Systematic Jump Risk*

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June 1, 2022

Abstract

In this paper we develop tests for detecting systematic jump risk in asset prices of general form and we further propose nonparametric estimates for it. The inference is based on a panel of high-frequency asset returns, with both the sampling frequency and the size of the cross-section increasing asymptotically. The feasible limit theory developed in the paper utilizes the different asymptotic roles played by diffusive versus jump risks and systematic versus idiosyncratic risks in statistics that involve cross-sectional averages of suitably chosen transforms of the high-frequency price increments. The rate of convergence of the statistics is determined by the two asymptotically increasing dimensions of the panel, without imposing restrictions on their relative size. In an empirical application, using the developed tools, we document the existence of systematic jump risk, that is not spanned by traditional (observable) risk factors, and we further show that this risk commands a nontrivial risk premium.

Keywords: asset pricing, dynamic factor models, high-frequency data, nonparametric inference, stable convergence, systematic risk.

JEL classification: C51, C52, G12.

[*]We would like to thank Dacheng Xiu for sharing with us the Fama-French risk factor high-frequency data as well as participants at various seminars and conferences for useful feedback.

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1 Introduction

Large moves in stock prices in the form of jumps play important role in asset pricing. Early work, following Merton (1976), models jump risk as idiosyncratic and without aggregate pricing implications. However, recent empirical evidence, based on nonparametric jump tests developed by Barndorff-Nielsen and Shephard (2004, 2006), Aït-Sahalia and Jacod (2009) and Lee and Mykland (2008) and applied to market indices, shows that some of the jump risk in asset prices is systematic. Additional nonparametric tests for co-jumps in Bollerslev, Law, and Tauchen (2008), Jacod and Todorov (2009), Mancini and Gobbi (2012) and Caporin, Kolokolov, and Renò (2017) further suggest that jumps cluster cross-sectionally.[1] Empirical evidence in Todorov and Bollerslev (2010), Bollerslev, Li, and Todorov (2016) and Aït-Sahalia, Jacod, and Xiu (2020) show that jumps in systematic risk factors are priced in the cross-section of asset prices.[2]

Can observable risk factors span all systematic jump risk in asset prices? In other words, is there cross-sectional jump clustering outside the jumps of observable risk factors? The goal of this paper is to design nonparametric methods for detecting presence of systematic jump risk in asset prices, without any assumption regarding the existence of factor structure of jump risk, and further provide measures for this risk and for the assets' sensitivity towards it.

A natural approach for studying systematic jump risk is to assume the existence of a linear latent factor structure (with constant factor loadings) and perform classical principal component analysis on asset jumps filtered from the data. Such an approach has been formally developed by Pelger (2019) and applied empirically by Pelger (2020), with Pelger (2020) documenting the existence of one (stable) jump factor associated with the jumps in the market portfolio. However, jump factor loadings can change rather rapidly or more generally jumps can cluster cross-sectionally without exhibiting a linear factor structure. In addition, due to the nontrivial idiosyncratic risk in asset prices, the threshold approach for jump detection, when applied on an individual asset level, can allow the identification of the relatively large jumps only. This makes difficult and practically impossible to infer latent jump factors, associated with large cross-sectional variation in the assets' response to them, from jumps detected on an individual asset level.

In this paper, therefore, we pursue a different approach of making inference for systematic jump risk which does not require an assumption for the existence of a linear jump factor model, or its

[1] Additional work on co-jumps can be found in Lahaye, Laurent, and Neely (2011), Gilder, Shackleton, and Taylor (2014), Bibinger and Winkelmann (2015), Dungey, Erdemlioglu, Matei, and Yang (2018), Novotnỳ and Urga (2018) and Corradi, Distaso, and Fernandes (2019), among others.

[2] Inference for jump factor models with observable factors has been developed by Todorov and Bollerslev (2010), Li, Todorov, and Tauchen (2017a,b, 2019) and Aït-Sahalia, Kalnina, and Xiu (2020). The case of latent jump factors has been considered by Pelger (2019, 2020). Other work on estimation of factor models using high-frequency returns includes Chang, Choi, Kim, and Park (2016), Fan, Furger, and Xiu (2016), Ait-Sahalia and Xiu (2017) and Dai, Lu, and Xiu (2019) among others.

temporal stability, and can further minimize the role of idiosyncratic risk as well as systematic diffusive risk in the inference. We first cross-sectionally average suitable transforms of the assets return increments and then take first difference of these cross-sectional averages. This approach allows for separating effectively systematic jump risk from the rest of the risks contained in asset prices. In particular, the cross-sectional averaging smooths out idiosyncratic risk in asset prices and by differencing consecutive cross-sectional average statistics, we further remove the leading component of idiosyncratic risk in our statistic and we also minimize the contribution to it that is due to the systematic diffusive risk. Our aggregate measure can thus identify systematic jump risks even in settings in which this type of risk is small relative to idiosyncratic risk on an individual asset level. Finally, the transform of the returns in our statistics allows us to separate systematic jump risk from systematic diffusive risk by utilizing the fact that jumps feature more prominently in higher powers of returns. Additional improvements can be achieved when subtracting from the return increments the cross-sectional average of the asset returns. The aggregate effect of this is to minimize the role of systematic diffusive risk in the inference.

We derive a Central Limit Theorem (CLT) for our aggregate systematic jump risk measure as well as for a measure that captures an asset's exposure towards this type of risk. The limit is mixed Gaussian and the convergence is in a joint asymptotic setting of increasing sampling frequency and growing cross-sectional dimension of the panel of return observations. The rate of convergence depends on both dimensions of the panel and is determined by two sources of error in the estimation. One is the diffusive systematic risk around the times of the systematic jumps (which depends on the sampling frequency) and the second is the cross-sectional dispersion in the systematic jump risk (which depends on the size of the cross-section). We extend these results to separately identify systematic jump risk that happens outside the jump times of observable systematic risk factors using jump detection techniques for the latter.

We further propose a test for deciding whether a time interval contains a systematic jump event that happens outside of the jump times of observable risk factors. The test utilizes the fact that our systematic jump risk measure shrinks to zero when this risk is absent from the cross-section of asset prices. We derive the limit in probability of the properly rescaled statistic in this case. This limit is determined by the systematic diffusive risk as well as by the idiosyncratic jump risk in the asset prices. Using these results, we propose a test for systematic jumps on the basis of the difference between our statistic and its truncated counterpart which removes the systematic jumps from it.

We implement the developed inference techniques on high-frequency data on the 500 largest stocks by market capitalization, traded in US, for the period 2001-2020. We provide nontrivial evidence for presence of systematic jumps that occur outside the jump times of the Fama-French (FF) three factors (market, size and value). The non-FF systematic jump risk exhibits significant time variation, with dynamics that differs from that of the market variance. Similarly, the assets' sensitivities to this risk differ from their exposures to traditional risk factors such as the market portfolio return. These differences are shown to have nontrivial pricing implications.

The rest of the paper is organized as follows. In Section 2 we introduce our setup and state the assumptions. In Section 3 we derive nonparametric estimates for systematic jump risk as well as for assets' sensitivity towards it. In Section 4, we develop tests for detecting presence of systematic jump risk from discrete returns. Section 5 and 6 contain a Monte Carlo study and our empirical application, respectively. Section 7 concludes. Proofs are given in the online Appendix.

2 Setup and Assumptions

We start with introducing our setup. We denote the log-price of an asset j at time t with X_t^j. Asset prices are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with the σ-field $\mathcal{F}$ being separable. The dynamics of all X^j's is given by the following continuous-time model

$$\begin{aligned} X_t^j = X_0^j + \int_0^t b_s^j\, ds + \int_0^t \sigma_s^j\, d\widetilde{W}_s^j + \int_{(0,t]\times\mathbb{R}} \theta^j(s,z)\,(\mu^j - \nu)(ds,dz) \\ + \sum_{l=1}^{K} \int_0^t \lambda_s^{j,l}\, dW_s^l + \sum_{p\geq 1} \beta_p^j\, 1_{\{\rho_p\leq t\}} + \sum_{p\geq 1} \delta_p^j\, 1_{\{\tau_p\leq t\}}. \end{aligned} \tag{1}$$

The driving terms above are as follows:

- $W^1,\ldots,W^K,\widetilde{W}^1,\widetilde{W}^2,\ldots$ are independent Brownian motions;
- $(\mu^j)_{j\geq 1}$ are independent Poisson measures on $\mathbb{R}_+\times\mathbb{R}_+$ with compensator $\nu(dt,dz) = dt\otimes dz$;
- $(\tau_p)_{p\geq 1}$ are the successive jump times of a Poisson process, independent of all μ^j's;
- $(\rho_p)_{p\geq 1}$ ia a strictly increasing sequence of positive stopping times going to ∞ and distinct from all τ_p's.

The other ingredients, $b_s^j, \sigma_s^j, \theta^j, \lambda_s^{j,l}, \beta_p^j$ and δ_p^j, will be specified in assumption (A1) below. Taking the measures μ^j on $\mathbb{R}_+\times\mathbb{R}$ instead of the more usual $\mathbb{R}_+\times E$ for some Polish space E, and all with the same compensator ν, is not a restriction for our model. This is because while the jump measures μ^j have the same compensator ν, the corresponding jump size functions can differ across the different assets. When the ρ_p's are the jump times of another Poisson-type process, each X^j is an Itô semimartingale. This is the standard setting in the literature. Here, however, we allow for an extra generality. Mainly, the ρ_p's can be predictable, or even deterministic, to account for jumps occurring at the times of economic announcements.

The first line on the right hand side of (1) captures the idiosyncratic risks in X^j. The terms in the second line of (1) are due to the systematic risk in X^j: the first one is the asset's response to the continuous systematic risk while the last two are the asset's response to systematic jumps (which are thus assumed to be of finite activity).

The reason to single out two distinct terms for the systematic jumps is as follows. In practice, one may observe some risk factors, e.g. the Fama-French factors, whose jumps occur at the times ρ_p, and which typically induce jumps for all or most stock prices, and the jump size for stock j is then β^j_p. However, we may also have systematic jumps at the times τ_p or τ'_p, that are not associated with an observable risk factor, inducing jumps of size δ^j_p or δ'^j_p for stock j. The aim of this paper is these hidden systematic jumps, which we simply refer to henceforth as *systematic jumps*.

If we do not observe risk factors or do not want to separate their jumps from the rest of the systematic jumps, then we may take $\rho_p = \infty$ for all p, and the second term on the second line of (1) then simply disappears.

Finally, we write W_t and λ^j_t for the K-dimensional vectors with components W^l_t and $\lambda^{j,l}_t$, so the continuous part of the systematic risk component of X^j is $\int_0^t \lambda^{j,\top}_s \, dW_s$, with $^\top$ denoting the transpose, whereas $\|\lambda^j_t\|$ is the usual norm on $\mathbb{R}^K$. We also write the last integral in the first line of (1) as $\theta^j * (\mu^j - \nu)_t$.

The assumptions on the coefficients $b^j, \sigma^j, \theta^j, \lambda^j, \beta^j_p$ and δ^j_p are as follows:

Assumption (A1). *(i) The processes λ^j and σ^j are càdlàg adapted, the processes b^j are optional, and the functions θ^j are predictable on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$, and each variable β^j_p, resp. δ^j_p is measurable with respect to $\mathcal{F}_{\rho_p}$, resp. $\mathcal{F}_{\tau_p}$.*

(ii) There are a sequence $T_1, T_2, \ldots$ of stopping times increasing to ∞, $[0,1]$-valued functions φ^j_m on $\mathbb{R}$ with $\int_{\mathbb{R}} \varphi^j_m(z)^w \, dz \le 1$, for some number $w \in [0,2)$, and $[1,\infty)$-valued random variables χ^j_m, satisfying $\sup_j \mathbb{E}\big(|\chi^j_m|^q\big) < \infty$ for any $q \in \mathbb{R}_+$, such that for all m, j, z, p, ω:

$$t < T_m \Rightarrow \|\lambda^j_t\| + |\sigma^j_t| + |b^j_t| \le \chi^j_m, \quad t \le T_m \Rightarrow |\theta^j(t,z)| \le \varphi^j_m(z)\chi^j_m, \quad \tau_p \le T_m \Rightarrow |\delta^j_p| \le \chi^j_m.$$

Moreover, for any $m, j \ge 1$ and any finite stopping time $S \le T_m$ we have

$$\sup_{j \ge 1} \mathbb{E}\Big(\sup_{s' \in [(S-s)^+, S)} \big(\|\lambda^j_{s'} - \lambda^j_{S-}\|^2 + |\sigma^j_{s'} - \sigma^j_{S-}|^2 \big) \Big) \to 0 \quad \text{as } s \downarrow 0,$$

and, for some constants $\rho > 0$ and C_m,

$$\mathbb{E}_{\mathcal{F}_S}\Big(\sup_{s' \in [S, S+s]} \Big(\frac{\|\lambda^j_{s' \wedge T_m} - \lambda^j_S\|^2 + |\sigma^j_{s' \wedge T_m} - \sigma^j_S|^2}{(\chi^j_m)^2} \Big) \le C_m \, s^\rho. \tag{2}$$

Part (i) of the above assumption is rather standard and ensures that all terms in (1) make sense. Part (ii) imposes various smoothness and integrability assumptions. For a fixed j, all of these conditions are standard, except for the degree of jump activity index which is assumed to be strictly below 2. This is a rather mild assumption, though, given the fact that the maximal possible value of the jump activity index is 2. In part (ii) of (A1), we require some kind of uniformity in j of the various smoothness and integrability conditions. Such uniformity will hold if, for example, λ^j and σ^j are identically distributed across j-s after conditioning on a set of common systematic shocks. We finally note that the last property in (ii) holds with $\rho = 1$ as soon as λ^j and σ^j are themselves Itô semimartingales, but here we simply assume $\rho > 0$, which is much weaker.

Besides this general assumption, we need an assumption about the cross-sectional behavior of the variables δ_p^j and the processes λ_t^j. For stating this assumption, we first introduce a number of σ-fields, sometimes depending on an asset index j:

$$\begin{array}{lll} \mathcal{J} = \sigma\big(\tau_p, \rho_p : p \geq 1\big), & \mathcal{J}_j = \mathcal{J} \bigvee \sigma(\delta_p^j : p \geq 1), & \mathcal{J}' = \mathcal{J} \bigvee \sigma\big(\delta_p^k : k, p \geq 1\big) \\ \overline{\mathcal{J}} = \mathcal{J} \bigvee \sigma(W_t, \lambda_t^k : t \geq 0,\, k \geq 1), & & \overline{\mathcal{J}}_j = \overline{\mathcal{J}} \bigvee \mathcal{J}_j \bigvee \sigma(\widetilde{W}_t^j, \sigma_t^j ;\, t \geq 0). \end{array}$$

For a σ-field $\mathcal{G} \subset \mathcal{F}$ we write $\mathbb{P}_{\mathcal{G},\omega}$, or simply $\mathbb{P}_{\mathcal{G}}$, for a regular version of the $\mathcal{G}$-conditional probability, which exists because $\mathcal{F}$ is separable, and $\mathbb{E}_{\mathcal{G},\omega}$ or $\mathbb{E}_{\mathcal{G}}$ for the associated conditional expectation.

Assumption (A2). *(i) For each j and $\mathbb{P}$-almost all ω the sequences $(\delta_p^k)_{p\geq 1}$ are i.i.d. under $\mathbb{P}_{\mathcal{J},\omega}$, resp. $\mathbb{P}_{\overline{\mathcal{J}},\omega}$, resp. $\mathbb{P}_{\overline{\mathcal{J}}_j,\omega}$, as k varies in $\mathbb{N}^*$, resp. $\mathbb{N}^*$, resp. $\mathbb{N}^*\backslash\{j\}$.*

(ii) For $\mathbb{P}$-almost all ω the processes W, $\widetilde{W}^j$ and the measures μ^j are still independent Brownian motions and Poisson measures under both $\mathbb{P}_{\mathcal{J},\omega}$ and $\mathbb{P}_{\mathcal{J}',\omega}$.

(iii) There are càdlàg processes $\overline{\lambda}^{(1)} = (\overline{\lambda}^{(1),l})_{1\leq l\leq K}$ and $\overline{\lambda}^{(2)} = (\overline{\lambda}^{(2),l,l'})_{1\leq l,l'\leq K}$ such that, with T_m as in (A1)-(ii) and as $N \to \infty$ and for any pair l, l':

$$\mathbb{E}\Big(\sup_{s\leq t\wedge T_m}\Big(\Big|\frac{1}{N}\sum_{j=1}^{N}\lambda_s^{j,l} - \overline{\lambda}_s^{(1),l}\Big| + \Big|\frac{1}{N}\sum_{j=1}^{N}\lambda_s^{j,l}\lambda_s^{j,l'} - \overline{\lambda}_s^{(2),l,l'}\Big|\Big)\Big) \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad \textit{for all } t \in \mathbb{R}_+,$$

and furthermore we have $\int_0^T \|\overline{\lambda}_s^{(1)}\|^2\, ds > 0$ a.s., where $T > 0$ is our time horizon.

As Lemma 2 in the Appendix shows, assumption (A2) implies that, for $\mathbb{P}$-almost all ω the law of δ_p^j is the same under $\mathbb{P}_{\mathcal{J},\omega}$ and $\mathbb{P}_{\overline{\mathcal{J}},\omega}$ and $\mathbb{P}_{\overline{\mathcal{J}}_k,\omega}$ if $k \neq j$, and when further (A1) holds δ_p^j has finite moments of all orders under these conditional probabilities. Then the law of large number implies the existence, for any Borel function h with at most polynomial growth, of $\mathcal{F}_{\tau_p}$-measurable variables $\overline{\delta}_{h,p}$ such that, as $N \to \infty$ and for any k and $k' \neq k$,

$$\overline{\delta}_{h,p}^N := \frac{1}{N}\sum_{j=1}^{N} h(\delta_p^j) \stackrel{\text{a.s.}}{\longrightarrow} \overline{\delta}_{h,p}, \qquad \text{with} \quad \overline{\delta}_{h,p} = \mathbb{E}_{\mathcal{J}}\big(h(\delta_p^k)\big) = \mathbb{E}_{\overline{\mathcal{J}}}\big(h(\delta_p^k)\big) = \mathbb{E}_{\overline{\mathcal{J}}_{k'}}\big(h(\delta_p^k)\big). \tag{3}$$

Recall that the variables δ_p^j capture the hidden systematic jump events and, as already mentioned, our goal in this paper is to identify and assess the impact they have on asset prices.

Finally, part (iii) of (A2) assumes that systematic diffusive risk "survives" cross-sectional aggregation. This condition is rather mild and is expected to be true in the data. The results that follow can be easily extended to accommodate situations without aggregate systematic diffusive risk but we do not do this here as such an extension seems empirically irrelevant.

We finish this section with introducing our sampling scheme. Our inference is based on a large panel of high-frequency return observations of N assets, each one being observed on the equidistant observation grid $0, \Delta_n, 2\Delta_n ....$, up to some *fixed* time horizon T, and with $\Delta_n = T/n$ for some integer n. The number of observations for each asset is then $n+1$. "Large panel" and "high-frequency" mean that we assume

$$n \to \infty, \qquad N \to \infty, \qquad \sqrt{\frac{n}{N}} \to \phi \in [0, \infty], \tag{4}$$

and we note that in our analysis we allow for $\phi = 0$ or $\phi = \infty$, i.e., one of the two dimensions of the panel may grow at a faster rate than the other one.

For $i = 1, \ldots, n$, we denote the ith time interval between two successive observations and the increment of a price, or more generally of any process Y, over this interval by

$$I(n,i) = \big((i-1)\Delta_n, i\Delta_n\big], \qquad \Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}.$$

Finally, throughout the paper we use many different statistics computed from the observed increments $\Delta_i^n X^j$ for $j = 1, \ldots, N$ and $i = 1, \ldots, n$. They depend on n and/or N, of course, but for ease of notation we simply write them with a "hat". For instance, our basic statistic below, depending on both n and N, is written as $\widehat{SJ}_g$ instead of $\widehat{SJ}_g^{n,N}$.

3 Inference for Systematic Jump Risk

We start in this section with designing measures that quantify systematic jump risk as well as assets' sensitivity towards it, and we further derive results for their asymptotic behavior.

3.1 Measuring Systematic Jump Risk

For our measures of systematic jump risk, we use various test functions g on $\mathbb{R}$ belonging to one of the following two sets:

$\mathcal{C}$: the set of all C^3 nonnegative functions with g'', g''' bounded, and $g(0) = g'(0) = 0$

$\mathcal{C}_+$: the set of all $g \in \mathcal{C}$ with $g''(0) > 0$ and $g(x) > 0$ for all $x \neq 0$.

Obvious examples of functions in $\mathcal{C}$ are $g(x) = x^2 e^{-ax^2}$, for some $a \geq 0$, or $g(x) = x^2\, 1\{x \geq 0\}$. Although $g(x) = x^2$ is perhaps the most natural test function, using a bounded function provides robustness in finite samples against outliers in the data.

The aggregate measure associated with g over the time interval $[0,T]$ of the systematic jump risk, outside the jump times of observed factors, will be, with the notation in (3) and omitting T since this time is fixed throughout:

$$SJ_g = \sum_{p\geq 1:\tau_p\leq T} (\overline{\delta}_{g,p})^2.$$

The function g in $\mathcal{C}$ is allowed to vanish outside a subset A of $\mathbb{R}$ in order to obtain an aggregate measure of systematic jumps whose size lies in A (for example of positive jumps if $A=(0,\infty)$).

