma_{t^*-}^2 + \phi_{t^*-}\int_{\mathbb{R}} z^2\nu(dz)), \\ &\mathbb{E}_t^{\mathbb{Q}}(QV_{t,T_3} - QV_{t,T_2}) = (T_3 - T_2)\mathbb{E}_{t^*-}^{\mathbb{Q}}(\sigma_{t^*}^2 + \phi_{t^*}\int_{\mathbb{R}} z^2\nu(dz)). \end{aligned} \tag{43}$$

Therefore, we can compare $\frac{1}{T_3-T_2}(\widehat{QV}_{t,T_3} - \widehat{QV}_{t,T_2})$ with $\frac{1}{N_t}\sum_{i:t_i\in(t+T_2,t+T_3]} \frac{1}{T_i}\widehat{QV}_{t_i,T_i}$, where the t_i-s in the summation are the times of market close on each of the trading days between $t + T_2$

and $t+T_3$, T_i is the shortest available tenor at time t_i, and N_t is the number of observations in the interval $(t+T_2, t+T_3]$. The sum $\frac{1}{N_t}\sum_{i:t_i\in(t+T_2,t+T_3]}\frac{1}{T_i}\widehat{QV}_{t_i,T_i}$ is a proxy for $\sigma^2_{t^*}+\phi_{t^*}\int_{\mathbb{R}}z^2\nu(dz)$. We note that the expected value $\mathbb{E}^{\mathbb{P}}\left(\sigma^2_{t^*}+\phi_{t^*}\int_{\mathbb{R}}z^2\nu(dz)-\mathbb{E}^{\mathbb{Q}}_{t^*-}(\sigma^2_{t^*}+\phi_{t^*}\int_{\mathbb{R}}z^2\nu(dz))\right)$ reflects risk premium for the jump in $\sigma^2_t+\phi_t\int_{\mathbb{R}}z^2\nu(dz)$ at the announcement time t^* demanded by investors. To study the potential existence of such premium, in Table 5, we report the time-series average of $\frac{1}{N_t}\sum_{i:t_i\in(t+T_2,t+T_3]}\frac{1}{T_i}\widehat{QV}_{t_i,T_i}$ (column Realized QV) and $\frac{1}{T_3-T_2}(\widehat{QV}_{t,T_3}-\widehat{QV}_{t,T_2})$ for each ticker. As the results of the table show, for most of the stocks, the option-implied expected value is above the realization on average.[10] This is an indication that this risk is priced by investors. The economic rationale for that is that earnings announcements contain information about systematic risk, particularly for big companies, and this carries over to the extra volatility triggered by the announcements.

Table 5: Earnings Announcement Volatility Jump Risk Premium

Ticker	Realized QV	QVD23	Ticker	Realized QV	QVD23	Ticker	Realized QV	QVD23
AAPL	0.2527	0.2693	GE	0.3105	0.3502	MSFT	0.2225	0.2230
AMZN	0.2724	0.3124	GOOG	0.2240	0.2374	NFLX	0.3608	0.3841
AXP	0.2258	0.2416	GS	0.2394	0.2608	NKE	0.2212	0.2239
BA	0.2931	0.3035	HD	0.2032	0.1940	NVDA	0.3832	0.4008
BAC	0.3089	0.3159	IBM	0.1972	0.1881	PFE	0.2751	0.2559
BLK	0.2269	0.2681	INTC	0.2762	0.2735	PG	0.1634	0.1658
C	0.2524	0.2710	JNJ	0.1680	0.1724	T	0.1906	0.2060
CAT	0.2635	0.2661	JPM	0.2152	0.2415	TSLA	0.4978	0.5597
COF	0.2824	0.2975	KO	0.1840	0.2065	V	0.2170	0.2166
CSCO	0.2104	0.2243	MCD	0.1750	0.1883	VZ	0.1647	0.1729
CVX	0.2366	0.2316	MMM	0.1997	0.2053	WMT	0.1893	0.1764
DIS	0.2107	0.2279	MRK	0.2153	0.1910	XOM	0.2348	0.2344
FB	0.2864	0.2964	MS	0.2653	0.2850			

Note: The table reports the time-series averages of $\frac{1}{N_t}\sum_{i:t_i\in(t+T_2,t+T_3]}\frac{1}{T_i}\widehat{QV}_{t_i,T_i}$ (column Realized QV) and $\frac{1}{T_3-T_2}(\widehat{QV}_{t,T_3}-\widehat{QV}_{t,T_2})$ (column QVD23) for each stock in the same sample.

[10] A formal test shows high statistical significance of this difference.

6 Conclusion

In this paper we consider the behavior of assets' volatility and jump intensity following the occurrence of pre-scheduled events. Our analysis is based on options with short time to maturity written on an underlying asset prior to the event and expiring before or shortly after it. We derive a functional relation between characteristic functions of returns with different horizons that can discriminate between the hypothesis that volatility and jump intensity are anticipated to remain unchanged following the event and the alternative hypothesis where this is not the case. Using functional limit theory, we derive an asymptotically valid test for no jump in volatility and/or jump intensity at the event time. In an empirical application to earnings announcements, we find evidence for volatility jump at the earnings announcement event times for many stocks and announcements. The implied earnings announcement volatility jump from options is on average higher than the realized one suggesting that investors are willing to pay in order to avoid this volatility jump risk.