In what follows, it is convenient to separate systematic jumps into two groups: one for which the average asset price jumps and one for which this is not the case. Towards this end, lets denote the identity function with $\mathbb{I}(x)=x$ on $\mathbb{R}$. We then split SJ_g into the sum of the following two quantities:

$$SJ'_g = \sum_{p\geq 1:\tau_p\leq T} (\overline{\delta}_{g,p})^2\, 1_{\{\overline{\delta}_{\mathbb{I},p}=0\}}, \qquad SJ''_g = \sum_{p\geq 1:\tau_p\leq T} (\overline{\delta}_{g,p})^2\, 1_{\{\overline{\delta}_{\mathbb{I},p}\neq 0\}}. \tag{5}$$

The reason for splitting SJ_g into SJ'_g and SJ''_g is that, in order to improve efficiency, their estimation will be done slightly differently. When the market portfolio is among the observed systematic factors, one may expect that if $\overline{\delta}_{\mathbb{I},p}\neq 0$ the price jumps at the time τ_p induce a jump of this market portfolio factor, so within our setting this time is indeed included among the jump times ρ_p's of the factors. In other words, in this case one should have $\overline{\delta}_{\mathbb{I},p}=0$ for all p, and thus SJ''_g vanishes identically.

We next define the following sets on which there is no systematic jump on $[0,T]$:

$$\begin{aligned}
\Omega_{noSJ} &= \big\{\omega:\ \delta^j_p(\omega)=0 \text{ for all } j\geq 1 \text{ and all } p\geq 1 \text{ with } \tau_p(\omega)\leq T\big\}\\
\Omega'_{noSJ} &= \big\{\omega:\ \delta^j_p(\omega)=0 \text{ for all } j\geq 1 \text{ and all } p\geq 1 \text{ with } \tau_p(\omega)\leq T \text{ and } \overline{\delta}_{\mathbb{I},p}(\omega)=0\big\}\\
\Omega''_{noSJ} &= \big\{\omega:\ \overline{\delta}_{\mathbb{I},p}(\omega)=0 \text{ for all } p\geq 1 \text{ with } \tau_p(\omega)\leq T\big\}.
\end{aligned}$$

By virtue of (3), we either have $\overline{\delta}_{g,p}=0$ for all $g\in\mathcal{C}_+$, implying $\delta^j_p=0$ a.s. for all $j\geq 1$, or $\overline{\delta}_{g,p}>0$ for all $g\in\mathcal{C}_+$, implying $\delta^j_p\neq 0$ a.s. for infinitely many j. Therefore

$$\Omega_{noSJ}=\{SJ_g=0\},\quad \Omega'_{noSJ}=\{SJ'_g=0\},\quad \Omega''_{noSJ}=\{SJ''_g=0\}\quad \text{a.s., for any } g\in\mathcal{C}_+. \tag{6}$$

For measuring the sensitivity of any particular asset j towards systematic jump risk, we use the following aggregated measures of sensitivity, where both functions g,h are in $\mathcal{C}$:

$$\begin{gathered}
SJ'^j_{g,h} = \sum_{p\geq 1:\,\tau_p\leq T} h(\delta^j_p)\,\overline{\delta}_{g,p}\, 1_{\{\overline{\delta}_{\mathbb{I},p}=0\}}, \qquad SJ''^j_{g,h} = \sum_{p\geq 1:\,\tau_p\geq T} h(\delta^j_p)\,\overline{\delta}_{g,p}\, 1_{\{\overline{\delta}_{\mathbb{I},p}\neq 0\}}\\
SJ^j_{g,h} = SJ'^j_{g,h} + SJ''^j_{g,h}.
\end{gathered} \tag{7}$$

For example, with g and h being equal to the square function, we can measure asset's exposure to systematic jump risk in terms of variance risk. If $g\in\mathcal{C}_+$ and h are vanishing on $\mathbb{R}_+$ or $\mathbb{R}_-$, $SJ^j_{g,h}$ allows us to measure also the direction in which the asset prices move during systematic jump events.

3.2 Preliminary Estimators

In this section we exhibit our preliminary estimators of the various quantities introduced above. We also aim to consistently recover from the data the systematic jump times, which will be needed for construction of confidence intervals and testing.

We start first with eliminating the systematic jump risk due to observable factors, i.e., the jumps occurring at the times ρ_p. The factors are observed also at the times $i\Delta_n$ for $i = 1, \dots, n$, and they are supposed to have finite jump activity and with continuous parts being Itô semimartingales. Therefore, by using standard truncation methods, see e.g. Jacod and Protter (2012) and our implementation in Section 5, and on the basis of the observation of these factors, one can construct a set $\widehat{I}$ of integers between 1 and n such that:

$$\mathbb{P}(\widehat{I} = I_n) \to 1, \qquad \text{where} \;\; I_n = \big\{ i = 1, \dots, n : \;\; I(n,i) \text{ contains no } \rho_p \big\}, \tag{8}$$

and for convenience we write the random elements of $\widehat{I}$, in increasing order, as $\widehat{i}_1, \widehat{i}_2, \dots, \widehat{i}_{\widehat{p}}$. Note that $\widehat{p} \leq n$ and $\widehat{p}/n \xrightarrow{\text{a.s.}} 1$. Of course, when no jumping factor is observed or if we do not want to separate these jumps from the rest of the systematic jumps, we simply take $\widehat{I} = \{1, 2, \dots, n\}$.

For our inference for the systematic jump times, instead of the raw asset returns, we will use excess returns over equally-weighted market proxy constructed from the cross-section of asset prices. More specifically, we define the cross-sectional average process

$$\overline{X}_t^N = \frac{1}{N} \sum_{j=1}^{N} X_t^j. \tag{9}$$

Then, with a fixed number ψ and a sequence $u_{n,N}$ satisfying the conditions below, we set

$$\begin{array}{c} \widehat{r}_i^j = \Delta_i^n X^j - \psi \widehat{r}_i, \qquad \text{with} \quad \widehat{r}_i = \Delta_i^n \overline{X}^N \, 1_{\{|\Delta_i^n \overline{X}^N| \leq u_{n,N}\}}, \\ \text{assuming for some } \varepsilon \in (0,1): \quad \psi \neq 1, \quad u_{n,N} \to 0, \quad u_{n,N}^2 (n^{1-\varepsilon} \wedge N) \to \infty. \end{array} \tag{10}$$

The reason for using $\widehat{r}_i^j$ in the inference, instead of $\Delta_i^n X^j$, is to remove (at least partially) the systematic diffusive risk in the asset prices. As we will see later on, this reduces the estimation error of our estimation procedure in a nontrivial way. Note that in defining $\widehat{r}_i$, we truncate $\Delta_i^n \overline{X}^N$ in order not to affect the systematic jumps in $\Delta_i^n X^j$.

In our analysis, we also need a truncated version of both $\Delta_i^n X^j$ and $\widehat{r}_i^j$, associated with a sequence u_n^j and a constant $C > 0$, by

$$\left.\begin{array}{l} \Delta_i^{n,\mathcal{T}} X^j = \Delta_i^n X^j \, 1_{\{|\Delta_i^n X^j| \leq u_n^j\}} \\ \widehat{r}_i^{\mathcal{T}j} = \widehat{r}_i^j \, 1_{\{|\widehat{r}_i^j| \leq u_n^j\}}, \end{array}\right\} \quad \text{assuming} \quad \frac{1}{C} \leq u_n^j \Delta_n^{-\varpi} \leq C, \quad \varpi \in \big(0, \frac{1}{2}\big). \tag{11}$$

The requirements on the threshold above are standard and are the same as those used in prior work on truncated volatility estimation. We note also that the threshold levels for $\Delta_i^n X^j$ and $\widehat{r}_i^j$ can differ but should both satisfy the growth condition above.

The key ingredient of our inference procedure is the following cross-sectional average quantity, for any function g on $\mathbb{R}$:

$$\widehat{a}(g)_i = \tfrac{1}{N}\sum_{j=1}^{N} g(\widehat{r}_i^j), \qquad \widehat{a}^{\mathcal{T}}(g)_i = \tfrac{1}{N}\sum_{j=1}^{N} g(\widehat{r}_i^{\mathcal{T}j}),$$

for $i \in \widehat{I}$. If $\tau_p \in I(n,i)$, then $\widehat{a}(g)_i$ will provide an estimate of $\overline{\delta}_{g,p}$. If $I(n,i)$ does not contain a systematic jump, then $\widehat{a}(g)_i$ will shrink asymptotically to zero. We will also make use of bias-corrected versions of $\widehat{a}(g)_i$, given for any $i \in \widehat{I}$ by:

$$\begin{gathered}\breve{a}(g)_i = \widehat{a}(g)_i - \tfrac{1}{2}\,\widehat{a}(g)_{i_-} - \tfrac{1}{2}\,\widehat{a}(g)_{i_+}, \qquad \breve{a}^{\mathcal{T}}(g)_i = \widehat{a}^{\mathcal{T}}(g)_i - \tfrac{1}{2}\,\widehat{a}^{\mathcal{T}}(g)_{i_-} - \tfrac{1}{2}\,\widehat{a}^{\mathcal{T}}(g)_{i_+} \\ \text{where, when } i = \widehat{i}_l\text{:} \qquad i_- = \begin{cases} \widehat{i}_{l-1} & \text{if } l > 1 \\ \widehat{i}_{l+1} & \text{if } l = 1 \end{cases} \qquad i_+ = \begin{cases} \widehat{i}_{l+1} & \text{if } l < \widehat{p} \\ \widehat{i}_{l-1} & \text{if } l = \widehat{p}. \end{cases}\end{gathered} \tag{12}$$

As will become clear later, this differencing of sequential cross-sectional averages has no impact on our ability to estimate the systematic jump risk but it reduces significantly the effect of the systematic diffusive risk as well as the idiosyncratic risks in the asset prices on our estimates of the systematic jump risk. A bias-corrected version of the variable $g(\widehat{r}_i^j)$ for any index j, analogous to $\breve{a}(g)_i$, is also defined as follows:

$$\breve{g}_i^j = g(\widehat{r}_i^j) - \frac{1}{2}\,g(\widehat{r}_{i-}^j) - \frac{1}{2}\,g(\widehat{r}_{i+}^j), \qquad \text{for } i \in \widehat{I} \text{ and } i_\pm \text{ as in (12).} \tag{13}$$

Finally, for identifying the systematic jump times, we are also going to use

$$\widehat{VJ}_g = \sum_{i \in \widehat{I}} |\breve{a}^{\mathcal{T}}(g)_i|.$$

With all this notation, we can now proceed to estimating the set of all integers i in I_n such that $I(n,i)$ contains at least one systematic jump time τ_p. As a matter of fact, in view of the decomposition $SJ_g = SJ'_g + SJ''_g$, we have two sets of interest (with some $g \in \mathcal{C}_+$ for $\mathcal{I}'_n$, which does not depend on the choice of such a g):

$$\begin{aligned} I'_n &= \big\{ i \in I_n : I(n,i) \text{ contains a } \tau_p \text{ with } \overline{\delta}_{\mathbb{I},p} = 0 \text{ and } \overline{\delta}_{g,p} > 0 \big\} \\ I''_n &= \big\{ i \in I_n : I(n,i) \text{ contains a } \tau_p \text{ with } \overline{\delta}_{\mathbb{I},p} \neq 0 \big\}. \end{aligned} \tag{14}$$

Note that the random set $I'_n \cap I''_n$ may be non-empty, but $\mathbb{P}(I'_n \cap I''_n = \emptyset) \to 1$.

For estimating I''_n we use the averaged prices $\overline{X}^n$ of (9) and set, with $u_{n,N}$ as in (10):

$$\widehat{I}'' = \big\{ i \in \widehat{I} : |\Delta_i^n \overline{X}^N| > u_{n,N} \big\}. \tag{15}$$

For I'_n we define a first estimator as follows, with some given $f \in \mathcal{C}_+$:

$$\widehat{I}'_1 = \big\{ i \in \widehat{I} \backslash \widehat{I}'' :\ \breve{a}(f)_i > \gamma_{n,N} \widehat{VJ}_f \big\}, \quad \text{with} \begin{cases} \gamma_{n,N} \ \text{ satisfying, for some } \varepsilon \in (0,1), \\ \gamma_{n,N} \to 0, \quad \gamma_{n,N}\big(n^{1-\varepsilon} \wedge N\big) \to \infty. \end{cases} \tag{16}$$

Intuitively, we label an interval $I(n,i)$ as one with systematic jump risk if $\breve{a}(f)_i$ is larger than a multiple of its average absolute value computed after removing jumps in the individual return series.

The estimator $\widehat{I}'_1$ is "asymptotically perfect" in the sense that $\mathbb{P}(\widehat{I}'_1 = I'_n) \to 1$. However, when N is relatively small, $\widehat{I}'_1$ could be bigger than I'_n, due to the presence of some large idiosyncratic jumps, which can increase the value of $\breve{a}(f)_i$ (this is contrast to the case of (15), because the idiosyncratic jumps tend to cancel each other in the average $\overline{X}^N$). To avoid this, we rather use the following:

$$\widehat{I}' = \widehat{I}'_1 \cap \widehat{I}'_2, \qquad \widehat{I}'_2 = \big\{ i \in \widehat{I} \backslash \widehat{I}'' :\ \widehat{a}(f)_i > \gamma'_{n,N} \sqrt{\widehat{a}(f^2)_i} \big\}, \qquad \text{assuming } \ \gamma'_{n,N} \to 0, \tag{17}$$

which will also satisfy $\mathbb{P}(\widehat{I}' = I'_n) \to 1$. Intuitively, $\widehat{I}'_2$ removes increments for which the cross-sectional average $\widehat{a}(f)_i$ is quite smaller than the cross-sectional standard deviation $\sqrt{\widehat{a}(f^2)_i}$. This will never be the case asymptotically if the interval contains a systematic jump, but may happen if the interval $I(n,i)$ contains no systematic jump but some large idiosyncratic jumps.

For obtaining the property $\mathbb{P}(\widehat{I}' = I'_n) \to 1$, which are crucial for our main results, we need that $\widehat{VJ}_f$ does not converge to 0. For deriving the probability limit of $\widehat{VJ}_f$, we need some additional notation. For any $K \times K$ matrix M with entries $M^{l,l'}$ and any vector $x \in \mathbb{R}^K$ with components x^l, we set

$$\begin{array}{c} F_\psi(M,x) = \mathbb{E}\Big(\Big| \sum_{l,l'=1}^{K} \big(M^{l,l'} + (\psi^2 - 2\psi) x^l x^{l'} \big) \big(\Phi_1^l \Phi_1^{l'} - \frac{1}{2}\, \Phi_2^l \Phi_2^{l'} - \frac{1}{2}\, \Phi_3^l \Phi_3^{l'} \big) \Big| \Big), \\ \text{where the } \Phi_m^l \text{ for } l = 1, \ldots, K \text{ and } m = 1,2,3 \text{ are independent } \mathcal{N}(0,1) \text{ variables.} \end{array} \tag{18}$$

The function $F_\psi(M,x)$ is not explicit, but we will see that it is non-vanishing as soon as $x \neq 0$ and $\psi \neq 1$ and $M - xx^\top$ is symmetric nonnegative. This property is indeed obvious when $K = 1$, because then $F(M,x,\psi)$ is simply $M - x^2 + (1-\psi)^2 x^2$ times the (positive) expectation of $|(\Phi_1)^2 - \frac{1}{2}\,(\Phi_2)^2 - \frac{1}{2}\,(\Phi_3)^2|$. Below, $\overline{\lambda}_t^{(2)}$ and $\overline{\lambda}_t^{(1)}$ are the matrix and vector with components $\overline{\lambda}_t^{(2),l,l'}$ and $\overline{\lambda}_t^{(1),l}$, defined in (A2).

Theorem 1 *Assume* (4)*, (A1) and (A2) and* $\psi \neq 1$*. If* $g \in \mathcal{C}_+$ *we have*

$$\widehat{VJ}_g \xrightarrow{\mathbb{P}} \overline{A}_g := \frac{g''(0)^2}{2} \int_0^T F_\psi(\overline{\lambda}_t^{(2)}, \overline{\lambda}_t^{(1)})\, dt. \tag{19}$$

and also $\mathbb{P}(\widehat{I}' = I'_n) \to 1$ *and* $\mathbb{P}(\widehat{I}'' = I''_n) \to 1$.

The above theorem establishes the consistency of $\widehat{I}'$ and $\widehat{I}''$ for estimating I'_n and I''_n, respectively.

3.3 Aggregate Measures of Systematic Jump Risk

For estimating SJ'_g and SJ''_g we propose the following estimators (recall the notation in (12)):

$$\widehat{SJ}^z_g = \sum_{i\in\widehat{I}^z} \left(\frac{1}{2}(\widehat{a}(g)_i - \widehat{a}(g)_{i_-})^2 + \frac{1}{2}(\widehat{a}(g)_i - \widehat{a}(g)_{i_+})^2\right), \quad z = \{'\},\{''\}. \tag{20}$$

For SJ_g we can use one the following three estimators:

$$\begin{gathered} \widehat{SJ}_g = \widehat{SJ}'_g + \widehat{SJ}''_g \\ \widehat{SJ}^\flat_g = \tfrac{1}{2}\sum_{i=2}^{\widehat{p}} \big(\widehat{a}(g)_{\widehat{i}_i} - \widehat{a}(g)_{\widehat{i}_{i-1}}\big)^2 + \tfrac{1}{2}\big(\widehat{a}(g)^2_{\widehat{i}_1} + \widehat{a}(g)^2_{\widehat{i}_{\widehat{p}}}\big), \qquad \widehat{SJ}^{\flat\flat}_g = \sum_{i=1}^{\widehat{p}} \big(\widehat{a}(g)_{\widehat{i}_i}\big)^2. \end{gathered} \tag{21}$$

These three estimators enjoy the same CLT, and we will comment on the best choice among them in the end of this section below.

For stating the CLT, we need some further notation. First, we define the following random variables (not depending on the choice of j):

$$\begin{aligned} V'_g &= 4\,\mathbb{E}_{\mathcal{J}}\Big(\Big(\sum_{p\geq 1:\tau_p\leq T} \overline{\delta}_{g,p}\big(g(\delta^j_p) - \overline{\delta}_{g,p} - \psi\overline{\delta}_{g',p}\delta^j_p\big)1_{\{\overline{\delta}_{\mathbb{I},p}=0\}}\Big)^2\Big) \\ V''_g &= 4\,\mathbb{E}_{\mathcal{J}}\Big(\Big(\sum_{p\geq 1:\tau_p\leq T} \overline{\delta}_{g,p}\big(g(\delta^j_p) - \overline{\delta}_{g,p}\big)1_{\{\overline{\delta}_{\mathbb{I},p}\neq 0\}}\Big)^2\Big). \end{aligned} \tag{22}$$

With the notation $\overline{\lambda}^{(1)}_t$ in (A2) we also set

$$\Gamma^-_{g,p} = \overline{\delta}_{g,p}\,\overline{\delta}_{g',p}\,\|\overline{\lambda}^{(1)}_{\tau_p-}\|, \qquad \Gamma^+_{g,p} = \overline{\delta}_{g,p}\,\overline{\delta}_{g',p}\,\|\overline{\lambda}^{(1)}_{\tau_p}\|. \tag{23}$$

Next, we need a collection of variables, all defined on an extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ of the original space $(\Omega,\mathcal{F},\mathbb{P})$ and independent of $\mathcal{F}$, and all being mutually independent. These are: $\mathcal{N}(0,1)$ variables Z', Z'', Z^+_p, Z^-_p, and uniform on $[0,1]$ variables κ_p.

Finally, let us also recall that if $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$, the $\mathcal{G}$-stable convergence of a sequence Y_n of variables to a limit Y defined on an extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ of $(\Omega,\mathcal{F},\mathbb{P})$ means that

$$\mathbb{E}\big(\Psi\,F(Y_n)\big) \to \widetilde{\mathbb{E}}\big(\Psi\,F(Y)\big), \tag{24}$$

for any bounded $\mathcal{G}$-measurable variable Ψ and any continuous bounded function F, and this convergence is denoted $Y_n \stackrel{\mathcal{L}_{\mathcal{G}}-\mathrm{s}}{\longrightarrow} Y$ (when $\mathcal{G}=\mathcal{F}$ this is simply called stable convergence in law, written $Y_n \stackrel{\mathcal{L}-\mathrm{s}}{\longrightarrow} Y$). This obviously implies the ordinary convergence in law, and its main interest for us is because, if it holds and if further $\zeta_n \stackrel{\mathbb{P}}{\longrightarrow} \zeta$ for some sequence of $\mathcal{F}$-measurable variables with a limit ζ that is $\mathcal{G}$-measurable, the pair (Y_n,ζ_n) converges in law to (Y,ζ).

Theorem 2 *Assume* (4)*, (A1) and (A2), and set* $\mathcal{G}=\overline{\mathcal{J}}$ *when* $\phi\in(0,\infty]$ *and* $\mathcal{G}=\mathcal{F}$ *when* $\phi=0$.

(a) If $g \in \mathcal{C}$ the joint convergence $\big(\sqrt{n\wedge N}(\widehat{SJ}'_g - SJ'_g), \sqrt{n\wedge N}(\widehat{SJ}''_g - SJ''_g)\big) \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} (\widetilde{Z}'_g, \widetilde{Z}''_g)$ *holds, where with the convention $1/\infty = 0$ and with matrix notation,*

$$\begin{aligned}
\widetilde{Z}'_g &= (1\bigwedge\phi)\sqrt{V'_g}\,Z' + \tfrac{1}{1\bigvee\phi}\,Z'_g, \qquad\qquad \widetilde{Z}''_g = (1\bigwedge\phi)\sqrt{V''_g}\,Z'' + \tfrac{1}{1\bigvee\phi}\,Z''_g,\\
Z'_g &= 2(1-\psi)\sqrt{T}\sum_{p\geq 1:\,\tau_p\leq T}\Big(\Gamma^-_{g,p}\sqrt{\kappa_p}\,Z^-_p + \Gamma^+_{g,p}\sqrt{1-\kappa_p}\,Z^+_p\Big)1_{\{\overline{\delta}_{\mathbb{I},p}=0\}}\\
Z''_g &= 2\sqrt{T}\sum_{p\geq 1:\,\tau_p\leq T}\Big(\Gamma^-_{g,p}\sqrt{\kappa_p}\,Z^-_p + \Gamma^+_{g,p}\sqrt{1-\kappa_p}\,Z^+_p\Big)1_{\{\overline{\delta}_{\mathbb{I},p}\neq 0\}},
\end{aligned} \tag{25}$$

(b) We have $\sqrt{n\wedge N}(\widehat{SJ}_g - SJ_g) \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} \widetilde{Z}'_g + \widetilde{Z}''_g$ *and* $\sqrt{n\wedge N}(\widehat{SJ}^\flat_g - SJ_g) \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} \widetilde{Z}'_g + \widetilde{Z}''_g$ *and* $\sqrt{n\wedge N}(\widehat{SJ}^{\flat\flat}_g - SJ_g) \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} \widetilde{Z}'_g + \widetilde{Z}''_g$.

(c) As soon as $g \in \mathcal{C}_+$, we have $\mathbb{P}\big(SJ'_g = 0 < \widehat{SJ}'_g\big) \to 0$ and $\mathbb{P}\big(SJ''_g = 0 < \widehat{SJ}''_g\big) \to 0$, hence $\mathbb{P}\big(SJ'_g = 0 < \widehat{SJ}_g\big) \to 0$ as well.

Note that if $g \in \mathcal{C}_+$, on the set $\Omega'_{noSJ} = \{SJ'_g = 0\}$ we have $V'_g = 0$ and $Z'_g = 0$ by construction, hence $\widetilde{Z}'_g = 0$ and the CLT in (a) above is degenerate, but (b) tells us that on this set $\widehat{SJ}'_g$ is a "perfect" estimator of SJ'_g (and of course the same applies for SJ''_g and $\widehat{SJ}''_g$, or for SJ_g and $\widehat{SJ}_g$).

The CLT in Theorem 2 (a) is governed by two sources of error. The first one is the error associated with measuring systematic jumps from discrete observations of the asset prices. Its size depends on the sampling frequency n and is determined by the systematic diffusive components of the assets' price increments containing the systematic jumps. The uncertainty of the jump location within the discrete interval adds an additional source of error captured by the κ_p's. We note that the size of $\Gamma^\pm_{g,p}$ depends on the average diffusive systematic risk in asset prices around the systematic jump times, $\overline{\lambda}_{\tau_p-}$ and $\overline{\lambda}_{\tau_p}$.

The second source of error driving the CLT result is due to the cross-sectional heterogeneity in the systematic jumps. Naturally, this error depends on the size of the cross-section N. Importantly, we note that in this theorem there is no restriction on the relative size of the two dimensions of the high-frequency panel, n and N, apart from $n/N \to \phi^2$, with $\phi \in [0,1]$. In particular, both $n \gg N$ and $N \gg n$ are allowed.

We note that the constant ψ, used in (10), appears in the limit in (25). This constant plays no role for the estimation of SJ''_g, but for estimating SJ'_g, taking ψ close to 1 improves the efficiency of the estimator. The optimal choice seems to be $\psi = 1$, but if $n \gg N$ (i.e., if $\phi = 0$) this would then lead to a degenerate limit of the CLT in Theorem 2-(a). We could have a proper CLT when $\psi = 1$, with the limit for $\widehat{SJ}'_g$ being $\sqrt{V'_g}\,Z'$, but only under an additional restriction on how fast N can grow relative to n.

Which estimator for SJ_g? All three estimators $\widehat{SJ}_g$, $\widehat{SJ}^\flat_g$, $\widehat{SJ}^{\flat\flat}_g$ are asymptotically equivalent. The simplest ones are $\widehat{SJ}^{\flat\flat}_g$ and $\widehat{SJ}^\flat_g$ as, unlike $\widehat{SJ}_g$, they do not require the choice of tuning parameters. Of course, if one wants to construct confidence intervals for SJ_g, then for all estimators we still need the set estimators $\widehat{I}'$ and $\widehat{I}''$ identifying the systematic jump times.

On the other hand, all three estimators have different asymptotically negligible biases. These biases are difficult to evaluate in general, but one can get a handle on them by looking at the asymptotic behavior when $SJ_g = 0$ in the case g in $\mathcal{C}_+$ satisfies $|g(x+y) - g(x) - g(y) - g'(x)y| \leq C|x|y^2$ for some constant C, and under the (mild) additional assumption that the processes $t \mapsto \theta(t,z)^j/\phi_m^j(z)$ satisfy for all j, z the same condition (2) as σ_t^j. Then

$$\text{in restriction to the set } \{SJ_g = 0\}: \quad \begin{cases} \widehat{SJ}_g = 0 \quad \text{with a probability going to 1} \\ (n \wedge N)(\widehat{SJ}^\flat_g - A_g^{n,N}) \stackrel{\mathbb{P}}{\longrightarrow} 0 \\ (n \wedge N)(\widehat{SJ}^{\flat\flat}_g - A_g^{n,N} - A_g'^{n,N}) \stackrel{\mathbb{P}}{\longrightarrow} 0, \end{cases} \tag{26}$$

where

$$A_g^{n,N} = \int_0^T \Big(\frac{T g''(0)^2}{2n} \Big(\frac{1}{N^2} \sum_{j,k=1}^N (\lambda_t^{j,\top} \lambda_t^k)^2 + ((\psi-1)^4 - 1) \Big\| \frac{1}{N} \sum_{j=1}^N \lambda_t^j \Big\|^4 \Big) + \frac{1}{N} \sum_{j=1}^N \int_{\mathbb{R}} g(\theta^j(t,z)^2\, dz \Big) dt,$$

$$A_g'^{n,N} = \int_0^T \Big(\frac{g''(0)}{2} \Big\| \frac{1}{N} \sum_{j=1}^N \big(\|\lambda_t^j\|^2 + (\sigma_t^j)^2 \big) + (\psi^2 - 2\psi) \Big\| \frac{1}{N} \sum_{j=1}^N \lambda_t^j \Big\|^2 + \frac{1}{N} \sum_{j=1}^N \int_{\mathbb{R}} g(\theta^j(t,z)\, dz \Big)^2 dt$$

Apart from the first line in (26), which is a part of Theorem 2, those properties are rather complicated to prove, and since they are not formally used in the sequel we omit the proof. Note that the first term under the integral $\int_0^T$, for both $A_g^{n,N}$ and $A_g'^{n,N}$, is nonnegative for all values of ψ, and minimal for $\psi = 1$.

The convergence result in (26) provides clear ranking of the estimators of SJ_g in terms of bias. On the set $\{SJ_g = 0\}$, where there are no systematic jumps in the asset prices, the bias is zero with a large probability for $\widehat{SJ}_g$, positive with probability 1 for $\widehat{SJ}^\flat_g$, and even bigger for $\widehat{SJ}^{\flat\flat}_g$. We note that ψ shows up in $A_g^{n,N}$ and $\widetilde{A}_g^{n,N}$, and with a choice of ψ close to one, we can minimize the bias in the statistics that is due to the systematic diffusive component of asset prices. The value of $\widetilde{A}_g^{n,N}$ reveals the gains from the sequential differencing of $\widehat{a}(g)_i$ in the construction of $\widehat{SJ}^\flat_g$ (and $\widehat{SJ}_g$). Indeed, when going from $\widehat{SJ}^{\flat\flat}_g$ to $\widehat{SJ}^\flat_g$, we minimize the effect on the measurement of SJ_g from all risks in asset prices. In particular, the asymptotic limit of the rescaled $\widehat{SJ}^\flat_g$ does not depend on the idiosyncratic diffusive risk, which is typically nontrivial.

On the set $\{SJ_g \neq 0\}$ one should add the sum of the biases induced by the estimation of $(\overline{\delta}_{g,p})^2$ at each systematic jump time $\tau_p \leq T$. One might expect those to be of the same order of magnitude for all three statistics, so the bias increases when going from $\widehat{SJ}_g$ to $\widehat{SJ}^\flat_g$, and from $\widehat{SJ}^\flat_g$ to $\widehat{SJ}^{\flat\flat}_g$.

3.4 Asset Sensitivity towards Systematic Jump Risk

We next estimate the sensitivity of any particular asset to systematic jump risk as expressed by $SJ'^j_{g,h}$, $SJ''^j_{g,h}$ and $SJ^j_{g,h}$. For the first two ones we use the following statistics, where both functions g,h are in $\mathcal{C}$ and $\widehat{I}', \widehat{I}''$ as in (17) and (15), plus the notation (12) and (13):

$$\widehat{SJ}'^j_{g,h} = \sum_{i \in \widehat{I}'} \breve{h}^j_i \, \breve{a}(g)_i, \qquad \widehat{SJ}''^j_{g,h} = \sum_{i \in \widehat{I}''} \breve{h}^j_i \, \breve{a}(g)_i.$$

For $SJ^j_{g,h}$, in accordance with (21), we have three possible estimators:

$$\begin{gathered}\widehat{SJ}^j_{g,h} = \widehat{SJ}'^j_{g,h} + \widehat{SJ}''^j_{g,h} \\ \widehat{SJ}^{j,\flat}_{g,h} = \tfrac{1}{2}\Big(\sum_{i=2}^{\widehat{p}} \big(h(\widehat{r}^j_{\widehat{i}_i}) - h(\widehat{r}^j_{\widehat{i}_{i-1}})\big)\big(\widehat{a}(g)_{\widehat{i}_i} - \widehat{a}(g)_{\widehat{i}_{i-1}}\big) + h(\widehat{r}^j_{\widehat{i}_1})\widehat{a}(g)_{\widehat{i}_1} + h(\widehat{r}^j_{\widehat{p}})\widehat{a}(g)_{\widehat{i}_{\widehat{p}}} \Big) \\ \widehat{SJ}^{j,\flat\flat}_{g,h} = \sum_{i=1}^{\widehat{p}} h(\widehat{r}^j_{\widehat{i}_i})\widehat{a}(g)_{\widehat{i}_i}.\end{gathered}$$

For stating the CLT enjoyed by these statistics, we define the following two variables, similar to (22) and not depending on the choice of k as soon as $k \neq j$:

$$\begin{aligned} V'^j_{g,h} &= \mathbb{E}_{\mathcal{J}_j}\Big(\Big(\sum_{p:\, \tau_p \leq T} \big(h(\delta^j_p)\big(g(\delta^k_p) - \overline{\delta}_{g,p}\big) - \psi h(\delta^j_p)\overline{\delta}_{g',p}\delta^k_p - \psi h'(\delta^j_p)\overline{\delta}_{g,p}\delta^k_p\big) 1_{\{\overline{\delta}_{\mathbb{I},p}=0\}}\Big)^2\Big) \\ V''^j_{g,h} &= \mathbb{E}_{\mathcal{J}_j}\Big(\Big(\sum_{p:\, \tau_p \leq T} h(\delta^j_p)\big(g(\delta^k_p) - \overline{\delta}_{g,p}\big) 1_{\{\overline{\delta}_{\mathbb{I},p}\neq 0\}}\Big)^2\Big). \end{aligned} \tag{27}$$

We also define the following analogues $\Gamma'^{+,j}_{g,h,p}$, $\Gamma'^{-,j}_{g,h,p}$, $\Gamma''^{+,j}_{g,h,p}$, $\Gamma''^{-,j}_{g,h,p}$ of $\Gamma^+_{g,p}$ and $\Gamma^-_{g,p}$ in (23), with the convention $\tau_{p+} = \tau_p$ below:

$$\begin{aligned} \Gamma'^{\pm,j}_{g,h,p} &= \Big(\big\| \big((1-\psi)h(\delta^j_p)\overline{\delta}_{g',p} - \psi h'(\delta^j_p)\overline{\delta}_{g,p}\big)\overline{\lambda}^{(1)}_{\tau_p\pm} + h'(\delta^j_p)\,\overline{\delta}_{g,p}\,\lambda^j_{\tau_p\pm}\big\|^2 + \big(\overline{\delta}_{g,p}\, h'(\delta^j_p)\,\sigma^j_{\tau_p\pm}\big)^2\Big)^{1/2} \\ \Gamma''^{\pm,j}_{g,h,p} &= \Big(\big\| h(\delta^j_p)\,\overline{\delta}_{g',p}\,\overline{\lambda}^{(1)}_{\tau_p\pm} + h'(\delta^j_p)\,\overline{\delta}_{g,p}\,\lambda^j_{\tau_p\pm}\big\|^2 + \big(\overline{\delta}_{g,p}\, h'(\delta^j_p)\,\sigma^j_{\tau_p\pm}\big)^2\Big)^{1/2}. \end{aligned} \tag{28}$$

Then, with $Z', Z'', Z^+_p, Z^-_p, \kappa_p$ as after (23), we have the following CLT:

Theorem 3 *Assume (4), (A1) and (A2), and set $\mathcal{G} = \overline{\mathcal{J}}_j$ if $\phi \in (0,\infty]$ and $\mathcal{G} = \mathcal{F}$ if $\phi = 0$. If $g,h \in \mathcal{C}$ the joint convergence $\big(\sqrt{n \wedge N}(\widehat{SJ}'^j_{g,h} - SJ'^j_{g,h}), \sqrt{n \wedge N}(\widehat{SJ}''^j_{g,h} - SJ''^j_{g,h}\big) \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} (\widetilde{Z}'^j_{g,h}, \widetilde{Z}''^j_{g,h})$ holds, where*

$$\begin{gathered} \widetilde{Z}'^j_{g,h} = (1 \textstyle\bigwedge \phi)\, \sqrt{V'^j_{g,h}}\, Z' + \frac{1}{1 \bigvee \phi} Z'^j_{g,h}, \qquad \widetilde{Z}''^j_{g,h} = (1 \textstyle\bigwedge \phi)\, \sqrt{V'''^j_{g,h}}\, Z'' + \frac{1}{1 \bigvee \phi} Z''^j_{g,h} \\ Z'^j_{g,h} = \sqrt{T} \sum_{p \geq 1:\, \tau_p \leq T} \Big(\Gamma'^{-,j}_{g,h,p}\,\sqrt{\kappa_p}\, Z^-_p + \Gamma'^{+,j}_{g,h,p}\,\sqrt{1-\kappa_p}\, Z^+_p\Big) 1_{\{\overline{\delta}_{\mathbb{I},p}=0\}} \\ Z''^j_{g,h} = \sqrt{T} \sum_{p \geq 1:\, \tau_p \leq T} \Big(\Gamma''^{-,j}_{g,h,p}\,\sqrt{\kappa_p}\, Z^-_p + \Gamma''^{+,j}_{g,h,p}\,\sqrt{1-\kappa_p}\, Z^+_p\Big) 1_{\{\overline{\delta}_{\mathbb{I},p}\neq 0\}}. \end{gathered} \tag{29}$$

Moreover, we have $\sqrt{n \wedge N}(\widehat{SJ}^j_{g,h} - SJ^j_{g,h}) \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} \widetilde{Z}'^j_g + \widetilde{Z}''^j_g$ and $\sqrt{n \wedge N}(\widehat{SJ}^{j,\flat}_{g,h} - SJ^j_{g,h}) \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} \widetilde{Z}'_g + \widetilde{Z}''_g$ and $\sqrt{n \wedge N}(\widehat{SJ}^{j,\flat\flat}_{g,h} - SJ^j_{g,h}) \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} \widetilde{Z}'_g + \widetilde{Z}''_g$.

This CLT is of similar type as Theorem 2. The only qualitative difference is that here, on top of the times τ_p, the limit also depends on the asset specific variables $\delta_p^j, \lambda_{\tau_p\pm}^j, \sigma_{\tau_p\pm}^j$. Again, the three statistics $\widehat{SJ}_{g,h}^j$, $\widehat{SJ}_{g,h}^{j,\flat}$ and $\widehat{SJ}_{g,h}^{j,\flat\flat}$ are asymptotically equivalent, and similar comparisons about their biases can be made as in the previous section.

3.5 Feasible Inference

For feasible inference based on the previous theorems, we need to know the variance or the quantiles of the limiting variables $\widetilde{Z}'_g$, $\widetilde{Z}''_g$, $\widetilde{Z}'^j_{g,h}$ or $\widetilde{Z}''^j_{g,h}$, at least asymptotically. However, those limits are in general not $\mathcal{G}$-conditionally Gaussian, so the best one can do is to find a sequence of observable variables which approach the limit. For this, besides the sets $\widehat{I}'$ and $\widehat{I}''$, we need three ingredients:

1) Consistent estimators for the conditional variances (22) or (27). Recalling (12) and (13) and with the notation $L_N^j = \{k \in \mathbb{N} : k \neq j, 1 \leq k \leq N\}$, we use

$$
\begin{array}{ll}
\widehat{V}'_g = \frac{4}{N} \sum\limits_{j=1}^N (\widehat{c}(g)'_j)^2, & \widehat{c}(g)'_j = \sum\limits_{i\in\widehat{I}'} \breve{a}(g)_i \big(\breve{g}_i^j - \breve{a}(g)_i - \psi \breve{a}(g')_i \breve{\mathbb{I}}_i^j\big) \\
\widehat{V}''_g = \frac{4}{N} \sum\limits_{j=1}^N (\widehat{c}(g)''_j)^2, & \widehat{c}(g)''_j = \sum\limits_{i\in\widehat{I}''} \breve{a}(g)_i \big(\breve{g}_i^j - \breve{a}(g)_i\big) \\
\widehat{V}'^j_{g,h} = \frac{1}{N-1} \sum\limits_{k\in L_N^j} \big(\widehat{c}(g,h)'^j_k\big)^2, & \widehat{c}(g,h)'^j_k = \sum\limits_{i\in\widehat{I}'} \Big(\breve{h}_i^j \big(\breve{g}_i^k - \breve{a}(g)_i\big) \\
 & \qquad -\psi\big(\breve{h}'^j_i \breve{a}(g)_i + \breve{h}_i^j \breve{a}(g')_i\big)\breve{\mathbb{I}}_i^k\Big) \\
\widehat{V}''^j_{g,h} = \frac{1}{N-1} \sum\limits_{k\in L_N^j} \big(\widehat{c}(g,h)''^j_k\big)^2, & \widehat{c}(g,h)''^j_k = \sum\limits_{i\in\widehat{I}''} \breve{h}_i^j \big(\breve{g}_i^k - \breve{a}(g)_i\big)
\end{array}
$$

2) Consistent estimators for the variables in (23) or (28). These involve the processes λ^j and σ^j at time τ_p, or their left limits at this time. We thus need the truncation procedure (11) for eliminating the jumps, and rolling windows of m_n successive increments, with m_n satisfying as $n \to \infty$:

$$
m_n \to \infty, \qquad \frac{m_n}{n} \to 0, \tag{30}
$$

For any $i \in \widehat{I}' \cup \widehat{I}''$, necessarily of the form $i = \widehat{i}_l$ for some l between 1 and $\widehat{p}$, a time $i'\Delta_n$ will belong the window on the left, resp. right, side of the time $i\Delta_n$ if i' belongs to the set $M_{n,i}^+ = \{\widehat{i}_{(l+m)\wedge\widehat{p}} : m = 1, \ldots, m_n\}$, resp. $M_{n,i}^- = \{\widehat{i}_{(l-m)\vee 1} : m = 0, \ldots, m_n - 1\}$. With this notation, we set

$$
\begin{aligned}
\widehat{\Gamma}_{g,i}^{\pm} &= \breve{a}(g)_i\, \breve{a}(g')_i \Big(\frac{n}{m_n} \sum_{i'\in M_{n,i}^{\pm}} \Big(\frac{1}{N}\sum_{j=1}^N \Delta_{i'}^{n,\mathcal{T}} X^j\Big)^2\Big)^{1/2} \\
\widehat{\Gamma}'^{\pm,j}_{g,h,i} &= \Big(\frac{n}{m_n} \sum_{i'\in M_{n,i}^{\pm}} \Big((1-\psi)\breve{a}(g')_i \breve{h}_i^j \frac{1}{N}\sum_{k=1}^N \Delta_{i'}^{n,\mathcal{T}} X^k + \breve{a}(g)_i \breve{h}'^j_i\, \Delta_{i'}^{n,\mathcal{T}} X^j\Big)^2\Big)^{1/2} \\
\widehat{\Gamma}''^{\pm,j}_{g,h,i} &= \Big(\frac{n}{m_n} \sum_{i'\in M_{n,i}^{\pm}} \Big(\breve{a}(g')_i \breve{h}_i^j \frac{1}{N}\sum_{k=1}^N \Delta_{i'}^{n,\mathcal{T}} X^k + \breve{a}(g)_i \breve{h}'^j_i\, \Delta_{i'}^{n,\mathcal{T}} X^j\Big)^2\Big)^{1/2}.
\end{aligned} \tag{31}
$$

Note that here we use the truncated increments $\Delta_i^{n,\mathcal{T}} X^j$ and not $\widehat{r}_i^{\mathcal{T}j}$. This is because the latter were introduced to eliminate the systematic diffusive part of asset prices, as much as possible, whereas here we want to estimate for example $\overline{\lambda}_t^{(1)}$ which depends on this component of asset prices. Note also that, when $i \in \widehat{I}' \cup \widehat{I}''$, the number of points in $M_{n,i}^{\pm}$ equals m_n with a probability going to 1. However, for small samples this number could actually be quite smaller than m_n.

3) An extra set of variables, all mutually independent, and independent of the observations: $\overline{Z}, \overline{Z}'$ and $\overline{Z}_i^+, \overline{Z}_i^-$ which are $\mathcal{N}(0,1)$, and $\overline{\kappa}_i$ which are uniformly distributed on $[0,1]$.