On a general level, the results of the paper can be used for deciding on the way option data can be incorporated in the estimation of spot volatility when options are "contaminated" by event risk. Our results, and extensions of them, can be used also for the study of the volatility risk generated by important events.

7 Proofs

7.1 Proof of Theorem 1

The proof of part(a) follows by use of successive conditioning and taking into account that on Ω_0, $\mathbb{Q}_{t^*-}(\sigma^2_{t^*} = \sigma^2_{t^*-}) = 1$ and $\mathbb{Q}_{t^*-}(\phi_{t^*} = \phi_{t^*-}) = 1$. The equality in (13) holds for values of u for which $\max\{T_1, T_3 - T_2\}|\Im(\psi_{t^*-}(u))| < \pi$. Note that $|\int_{\mathbb{R}} \sin(uz)\nu(dz)| \leq C|u|$, for some positive constant C. Therefore, $\max\{T_1, T_3 - T_2\}|\Im(\psi_{t^*-}(u))| < \pi$ holds for $|u| < u^*_T = \frac{\pi}{\max\{T_1, T_3-T_2\}C\phi_{t^*-}}$. From here, since $u^2_T T \to 0$, we have the result in part (a).

We proceed with part (b). A first-order Taylor expansion, yields

$$e^{T_a\psi_{t^*}(u)} = 1 + T_a\psi_{t^*}(u) + R_{T_a}(u), \tag{44}$$

for some $R_{T_a}(u)$ satisfying $|R_{T_a}(u)| \leq C(u^2 T_a)^2$ with C being a positive constant. Similarly,

$$e^{T_b\psi_{t^*-}(u)} = 1 + T_b\psi_{t^*-}(u) + R_{T_b}(u), \tag{45}$$

for some $R_{T_b}(u)$ satisfying $|R_{T_b}(u)| \leq C_{t^*-}(u^2 T_b)^2$ with C being a positive constant.

Using these two results and (12), we can write for $|u| < u^*_T$ (u^*_T being the quantity defined in the proof of part (a) above):

$$\mathcal{L}_{t,T_1}(u)^{\frac{T_3-T_2}{T_1}}\mathcal{L}_{t,T_2}(u)-\mathcal{L}_{t,T_3}(u)=e^{(t^*-t)\psi_{t^*-}(u)} \times\left\{\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*}+(t+T_2-t^*)\psi_{t^*}(u)+(T_3-T_2)\psi_{t^*-}(u)}\right)-\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*}+(t+T_3-t^*)\psi_{t^*}(u)}\right)\right\}. \tag{46}$$

Next, using the above expansion results above, we get

$$\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*}+(t+T_2-t^*)\psi_{t^*}(u)+(T_3-T_2)\psi_{t^*-}(u)}\right)-\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*}+(t+T_3-t^*)\psi_{t^*}(u)}\right) = -(T_3-T_2)\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*}}(\psi_{t^*}(u)-\psi_{t^*-}(u))\right)+R_T(u), \tag{47}$$

for $R_T(u)$ satisfying $|R_T(u)|\le CT_3^2u^4$ and some positive constant C. Using the $\mathcal{F}_{t^*-}$-independence of Δx_{t^*} from $\sigma^2_{t^*}-\sigma^2_{t^*-}$ and $\phi_{t^*}-\phi_{t^*-}$, we have

$$\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*}}(\psi_{t^*}(u)-\psi_{t^*-}(u))\right)=\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*}}\right)\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(\psi_{t^*}(u)-\psi_{t^*-}(u)\right). \tag{48}$$

For u sufficiently high, $\psi_{t^*}(u)-\psi_{t^*-}(u)$ is dominated by $-\frac{u^2}{2}(\sigma^2_{t^*}-\sigma^2_{t^*-})$, and hence if $\mathbb{E}^{\mathbb{Q}}_{t^*-}(\sigma^2_{t^*}-\sigma^2_{t^*-})\neq 0$, we have

$$\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*}+(t+T_2-t^*)\psi_{t^*}(u)+(T_3-T_2)\psi_{t^*-}(u)}\right)-\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(e^{iu\Delta x_{t^*}+(t+T_3-t^*)\psi_{t^*}(u)}\right)\asymp T. \tag{49}$$

If $\mathbb{E}^{\mathbb{Q}}_{t^*-}(\sigma^2_{t^*}-\sigma^2_{t^*-})=0$, then since $\mathbb{E}^{\mathbb{Q}}_{t^*-}(\psi_{t^*}-\psi_{t^*-})\neq 0$ by the definition of Ω_A, we have again $\mathbb{E}^{\mathbb{Q}}_{t^*-}\left(\psi_{t^*}(u)-\psi_{t^*-}(u)\right)\neq 0$, and therefore the above result holds again.