With all these ingredients, we can now set

$$\begin{array}{lcl} \widehat{Z}'_g &=& \sqrt{\frac{n\wedge N}{N}}\,\sqrt{\widehat{V}'_g}\,\overline{Z}' + 2(1-\psi)\sqrt{\frac{N}{n\vee N}}\sum_{i\in\widehat{I}'}\big(\widehat{\Gamma}^-_{g,i}\sqrt{\overline{\kappa}_i}\,\overline{Z}_i^- + \widehat{\Gamma}^+_{g,i}\sqrt{1-\overline{\kappa}_i}\,\overline{Z}_i^+\big) \\ \widehat{Z}''_g &=& \sqrt{\frac{n\wedge N}{N}}\,\sqrt{\widehat{V}''_g}\,\overline{Z}'' + 2\sqrt{\frac{N}{n\vee N}}\sum_{i\in\widehat{I}''}\big(\widehat{\Gamma}^-_{g,i}\sqrt{\overline{\kappa}_i}\,\overline{Z}_i^- + \widehat{\Gamma}^+_{g,i}\sqrt{1-\overline{\kappa}_i}\,\overline{Z}_i^+\big) \\ \widehat{Z}'^j_{g,h} &=& \sqrt{\frac{n\wedge N}{N}}\,\sqrt{\widehat{V}'^j_{g,h}}\,\overline{Z}' + \sqrt{\frac{N}{n\vee N}}\sum_{i\in\widehat{I}'}\big(\widehat{\Gamma}'^{-,j}_{g,h,i}\sqrt{\overline{\kappa}_i}\,\overline{Z}_i^- + \widehat{\Gamma}'^{+,j}_{g,h,i}\sqrt{1-\overline{\kappa}_i}\,\overline{Z}_i^+\big) \\ \widehat{Z}''^j_{g,h} &=& \sqrt{\frac{n\wedge N}{N}}\,\sqrt{\widehat{V}''^j_{g,h}}\,\overline{Z}'' + \sqrt{\frac{N}{n\vee N}}\sum_{i\in\widehat{I}''}\big(\widehat{\Gamma}''^{-,j}_{g,h,i}\sqrt{\overline{\kappa}_i}\,\overline{Z}_i^- + \widehat{\Gamma}'^{+,j}_{g,h,i}\sqrt{1-\overline{\kappa}_i}\,\overline{Z}_i^+\big). \end{array} \tag{32}$$

Theorem 4 *Under the assumptions of Theorem 2 the $\mathcal{F}$-conditional law of the two-dimensional variables $(\widehat{Z}'_g, \widehat{Z}''_g)$ (resp. $(\widehat{Z}'^j_{g,h}, \widehat{Z}''^j_{g,h})$) converge to the $\mathcal{F}$-conditional law of $(\widetilde{Z}'_g, \widetilde{Z}''_g)$ (resp. $(\widetilde{Z}'^j_{g,h}, \widetilde{Z}''^j_{g,h})$).*

This result allows us to conduct feasible inference, as will be shown below. We note in this regard that the estimates in (31) do not require prior knowledge regarding the number of systematic diffusive risk factors and they do not need prior estimates of the factor loadings λ_t^j. This is despite of the fact that the limits in Theorems 2 and 3 depend on these factor loadings, and is very convenient for applications.

3.6 Estimation of Aggregated Measures of Systematic Jumps

We now show how the previous theorems can be put in use for feasible inference of the quantities SJ_g, SJ'_g, SJ''_g, $SJ^j_{g,h}$, $SJ'^j_{g,h}$ or $SJ''^j_{g,h}$, when g and h are in $\mathcal{C}$.

Let us consider the case of our aggregate systematic jump measure SJ'_g. We construct a confidence interval for SJ'_g, with asymptotic level $\alpha \in (0,1)$. For this, note that the $\mathbb{P}_{\mathcal{F}}$-conditional law of $\widetilde{Z}'_g$ has clearly no atom, hence $\mathbb{P}_{\mathcal{F}}(\widehat{Z}'_g > \alpha) \to \mathbb{P}_{\mathcal{F}}(\widetilde{Z}'_g > \alpha)$ by Theorem 4. Recall also that $\widehat{V}'_g$ and $\widehat{\Gamma}'^{\pm}_{g,i}$ and $\widehat{I}'$ are known to the econometrician. Then, one can simulate M copies of the variables $\overline{Z}'$, $\overline{Z}_i^{\pm}$ and $\overline{\kappa}_i$ for all $i \in \widehat{I}'$, and compute the associated $\widehat{Z}(m)'_g$ for $m = 1, \ldots, M$ by the first formula in (32). The confidence interval is then constructed from

$$\begin{array}{l} \widehat{A}^-_\alpha = \sup\big(x:\, x>0,\, \frac{1}{M}\sum_{m=1}^M 1_{\{\widehat{Z}(m)'_g \le -x\}} > \frac{\alpha}{2}\big) \\ \widehat{A}^+_\alpha = \inf\big(x:\, x>0,\, \frac{1}{M}\sum_{m=1}^M 1_{\{\widehat{Z}(m)'_g \ge x\}} > \frac{\alpha}{2}\big). \end{array}$$

In the important special case in which the systematic jump times are not accompanied by jumps in the diffusive factor loadings λ_t^j, then the limit in Theorem 2 is $\mathcal{F}$-conditionally Gaussian. Therefore, $\widehat{A}_\alpha^\pm$ can be computed without simulation as

$$\widehat{A}_\alpha^\pm = Q_{1-\alpha/2}\sqrt{\frac{1}{N}\,\widehat{V}'_g + 2\frac{1}{n}(1-\psi)^2 \sum_{i\in\widehat{I}'} \big((\widehat{\Gamma}^-_{g,i})^2 + (\widehat{\Gamma}^+_{g,i})^2\big)}, \tag{33}$$

where Q_x is the x-quantile of a standard normal random variable. Given $\widehat{A}_\alpha^\pm$, a confidence interval is given by

$$\widehat{I}(SJ'_g)_\alpha = \big[\widehat{SJ}'_g - \widehat{A}_\alpha^-, \widehat{SJ}'_g + \widehat{A}_\alpha^+\big],$$

and the following is easily deduced from Theorems 2 and 4:

Theorem 5 *Under the assumptions of Theorem 2 we have $\mathbb{P}(SJ'_g \notin \widehat{I}(SJ'_g)_\alpha) \to \alpha$ as $n, N, M \to \infty$.*

The estimation of SJ_g, SJ''_g, $SJ^j_{g,h}$, $SJ'^j_{g,h}$ or $SJ''^j_{g,h}$ can be conducted in a similar way.

4 Testing for the Presence of Systematic Jump Risk

We next design tests for deciding whether systematic jumps occur within the time interval $[0, T)$. A *systematic jump time* is a jump time τ_p at which the jump sizes δ_p^j significantly impact most of the assets. By this, we mean that the averaged absolute jump sizes $\frac{1}{N}\sum_{j=1}^N |\delta_p^j|$ do not go to 0 as $N \to \infty$. Under (A2) and by (6), this amounts to saying that we are not in the set Ω_{noSJ}.

Therefore, below we derive a test for the following (random) null and alternative hypotheses:

$$\Omega^{null} = \Omega\backslash\Omega_{noSJ}, \qquad \Omega^{alt} = \Omega_{noSJ}. \tag{34}$$

Let us recall that a sequence of tests with critical regions $\mathcal{C}_n$ is called asymptotically null-consistent, resp. alternative-consistent, if $\mathbb{P}(\mathcal{C}_n \cap \Omega^{null}) \to 0$, resp. $\mathbb{P}((\mathcal{C}_n)^c \cap \Omega^{alt}) \to 0$. If further $\mathbb{P}(\mathcal{C}_n|B)$ converges to some $\alpha > 0$ for all measurable subsets B of Ω^0 with $\mathbb{P}(B) > 0$, one says that the sequence of tests has the *strong asymptotic size* α.

For testing the null Ω^{null} against the alternative Ω^{alt} of (34), we can use the statistics $\widehat{SJ}_g$ of (21) and define the critical (rejection) regions

$$\widehat{\mathcal{C}}_g = \big\{\widehat{SJ}_g > \zeta_{n,N}\big\},$$

for a double sequence $\zeta_{n,N}$ of positive reals going to 0 as $n, N \to \infty$. An obvious consequence of Theorems 2 is as follows:

Theorem 6 *Under (A1), (A2) and if $g \in \mathcal{C}_+$, the critical regions $\widehat{\mathcal{C}}_g$ are asymptotically null-consistent and alternative-consistent.*

The test above has asymptotic size of $\alpha = 0$. However, the main problem with this test is that we have no guidance about how to choose $\zeta_{n,N}$ for a given pair (n, N). Obviously, if we increase $\zeta_{n,N}$ the error of the first kind (the conditional probability of rejection under the null) decreases, whereas the error of the second kind increases.

A natural way for constructing classical tests with some prescribed asymptotic size $\alpha > 0$ would be to derive a CLT for $\widehat{SJ}_g$ under the null. However, part (b) of Theorem 2 shows that such a CLT cannot exist. Another approach is to use $\widehat{SJ}^\flat_g$ instead of $\widehat{SJ}_g$, computed for $(n/2, N/2)$ and for (n, N). It is easy to show, based on the results derived above, that a ratio of $\widehat{SJ}^\flat_g$, computed for $(n/2, N/2)$ and for (n, N), converges to 2 under the null and to 1 under the alternative. Deriving a CLT under the null for $\widehat{SJ}^\flat_g$, however, is challenging.

We will, therefore, develop an alternative and simpler strategy, using the statistic $\widehat{SJ}^\flat_g$. In fact, $\widehat{SJ}^\flat_g - \widehat{SJ}_g$ is basically the same as $\widehat{SJ}^\flat_g$ would be if we deleted all systematic jumps from the X^j's. Therefore, from the result in (26), $\widehat{SJ}^\flat_g - \widehat{SJ}_g$ will be shrinking asymptotically, both under the null and under the alternative hypothesis. As a result, for any $\alpha \in (0, 1)$ we propose a systematic jump test with the following critical region:

$$\widehat{\mathcal{C}}(\alpha) = \left\{ \widehat{SJ}_g > |\widehat{SJ}^\flat_g - \widehat{SJ}_g| (Q_{1-\alpha} - Z^*) \right\},$$

where Z^* is a $N(0, 1)$ random variable, which is simulated independently from the observations, and Q_x denotes again the x-quantile of the standard normal distribution. We take the absolute value of $\widehat{SJ}^\flat_g - \widehat{SJ}_g$ because this might be negative, although it is positive with a probability going to 1.

Theorem 7 *Assume* (4), *(A1), (A2), and also $g \in \mathcal{C}_+$. Then, the critical regions $\widehat{\mathcal{C}}(\alpha)$ have the strong asymptotic size α and are asymptotically alternative consistent.*

One can also conduct tests for deciding whether $SJ'_g > 0$ or $SJ''_g > 0$ in pretty much the same way. We leave this to the reader.

5 Monte Carlo Study

We next evaluate the finite sample performance of the developed testing and inference procedures.

5.1 Setup

We use the following model for the log-prices of the assets in our Monte Carlo:

$$dX^j_t = \lambda_j dX_t + \sqrt{V_t} \left(d\widetilde{W}^j_t + d\widetilde{L}^j_t + \phi \beta_j dS_t \right), \qquad dX_t = \sqrt{V_{t-}} (dW_t + dL_t),$$

$$dV_t = 8.3(0.02 - V_t)dt + \sqrt{V_t}(-0.1 dW_t + 0.2\sqrt{0.75} dB_t),$$

where W_t, B_t and $\{\widetilde{W}_t^j\}_{j=1,\ldots,N}$ are independent standard Brownian motions, L_t and $\{\widetilde{L}_t^j\}_{j=1,\ldots,N}$ are independent pure-jump Lévy martingales with respective Lévy measures $c_L\, e^{-\lambda_L|x|}\, dx$ and $\widetilde{c}\, e^{-\widetilde{\lambda}|x|}\, dx$, with

$$\frac{4c_L}{\lambda_L^3} = 0.2 \text{ and } \frac{2c_L}{\lambda_L} = 52,\ \frac{4\widetilde{c}}{\widetilde{\lambda}^3} = 0.4 \text{ and } \frac{2\widetilde{c}}{\widetilde{\lambda}} = 52.$$

S_t is systematic jump in asset prices that is independent from L_t and $\{\widetilde{L}_{t,j}\}_{j=1,\ldots,N}$. It is given by

$$S_t = \begin{cases} \Delta S_\tau, & \text{for } s \geq \tau \\ 0, & \text{otherwise} \end{cases}, \quad \tau \sim U\left(\frac{T}{4}, \frac{3T}{4}\right).$$

Finally, $\{\lambda_j\}_{j=1,\ldots,N}$ is an i.i.d. sequence with $\lambda_j \sim N(1, 0.5/3)$ and $\{\beta_j\}_{j=1,\ldots,N}$ is another i.i.d. sequence with $\beta_j \sim U(-2, 2)$.

In the specification above, x_t can be thought of as the market index and therefore λ_j is the market beta of asset j, $\widetilde{W}_t^j$ and $\widetilde{L}_t^j$ are idiosyncratic diffusive and jump risks, and S_t captures non-market systematic jump risk, with β_j being the exposure of asset j to such risk. All jump processes in the model, but S_t, are double-exponential and for all of them, the parameters are set so that their expected jump arrival is approximately once every week and their contribution to the quadratic variation of X_t and $\{X_t^j\}_{j=1,\ldots,N}$ is around 16%. All risks in asset prices have time-varying volatility proportional to V_t, which is modeled as a square-root diffusion process with half-life of mean-reversion coefficient corresponding to one month (our unit of time is one year) and mean of 0.02. Finally, the parameter ϕ takes two values: zero (corresponding to null hypothesis of no systematic jump risk other than market jump risk) and one (corresponding to presence of non-market systematic jump risk in asset prices).

The size of the systematic jump, ΔS_τ, is set to $\sqrt{V_\tau} \times 0.015$. This means that the contribution of the systematic risk due to S in the average cross-sectional daily realized variance is approximately 2.8%. This is rather small and therefore inference for such a jump is challenging. To illustrate this, if we use a standard truncation method to identify the jumps in the individual assets, see equation (35) above, then on average only three stocks will be detected to have jumped at the time of the systematic jump introduced here for ten minute sampling frequency ($n = 40$).

Turning next to the sampling scheme, we will set T to one day and we will use sampling frequency of $n = 80$ and $n = 40$. This corresponds to sampling asset prices approximately every five and ten minutes, respectively, in a 6.5 hours trading day. We will experiment with size of the cross-section of $N = 300$ and $N = 500$.

5.2 Choice of Tuning Parameters

For implementing the developed inference procedures, we need to choose the function g and also set the tuning parameters for identifying the set of systematic jumps. This is what we describe in this section.

Throughout the numerical analysis, we use the following function for g:

$$g_a(x) = \begin{cases} \frac{1}{2}x^2 \exp\left(-\frac{x^2}{2a^2-x^2}\right) + a^2\left(1-\exp\left(-\frac{x^2}{2a^2-x^2}\right)\right), & \text{if } |x| < a\sqrt{2}, \\ a^2, \quad \text{if } |x| \geq a\sqrt{2}, \end{cases}$$

for some $a > 0$. The function g_a can be viewed as a smooth approximation of $x^2 \wedge a^2$ and it can be shown that it belongs to $\mathcal{C}_+$. We set $a = 0.025$, which can be viewed as "smooth winsorizing" of the square function at the level of 2.5% return size. This should provide robustness in the analysis against extreme return observations (mostly due to idiosyncratic jumps).

Our interest is in the systematic jumps, which happen outside the times of observable systematic risk factors (such as the market portfolio jumps). As common in the literature, we identify the set of jumps in a process X observed at high frequencies via

$$\{i = 1, ..., n : |\Delta_i^n X| > u_n(X)\}, \quad u_n(X) = 3\sqrt{\min\{BV_T^n,\, RV_T^n\}}\Delta_n^{0.49}, \tag{35}$$

where $BV_T^n = \frac{\pi}{2}\sum_{i=2}^n |\Delta_i^n X||\Delta_{i-1}^n X|$ and $RV_T^n = \sum_{i=1}^n (\Delta_i^n X)^2$. The set $\widehat{I}$ is then formed as the intersection of the complements of the above sets for the set of observable systematic factors (which in the model in our Monte Carlo is just one). We set the truncation levels u_n^j in (11) for the price increments to $u_n(X^j)$. The truncation of the cross-sectional average of the returns, $u_{n,N}$ in (10) is then set to $u_{n,N} = \left(1 \vee \frac{1}{N^{0.49}\Delta_n^{0.49}}\right) \frac{1}{N}\sum_{j=1}^N u_n^j$. Finally, the excess returns $\widehat{r}_i^j$ in (10) are constructed by setting $\psi = 1.01$.

Turning next to the set $\widehat{I}'$ in (17), we set $f = g$ and we need to select $\gamma_{n,N}$ and $\gamma'_{n,N}$. For $\gamma'_{n,N}$, we use

$$\gamma'_{n,N} = \frac{4}{\sqrt{N}}.$$

Note that, if $I(n,i)$ contains a systematic jump, $\frac{1}{\sqrt{N}}\sqrt{\widehat{a}(f^2)_i}$ is an estimate of the asymptotic standard deviation of measuring the systematic jump in this increment that is due to the cross-sectional dispersion of the systematic jump risk. Therefore, with the above choice of $\gamma'_{n,N}$, we require $\widehat{a}(f)_i$ to be four standard deviations away from zero. Next, we set

$$\gamma_{n,N} = \overline{\gamma} \times \left(n^{-1+0.01} \vee N^{-1+0.01}\right),$$

for some constant $\overline{\gamma} > 0$. We experiment with several choices of $\overline{\gamma}$: 2.5, 3.0 and 3.5. Intuitively, $\frac{1}{n}\widehat{VJ}_f$ is an estimate of the "normal" behavior of $\breve{a}(f)_i$, and our choice for $\gamma_{n,N}$ requires $\breve{a}(f)_i$ to be several times bigger than this average value. Of course, lower value of $\overline{\gamma}$ means that we can erroneously classify some increments as containing systematic jumps when they do not while a higher value of $\overline{\gamma}$ means that we can omit some increments that do contain systematic jumps.

Finally, for constructing the confidence intervals, we set m_n to 10 for $n = 40$ and to 14 for $n = 80$.

5.3 Results

On the simulated data, we evaluate the precision in recovering SJ_g, the finite sample behavior of a test for systematic jumps as well as the accuracy of confidence intervals for SJ_g. Since in the simulated model the diffusive volatility of the assets do not jump at the time of the systematic jump time τ, in order to save on computational time, we construct confidence intervals using $\widehat{A}_{\alpha}^{\pm}$ computed as in (33).

In Table 5.3, we report the quantiles of the ratios $\widehat{SJ}_g/SJ_g$ and $\widehat{SJ}_g^{\flat}/SJ_g$ (note that SJ_g is a random variable) when asset prices contain a systematic jump. For brevity, we report only the results for the threshold level of $\overline{\gamma} = 3.0$ as the results for the other two values of $\overline{\gamma}$ are very similar. As seen from the reported results, $\widehat{SJ}_g^{\flat}$ is slightly upward biased. This is to be expected as the size of the systematic jump is small relative to the other risks in the asset prices. Not surprisingly, the bias in $\widehat{SJ}_g^{\flat}$ is larger for lower values of n and N. We note also that the bias in $\widehat{SJ}_g^{\flat}$ is significantly smaller (by a factor of at least ten) than the one in the "raw" statistic $\widehat{SJ}_g^{\flat\flat}$. After the truncation, i.e., when switching from $\widehat{SJ}_g^{\flat}$ to $\widehat{SJ}_g$, the bias gets effectively eliminated.

Table 1: Monte Carlo Results, Part I

		Quantiles of $\widehat{SJ}_g^{\flat}/SJ_g$			*Quantiles of* $\widehat{SJ}_g/SJ_g$		
n	N	25%	50%	75%	25%	50%	75%
80	500	1.084	1.229	1.466	0.894	0.992	1.100
40	500	1.140	1.357	1.719	0.886	1.012	1.177
80	300	1.143	1.349	1.708	0.861	0.973	1.101
40	300	1.201	1.489	1.991	0.835	0.989	1.176

Note: $\widehat{SJ}_g^{\flat}$ and $\widehat{SJ}_g$ are computed using $\overline{\gamma} = 3.0$. The reported results are based on 5000 Monte Carlo replications.

We continue next with evaluating the performance of the test for systematic jumps and the accuracy of a confidence interval for SJ_g. These results are reported in Table 2. The test for systematic jumps performs overall well under the null hypothesis in the various configurations and for the different choices of $\overline{\gamma}$ for detecting systematic jump occurrences. We notice only some mild over-rejection in the case $N = 500$ and $n = 40$ when $\overline{\gamma} = 2.5$ and $\overline{\gamma} = 3.0$. This is because for a lower level of the threshold, there is a larger number of increments (in relative terms) which are misclassified to contain systematic jumps. Table 2 shows that the test has good power against the considered alternative. We recall that the systematic jump in our setting is very small, particularly relative to the remaining risks in the individual asset prices. Our aggregate measure $\widehat{SJ}_g$ is able, nevertheless, to identify the systematic jumps by utilizing the different cross-sectional and pathwise properties of the various risks embedded in the asset prices. Consistent with the asymptotic theory, the power of the test tends to increase when either of the two dimensions of the high-frequency return panel increases, with N playing a slightly more important role in this regard.

We finish this section with analyzing the precision of the confidence intervals for SJ_g. Consistent with earlier work on realized volatility, we construct confidence intervals for the log of SJ_g using the Delta method and our theoretical results. We see from Table 2 that the constructed confidence intervals have empirical coverage rates that are similar to the nominal ones for all considered cases and for the different choices of $\overline{\gamma}$.

Table 2: Monte Carlo Results, Part II

		Test for $SJ_g = 0$				CI Coverage Rate	
		Size		Power		for SJ_g	
n	N	10%	5%	10%	5%	90%	95%
				Panel A: $\overline{\gamma} = 2.5$			
80	500	0.118	0.058	0.958	0.935	0.901	0.954
40	500	0.145	0.078	0.913	0.869	0.888	0.942
80	300	0.111	0.056	0.822	0.754	0.899	0.947
40	300	0.127	0.066	0.816	0.735	0.905	0.953
				Panel B: $\overline{\gamma} = 3.0$			
80	500	0.115	0.056	0.956	0.930	0.903	0.954
40	500	0.141	0.076	0.902	0.845	0.892	0.945
80	300	0.107	0.052	0.825	0.754	0.899	0.947
40	300	0.119	0.061	0.805	0.725	0.907	0.954
				Panel C: $\overline{\gamma} = 3.5$			
80	500	0.102	0.051	0.948	0.921	0.905	0.955
40	500	0.104	0.053	0.874	0.809	0.895	0.947
80	300	0.099	0.051	0.821	0.749	0.898	0.946
40	300	0.102	0.050	0.789	0.706	0.906	0.954

Note: the reported results are based on 5000 Monte Carlo replications.

6 Empirical Study

In this section we provide empirical evidence regarding systematic jump risk in asset prices using the developed techniques. Our sample covers the period from January 1, 2001 till December 31, 2020, and we sample asset prices every five minutes, excluding the first minutes of the trading day. The composition of our cross-section of stocks varies over the sample. In each year, we select the 500 largest stocks by market capitalization as of the end of the previous year. In our analysis, we use high-frequency data for the three Fama-French (FF) systematic risk factors (which are market, HML and SMB).[3]

Our interest in the empirical study is the existence and importance of systematic jump risk outside that of the FF systematic risk factors. We rescale the high-frequency returns by a time of day factor to account for the well-known intraday seasonality in volatility. Finally, the selection of the tuning parameters for conducting the test for systematic jump risk as well as for the computation of the standard errors is done exactly as in the Monte Carlo. For brevity, we report only results with $\overline{\gamma} = 3.0$, with the results for $\overline{\gamma} = 2.5$ and $\overline{\gamma} = 3.5$ being similar.

On Figure 1, we plot the test statistic for presence of systematic jump risk not spanned by the three FF factors. As seen from the figure, there is nontrivial evidence for such type of systematic jump risk in asset prices. Moreover, the reported results suggest that the systematic jump risk is scattered over the entire sample. Formally, the percentage of days where evidence for non-FF systematic jump risk is detected, based on a test with size of 5%, is 15.88%. We also notice from the figure that the median value of the statistic is slightly above zero. This is indicative of a lot of systematic jumps of smaller size.

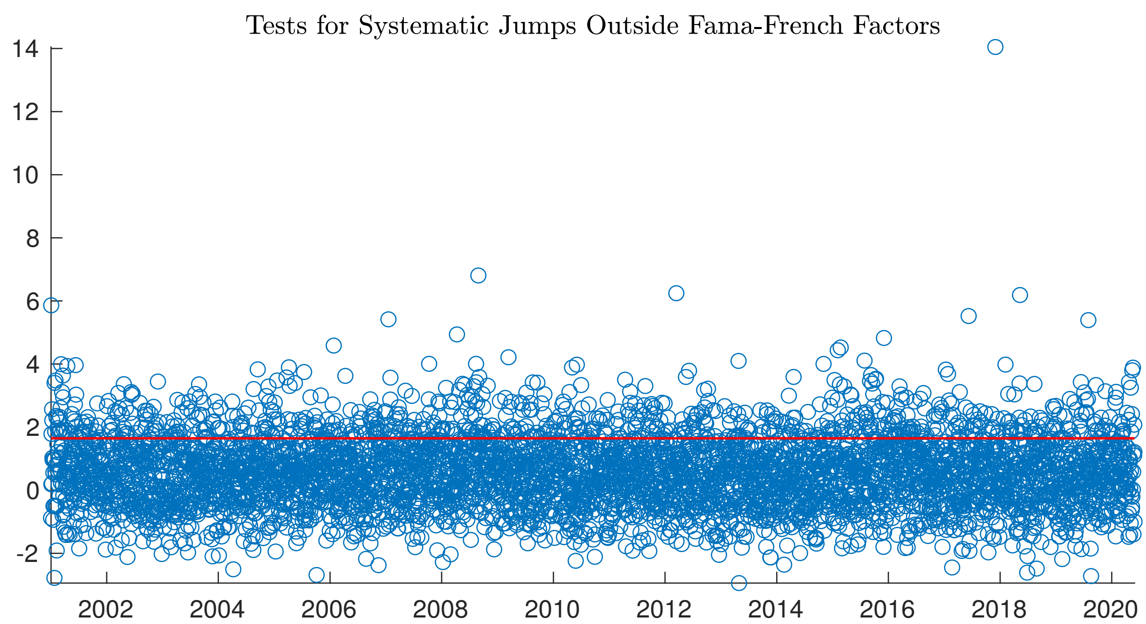

Figure 1: Daily Test for non-FF Systematic jumps. Solid line corresponds to the critical value of the test for non-FF systematic jumps at 5% significance level.

[3]We also performed the analysis using the Fama-French five factor model, with the results being qualitatively the same as the ones reported here.

On Figure 2, we plot the time series of $\sqrt{\widehat{SJ}_g^\flat}$ against that of market realized variance. The non-FF systematic jump risk exhibits a lot of time series variation. Similar to market variance, our aggregate systematic jump risk measure spikes typically during and in the aftermath of periods of market turbulence. That said, Figure 2 reveals nontrivial differences in the two series. Mainly, $\sqrt{\widehat{SJ}_g^\flat}$ spikes less (in relative terms) during the 2008 financial crisis and the pandemic-triggered market turmoil of 2020. It has also higher elevated level during prolonged periods when the market is relatively calm as in 2001 and 2016-2017.

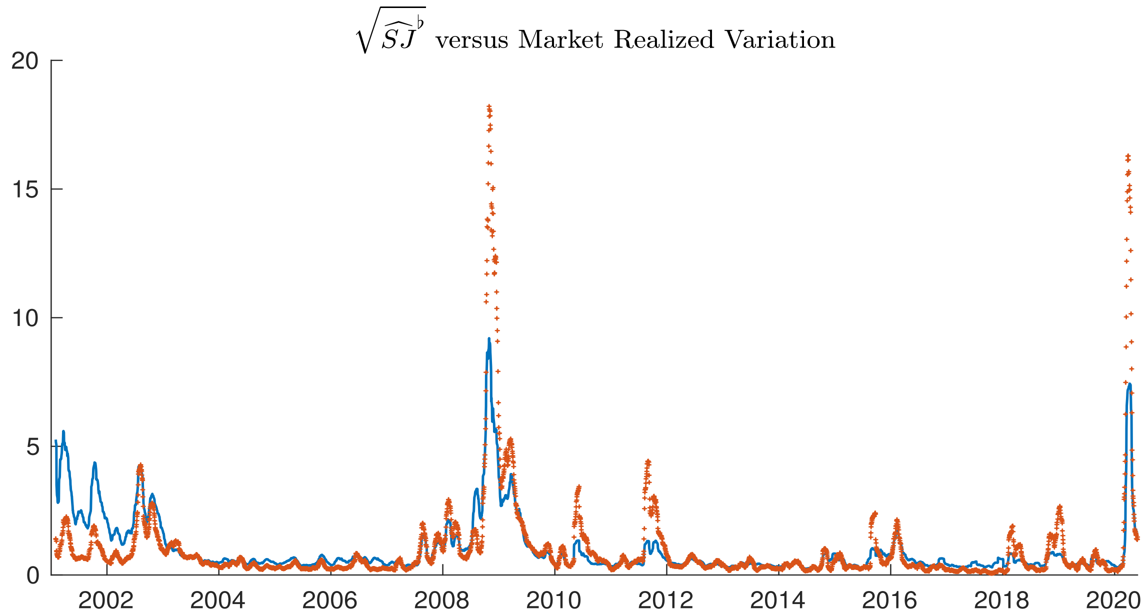

Figure 2: Systematic non-FF Jump Risk versus Market Realized Variance. The series are normalized by dividing each of them by their sample means. Blue line corresponds to $\sqrt{\widehat{SJ}_g^\flat}$ and the red line to market realized volatility. Both series are computed on a rolling windows basis with window length of one month.

Given the strong evidence for non-FF systematic jump risk, it is interesting to analyze the assets' sensitivity to this risk and to contrast this sensitivity to squared market betas. Towards this end, we compute the ratio of $\widehat{SJ}_{g,g}^{j,\flat}/\widehat{SJ}_g^\flat$ on a rolling window basis, with window length of one month, and we compare this quantity with squared market beta. To reduce the effect of measurement error, we aggregate the sensitivity measures over industry portfolios (using the industry classification on Ken French's website). On Figure 3, we plot time series of these quantities for three representative industry sectors. The figure reveals persistent patterns in $\widehat{SJ}_{g,g}^{j,\flat}/\widehat{SJ}_g^\flat$ which differ from market betas. For example, the non-FF systematic jump risk sensitivity of the financial sector is significantly lower than its market exposure for long periods prior to 2008 and after 2010. Similarly, the non-FF systematic jump risk sensitivity of the energy sector increases after 2014 in contrast to the behavior of the market beta over the same period. Overall, Figure 3 suggests significant differences in the behavior of assets' sensitivity towards non-FF systematic jump risk and market risk.

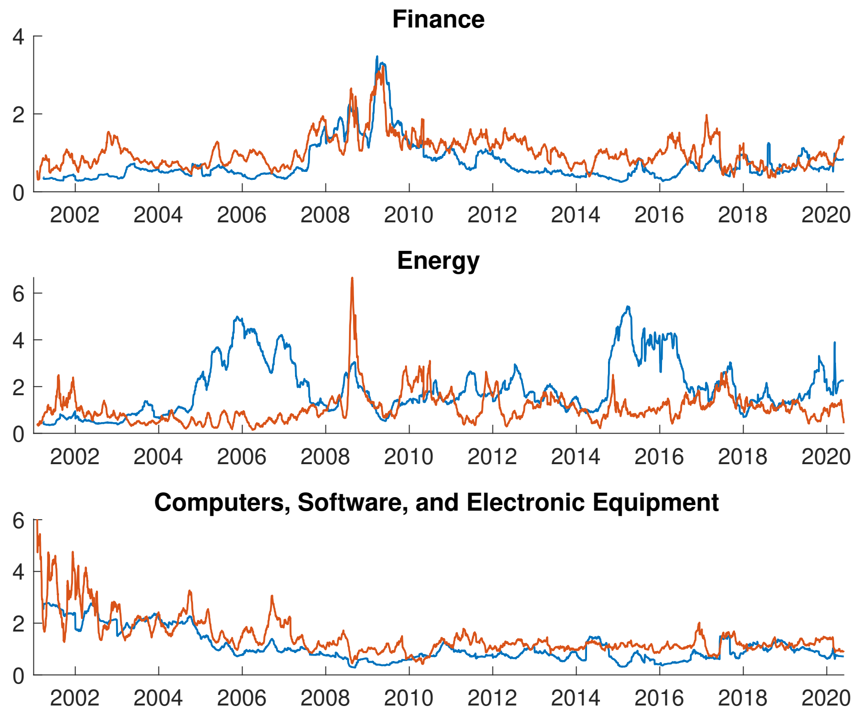

Figure 3: Sensitivity to non-FF systematic jump risk. On each plot, we display average $\widehat{SJ}^{j,\flat}_{g,g}/\widehat{SJ}^{\flat}_{g}$ across stocks in the corresponding sector (blue line) against that of squared market beta (red line). Both series are computed on a rolling windows basis with window length of one month.

A natural question is whether these differences have pricing implications. Given the bigger role played by jumps in higher asset return moments, it is easier to study the pricing of systematic jump risk as manifest in the variance risk premium, which is the compensation demanded by investors for bearing variance risk. Proxies for variance risk premium can be constructed using volatility estimates from asset returns and options written on them. Sorting stocks into quintiles according to their systematic jump exposures, $\widehat{SJ}^{j,\flat}_{g,g}/\widehat{SJ}^{\flat}_{g}$, generates big spreads in their variance risk premia. Our results show that these spreads cannot be rationalized by the stocks' exposure to FF jump risk. This indicates that the systematic jump risk that is not spanned by the FF factors is priced in the cross-section of asset prices. For brevity we do not present these results here and we leave a complete exploration of the pricing implications of systematic jump risk for future work.

7 Conclusion

In this paper, we develop nonparametric measures for systematic jump risk in asset prices using a panel of high-frequency returns. The asymptotic setting is of joint type: both the number of assets and the sampling frequency increase while the time span of the data remains fixed. The developed statistics use sequential differences of cross-sectionally averaged transforms of individual asset returns to disentangle systematic from idiosyncratic risk and utilize the leading role in higher-order moments played by jump risk. We derive Central Limit Theorems for our statistics, whose rate of convergence depends on the two asymptotically increasing dimensions of the return panel. We further derive the probability limit of the properly rescaled statistics when no systematic jump risk is present in the asset prices (outside the jump times of observable systematic risk factors). Using these limit results, we further propose a test for presence of systematic jump risk that is not spanned by observable risk factors. The empirical analysis reveals the existence of nontrivial systematic jump risk, which is not spanned by the three Fama-French factors (market, size and value), as well as its importance for asset pricing.

8 Proofs

1) Preliminaries. Using the same localization procedure as in the proof of Lemma 4.4.9 (see part 1 of that proof) of Jacod and Protter (2012), one easily sees that it suffices to prove all theorems when in (A1) and (A2) we have $T_1 = \infty$. So below we always assume $T_1 = \infty$, and also (4). Therefore, there are $[0,1]$-valued functions φ^j on $\mathbb{R}$ and $[1,\infty)$-valued random variables χ^j and a number $w \in (1,2)$, such that for any finite stopping time S we have

$$
\begin{aligned}
&\bullet\ \|\lambda_t^j\| + |b_t^j| + |\sigma_t^j| + |\delta^j(t,z)| + |\delta_p^j| \le \chi^j \qquad |\theta^j(t,z)| \le \varphi^j(z)\chi^j, \qquad \textstyle\int_{\mathbb{R}} \varphi^j(z)^w\,dz \le 1 \\
&\bullet\ q \in \mathbb{R}_+ \ \Rightarrow\ \sup_{j\ge1} \mathbb{E}\big((\chi^j)^q\big) < \infty, \text{ hence } \sup_{N\ge1} \mathbb{E}\big(\overline{\chi}_N^{(q)})^q\big) < \infty \text{ if } \overline{\chi}_N^{(q)} = \frac{1}{N}\sum_{j=1}^N (\chi^j)^q \\
&\bullet\ \sup_{j\ge1} \mathbb{E}\Big(\sup_{s\in[(S-s)^+,S)} \big(\|\lambda_s^j - \lambda_{S-}^j\|^2 + |\sigma_s^j - \sigma_{S-}^j|^2\big)\Big) \to 0 \quad \text{as } s \downarrow 0 \\
&\bullet\ \sup_{j\ge1} \mathbb{E}\Big(\sup_{s\in[S,S+s)} \big(\|\lambda_s^j - \lambda_S^j\|^2 + |\sigma_s^j - \sigma_S^j|^2\big)\Big) \to 0 \quad \text{as } s \downarrow 0 \\
&\bullet\ \mathbb{E}_{\mathcal{F}_S}\Big(\sup_{s'\in[S,S+s]} \Big(\frac{\|\lambda_{s'}^j - \lambda_S^j\|^2 + |\sigma_{s'}^j - \sigma_S^j|^2}{(\chi^j)^2}\Big) \le C\, s^\rho.
\end{aligned}
\tag{36}
$$

Note that the fourth line in (36) is implied by the second and fifth ones.

The functions g, h are always in $\mathcal{C}$, and the function f used for defining $\widehat{I}'$ is in $\mathcal{C}_+$. Below, we consider variables indexed by n and/or N and sometimes an extra index j. We write $Y^{n,N} \xrightarrow{\mathbb{P}} Y$ for the convergence in probability, as both n and N go to ∞. We denote with C a generic positive constant which might change from one equation to another (written C_q if it depends on an extra parameter q). The Cauchy-Schwarz inequality being used very often, we abbreviate it by "C-S".

We use the original filtration $(\mathcal{F}_t)$, the initially enlarged one $\mathcal{F}'_t = \mathcal{F}_t \bigvee \mathcal{J}'$, and the notation

$$\mathbb{E}^n_i(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_{i\Delta_n}), \qquad \mathbb{E}'^n_i(\cdot) = \mathbb{E}(\cdot|\mathcal{F}'_{i\Delta_n}).$$

We also write, with m_n as in (30) and the convention $\sup(\emptyset) = 0$ and I_n, I'_n, I''_n as in(8) and (14):

$$\left.\begin{array}{l} P = \sup(p \in \mathbb{N}:\ \tau_p \leq T), \qquad 1 \leq p \leq P\ \Rightarrow\ i_{n,p} = \text{the integer such that } \tau_p \in I(n, i_{n,p}) \\ \overline{P} = \sup(p \in \mathbb{N}:\ \rho_p \leq T), \qquad 1 \leq p \leq \overline{P}\ \Rightarrow\ \bar{i}_{n,p} = \text{the integer such that } \rho_p \in I(n, \bar{i}_{n,p}) \\ \qquad\qquad \widetilde{I}_n = I_n \backslash (I'_n \cup I''_n) \\ \qquad \Omega_n = \text{the set on which } \widehat{I} = I_n \text{ and the distance between any two of the} \\ \text{numbers } 0, \tau_1, \dots, \tau_P, \rho_1, \dots, \rho_{\overline{P}} \text{ is at least } (3 \vee m_n)\Delta_n, \text{ and also } T - \tau_P \geq (3 \vee m_n)\Delta_n, \end{array}\right\} \tag{37}$$

(so $I'_n = \{i_{n,p} :\ 1 \leq p \leq P,\ \overline{\delta}_{f,p} > 0,\ \overline{\delta}_{\mathbb{I},p} = 0\}$ and $I''_n = \{i_{n,p} :\ 1 \leq p \leq P,\ \overline{\delta}_{\mathbb{I},p} \neq 0\}$, for any $f \in \mathcal{C}_+$). In particular, on the set Ω_n any $I(n, i)$ with $i \leq n$ contains at most either one τ_p or one ρ_p. Since the τ_p's for $p \leq P$ and the ρ_p's for $p \leq \overline{P}$ are a.s. all distinct and different from 0, and $\tau_P < T$ a.s., whereas $(3 \vee m_n)\Delta_n \to 0$, we deduce that $\mathbb{P}(\Omega_n) \to 1$.

Lemma 1 *Let $\mathcal{G}, \mathcal{G}'$ be two sub-σ-fields of $\mathcal{F}$ and $(\zeta_j)_{j\geq 1}$ be a sequence of E-valued variables for some Polish space E. If this sequence is i.i.d. under $\mathbb{P}_{\mathcal{G},\omega}$ and $\mathbb{P}_{\mathcal{G}',\omega}$ for $\mathbb{P}$-almost all ω, the laws of ζ_j under $\mathbb{P}_{\mathcal{G},\omega}$ and $\mathbb{P}_{\mathcal{G}',\omega}$ (not depending on j) are the same for $\mathbb{P}$-almost all ω.*

Lemma 2 *For any Borel function h with at most polynomial growth we have for all $p \geq 1$:*

$$\text{on } \{\tau_p \leq T\}: \qquad \frac{1}{N}\sum_{j=1}^N h(\delta^j_p)\lambda^j_{\tau_p} \xrightarrow{\mathbb{P}} \overline{\delta}_{h,p}\overline{\lambda}^{(1)}_{\tau_p}, \qquad \frac{1}{N}\sum_{j=1}^N h(\delta^j_p)\lambda^j_{\tau_p-} \xrightarrow{\mathbb{P}} \overline{\delta}_{h,p}\overline{\lambda}^{(1)}_{\tau_p-}. \tag{38}$$

2) Some estimates. It is convenient to single out some of the constituents of X^j as

$$X^{(\lambda),j}_t = \int_0^t \lambda^{j\top}\, dW_s, \qquad X^{(\sigma),j}_t = \int_0^t \sigma^j_s\, d\widetilde{W}^j_s, \qquad X^{(\theta),j}_t = \theta^j * (\mu^j - \nu)_t, \qquad X^{(b),j}_t = \int_0^t b^j_s\, ds$$
$$X^{(c),j}_t = X^{(\lambda),j}_t + X^{(\sigma),j}_t + X^{(b),j}_t, \quad X^{(I),j}_t = X^{(\sigma),j}_t + X^{(\theta),j}_t + X^{(b),j}_t, \quad X^{(A),j}_t = X^{(c),j}_t + X^{(\theta),j}_t.$$

With ψ as in (10), u^j_n as in (11) and $(\mathcal{S})$ being (λ), (σ), (b), (θ), (c), (I) or (A), set

$$\begin{array}{ll} \overline{X}^{N,(\mathcal{S})} = \frac{1}{N}\sum\limits_{j=1}^N X^{(\mathcal{S}),j}, & X^{(\mathcal{S}),N,j} = X^{(\mathcal{S}),j} - \psi\overline{X}^{N,(\mathcal{S})} \\ \widehat{X}^{(\mathcal{S}),\mathcal{T},j}_i = \Delta^n_i X^{(\mathcal{S}),N,j}\, 1_{\{|\Delta^n_i X^{(\mathcal{S}),N,j}| \leq u^j_n\}}, & \Delta^{n,\mathcal{T}}_i X^{(\mathcal{S}),j} = \Delta^n_i X^{(\mathcal{S}),j}\, 1_{\{|\Delta^n_i X^{(\mathcal{S}),j}| \leq u^j_n\}} \\ \widetilde{a}(g)^{n,N}_i = \frac{1}{N}\sum_{j=1}^N g(\Delta^n_i X^{(A),N,j}), & \widetilde{a}^{\mathcal{T}}(g)^{n,N}_i = \frac{1}{N}\sum_{j=1}^N g(\widehat{X}^{(A),\mathcal{T},j}_i). \end{array} \tag{39}$$

For any process Y (such as X^j, $X^{(\lambda),j}$, and so on) and positive integers n, i we write $Y(n,i)^* = \sup_{t\in I(n,i)} |Y_t - Y_{(i-1)\Delta_n}|$. By virtue of (2.1.33), (2.1.34) and Lemma 2.1.5 of Jacod and Protter