7.2 Proof of Theorem 2

Throughout the proof we do the normalization $X_t=1$ and we set $\tau=\frac{T_3-T_2}{T_1}$. We start with establishing some preliminary results. We can decompose

$$\frac{1}{\sqrt{\Delta}}\left(\widehat{\mathcal{L}}_{t,T}(u)-\mathcal{L}_{t,T}(u)\right)=\overline{\mathcal{L}}_{t,T}(u)+R^{(1)}_{t,T}(u)+R^{(2)}_{t,T}(u), \tag{50}$$

where we denote

$$\overline{\mathcal{L}}_{t,T}(u)=-\frac{1}{\sqrt{\Delta}}(u^2+iu)\sum_{j=2}^{N_{t,T}}e^{iu\log(K_{j-1})}\frac{\epsilon_{t,T}(K_{j-1})}{K^2_{j-1}}(K_j-K_{j-1}), \tag{51}$$

$$R^{(1)}_{t,T}(u)=\frac{1}{\sqrt{\Delta}}(u^2+iu)\sum_{j=2}^{N_{t,T}}\int_{K_{j-1}}^{K_j}\left(\frac{e^{iu\log(K_{j-1})}O_{t,T}(K_{j-1})}{K^2_{j-1}}-\frac{e^{iu\log(K)}O_{t,T}(K)}{K^2}\right)dK, \tag{52}$$

$$R^{(2)}_{t,T}(u)=-\frac{1}{\sqrt{\Delta}}(u^2+iu)\int_0^{K_1}\frac{e^{iu\log(K)}O_{t,T}(K)}{K^2}dK-\frac{1}{\sqrt{\Delta}}\int_{K_{N_{t,T}}}^{\infty}\frac{e^{iu\log(K)}O_{t,T}(K)}{K^2}dK. \tag{53}$$

We start with analyzing $R^{(1)}_{t,T}(u)$ and $R^{(2)}_{t,T}(u)$. Using the definition of $O_{t,T}(K)$, we have

$$O_{t,T}(K)\le C_t\left(\mathbb{E}^{\mathbb{Q}}_t\left(e^{2|x_{t+T}-x_t|}\right)+1\right)\left(e^{-\log(K/X_t)}1_{\{K>X_t\}}+e^{3\log(K/X_t)}1_{\{K\le X_t\}}\right), \tag{54}$$

and

$$
\begin{aligned}
&|O_{t,T}(K_2) - O_{t,T}(K_1)| \le |K_2 - K_1| \mathbb{Q}(x_{t+T} - x_t \le \log(K_1/X_t)), \quad \text{for } K_1 \le K_2 \le X_t, \\
&|O_{t,T}(K_2) - O_{t,T}(K_1)| \le |K_2 - K_1| \mathbb{Q}(x_{t+T} - x_t \ge \log(K_2/X_t)), \quad \text{for } X_t \le K_1 \le K_2.
\end{aligned}
\tag{55}
$$

From here, we have altogether:

$$
|R^{(1)}_{t,T}(u)| \le C_t(|u|^3 \vee |u|)\sqrt{\Delta}, \quad |R^{(2)}_{t,T}(u)| \le C_t(|u|^3 \vee |u|)\left(K_1^2 + K_{N_{t,T}}^{-2}\right). \tag{56}
$$

Next, using assumption B1 (the $\mathcal{F}^{(0)}$-conditional independence of the observation errors in particular), it is easy to derive

$$
\mathbb{E}\big(|\overline{\mathcal{L}}_{t,T}(u)|^p \big| \mathcal{F}^{(0)}\big) \le C_t s_\Delta^p (|u|^2 \vee |u|)^p, \text{ for } p > 1, \tag{57}
$$

where the $\mathcal{F}^{(0)}_t$-adapted random variable C_t depends on p.

We are now ready to prove the limit result of the theorem. We denote

$$
\overline{Z}_{t,\mathbb{T}}(u) = \sqrt{\Delta}\tau \mathcal{L}_{t,T_2}(u)\overline{\mathcal{L}}_{t,T_1}(u) + \sqrt{\Delta}\overline{\mathcal{L}}_{t,T_2}(u) - \sqrt{\Delta}\overline{\mathcal{L}}_{t,T_3}(u). \tag{58}
$$

With the notation of u^*_T as in the proof of Theorem 1, we have the following decomposition:

$$
\begin{aligned}
\widehat{Z}_{t,\mathbb{T}}(u) - \overline{Z}_{t,\mathbb{T}}(u) &= \overline{Z}_{t,\mathbb{T}}(u) 1_{\{|u|>u_T\}} + (\widehat{\mathcal{L}}_{t,T_1}(u)^\tau - \mathcal{L}_{t,T_1}(u)^\tau)(\widehat{\mathcal{L}}_{t,T_2}(u) - \mathcal{L}_{t,T_2}(u))1_{\{|u|\le u_T\}} \\
&+ (R^{(1)}_{t,T_3}(u) + R^{(2)}_{t,T_3}(u))1_{\{|u|\le u_T\}} + \mathcal{L}_{t,T_1}(u)^\tau (R^{(1)}_{t,T_2}(u) + R^{(2)}_{t,T_2}(u))1_{\{|u|\le u_T\}} \\
&+ \tau\mathcal{L}_{t,T_1}(u)^{\tau-1}\mathcal{L}_{t,T_2}(u)(R^{(1)}_{t,T_1}(u) + R^{(2)}_{t,T_1}(u))1_{\{|u|\le u_T\wedge u^*_T\}}1_{\{|\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)|\le\frac{1}{2}\mathcal{L}_{t,T_1}(u)\}} \\
&+ (\tau(\mathcal{L}_{t,T_1}(u)^{\tau-1} - 1)\mathcal{L}_{t,T_2}(u)\sqrt{\Delta}\overline{\mathcal{L}}_{t,T_1}(u) + (\mathcal{L}_{t,T_1}(u)^\tau - 1)\sqrt{\Delta}\overline{\mathcal{L}}_{t,T_2}(u))1_{\{|u|\le u_T\}} \\
&+ \tau(\tau-1)\widetilde{\mathcal{L}}_{t,T_1}(u)^{\tau-2}\mathcal{L}_{t,T_2}(u)(\widehat{\mathcal{L}}_{t,T_1}(u) - \mathcal{L}_{t,T_1}(u))^2 1_{\{|u|\le u_T\wedge u^*_T\}}1_{\{|\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)|\le\frac{1}{2}\mathcal{L}_{t,T_1}(u)\}} \\
&+ \tau\mathcal{L}_{t,T_1}(u)^{\tau-1}\mathcal{L}_{t,T_2}(u)\sqrt{\Delta}\,\overline{\mathcal{L}}_{t,T_1}(u)1_{\{|u|\le u_T\}}1_{\{(|u|\le u^*_T \cap |\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)|\le\frac{1}{2}\mathcal{L}_{t,T_1}(u))^c\}} \\
&+ (\widehat{\mathcal{L}}_{t,T_1}(u)^\tau - \mathcal{L}_{t,T_1}(u)^\tau)1_{\{|u|\le u_T\}}1_{\{(|u|\le u^*_T \cap |\widehat{\mathcal{L}}_{t,T_1}(u)-\mathcal{L}_{t,T_1}(u)|\le\frac{1}{2}\mathcal{L}_{t,T_1}(u))^c\}},
\end{aligned}
\tag{59}
$$

where $\widetilde{\mathcal{L}}_{t,T_1}(u)$ is an intermediate value between $\widehat{\mathcal{L}}_{t,T_1}(u)$ and $\mathcal{L}_{t,T_1}(u)$. Using the bounds for $\overline{\mathcal{L}}_{t,T}(u)$, $R^{(1)}_{t,T}(u)$ and $R^{(2)}_{t,T}(u)$, derived above and taking into account the rate condition in assumption B3, we can then show

$$
\frac{1}{s_\Delta\sqrt{\Delta}}||\widehat{Z}_{t,\mathbb{T}} - \overline{Z}_{t,\mathbb{T}}|| = o_p(1). \tag{60}
$$

We are thus left with showing $\frac{1}{s_\Delta\sqrt{\Delta}}\overline{Z}_{t,\mathbb{T}} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} Z$, where Z is the limit process of the theorem with kernels of the covariance and relation operators given by

$$
k(z,u) = k_2(z,u) - k_3(z,u), \text{ and } \; c(z,u) = c_2(z,u) + c_3(z,u), \tag{61}
$$

where

$$
\begin{aligned}
k_l(z,u) &= \overline{v}_{t,l}\int_0^\infty (z^2 + iz)e^{iz\log(K)}\overline{(u^2+iu)e^{iu\log(K)}}\frac{O^2_{t^*}(K)}{K^4}dK, \\
c_l(z,u) &= \overline{v}_{t,l}\int_0^\infty (z^2 + iz)e^{iz\log(K)}(u^2+iu)e^{iu\log(K)}\frac{O^2_{t^*}(K)}{K^4}dK, \; l = 2,3,
\end{aligned}
\tag{62}
$$

and $O_{t^*}(K) = \mathbb{E}_{t^*-}(e^{\Delta x_{t^*}} - K)^+ \bigwedge \mathbb{E}_{t^*-}(K - e^{\Delta x_{t^*}})^+$ with the notation $x^+ = \max\{0, x\}$ for $x \in \mathbb{R}$.

Using a subsequence criterion for convergence in probability, we need to show that for all $\omega^{(0)}$ and every subsequence, there is a further subsequence along which we have $\frac{1}{s_\Delta\sqrt{\Delta}}\overline{Z}_{t,\mathbb{T}}(\omega^{(0)})$ converge in distribution to Z.

To show the functional convergence, we first show that the sequence is asymptotically finite-dimensional, see Section 1.8 in van der Vaart and Wellner (1996). That is, we need to establish that for arbitrary $\delta > 0$ and $\epsilon > 0$, there is $J > 0$ big enough such that

$$\limsup_{\Delta \downarrow 0} \mathbb{P}\left(\frac{1}{s_\Delta^2 \Delta} \sum_{j>J} \langle \overline{Z}_{t,\mathbb{T}}, e_j \rangle^2 > \delta \middle| \mathcal{F}^{(0)} \right) < \epsilon, \tag{63}$$

where $\{e_j\}_{j\geq 1}$ denotes an orthonormal basis in $\mathcal{L}^2(w)$. We first note that by Bessel's inequality,

$$\sum_{j>J} \langle \overline{Z}_{t,\mathbb{T}}, e_j \rangle^2 \leq ||\overline{Z}_{t,\mathbb{T}}||^2. \tag{64}$$

In addition, using our assumptions for the option observation error as well as the fact that the option price is monotonic for $K < 1$ and $K > 1$, we have

$$\frac{1}{s_\Delta^2 \Delta} \mathbb{E}\left(||\overline{Z}_{t,\mathbb{T}}||^2 \middle| \mathcal{F}^{(0)} \right) \leq C_t \sum_{i=1,2,3} \int_0^\infty \frac{O_{t,T_i}^2(K)}{K^4} dK, \tag{65}$$

for some $\mathcal{F}_t^{(0)}$-adapted random variable $C_t > 0$.