(2012)), plus (36), we easily obtain that for any $\mathcal{F}'_0$-measutable random integer i:

$$\begin{array}{lll} q>0 \;\Rightarrow & \mathbb{E}'^n_{i-1}(X^{(b),j}(n,i)^{*q}) \leq C_q\Delta_n^q\,\mathbb{E}'^n_{i-1}((\chi^j)^q), & \mathbb{E}(X^{(b),j}(n,i)^{*q}) \leq C_q\Delta_n^q \\ q\geq 1 \;\Rightarrow & \mathbb{E}'^n_{i-1}(X^{(\lambda),j}(n,i)^{*q}) \leq C_q\Delta_n^{q/2}\,\mathbb{E}'^n_{i-1}((\chi^j)^q), & \mathbb{E}(X^{(\lambda),j}(n,i)^{*q}) \leq C_q\Delta_n^{q/2} \\ q\geq 1 \;\Rightarrow & \mathbb{E}'^n_{i-1}(X^{(\sigma),j}(n,i)^{*q}) \leq C_q\Delta_n^{q/2}\,\mathbb{E}'^n_{i-1}((\chi^j)^q)), & \mathbb{E}(X^{(\sigma),j}(n,i)^{*q}) \leq C_q\Delta_n^{q/2} \\ q\geq w \;\Rightarrow & \mathbb{E}'^n_{i-1}(X^{(\theta),j}(n,i)^{*q}) \leq C_q\Delta_n\,\mathbb{E}'^n_{i-1}((\chi^j)^q)), & \mathbb{E}(X^{(\theta),j}(n,i)^{*q}) \leq C_q\Delta_n. \end{array} \tag{40}$$

Thus, by $\sup_j \mathbb{E}((\chi^j)^q) < \infty$ for all $q>0$ and Hölder's inequality, we get

$$\mathbb{E}(|\Delta_i^n X^{(\mathcal{S}),j}|^q + |\Delta_i^n \overline{X}^{N,(\mathcal{S})}|^q + |\Delta_i^n X^{(\mathcal{S}),N,j}|^q) \leq \begin{cases} C_q\Delta_n^{q/2} & \text{if } q\geq 1,\ (\mathcal{S})=(b),(\lambda),(\sigma),(c) \\ C_q\Delta_n & \text{if } q\geq 2,\ (\mathcal{S})=(\theta),(A),(I) \\ C_q\Delta_n & \text{if } q\geq w,\ (\mathcal{S})=(\theta),(b). \end{cases} \tag{41}$$

Lemma 3 *For any $\mathcal{J}'$-measurable random integer $i\geq 1$ we have (recall* (36) *for $\overline{\chi}_N^{(q)}$):*

$$\big|\mathbb{E}'^n_{i-1}\big(|\Delta_i^n\overline{X}^{N,(I)}|^2\big)\big| \leq C\Big(\Delta_n^{3/2} + \frac{\Delta_n}{N}\Big)\mathbb{E}'^n_{i-1}(\overline{\chi}_N^{(2)}), \tag{42}$$

$$j\neq k \;\Rightarrow\; \begin{cases} \big|\mathbb{E}'^n_{i-1}\big(\Delta_i^n X^{(I),j}\Delta_i^n X^{(I),k}\big)\big| \leq C\Delta_n^{3/2}\,\mathbb{E}'^n_{i-1}\big((\chi^j)^2+(\chi^k)^2\big) \\ \big|\mathbb{E}'^n_{i-1}\big(\Delta_i^n X^{(I),N,j}\Delta_i^n X^{(I),N,k}\big)\big| \leq \Big(\Delta_n^{3/2}+\frac{\Delta_n}{N}\Big)\mathbb{E}'^n_{i-1}\big((\chi^j)^2+(\chi^k)^2+\overline{\chi}_N^{(2)}\big). \end{cases} \tag{43}$$

Lemma 4 *For any $g,h\in\mathcal{C}$ and $\mathcal{J}'$-measurable integer i we have (with $w\in(1,2)$ as in* (36)*):*

$$\mathbb{E}\big((\widetilde{a}(g)_i^{n,N})^2\big) + \mathbb{E}\big(h(\Delta_i^n X^{(A),N,j})\,\widetilde{a}(g)_i^{n,N}\big) \leq C\big(\Delta_n^{1+1/w} + \Delta_n/N\big).$$

Lemma 5 *If i is any $\mathcal{J}'$-measurable positive (random) integer we have for all $q\geq 1$ and $\gamma>0$:*

$$\mathbb{P}\big(|\Delta_i^n\overline{X}^{N,(A)}|>\gamma\big) \leq C_q\,\frac{\Delta_n^q}{\gamma^{2q}} + C\,\frac{\Delta_n}{\gamma^2\,N}, \qquad \mathbb{P}\big(|\widetilde{a}(g)_i^{n,N}|>\gamma\big) \leq C_q\,\frac{\Delta_n^q}{\gamma^q} + C\,\frac{\Delta_n}{\gamma\,N}. \tag{44}$$

Lemma 6 *There is $\varepsilon>0$ such that, for any $\mathcal{J}'$-measurable integer i,*

$$\mathbb{E}\big(\big|\widehat{X}_i^{(A),\mathcal{T},j}\widehat{X}_i^{(A),\mathcal{T},k} - \Delta_i^n(X^{(\lambda),N,j}+X^{(\sigma),N,j})\,\Delta_i^n(X^{(\lambda),N,k}+X^{(\sigma),N,k})\big|\big) \leq C\Delta_n^{1+\varepsilon},$$

$$\mathbb{E}\big(\big|\Delta_i^{n,\mathcal{T}}X^{(A),j}\,\Delta_i^{n,\mathcal{T}}X^{(A),k} - \Delta_i^n(X^{(\sigma),j}+X^{(\sigma),j})\,\Delta_i^n(X^{(\lambda),k}+X^{(\lambda),k})\big|\big) \leq C\Delta_n^{1+\varepsilon}.$$

3) The proof of Theorem 1. Theorem 1 is a trivial consequence of the next three lemmas.

Lemma 7 *We have $\mathbb{P}(\overline{\Omega}_n)\to 1$ and $\mathbb{P}(\widehat{I}''=I''_n)\to 1$, where*

$$\overline{\Omega}_n \quad \textit{is the subset of } \Omega_n \textit{ on which} \quad \widehat{r}_i = \begin{cases} \Delta_i^n\overline{X}^{N,(A)} & \textit{if } i\in\widetilde{I}_n \\ \Delta_i^n\overline{X}^{N,(A)} - \overline{\delta}_{\mathbb{I},p}^N & \textit{if } i=i_{n,p}\in I'_n \\ 0 & \textit{if } i\in I''_n. \end{cases}$$

Lemma 8 *We have* (19).

Lemma 9 *We have* $\mathbb{P}(\widehat{I}' = I'_n) \to 1$.

4) Proof of Theorem 2. We need a few preliminary lemmas. To start with, we also set, for $(\mathcal{S})$ being any of our usual symbols $(\lambda,), (\sigma), (\theta), (b), (I), (A)$,

$$\begin{array}{ll} \mathcal{P}' = \{p = 1, \dots, P: \ \overline{\delta}_{\mathbb{I},p} = 0\}, & \mathcal{P}'' = \{p = 1, \dots, P: \ \overline{\delta}_{\mathbb{I},p} \neq 0\} \\ \delta_p^{N,j} = \begin{cases} \delta_p^j - \psi \overline{\delta}_{\mathbb{I},p}^N & \text{if } p \in \mathcal{P}' \\ \delta_p^j & \text{if } p \in \mathcal{P}'', \end{cases} & \widehat{X}_p^{(\mathcal{S}),j} = \begin{cases} \Delta_{i_{n,p}}^n X^{(\mathcal{S}),N,j} & \text{if } p \in I'_n \\ \Delta_{i_{n,p}}^n X^{(\mathcal{S}),j} & \text{if } p \in I''_n. \end{cases} \end{array} \tag{45}$$

Then the variable $\widehat{r}_i^j$ of (10) satisfy

$$\text{on the set } \overline{\Omega}'_n = \overline{\Omega}_n \cap \{\widehat{I}' = I'_n, \widehat{I}'' = I''_n\}: \quad \widehat{r}_i^j = \begin{cases} \Delta_i^n X^{(A),N,j} & \text{if } i \in \widetilde{I}_n, \\ \widehat{X}_p^{(A),j} + \delta_p^{N,j} & \text{if } i = i_{n,p}, \ p = 1, \dots, P, \end{cases} \tag{46}$$

and by Lemmas 7 and 9 we have $\mathbb{P}(\overline{\Omega}'_n) \to 1$. We also consider the modified statistics

$$\widehat{SJ}'^*_g = \sum_{i \in \widehat{I}'} \big(\widehat{a}(g)_i\big)^2, \qquad \widehat{SJ}''^*_g = \sum_{i \in \widehat{I}''} \big(\widehat{a}(g)_i\big)^2, \qquad \widehat{SJ}^*_g = \widehat{SJ}'^*_g + \widehat{SJ}''^*_g.$$

Lemma 10 *We have* $\sqrt{n}(\widehat{SJ}'_g - \widehat{SJ}'^*_g) \stackrel{\mathbb{P}}{\longrightarrow} 0$ *and* $\sqrt{n}(\widehat{SJ}''_g - \widehat{SJ}''^*_g) \stackrel{\mathbb{P}}{\longrightarrow} 0$ *and* $\sqrt{n \wedge N}(\widehat{SJ}^\flat_g - \widehat{SJ}^*_g) \stackrel{\mathbb{P}}{\longrightarrow} 0$ *and* $\sqrt{n \wedge N}(\widehat{SJ}^{\flat\flat}_g - \widehat{SJ}^*_g) \stackrel{\mathbb{P}}{\longrightarrow} 0$.

When $g, h \in \mathcal{C}$ one can expand those functions around $\delta_p^{N,j}$ or $\delta_p^{N,k}$ to get, in restriction to $\overline{\Omega}'_n$ and in view of (46) and $\widehat{X}_p^{(A),j} = \widehat{X}_p^{(\lambda),j} + \widehat{X}_p^{(I),j}$:

$$\begin{aligned} h(\widehat{r}^j_{i_{n,p}}) g(\widehat{r}^k_{i_{n,p}}) &= h(\delta_p^{N,j}) g(\delta_p^{N,k}) + h(\delta_p^{N,j}) g'(\delta_p^{N,k}) \widehat{X}_p^{(\lambda),k} + h'(\delta_p^{N,j}) g(\delta_p^{N,k}) \widehat{X}_p^{(\lambda),j} \\ &\quad + h(\delta_p^{N,j}) g'(\delta_p^{N,k}) \widehat{X}_p^{(I),k} + h'(\delta_p^{N,j}) g(\delta_p^{N,k}) \widehat{X}_p^{(I),j} + h'(\delta_p^{N,j}) g'(\delta_p^{N,k}) \widehat{X}_p^{(A),j} \widehat{X}_p^{(A),k} \\ &\quad + \alpha_p^{n,k} h(\delta_p^{N,j}) (\widehat{X}_p^{(A),k})^2 + \alpha_p^{n,j} g(\delta_p^{N,k}) (\widehat{X}_p^{(A),j})^2 + \alpha_p^{n,k} h'(\delta_p^{N,j}) \widehat{X}_p^{(A),j} (\widehat{X}_p^{(A),k})^2 \\ &\quad + \alpha_p^{n,j} g'(\delta_p^{N,k}) \widehat{X}_p^{(A),k} (\widehat{X}_p^{(A),j})^2 + \alpha_p^{n,j} \alpha_p^{n,k} (\widehat{X}_p^{(A),j})^2 (\widehat{X}_p^{(A),k})^2, \end{aligned} \tag{47}$$

where $\alpha_p^{n,j}$ is a family of random variables, bounded uniformly in ω, n, j, p. We also set, with $\mathcal{P}$ being either $\mathcal{P}'$ or $\mathcal{P}''$,

$$\mathcal{T}^N_{\mathcal{P}} = \sum_{p \in \mathcal{P}} \frac{1}{N^2} \sum_{j,k=1}^N g(\delta_p^{N,j}) g(\delta_p^{N,k}), \qquad \mathcal{U}^{n,N}_{\mathcal{P}} = \sum_{p \in \mathcal{P}} \frac{2}{N^2} \sum_{j,k=1}^N g(\delta_p^{N,j}) g'(\delta_p^{N,k}) \widehat{X}_p^{(\lambda),k}.$$

Then (47) with $h = g$, the properties of g and $\sum_{j=1}^N |\Delta_i^n X^{(A),N,j}| \leq (1 + |\psi|) \sum_{j=1}^N |\Delta_i^n X^{(A),j}|$ yield

$$\begin{gathered} \big|\widehat{SJ}'^*_g - \mathcal{T}^N_{\mathcal{P}'} - \mathcal{U}^{n,N}_{\mathcal{P}'}\big| + \big|\widehat{SJ}''^*_g - \mathcal{T}^N_{\mathcal{P}''} - \mathcal{U}^{n,N}_{\mathcal{P}''}\big| \leq C \sum_{m=1}^3 |\Psi^{n,N}_m| \quad \text{on } \overline{\Omega}'_n, \text{ where} \\ \Psi^{n,N}_1 = \sum_{p=1}^P \tfrac{1}{N} \sum_{j=1}^N (\widehat{X}_p^{(A),j})^4, \qquad \Psi^{n,N}_2 = \sum_{p=1}^P \tfrac{1}{N^2} \sum_{j,k=1}^N g(\delta_p^{N,j}) g'(\delta_p^{N,k}) \, \widehat{X}_p^{(I),k}, \\ \Psi^{n,N}_3 = \sum_{p=1}^P \tfrac{1}{N^2} \sum_{j,k=1}^N (\delta_p^{N,j})^2 (\widehat{X}_p^{(A),k})^2. \end{gathered} \tag{48}$$

Lemma 11 *We have* $\sqrt{n}\,\Psi_m^{n,N} \stackrel{\mathbb{P}}{\longrightarrow} 0$ *for* $m=1,2,3$.

Lemma 12 *If* $Y'^N = \sqrt{N}(\mathcal{T}_{\mathcal{P}'}^N - SJ'_g)$ *and* $Y''^N = \sqrt{N}(\mathcal{T}_{\mathcal{P}''}^N - SJ''_g)$*, and under* $\mathbb{P}_{\overline{\mathcal{J}},\omega}$ *for* $\mathbb{P}$*-almost all* ω*, the two-dimensional sequence* (Y'^N, Y''^N) *converges in law to* $(\sqrt{V'_g(\omega)}\,Z', \sqrt{V''_g(\omega)}\,Z'')$.

Lemma 13 *For* $\mathcal{P}=\mathcal{P}',\mathcal{P}''$ *one can write* $\sqrt{n}\,\mathcal{U}_{\mathcal{P}}^{n,N} = \mathcal{Y}_{\mathcal{P}}^{n,N} + \widetilde{Y}_{\mathcal{P}}^{n,N}$*, where* $\widetilde{Y}_{\mathcal{P}}^{n,N}$ *is* $\overline{\mathcal{J}}$*-measurable and goes to 0 in probability, and further the pair* $(\mathcal{Y}_{\mathcal{P}'}^{n,N}, \mathcal{Y}_{\mathcal{P}''}^{n,N})$ *converges stably in law to* (Z'_g, Z''_g)*, as given in* (25).

We can now turn to the proof of Theorem 2. Recalling $I'_n=\emptyset$ if $SJ'_g=0$ and $I''_n=\emptyset$ if $SJ''_g=0$, when $g\in\mathcal{C}_+$, (c) of Theorem 2 follows from Theorem 1 and the definitions of $\widehat{SJ}'_g$ and $\widehat{SJ}''_g$.

For (a) and (b), in view of (48) and Lemmas 10 and 11, it is enough to show that

$$\Big(\sqrt{n\wedge N}\big(\mathcal{T}_{\mathcal{P}'}^N + \mathcal{U}_{\mathcal{P}'}^{n,N} - SJ'_g\big), \sqrt{n\wedge N}\big(\mathcal{T}_{\mathcal{P}''}^N + \mathcal{U}_{\mathcal{P}''}^{n,N} - SJ''_g\big)\Big) \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} (\widetilde{Z}'_g, \widetilde{Z}''_g).$$

With the notation of Lemmas 12 and 13 and

$$Y^N = \begin{pmatrix} Y'^N \\ Y''^N \end{pmatrix}, \qquad \mathcal{Y}^{n,N} = \begin{pmatrix} \mathcal{Y}_{\mathcal{P}'}^{n,N} \\ \mathcal{Y}_{\mathcal{P}''}^{n,N} \end{pmatrix}, \qquad Y = (V_g)^{1/2}\begin{pmatrix} Z' \\ Z'' \end{pmatrix}, \qquad \mathcal{Y} = \begin{pmatrix} Z'_g \\ Z''_g \end{pmatrix},$$

and $\phi_{n,N}=\sqrt{n/N}$, this amounts to proving that

$$(1\wedge\phi_{n,N})Y^N + \frac{1}{1\vee\phi_{n,N}}\,\mathcal{Y}^{n,N} \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} (1\wedge\phi)Y + \frac{1}{1\vee\phi}\,\mathcal{Y}. \tag{49}$$

Toward this aim, we single out the two cases $\phi=0$ and $0<\phi\le\infty$.

(i) The case of $\phi=0$*:* We have $\mathcal{G}=\mathcal{F}$, so Lemma 13 implies $\mathcal{Y}^{n,N} \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} \mathcal{Y}$. Moreover, the sequence Y^N is bounded in probability by Lemma 12, whereas $\phi_{n,N}\to 0$. We thus deduce (49).

(ii) The case of $\phi\in(0,\infty]$*:* We have $\mathcal{G}=\overline{\mathcal{J}}$, and (49) clearly follows from the $\mathcal{G}$-stable convergence

$$\big(Y^N, \mathcal{Y}^{n,N}\big) \stackrel{\mathcal{L}_{\mathcal{G}}-s}{\longrightarrow} (Y,\mathcal{Y}).$$

Recalling the definition (24) and by a density argument, it suffices to show that, for any bounded $\overline{\mathcal{J}}$-measurable variable Φ and any two continuous bounded functions Ψ,Ψ' on $\mathbb{R}^2$ we have, as $n,N\to\infty$:

$$\mathbb{E}\big(\Phi\,\Psi(Y^N)\,\Psi'(\mathcal{Y}^{n,N})\big) \;\to\; \widetilde{\mathbb{E}}\big(\Phi\,\Psi(Y)\,\Psi'(\mathcal{Y})\big). \tag{50}$$

Now, since $\mathcal{Y}^{n,N}$ is $\overline{\mathcal{J}}$-measurable, the left hand side above equals $\mathbb{E}\big(\Phi\,\Psi'(\mathcal{Y}^{n,N})\,\mathbb{E}_{\overline{\mathcal{J}}}(\Psi(Y^N))\big)$, whereas Lemma 12 yields $\mathbb{E}_{\overline{\mathcal{J}}}(\Psi(Y^N))\to\widetilde{\mathbb{E}}_{\overline{\mathcal{J}}}(\Psi(Y))$ (with $\widetilde{\mathbb{E}}_{\overline{\mathcal{J}}}$ denoting the $\overline{\mathcal{J}}$-conditional expectation for $\widetilde{\mathbb{P}}$). Because of the boundedness of Φ, f, f' we deduce

$$\mathbb{E}\big(\Phi\,\Psi(Y^N)\,\Psi'(\mathcal{Y}^{n,N})\big) - \mathbb{E}\big(\Phi\,\Psi'(\mathcal{Y}^{n,N})\,\widetilde{\mathbb{E}}_{\overline{\mathcal{J}}}(\Psi(Y))\big) \;\to\; 0.$$

An application of Lemma 13 yields

$$\mathbb{E}\big(\Phi\,\Psi'(\mathcal{Y}^{n,N})\,\widetilde{\mathbb{E}}_{\overline{\mathcal{J}}}(\Psi(Y))\big) \;\to\; \widetilde{\mathbb{E}}\big(\Phi\,\Psi'(\mathcal{Y})\,\widetilde{\mathbb{E}}_{\overline{\mathcal{J}}}(\Psi(Y))\big) \;=\; \widetilde{\mathbb{E}}\big(\Phi\,\Psi(Y)\,\Psi'(\mathcal{Y})\big)$$

(because under $\widetilde{\mathbb{P}}$ the variables Y and $\mathcal{Y}$ are $\overline{\mathcal{J}}$-conditionally independent), and (50) follows. □

5) Proof of Theorem 3. The proof follows similar steps as the ones of Theorem 2. We fix the index $j \geq 1$ and suppose that $N \geq j$. We consider the modified statistics

$$\widehat{SJ}'^{*,j}_{g,h} = \sum_{i \in \widehat{I}'} h(\widehat{r}^j_i)\,\widehat{a}(g)_i, \qquad \widehat{SJ}''^{*,j}_{g,h} = \sum_{i \in \widehat{I}''} h(\widehat{r}^j_i)\,\widehat{a}(g)_i, \qquad \widehat{SJ}^{*,j}_{g,h} = \widehat{SJ}'^{*,j}_{g,h} + \widehat{SJ}^{*,j}_{g,h}.$$

Lemma 14 *We have* $\sqrt{n}(\widehat{SJ}'^j_{g,h} - \widehat{SJ}'^{*,j}_{g,h}) \xrightarrow{\mathbb{P}} 0$ *and* $\sqrt{n}(\widehat{SJ}''_g - \widehat{SJ}''^*_g) \xrightarrow{\mathbb{P}} 0$ *and* $\sqrt{n \wedge N}(\widehat{SJ}^{j,\flat}_{g,h} - \widehat{SJ}^*_{g,h}) \xrightarrow{\mathbb{P}} 0$ *and* $\sqrt{n \wedge N}(\widehat{SJ}^{j,\flat\flat}_{g,h} - \widehat{SJ}^*_{g,h}) \xrightarrow{\mathbb{P}} 0$.

Next, we replace $\mathcal{T}^N_{\mathcal{P}}$ and $\mathcal{U}^{n,N}_{\mathcal{P}}$ for $\mathcal{P} = \mathcal{P}'$ and $\mathcal{P} = \mathcal{P}''$ by

$$\begin{aligned} \mathcal{T}^{N,j}_{\mathcal{P}} &= \sum_{p \in \mathcal{P}} \tfrac{1}{N} \sum_{k=1}^{N} h(\delta^{N,j}_p) g(\delta^{N,k}_p) \\ \mathcal{U}^{n,N,j}_{\mathcal{P}} &= \sum_{p \in \mathcal{P}} \tfrac{1}{N} \sum_{k=1}^{N} \Big(h(\delta^{N,j}_p) g'(\delta^{N,k}_p) \widehat{X}^{(\lambda),k}_p + h'(\delta^{N,j}_p) g(\delta^{N,k}_p) \widehat{X}^{(A),j}_p \Big). \end{aligned}$$

Upon using (47) and $\widehat{X}^{(A),j}_p = \widehat{X}^{(\lambda),j}_p + \widehat{X}^{(I),j}_p$, we obtain instead of (48):

$$\begin{gathered} \big|\widehat{SJ}'^{*,j}_{g,h} - \mathcal{T}^{N,j}_{\mathcal{P}'} - \mathcal{U}^{n,N,j}_{\mathcal{P}'}\big| + \big|\widehat{SJ}''^{*,j}_{g,h} - \mathcal{T}^{N,j}_{\mathcal{P}''} - \mathcal{U}^{n,N_j}_{\mathcal{P}''}\big| \leq C \textstyle\sum_{m=1}^3 |\Psi^{n,N,j}_m| \quad \text{on } \overline{\Omega}'_n, \text{ where} \\ \Psi^{n,N,j}_1 = \sum_{p=1}^{P} \tfrac{1}{N} \sum_{k=1}^{N} \big((\widehat{X}^{(A),j}_p)^4 + (\widehat{X}^{(A),k}_p)^4\big), \qquad \Psi^{n,N,j}_2 = \sum_{p=1}^{P} \tfrac{1}{N} \sum_{k=1}^{N} h(\delta^{N,j}_p) g'(\delta^{N,k}_p)\,\widehat{X}^{(I),k}_p \\ \Psi^{n,N,j}_3 = \sum_{p=1}^{P} \tfrac{1}{N} \sum_{k=1}^{N} \big((\delta^{N,j}_p)^2 (\widehat{X}^{(A),k}_p)^2 + (\delta^{N,k}_p)^2 (\widehat{X}^{(A),j}_p)^2\big). \end{gathered} \tag{51}$$

Lemma 15 *We have* $\sqrt{n}\,\overline{\Psi}^{n,N,j}_m \xrightarrow{\mathbb{P}} 0$ *for* $m = 1, 2, 3$.

Lemma 16 *If* $Y'^{N,j} = \sqrt{N}(\mathcal{T}^{N,j}_{\mathcal{P}'} - SJ'^j_{g,h})$ *and* $Y''^{N,j} = \sqrt{N}(\mathcal{T}^{N,j}_{\mathcal{P}''} - SJ''^j_{g,h})$*, under* $\mathbb{P}_{\overline{\mathcal{J}}_j,\omega}$ *for* $\mathbb{P}$*-almost all* ω*, the* 2*-dimensional sequence* $(Y'^{N,j}, Y''^{N,j})$ *converges in law to* $(\sqrt{V'^j_g(\omega)}\,Z', \sqrt{V''^j_g(\omega)}\,Z'')$.

Lemma 17 *For* $\mathcal{P} = \mathcal{P}', \mathcal{P}''$ *one can write* $\sqrt{n}\,\mathcal{U}^{n,N,j}_{\mathcal{P}} = \mathcal{Y}^{n,N,j}_{\mathcal{P}} + \widetilde{Y}^{n,N,j}_{\mathcal{P}}$*, where* $\widetilde{Y}^{n,N,j}_{\mathcal{P}}$ *is* $\overline{\mathcal{J}}_j$*-measurable and goes to* 0 *in probability, and further the pair* $(\mathcal{Y}^{n,N,j}_{\mathcal{P}'}, \mathcal{Y}^{n,N,j}_{\mathcal{P}''})$ *converges stably in law to* $(Z'^j_{g,h}, Z''^j_{g,h})$*, as given in* (29).

At this stage, in view of (51) and Lemmas 14, 15 and 16, and with $\overline{\mathcal{J}}_j$ instead of $\overline{\mathcal{J}}$, the proof of Theorem 2 can be reproduced word for word for showing Theorem 3.

6) Proof of Theorem 4. Since $\frac{n \wedge N}{N} \to 1 \wedge \phi^2$ and $\frac{N}{n \vee N} \to \frac{1}{1 \vee \phi^2}$, by comparing (32) with the definitions of $\widetilde{Z}'_g, \widetilde{Z}''_g, \widetilde{Z}'^j_{g,h}, \widetilde{Z}''^j_{g,h}$ and by Lemmas 7 and 9, it is enough to show the following properties:

$$\begin{aligned}
&(a): \quad \widehat{V}'_g \xrightarrow{\mathbb{P}} V_g, \qquad \widehat{V}''_g \xrightarrow{\mathbb{P}} V''_g, \qquad \widehat{V}'^j_{g,h} \xrightarrow{\mathbb{P}} V'^j_{g,h}, \qquad \widehat{V}''^j_{g,h} \xrightarrow{\mathbb{P}} V''^j_{g,h} \\
&(b): \quad (\widehat{\Gamma}^{\pm}_{g,i_{n,p}})^2 \xrightarrow{\mathbb{P}} T\,(\Gamma^{\pm}_{g,p})^2 \quad \text{on } \{p \leq P\} \\
&(c): \quad (\widehat{\Gamma}'^{\pm,j}_{g,h,i_{n,p}})^2 \xrightarrow{\mathbb{P}} T\,(\Gamma'^{\pm,j}_{g,h,p})^2 \ \text{ on } \{p \in \mathcal{P}'\}, \quad (\widehat{\Gamma}''^{\pm,j}_{g,h,i_{n,p}})^2 \xrightarrow{\mathbb{P}} T\,(\Gamma''^{\pm,j}_{g,h,p})^2 \ \text{ on } \{p \in \mathcal{P}''\}.
\end{aligned} \tag{52}$$

Those three properties are proven in the next section.

7) Proof of Theorem 7. If G_j is the set where $\lambda^j_t = 0$ for all $t \in [0,T]$, it is obvious that $\mathbb{P}\big((G_j)^c \cap \{\Delta^n_i X^j = 0 \text{ for all } i = 1,\dots,n\}\big) = 0$. Since $g \in \mathcal{C}_+$ we thus have $\mathbb{P}\big((G_j)^c \cap \{\widehat{SJ}^\flat_g = 0\}\big) = 0$ as soon as $j \leq N$. Now, (A2) implies that $\mathbb{P}\big(\cap_{1\leq j\leq N} G^j\big) \to 0$, and thus $\mathbb{P}(\widehat{SJ}^\flat_g = 0) \to 0$. Then, in view (b) of Theorem 2 we deduce that the asymptotic size of the test is α.

For the asymptotic alternative consistency, we use the fact that by Theorem 2 both $\widehat{SJ}_g$ and $\widehat{SJ}^\flat_g$ converge in probability to SJ_g, so $\widehat{SJ}^\flat_g - \widehat{SJ}_g \xrightarrow{\mathbb{P}} 0$. Since $SJ_g > 0$ on Ω^{alt}, the claim is obvious. $\square$

9 Proof of the Technical Lemmas

Proof of Lemma 1. For any Borel subset A of E set $H_A = \mathbb{P}_{\mathcal{G}}(\zeta_1 \in A)$ and $H'_A = \mathbb{P}_{\mathcal{G}'}(\zeta_1 \in A)$. Our assumptions and the law of large numbers imply $S_N = \frac{1}{N}\sum_{j=1}^N 1_A(\zeta_j) \xrightarrow{\text{a.s.}} H_A(\omega)$ under $\mathbb{P}_{\mathcal{G},\omega}$ for $\mathbb{P}$-almost all ω, and since H is $\mathcal{G}$-measurable we deduce

$$\mathbb{P}(S_N \to H_A) = \int_\Omega \mathbb{P}_{\mathcal{G},\omega}\big(\{\omega' : S_n(\omega') \to H_A(\omega)\}\big)\,\mathbb{P}(d\omega) = 1,$$

and analogously $\mathbb{P}(S_N \to H'_A) = 1$. Therefore $H_A = H'_A$ $\mathbb{P}$-a.s., implying the claim. $\square$

Proof of Lemma 2. By (A2)-(iii), for the first claim it is enough to show that

$$Y_N := \frac{1}{N}\sum_{j=1}^N \xi^j \xrightarrow{\mathbb{P}} 0, \qquad \xi^j = \big(h(\delta^j_p) - \overline{\delta}_{h,p}\big)\lambda^{j,l}_{\tau_p},$$

for any $l = 1,\dots,K$. By (A2)-(i), (3) and the $\overline{\mathcal{J}}$-measurability of $\lambda^j_{\tau_p}$, for $\mathbb{P}$-almost all ω, under $\mathbb{P}_{\overline{\mathcal{J}},\omega}$ the variables ξ^j are independent, centered, with variances $v^j = (\lambda^{j,l}_{\tau_p})^2\big(\overline{\delta}_{h^2,p} - (\overline{\delta}_{h,p})^2\big)$ smaller than $C(\chi^j)^2\,\mathbb{E}_{\overline{\mathcal{J}}}((\chi^j)^q)$ for some $q > 0$ because of (36) and the polynomial growth of h. Then, successive conditioning yields

$$\mathbb{E}\big((Y_N)^2\big) = \frac{1}{N^2}\,\mathbb{E}\Big(\sum_{j=1}^N v^j\Big) \leq \frac{1}{N^2}\sum_{j=1}^N \mathbb{E}\big((\chi^j)^{q+2}\big) \to 0,$$

by (36), which implies the first claim. The second claim is proved analogously. $\square$

Proof of Lemma 3. We start with the first line of (43), for which it clearly suffices to prove the bound for each product $\Delta_i^n X^{(\mathcal{S}),j}\Delta_i^n X^{(\mathcal{S}'),k}$, when both $(\mathcal{S})$ and $(\mathcal{S}')$ are either (b) or (σ) or (θ). When at least one of $(\mathcal{S})$ and $(\mathcal{S}')$ is (b), the bound trivially follows from (40) and C-S. In the other cases, both $\Delta_i^n X^{(\mathcal{S}),j}$ and $\Delta_i^n X^{(\mathcal{S}'),k}$ are the increments over $I(n,i)$ of square-integrable martingales, which further are orthogonal because $\widetilde{W}^j$, $\widetilde{W}^k$, μ^j and μ^k are independent (so the latter two have no common jumps). Therefore in those cases $\mathbb{E}'^n_{i-1}\big(\Delta_i^n X^{(\mathcal{S}),j}\Delta_i^n X^{(\mathcal{S}'),k}\big) = 0$, and the claim follows.

We have $(\Delta_i^n \overline{X}^{N,(I)})^2 = \frac{1}{N^2}\sum_{j,k=1}^N \Delta_i^n X^{(I),j}\Delta_i^n X^{(I),k}$, and (40) yields $\mathbb{E}'^n_{i-1}\big((\Delta_i^n X^{(I),j})^2\big) \leq C\Delta_n\,\mathbb{E}'^n_{i-1}((\chi^j)^2)$. Then (42) follows from the first line of (43). Finally, since $X^{(I),N,j} = X^{(I),j} - \psi\overline{X}^{N,(I)}$, the second line of (43) follows from the first line, plus (42). $\square$

Proof of Lemma 4. Suppose for a while that we have, for any indices j,k,

$$\mathbb{E}\big((\Delta_i^n X^{(A),j})^2(\Delta_i^n X^{(A),k})^2\big) \leq C\big(\Delta_n^{1+1/w} + \Delta_n\,1_{\{j=k\}}\big). \tag{53}$$

In view of the definitions of $\overline{X}^{N,(A)}$ and $X^{(A),N,j}$, this successively implies

$$\mathbb{E}\big((\Delta_i^n \overline{X}^{N,(A)})^4\big) \leq C\big(\Delta_n^{1/1/w} + \Delta_n/N\big)$$
$$\mathbb{E}\big((\Delta_i^n X^{(A),N,j})^2(\Delta_i^n X^{(A),N,k})^2\big) \leq C\big(\Delta_n^{1/1/w} + \Delta_n/N + \Delta_n\,1_{\{j=k\}}\big).$$

Since both $g(x)$ and $h(x)$ are smaller than Cx^2, and in view of the definition of $\widetilde{a}(g)_i^{n,N}$, the claim readily from the second bound above. It thus remains to prove (53). This follows from (41) when $j=k$, so below we assume $j\neq k$. Using $(x+y)^2 \leq 2x^2+2y^2$ and $X^{(A),j} = X^{(c),j} + X^{(\theta,j}$, to obtain (53) it suffices to show that $\mathbb{E}(U_m) \leq C\Delta_n^{1+1/w}$ for $m=1,2,3$, where

$$U_1 = (\Delta_i^n X^{(c),j})^2(\Delta_i^n X^{(c),k})^2, \quad U_2 = (\Delta_i^n X^{(c),j})^2(\Delta_i^n X^{(\theta),k})^2, \quad U_3 = (\Delta_i^n X^{(\theta),j})^2(\Delta_i^n X^{(\theta),k})^2.$$

(41) and C-S yield $\mathbb{E}(U_1) \leq C\Delta_n^2$. (41) and Hölder's inequality (with the second conjugate exponent $q=w$) yields $\mathbb{E}(U_2) \leq C\Delta_n^{1+1/w}$. An integration by parts yields $(\Delta_i^n X^{(\theta),j})^2 = \zeta^j + \zeta'^j$, where

$$\zeta^j = \int_{I(n,i)\times\mathbb{R}} \big(2X^{(\theta),j}(n,i)_t\,\theta^j(t,z) + \theta^j(t,z)^2\big)(\mu^j-\nu)(dt,dz), \qquad \zeta'^j = \int_{I(n,i)\times\mathbb{R}} \theta^j(t,z)^2\,dt\,dz.$$

The bound $|\zeta'^j| \leq C(\chi^j)^2\Delta_n$ is obvious, and Lemma 2.1.5 of Jacod and Protter (2012) implies $\mathbb{E}(|\zeta^j|^w) \leq C_q\Delta_n$. Moreover $\mathbb{E}(\zeta^j\zeta^k)=0$ if $j\neq k$ because then ζ^j and ζ^k are increments over $I(n,i)$ of two orthogonal martingales, so Hölder's inequality again implies $\mathbb{E}(U_3) \leq C\Delta_n^{1+1/w}$. $\square$

Proof of Lemma 5. As seen in the previous proof, $\mathbb{E}(\Delta_i^n X^{(\theta),j}\Delta_i^n X^{(\theta),k}) = 0$ when $j\neq k$. We then deduce from (41) and $|\Delta_i^n \overline{X}^{N,(\theta)}|^2 = \frac{1}{N^2}\sum_{j,k=1}^N \Delta_i^n X^{(\theta),j}\Delta_i^n X^{(\theta),k}$ that

$$\mathbb{E}\big(|\Delta_i^n \overline{X}^{N,(c)}|^{2q}\big) \leq C_q\Delta_n^q, \qquad \mathbb{E}\big(|\Delta_i^n \overline{X}^{N,(\theta)}|^2\big) \leq C\,\frac{\Delta_n}{N}, \tag{54}$$

for all $q \geq 1$. Since $\Delta_i^n \overline{X}^{N,(A)} = \Delta_i^n \overline{X}^{N,(c)} + \Delta_i^n \overline{X}^{N,(\theta)}$, the first claim then follows from the above and Markov's inequality. Set $\xi^{(c)} = \frac{1}{N}\sum_{j=1}^N |\Delta_i^n X^{(c),j}|^2$ and $\xi^{(\theta)} = \frac{1}{N}\sum_{j=1}^N |\Delta_i^n X^{(\theta),j}|^2$. By integration by parts we have $(\Delta_i^n X^{(\theta),j})^2 = \zeta^j + \zeta'^j$, where

$$\zeta^j = \int_{I(n,i)\times\mathbb{R}} \big(2X^{(\theta),j}(n,i)_t\,\theta^j(t,z) + \theta^j(t,z)^2\big)(\mu^j - \nu)(dt,dz), \qquad \zeta'^j = \int_{I(n,i)\times\mathbb{R}} \theta^j(t,z)^2\,dt\,dz.$$

Any two variables ζ^j and ζ^k are increments over $I(n,i)$ of square-integrable martingales, which are orthogonal when $j \neq k$. Moreover $|\zeta'^j| \leq C(\chi^j)^2\Delta_n$ is obvious, hence $\mathbb{E}(|\zeta'^j|^q) \leq C_q\Delta_n^q$, and by (40) we deduce $\mathbb{E}(|\zeta^j|^2) \leq C\Delta_n$. Thus we have $\xi^{(\theta)} = \overline{\zeta} + \overline{\zeta}'$ with the properties

$$\mathbb{E}(|\overline{\zeta}|) \leq C\,\frac{\Delta_n}{N}, \qquad \mathbb{E}(|\overline{\zeta}'|^q) \leq C_q\,\Delta_n^q,$$

for any $q \geq 1$, whereas $\mathbb{E}(|\xi^{(c)}|^q) \leq C_q\Delta_n^q$ as well by (40). These bounds and (54), plus Markov's inequality, allow us to deduce the second claim from the following property, due to the definition (39) and $g(x) \leq Cx^2$:

$$\widetilde{a}(g)_i^{n,N} \leq C\big(\overline{\xi}^{(c)} + |\overline{\zeta}| + |\overline{\zeta}'_N| + |\Delta_i^n \overline{X}^{N,(c)}|^2 + |\Delta_i^n \overline{X}^{N,(\theta)}|^2\big). \qquad \square$$

Proof of Lemma 6. We focus on the first bound, the second one being proven in exactly the same way. Set $U_j = \Delta_i^n(X^{(\lambda),N,j} + X^{(\sigma),N,j})$ and $V_j = \Delta_i^n(X^{(\theta),N,j} + X^{(b),N,j})$, so $\widehat{X}^{(A),\mathcal{T},j} = (U_j + V_j)1_{\{|U_j+V_j|\leq u_n^j\}}$. With the help of the properties $|xy| \leq x^2 + y^2$ and $|x| \leq |x|\,1_{\{|x|\leq 2u\}} + 2|y|\,1_{\{|y|>u\}}$ if $|x+y| \leq u$ (because then either $|x| \leq 2u$ or $2u < |x| < 2|y|$), it is (relatively) simple to check that

$$\big|\widehat{X}_i^{(A),\mathcal{T},j}\,\widehat{X}_i^{(A),\mathcal{T},k} - U_jU_k\big| \leq C\textstyle\sum_{l=1}^4 \Phi_l, \qquad \text{where}$$

$$\Phi_1 = (U_j^2 + |U_jU_k|)\,1_{\{|U_j|>u_n^j/2\}} + (U_k^2 + |U_jU_k|)\,1_{\{|U_k|>u_n^k/2\}}$$

$$\Phi_2 = |U_jU_k|\big(1_{\{|V_j|>u_n^j/2\}} + 1_{\{|V_k|>u_n^k/2\}}\big), \qquad \Phi_3 = V_j^2\,1_{\{|V_j|\leq 2u_n^j\}} + V_k^2\,1_{\{|V_k|\leq 2u_n^k\}}$$

$$\Phi_4 = |U_jV_j|\,1_{\{|V_j|\leq 2u_n^j\}} + |U_kV_k|\,1_{\{|V_k|\leq 2u_n^k\}}.$$

By Markov's inequality, C-S, (41) and (11) we get $\mathbb{E}(\Phi_l) \leq C\Delta_n^{3/2-\varpi}$ for $l = 1,2$. Since $\Phi_3 \leq (2u_j^n)^{2-w}|V_j|^w + (2u_k^n)^{2-w}|V_k|^w$ and $\Phi_4 \leq (2u_n^j)^{(2-w)/2}\,|V_j|^{w/2}|U_k| + (2u_n^k)^{(2-w)/2}\,|V_k|^{w/2}|U_j|$, we get in the same way $\mathbb{E}(\Phi_l) \leq C\Delta_n^{1+(2-w)\varpi/2}$ for $l = 3,4$. This completes the proof, upon taking $\varepsilon = \big(\frac{1}{2} - \varpi\big)\bigwedge \frac{(2-w)\varpi}{2}$. $\square$

Proof of Lemma 7. The properties of $u_{n,N}$ in (10) and (44) used with $q \geq 1/\varepsilon$ yield

$$\mathbb{P}\Big(\sup_{i=1,\ldots,n} |\Delta_i^n \overline{X}^{N,(A)}| > \frac{u_{n,N}}{2}\Big) \leq \sum_{i=1}^n \mathbb{P}\Big(|\Delta_i^n \overline{X}^{N,(A)}| > \frac{u_{n,N}}{2}\Big) \leq C\Big(\frac{1}{n^{q-1}u_{n,N}^{2q}} + \frac{1}{Nu_{n,N}^2}\Big) \to 0.$$

Next, (A2)-(i) implies that for $\mathbb{P}$-almost all ω we have $\mathbb{E}_{\mathcal{J},\omega}\big(|\overline{\delta}_{\mathbb{I},p}^N - \overline{\delta}_{\mathbb{I},p}|^2\big) = \frac{1}{N}\big(\overline{\delta}_{\mathbb{I}^2,p}(\omega) - \overline{\delta}_{\mathbb{I},p}(\omega)^2\big)$, so (36) and $P < \infty$ yield $\mathbb{P}\big(|\overline{\delta}_{\mathbb{I},p}^N - \overline{\delta}_{\mathbb{I},p}| \leq u_{n,N}/2 \text{ for } p = 1,\ldots,P\big) \to 1$. Therefore, with a probability going to 1, we have $|\Delta_i^n \overline{X}^{N,(A)}| \leq u_{n,N}/2$ for all $i = 1, \ldots, n$ and $|\Delta_{i_{n,p}}^n \overline{X}^{N,(A)} + \overline{\delta}_{\mathbb{I},p}^N - \overline{\delta}_{\mathbb{I},p}| \leq u_{n,N}$ for all $p = 1, \ldots, P$. Moreover, in restriction to Ω_n we have $\Delta_i^n \overline{X}^N = \Delta_i^n \overline{X}^{N,(A)}$ when $i \in \widetilde{I}_n$ and $\Delta_{i_{n,p}}^n \overline{X}^N = \Delta_{i_{n,p}}^n \overline{X}^{N,(A)} + \overline{\delta}_{\mathbb{I},p}^N$ when $p = 1, \ldots, P$. Since $\mathbb{P}(\Omega_n) \to 1$ and $u_{n,N} \to 0$ and $\overline{\delta}_{\mathbb{I},p} \neq 0$ if and only if $i_{n,p} \in I_n''$, we deduce the first claim, and also the second one because of (8). $\square$

Proof of Lemma 8. 1) With the notation (39), write

$$1 < i < n \quad \Rightarrow \quad \widehat{\alpha}(g)_i = \widetilde{a}^{\mathcal{T}}(g)_i^{n,N} - \frac{1}{2}\big(\widetilde{a}^{\mathcal{T}}(g)_{i-1}^{n,N} + \widetilde{a}^{\mathcal{T}}(g)_{i+1}^{n,N}\big), \qquad \overline{VJ}_g^{n,N} = \sum_{i=2}^{n-1} |\widehat{\alpha}(g)_i|.$$

When $i \in \widetilde{I}_n$ we have $\widehat{r}_i^{\mathcal{T}j} = \widehat{X}_i^{(A),\mathcal{T},j}$, hence $\widehat{a}^{\mathcal{T}}(g)_i = \widetilde{a}^{\mathcal{T}}(g)_i^{n,N}$. Since $\{1, \ldots, n\} \backslash \widetilde{I}_n$ contains at most $P + \overline{P}$ points, the difference $|\widehat{VJ}_g - \overline{VJ}_g^{n,N}|$ is thus smaller than the sum of the variables $\xi_i = \widetilde{a}^{\mathcal{T}}(g)_i^{n,N} + \widehat{a}^{\mathcal{T}}(g)_i$ for at most $2 + 3P + 3\overline{P}$ values of i. Since $\xi_i \leq C\Delta_n^{2\varpi}$ by (11) plus $g(x) \leq Cx^2$, we have $\widehat{VJ}_g - \overline{VJ}_g^{n,N} \to 0$ pointwise.