Using our assumptions for the option observation error

$$\frac{1}{s_\Delta^2 \Delta} \mathbb{E}\left(\sum_{j>J} \langle \overline{Z}_{t,\mathbb{T}}, e_j \rangle^2 \middle| \mathcal{F}^{(0)} \right) \leq C_t \sum_{i=1,2,3} \int_0^\infty \sum_{j>J} \left(\int_{\mathbb{R}} f_{t,i}(u,K) e_j(u) w(u) du \right)^2 \frac{O_{t,T_i}^2(K)}{K^4} dK + \overline{R}_{t,\Delta}, \tag{66}$$

for some functions $f_{t,i}(u, K)$ which depend on u, K, t and $\mathbb{T}$, and where $C_t > 0$ is $\mathcal{F}_t^{(0)}$-adapted random variable and $\overline{R}_{t,\Delta}$ satisfies $\limsup_{\Delta\downarrow 0} \overline{R}_{t,\Delta} = 0$. For showing the negligibility of $\overline{R}_{t,\Delta}$, we use the fact that for $l = 1, 2, 3$:

$$\begin{aligned} &\sum_{i=1}^{N_{t,T_l}} \int_{K_{i-1}}^{K_i} \sum_{j>J} \left(\int_{\mathbb{R}} (f_{t,l}(u,K) - f_{t,l}(u,K_{i-1})) e_j(u) w(u) du \right)^2 \frac{O_{t,T_l}^2(K)}{K^4} dK \\ &\leq \sum_{i=1}^{N_{t,T_l}} \int_{K_{i-1}}^{K_i} ||f_{t,l}(\cdot,K) - f_{t,l}(\cdot,K_{i-1})||^2 \frac{O_{t,T_l}^2(K)}{K^4} dK \to 0, \quad \text{as } \Delta \downarrow 0 \end{aligned} \tag{67}$$

as well as

$$\sum_{i=1}^{N_{t,T_l}} \int_{K_{i-1}}^{K_i} \left| \sum_{j>J} \langle f_{t,l}(\cdot,K) - f_{t,l}(\cdot,K_{i-1}), e_j\rangle \langle f_{t,l}(\cdot,K_{i-1}), e_j\rangle \right| \frac{O^2_{t,T_l}(K)}{K^4} dK$$

$$\leq \sum_{i=1}^{N_{t,T_l}} \int_{K_{i-1}}^{K_i} \sqrt{\sum_{j>J} \langle f_{t,l}(\cdot,K) - f_{t,l}(\cdot,K_{i-1}), e_j\rangle^2} \sqrt{\sum_{j>J} \langle f_{t,l}(\cdot,K_{i-1}), e_j\rangle^2} \frac{O^2_{t,T_l}(K)}{K^4} dK \tag{68}$$

$$\leq \sum_{i=1}^{N_{t,T_l}} \int_{K_{i-1}}^{K_i} ||f_{t,l}(\cdot,K) - f_{t,l}(\cdot,K_{i-1})|| ||f_{t,l}(\cdot,K_{i-1})|| \frac{O^2_{t,T_l}(K)}{K^4} dK \to 0, \quad \text{as } \Delta \downarrow 0$$

where the last inequality follows from application of Cauchy-Schwarz and Bessel inequalities. We further use

$$\int_0^{K_1} \sum_{j>J} \langle f_{t,l}(\cdot,K), e_j\rangle^2 \frac{O^2_{t,T_l}(K)}{K^4} dK \leq \int_0^{K_1} ||f_{t,l}(\cdot,K)||^2 \frac{O^2_{t,T_l}(K)}{K^4} dK \to 0, \quad \text{as } \Delta \downarrow 0, \tag{69}$$

and a similar result for the right tail.

From here, the asymptotic finite dimensionality result to be proved follows by an application of Lebesgue's dominated convergence theorem. Therefore, the limit result of the theorem will follow from Theorem 1.8.4 in van der Vaart and Wellner (1996) if we can establish

$$\frac{1}{s_\Delta \sqrt{\Delta}} \langle \overline{Z}_{t,\mathbb{T}}, h\rangle \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \langle Z, h\rangle, \tag{70}$$

h being an arbitrary element in $\mathcal{L}^2(w)$. Note that this is $\mathcal{F}^{(0)}$-conditional convergence in distribution of a bivariate random vector, conditions for which are readily available. In addition, since Z is $\mathcal{F}^{(0)}$-conditionally $\mathcal{CN}(0, K, C)$, we have

$$\mathbb{E}\left(\langle Z,h\rangle \overline{\langle Z,h\rangle} \middle| \mathcal{F}^{(0)}\right) = \langle Kh, h\rangle, \quad \mathbb{E}\left(\langle Z,h\rangle^2 \middle| \mathcal{F}^{(0)}\right) = \langle C\overline{h}, h\rangle. \tag{71}$$

The finite-dimensional CLT, in turn, will therefore hold by application of Theorem VIII.5.25 of Jacod and Shiryaev (2003), if we can establish the following convergence results:

$$\frac{1}{s_\Delta^2 \Delta} \mathbb{E}\left(\langle \overline{Z}_{t,\mathbb{T}}, h\rangle \overline{\langle \overline{Z}_{t,\mathbb{T}}, h\rangle} \middle| \mathcal{F}^{(0)}\right) \xrightarrow{\mathbb{P}} \langle Kh, h\rangle, \quad \frac{1}{s_\Delta^2 \Delta} \mathbb{E}\left(\langle \overline{Z}_{t,\mathbb{T}}, h\rangle^2 \middle| \mathcal{F}^{(0)}\right) \xrightarrow{\mathbb{P}} \langle C\overline{h}, h\rangle, \tag{72}$$

$$\frac{1}{s_\Delta^{2+\varepsilon} \Delta^{1+\varepsilon/2}} \mathbb{E}\left(|\langle \overline{Z}_{t,\mathbb{T}}, h\rangle|^{2+\varepsilon} \middle| \mathcal{F}^{(0)}\right) \xrightarrow{\mathbb{P}} 0, \quad \text{for some } \varepsilon \in (0,1). \tag{73}$$

These results can be shown using our assumption for the option observation error.

7.3 Proof of Theorem 3

As in the proof of Theorem 2, we do the normalization $X_t = 1$ and set $\tau = \frac{T_3 - T_2}{T_1}$. We define for $l = 1, 2, 3$:

$$\overline{\mathcal{L}}^*_{t,T_l}(u) = -\frac{1}{\sqrt{\Delta}}(u^2 + iu) \sum_{j=2}^{N_{t,T_l}} e^{iu\log(K_{j-1})} \frac{\widehat{\epsilon}_{t,T}(K_{j-1}) z_{l,j}}{K_{j-1}^2} (K_j - K_{j-1}). \tag{74}$$

Next, with the notation

$$\breve{\mathcal{L}}^*_{t,T_l}(u) = -\frac{1}{\sqrt{\Delta}}(u^2+iu)\sum_{j=2}^{N_{t,T_l}} e^{iu\log(K_{j-1})}\frac{s_\Delta\sqrt{\overline{v}_{t,l}}O_{t,T_l}(K_{j-1})z_{l,j}}{K_{j-1}^2}(K_j-K_{j-1}), \tag{75}$$

where we recall the definition of $\overline{v}_{t,l}$ in assumption B1 and using this assumption for the option observation error, we have

$$\begin{aligned}\mathbb{E}\left(||\overline{\mathcal{L}}^*_{t,T_l}-\breve{\mathcal{L}}^*_{t,T_l}||^2\big|\mathcal{F}\right) &\le C_t\sum_{j=2}^{N_{t,T_l}}\frac{(\widehat{s}_{l,\Delta}\widehat{O}_{t,T_l}(K_{j-1})-s_\Delta\sqrt{\overline{v}_{t,l}}O_{t,T_l}(K_{j-1}))^2}{K_{j-1}^4}(K_j-K_{j-1})\\ \le C_t(\widehat{s}_{l,\Delta}-s_\Delta\sqrt{\overline{v}_{t,l}})^2\Delta\sum_{j=2}^{N_{t,T_l}}\frac{\widehat{O}_{t,T_l}(K_{j-1})^2}{K_{j-1}^4} &+ C_t s_\Delta^2\Delta\sum_{j=2}^{N_{t,T_l}}\frac{(\widehat{O}_{t,T_l}(K_{j-1})-O_{t,T_l}(K_{j-1}))^2}{K_{j-1}^4}.\end{aligned} \tag{76}$$

Using again assumption B1 for the observation error and the integrability assumption in B2, we get

$$\Delta\sum_{j=2}^{N_{t,T_1}}\frac{\widehat{O}_{t,T_1}(K_{j-1})^2}{K_{j-1}^4} = O_p(T_1^{3/2}),\ \ \Delta\sum_{j=2}^{N_{t,T_1}}\frac{(\widehat{O}_{t,T_1}(K_{j-1})-O_{t,T_1}(K_{j-1}))^2}{K_{j-1}^4} = O_p(T_1^{3/2}s_\Delta^2), \tag{77}$$

$$\Delta\sum_{j=2}^{N_{t,T_l}}\frac{\widehat{O}_{t,T_l}(K_{j-1})^2}{K_{j-1}^4} = O_p(1),\ \ \Delta\sum_{j=2}^{N_{t,T_l}}\frac{(\widehat{O}_{t,T_l}(K_{j-1})-O_{t,T_l}(K_{j-1}))^2}{K_{j-1}^4} = O_p(s_\Delta^2),\ \ l=2,3. \tag{78}$$

Next, using the algebraic inequality $|\sqrt{y}-\sqrt{x}|\le\sqrt{|y-x|}$, for $x,y\in\mathbb{R}_+$, we need to analyze $|\widehat{s}^2_{l,\Delta}-s^2_\Delta\overline{v}_{t,l}|$. First, we note that using Lemma 2 of Todorov (2019), we have that for $j\in I_t^1$, $O_{t,T_1}(K_j) > C_t\sqrt{T_1}$ for some strictly positive $\mathcal{F}_t^{(0)}$-adapted random variable C_t (this is because the diffusive volatility $\sigma_{t^*-}>0$). Similarly, because of the boundedness of the option observation error assumed in B1, we have $O_{t,T_2}(K_j) > C_{t,2}/2$ and $O_{t,T_3}(K_j) > C_{t,3}/2$, for $j\in I_t^2$ and $j\in I_t^3$, respectively ($C_{t,2}$ and $C_{t,3}$ are the variables appearing in the definition of I_t^2 and I_t^3). Combining these results, we have

$$\left|\log(O_{t,T_l}(K_j))-\frac{1}{2}(\log(O_{t,T_l}(K_{j-1}))+\log(O_{t,T_l}(K_{j+1})))\right|\le C_t\Delta,\ \text{ for } j\in I_t^l \text{ and } l=1,2,3. \tag{79}$$

Next, utilizing the boundedness of the observation error and the fact that $s_\Delta\to 0$ as $\Delta\downarrow 0$, we easily have that for Δ sufficiently small:

$$|\log(1+s_\Delta\overline{\epsilon}_{t,l,j})-s_\Delta\overline{\epsilon}_{t,l,j}|\le\frac{1}{4}s_\Delta^2\overline{\epsilon}^2_{t,l,j},\ \text{ for } j\in I_t^l \text{ and } l=1,2,3. \tag{80}$$

Altogether, since $|\log(1+s_\Delta\overline{\epsilon}_{t,1,j})|$ is bounded uniformly over j, we get

$$|\widehat{s}^2_{1,\Delta}-s^2_\Delta\overline{v}_{t,1}| = \begin{cases} O_p(1), & \text{if } \sqrt{T_1}/\Delta \text{ is bounded},\\ O_p(s_\Delta^2\sqrt{\Delta}/T_1^{1/4}\bigvee s_\Delta^4), & \text{if } \sqrt{T_1}/\Delta\to\infty.\end{cases} \tag{81}$$

Further, since $\mathrm{var}_{t^*-}^{\mathbb{Q}}(\Delta x_{t^*}) > 0$, this means that $O_{t,T_l}(K) > C_{t,T_l}$ for K in a neighborhood of X_t and $l = 2, 3$. Therefore,

$$|\widehat{s}_{l,\Delta}^2 - s_\Delta^2 \overline{v}_{t,l}| = O_p(s_\Delta^2 \sqrt{\Delta} \bigvee s_\Delta^4), \quad \text{for } l = 2, 3. \tag{82}$$

Combining these results, and taking into account that $s_\Delta/\sqrt{\Delta} \to \infty$ by assumption B1, we get

$$\mathbb{E}\left(||\overline{\mathcal{L}}_{t,T_l}^* - \check{\mathcal{L}}_{t,T_l}^*||^2 \big| \mathcal{F}\right) \leq C_t l(\Delta) s_\Delta^2, \tag{83}$$

where C_t is a positive-valued $\mathcal{F}_t^{(0)}$-adapted random variable and $l(\Delta)$ is a deterministic sequence with $l(\Delta) \to 0$ when $\Delta \to 0$. We can similarly establish

$$\mathbb{E}\left(|\overline{\mathcal{L}}_{t,T_l}^*(u) - \check{\mathcal{L}}_{t,T_l}^*(u)|^2 \big| \mathcal{F}\right) \leq C_t(|u|^2 \vee |u|) l(\Delta) s_\Delta^2. \tag{84}$$

Further, exactly as in the proof of Theorem 2, we can show

$$\mathbb{E}\left(|\check{\mathcal{L}}_{t,T_l}^*(u)|^p | \mathcal{F}^{(0)}\right) \leq C_t(|u|^2 \vee |u|)^p s_\Delta^p, \tag{85}$$

for some $\mathcal{F}_t^{(0)}$-adapted random variable C_t that depends on p.

With this we are ready to complete the proof of the theorem. We denote

$$\overline{Z}_{t,\mathbb{T}}^*(u) = \tau \mathcal{L}_{t,T_1}(u)^{\tau-1} \mathcal{L}_{t,T_2}(u) \sqrt{\Delta}\, \overline{\mathcal{L}}_{t,T_1}^*(u) + \mathcal{L}_{t,T_1}(u)^\tau \sqrt{\Delta}\, \overline{\mathcal{L}}_{t,T_2}^*(u) - \sqrt{\Delta}\, \overline{\mathcal{L}}_{t,T_3}^*(u), \tag{86}$$

and the set

$$\widehat{\mathcal{U}}_T = \{u : |u| \leq u_T^*,\ |\widehat{\mathcal{L}}_{t,T_1}(u) - \mathcal{L}_{t,T_1}(u)| \leq \frac{1}{2}|\mathcal{L}_{t,T_1}(u)|,\ |\widehat{\mathcal{L}}_{t,T_1}^*(u) - \widehat{\mathcal{L}}_{t,T_1}(u)| \leq \frac{1}{2}|\widehat{\mathcal{L}}_{t,T_1}(u)|\}. \tag{87}$$

With this notation, using first-order Taylor series expansion, we have

$$\begin{aligned}
\widehat{Z}_{t,\mathbb{T}}^*(u) - \overline{Z}_{t,\mathbb{T}}^*(u) &= -\overline{Z}_{t,\mathbb{T}}^*(u) 1_{\{|u|>u_T\}} + (\widehat{\mathcal{L}}_{t,T_1}^*(u)^\tau - \mathcal{L}_{t,T_1}(u)^\tau)(\widehat{\mathcal{L}}_{t,T_2}^*(u) - \widehat{\mathcal{L}}_{t,T_2}(u)) 1_{\{|u|\leq u_T\}} \\
&+ (\widehat{\mathcal{L}}_{t,T_1}^*(u)^\tau - \widehat{\mathcal{L}}_{t,T_1}(u)^\tau)(\widehat{\mathcal{L}}_{t,T_2}(u) - \mathcal{L}_{t,T_2}(u)) 1_{\{|u|\leq u_T\}} \\
&+ (\widehat{\mathcal{L}}_{t,T_1}^*(u)^\tau - \widehat{\mathcal{L}}_{t,T_1}(u)^\tau) \mathcal{L}_{t,T_2}(u) 1_{\{|u|\leq u_T \,\&\, u \notin \widehat{\mathcal{U}}_T\}} \\
&+ \tau(\widetilde{\mathcal{L}}_{t,T_1}^*(u)^{\tau-1} - \mathcal{L}_{t,T_1}(u)^{\tau-1})(\widehat{\mathcal{L}}_{t,T_1}^*(u) - \widehat{\mathcal{L}}_{t,T_1}(u)) \mathcal{L}_{t,T_2}(u) 1_{\{|u|\leq u_T \,\&\, u \in \widehat{\mathcal{U}}_T\}} \\
&- \tau \mathcal{L}_{t,T_1}(u)^{\tau-1} \sqrt{\Delta}\, \overline{\mathcal{L}}_{t,T_1}^*(u) \mathcal{L}_{t,T_2}(u) 1_{\{|u|\leq u_T \,\&\, u \notin \widehat{\mathcal{U}}_T\}} \\
&+ (\tau \mathcal{L}_{t,T_1}(u)^{\tau-1} \mathcal{L}_{t,T_2}(u)(R_{t,T_1}^{(1)} + R_{t,T_1}^{(2)}) + \mathcal{L}_{t,T_1}(u)^\tau (R_{t,T_2}^{(1)} + R_{t,T_2}^{(2)}) - (R_{t,T_3}^{(1)} + R_{t,T_3}^{(2)})) 1_{\{|u|\leq u_T\}},
\end{aligned} \tag{88}$$

and $\widetilde{\mathcal{L}}_{t,T_1}^*(u)$ is an intermediate value between $\widehat{\mathcal{L}}_{t,T_1}^*(u)$ and $\widehat{\mathcal{L}}_{t,T_1}(u)$.

Similar to the proof of Theorem 2, using the bounds for $\overline{\mathcal{L}}_{t,T_l}^*(u)$, $\check{\mathcal{L}}_{t,T_l}^*(u)$, $R_{t,T}^{(1)}(u)$ and $R_{t,T}^{(2)}(u)$, we can establish

$$\frac{1}{s_\Delta \sqrt{\Delta}} ||\widehat{Z}_{t,\mathbb{T}}^* - \overline{Z}_{t,\mathbb{T}}^*|| = o_p(1). \tag{89}$$

Since from the bounds above $\mathbb{E}\left(||\overline{\mathcal{L}}^*_{t,T_l} - \widetilde{\mathcal{L}}^*_{t,T_l}||^2 \big| \mathcal{F}\right) \leq C_t l(\Delta) s_\Delta^2$ and also $||\mathcal{L}^{\tau-1}_{t,T_1} - 1|| + ||\mathcal{L}^{\tau}_{t,T_1} - 1|| = o_P(1)$ (recall that $t + T_1 < t^*$), we further have

$$\frac{1}{s_\Delta \sqrt{\Delta}} ||\overline{Z}^*_{t,\mathbb{T}} - \check{Z}^*_{t,\mathbb{T}}|| = o_p(1), \tag{90}$$

where

$$\check{Z}^*_{t,\mathbb{T}}(u) = \tau \mathcal{L}_{t,T_2}(u) \sqrt{\Delta}\, \check{\mathcal{L}}^*_{t,T_1}(u) + \sqrt{\Delta}\, \check{\mathcal{L}}^*_{t,T_2}(u) - \sqrt{\Delta}\, \check{\mathcal{L}}^*_{t,T_3}(u). \tag{91}$$

Hence, we are left with establishing a CLT for $\frac{1}{s_\Delta \sqrt{\Delta}} \check{Z}^*_{t,\mathbb{T}}$. This can be done in exactly the same way as the CLT for $\frac{1}{s_\Delta \sqrt{\Delta}} \overline{Z}_{t,\mathbb{T}}$ is shown in the proof of Theorem 2.

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