Moreover we have $\big|g(x) - g''(0)x^2/2\big| \leq C|x|^3$ and $|\widehat{X}_i^{(A),\mathcal{T},j}|^3 \leq C\Delta_n^{\varpi}(\Delta_i^n X^{(A),N,j})^2$, hence

$$\mathbb{E}\Big(\big|g(\widehat{X}_i^{(A),\mathcal{T},j}) - \frac{g''(0)}{2}\,(\widehat{X}_i^{(A),\mathcal{T},j})^2\big|\Big) \leq C\Delta_n^{1+\varpi},$$

by (41). Thus if $h(x) = x^2$ we have $\mathbb{E}\big(\big|\overline{VJ}_g^{n,N} - g''(0)\overline{V}_h^{n,N}/2\big|\big) \leq C\Delta_n^{\varpi}$, implying $\overline{VJ}_g^{n,N} - g''(0)\overline{V}_h^{n,N}/2 \xrightarrow{\mathbb{P}} 0$, and it suffices to prove that $\overline{V}_h^{n,N} \xrightarrow{\mathbb{P}} \overline{A}_h$.

Next, Lemma 6 readily implies (since $h(x) = x^2$) that

$$\mathbb{E}\big(|\widetilde{a}^{\mathcal{T}}(h)_i^{n,N} - \widehat{Y}_i|\big) \leq C\Delta_n^{1+\varepsilon}, \quad \text{where} \quad \widehat{Y}_i = \frac{1}{N}\sum_{j=1}^{N}(\widehat{Y}_i^j)^2, \quad \widehat{Y}_i^j = \Delta_i^n X^{(\lambda),N,j} + \Delta_i^n X^{(\sigma),N,j}.$$

So indeed it is enough to prove that $\widehat{V} := \sum_{i=2}^{n-1} \big|\widehat{Y}_i - \frac{1}{2}(\widehat{Y}_{i-1} + \widehat{Y}_{i+1})\big| \xrightarrow{\mathbb{P}} \overline{A}_h$.

2) In this step we use the following simple property of the processes $X^{(\lambda),j} + X^{(\sigma),j}$. Namely, by the second and the last parts of (36) plus C-S we have for $m = 0, 1, 2$ and any $i \geq 1$:

$$\mathbb{E}\big(|\Delta_{i+m}^n X^{(\lambda),j} + \Delta_{i+m}^n X^{(\sigma),j} - (\lambda_{(i-1)\Delta_n}^j)^\top \Delta_{i+m}^n W - \sigma_{(i-1)\Delta_n}^j \Delta_{i+m}^n \widetilde{W}^j|^2 \leq C\Delta_n^{1+\rho/2}.$$

Thus if $\lambda_t^{N,j} = \lambda_t^j - \frac{\psi}{N}\sum_{k=1}^N \lambda_t^k$ (with components $\lambda_t^{N,j,l}$) and $\sigma_t^{N,j} = \sigma_t^j - \frac{\psi}{N}\sum_{k=1}^N \sigma_t^k$, we have

$$\mathbb{E}\Big(\big|\widehat{Y}_{i+m}^j - (\lambda_{(i-1)\Delta_n}^{N,j})^\top \Delta_{i+m}^n W - \sigma_{(i-1)\Delta_n}^{N,j} \Delta_{i+m}^n \widetilde{W}^j\big|^2\Big) \leq C\Delta_n^{1+\rho/2}.$$

Since $\mathbb{E}\big((\widehat{Y}_i^j)^q + \|\Delta_i^n W\|^q + (\Delta_i^n \widetilde{W}^j)^q\big) \leq C\Delta_n^{q/2}$, we then deduce from C-S and (36) again that

$$\mathbb{E}\Big(\big|(\widehat{Y}_{i+m}^j)^2 - \big((\lambda_{(i-1)\Delta_n}^{N,j})^\top \Delta_{i+m}^n W + \sigma_{(i-1)\Delta_n}^{N,j} \Delta_{i+m}^n \widetilde{W}^j\big)^2\big|\Big) \leq C\Delta_n^{1+\rho/4}.$$

As a consequence, with the simplifying notation $U_i^l = \Delta_i^n W^l/\sqrt{\Delta_n}$ and $\widetilde{U}_i^j = \Delta_i^n \widetilde{W}^j/\sqrt{\Delta_n}$, implicitly depending on n and which are $\mathcal{N}(0,1)$ variables, all independent as i,l,j vary, we have

$$\mathbb{E}(|\widehat{V}-\widehat{V}_1-\widehat{V}_2-\widehat{V}_3|)\leq C\Delta_n^{\rho/4},\qquad \text{where}\qquad \widehat{V}_m=\sum_{i=2}^{n-1}|\widehat{\zeta}_{i,m}|,\qquad \widehat{\zeta}_{i,m}=\tfrac{1}{N}\sum_{j=1}^{N}\widehat{\mu}_{i,m}^{j}$$
$$\widehat{\mu}_{i,1}^{j}=\sum_{l,l'=1}^{K}\lambda_{(i-2)\Delta_n}^{N,j,l}\lambda_{(i-2)\Delta_n}^{N,j,l'}\nu_{i,1}^{n,l,l'},\qquad \nu_{i,1}^{n,l,l'}=\big(U_i^lU_i^{l'}-\tfrac{1}{2}\,U_{i-1}^lU_{i-1}^{l'}-\tfrac{1}{2}\,U_{i+1}^lU_{i+1}^{l'}\big)\Delta_n$$
$$\widehat{\mu}_{i,2}^{j}=\sum_{l=1}^{K}\lambda_{(i-2)\Delta_n}^{N,j,l}\sigma_{(i-2)\Delta_n}^{N,j}\nu_{i,2}^{n,l,j},\qquad \nu_{i,2}^{n,l,j}=\big(2U_i^l\widetilde{U}_i^j-U_{i-1}^l\widetilde{U}_{i-1}^j-U_{i+1}^l\widetilde{U}_{i+1}^{\prime j}\big)\Delta_n$$
$$\widehat{\mu}_{i,3}^{j}=(\sigma_{(i-2)\Delta_n}^{N,j})^2\,\nu_{i,3}^{n,j},\qquad \nu_{i,3}^{n,j}=\big((\widetilde{U}_i^j)^2-\tfrac{1}{2}(\widetilde{U}_{i-1}^j)^2-\tfrac{1}{2}(\widetilde{U}_{i+1}^j)^2\big)\Delta_n.$$

Moreover, $\mathbb{E}_{i-2}'^{n}(\nu_{i,2}^{n,l,j}\nu_{i,2}^{n,l,k})$ and $\mathbb{E}_{i-2}'^{n}(\nu_{i,3}^{n,j}\nu_{i,3}^{n,k})$ are equal $6\Delta_n^2$ and $9\Delta_n^2/2$, respectively, if $j=k$, and both are 0 when $k\neq j$. Hence (36) and successive conditioning imply $\mathbb{E}(|\widehat{\zeta}_{i,m}|^2)\leq C\Delta_n^2/N$, yielding $\mathbb{E}(|\widehat{V}_m|^2)\leq C/N$ and thus $\widehat{V}_m\xrightarrow{\mathbb{P}}0$, if $m=2,3$. and we are left to proving the claim for $\widehat{V}'$. Therefore, it remains to prove that $\widehat{V}_1\xrightarrow{\mathbb{P}}\overline{A}_h$.

3) With $\overline{\lambda}_t^{(1),N,l}=\frac{1}{N}\sum_{j=1}^N\lambda_t^{j,l}$ and $\overline{\lambda}_t^{(2),N,l,l'}=\frac{1}{N}\sum_{j=1}^N\lambda_t^{j,l}\lambda_t^{j,l'}$, a simple calculation yields

$$\widehat{\zeta}_{i,1}=\widehat{\zeta}_i^1+\widehat{\zeta}_i^2,\qquad \widehat{\zeta}_i^m=\sum_{l,l'=1}^{K}\widehat{b}_{(i-2)\Delta_n}^{m,l,l'}\,\nu_{i,1}^{n,l,l'},\qquad \widehat{b}_t^{1,l,l'}=\overline{\lambda}_t^{(2),l,l'}+(\psi^2-2\psi)\overline{\lambda}_t^{(1),l}\overline{\lambda}_t^{(1),N,l'}$$
$$\widehat{b}_t^{2,l,l'}=\overline{\lambda}_t^{(2),N,l,l'}+(\psi^2-2\psi)\overline{\lambda}_t^{(1),N,l}\overline{\lambda}_t^{(1),N,l'}-\widehat{b}_t^{1,l,l'}.$$

(A2)-(iii) with $T_1=\infty$ implies $|\widehat{b}_t^{2,l,l'}|\leq\alpha_t^N$ for all l,l' for some $\mathcal{F}_t$-measurable variables α_t^N satisfying $\mathbb{E}\big(\sup_{t\leq T}\,\alpha_t^N\big)\to0$ as $N\to\infty$. Since further $\mathbb{E}_{i-2}^n(|\nu_{i,1}^{n,l,l'}|)\leq6\Delta_n$, we deduce

$$\mathbb{E}\Big(\sum_{i=2}^{n-1}|\widehat{\zeta}_i^2|\Big)\leq 6K^2\Delta_n\,\mathbb{E}\Big(\sum_{i=2}^{n-1}\alpha_{(i-2)\Delta_n}^N\Big)\leq 6nK^2\Delta_n\,\mathbb{E}\Big(\sup_{t\leq T}\alpha_t^N\Big)\to0.$$

We are thus left to proving that $V'^n:=\sum_{i=2}^{n-1}|\widehat{\zeta}_i^1|\xrightarrow{\mathbb{P}}\overline{A}_h$.

Since $\overline{\lambda}_t^{(1),l}$ and $\overline{\lambda}_t^{(2),l,l'}$ are càdlàg, by a localization argument it is no restriction to assume that they are bounded, so $Y_t=F_\psi(\overline{\lambda}_t^{(2)},\overline{\lambda}_t^{(1)})$ is also càdlàg and bounded. Then $\mathbb{E}(|\widehat{\zeta}_i^1|^2)\leq C\Delta_n^2$ and, in view of the definition (18), we have $\mathbb{E}_{i-2}^n(|\widehat{\zeta}_i^1|)=\Delta_nY_{(i-2)\Delta_n}$. By Riemann integration we get $\sum_{i=2}^{n-1}\mathbb{E}_{i-2}^n(|\widehat{\zeta}_i^1|)\xrightarrow{\mathbb{P}}\overline{A}_h$. Since $\widehat{\zeta}_i^1$ is $\mathcal{F}_{(i+1)\Delta_n}$-measurable, it remains to apply a classical martingale convergence theorem for triangular arrays (upon singling out the sums of the $|\widehat{\zeta}_i^1|$'s for all i are equal to m modulo 3, separately for $m=0,1,2$), and we get $V'^n\xrightarrow{\mathbb{P}}\overline{A}_h^N$. $\square$

Proof of Lemma 9. 1) We first prove the following, already mentioned before Theorem 1:

$$\psi\neq1,\quad x\in\mathbb{R}^K\backslash\{0\},\quad M-xx^\top\ \text{ is symmetric nonnegative}\quad\Rightarrow\quad F_\psi(M,x)>0,\tag{55}$$

Set $M'=M-xx^\top$ and $M''=M+(\psi^2-2\psi)xx^\top=M'+(1-\psi)^2xx^\top$, so $F_\psi(M,x)$ can be rewritten as follows, with Φ^l,Φ'^l,Φ''_l as in (18):

$$F_\psi(M,x)=\mathbb{E}(|V|),\qquad\text{with}\quad V=\sum_{l,l'=1}^{K}\alpha^{l,l'}\big(\Phi_1^l\Phi_1^{l'}-\frac{1}{2}\,\Phi_2^l\Phi_2^{l'}-\frac{1}{2}\,\Phi_3^l\Phi_3^{l'}\big)=0.$$

Let $C = \sup_{l,l'} |M'^{l,l'}|$. Under the conditions of (55) there is an l_0 with $x^{l_0} \neq 0$, and $M'^{l_0,l_0} \geq 0$. If $|\Phi^l_m| \leq 1$ for all $(l,m) \neq (l_0,1)$ we have $\big|V - M''^{l_0,l_0}(\Phi^{l_0}_1)^2\big| \leq 2K^2C$, hence $V \geq (1-\psi)^2(x^{l_0})^2(\Phi^{l_0}_1)^2 - 2K^2C > 0$ on the set on which $|\Phi^l_m| \leq 1$ for all $(l,m) \neq (l_0,1)$ and $(\Phi^{l_0}_1)^2 > 2K^2C/(1-\psi)^2(x^{l_0})^2$. This set has a positive probability, hence necessarily $\mathbb{E}(|V|) > 0$, and (55) is proven. Thus, by the very definition of $\overline{A}_f$ and $f \in \mathcal{C}_+$ and the last part of (iii) of (A2) we have $\overline{A}_f > 0$ a.s., because the matrix $\overline{\lambda}^{(2}_t - \overline{\lambda}^{(1)}_t(\overline{\lambda}^{(1)}_t)^\top$ is obviously symmetric nonnegative.

2) Next, we prove $\mathbb{P}(\widehat{I}'_1 = I'_n) \to 1$. Since $\widehat{VJ}_f \stackrel{\mathbb{P}}{\longrightarrow} \overline{A}_f$ by Lemma 8 and $\overline{A}_f > 0$ a.s. by Step 1 and $\mathbb{P}(\Omega_n) \to 1$, it is enough to show that, for any $y \in (0,1)$ and with $\Omega^{n,N}_y = \Omega_n \cap \{\widehat{VJ}_f > y\}$:

$$\mathbb{P}\big(I'_n \not\subset \widehat{I}'_1\big) \to 0, \qquad \mathbb{P}\big(\{\widehat{I}'_1 \not\subset I_n\} \cap \Omega^{n,N}_y\big) \to 0. \tag{56}$$

Since $\psi\overline{\delta}^N_{\mathbb{I},p}\,1_{\{\overline{\delta}_{\mathbb{I},p}=0\}} \stackrel{\mathbb{P}}{\longrightarrow} 0$ as $N \to \infty$, (41) and $\mathbb{P}(\overline{\Omega}_n) \to 1$ imply $\widehat{a}(f)_{i_{n,p}} \stackrel{\mathbb{P}}{\longrightarrow} \overline{\delta}_{f,p}$ for $p = 1,\dots,P$. We also have $\widehat{a}(f)_{i_{n,p}\pm1} = \widetilde{a}(f)^{n,N}_{i_{n,p}\pm1}$ on Ω_n, hence $\mathbb{P}(\Omega_n) \to 1$ and (40) yield $\widehat{a}(f)_{i_{n,p}\pm1} \stackrel{\mathbb{P}}{\longrightarrow} 0$. Thus $\breve{a}(f)_{i_{n,p}} \stackrel{\mathbb{P}}{\longrightarrow} \overline{\delta}_{f,p}$, and $\gamma_{n,N} \to 0$ and $P < \infty$ and $\mathbb{P}(\Omega_n) \to 1$ imply the first part of (56).

Next, on Ω_n and for any $i \in \widetilde{I}_n$ we have $\widehat{a}(f)'_i \leq \widehat{a}(f)_i = \widetilde{a}(f)^{n,N}_i$, hence

$$\mathbb{P}\big(\{\widehat{I}'_1 \not\subset I_n\} \cap \Omega^{n,N}_y\big) \leq \mathbb{P}(\widehat{I} \neq I_n) + \sum_{i=1}^n \mathbb{P}(\widetilde{a}(f)^{n,N}_i > y\gamma_{n,N}).$$

The first term of the right side above go to 0 by (8). The last term goes to 0 with our choice (16) for $\gamma_{n,N}$, by (44) with the choice $q = 1/\varepsilon$. Hence the second part of (56) holds true.

3) It thus remains to show that $\mathbb{P}(\Omega^*_n) \to 0$, where Ω^*_n is the set on which $\widehat{I}'_1 = I'_n$ and $\widehat{I}'_1$ is not a subset of $\widehat{I}'_2$. On this set Ω^*_n there is necessarily some p between 1 and P such that $i_{n,p} \in I'_n$ and $(\widehat{a}(f)_{i_{n,p}})^2 \leq \gamma'^2_{n,N}\,\widehat{a}(f^2)_{i_{n,p}}$. However, as seen in Step 2 we have $\widehat{a}(f)_{i_{n,p}} \stackrel{\mathbb{P}}{\longrightarrow} \overline{\delta}_{f,p}$, and we analogously have $\widehat{a}(f^2)_{i_{n,p}} \stackrel{\mathbb{P}}{\longrightarrow} \overline{\delta}_{f^2,p}$. Since $\overline{\delta}_{f,p} > 0$ and $\gamma'_{n,N} \to 0$, the probability that $(\widehat{a}(f)_{i_{n,p}})^2 \leq \gamma'^2_{n,N}\,\widehat{a}(f^2)_{i_{n,p}}$ goes to 0, and we deduce the claim. □

Proof of Lemma 10. Since $w < 2$, Lemma 4 implies for any $\mathcal{J}'$-measurable index i:

$$\sqrt{n}\,\widetilde{a}(g)^{n,N}_i \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad \sqrt{n \wedge N}\,\widehat{Y} \stackrel{\mathbb{P}}{\longrightarrow} 0, \quad \text{where} \quad \widehat{Y} = \sum_{i=1}^n \big(\widetilde{a}(g)^{n,N}_i\big)^2. \tag{57}$$

We also observe that, in restriction to the set $\overline{\Omega}'_n$ and because $2xy \leq x^2 + y^2$:

$$|\widehat{SJ}'_g - \widehat{SJ}'^*_g| + |\widehat{SJ}'_g - \widehat{SJ}''^*_g| \leq \sum_{p=1}^P (z_p + z'_p), \quad |\widehat{SJ}^\flat_g - \widehat{SJ}^{\flat\flat}_g| \leq \widehat{Y} + 2\sum_{p=1}^P z'_p, \quad |\widehat{SJ}^{\flat\flat}_g - \widehat{SJ}^*_g| \leq \widehat{Y}$$
$$z_p = \big(\widetilde{a}(g)^{n,N}_{i_{n,p}-1}\big)^2 + \big(\widetilde{a}(g)^{n,N}_{i_{n,p}+1}\big)^2, \qquad z'_p = \widehat{a}(g)_{i_{n,p}}\big(\widetilde{a}(g)^{n,N}_{i_{n,p}-1} + \widetilde{a}(g)^{n,N}_{i_{n,p}+1}\big).$$

Since $\mathbb{P}(\overline{\Omega}'_n) \to 1$ and $P < \infty$, all claims follow from (57) and the properties $\sqrt{n}\,z_p \stackrel{\mathbb{P}}{\longrightarrow} 0$ and $\sqrt{n}\,z'_p \stackrel{\mathbb{P}}{\longrightarrow} 0$ for each p. Those two properties again follows from (57), plus the fact (for z'_p) that the variables $\widehat{a}(g)_{i_{n,p}}$ are bounded in probability, as easily deduced from $|\delta^j_p| \leq \chi^j$, (36), (41) and (45).□

Proof of Lemma 11. Since $P < \infty$, it is enough to show that $\sqrt{n}\,\Phi^{n,N}_{m,p} \xrightarrow{\mathbb{P}} 0$ for each fixed p, where $\Phi^{n,N}_{m,p}$ denotes the pth summand in the definition of $\Phi^{n,N}_m$. (41) yields $\mathbb{E}\big((\widehat{X}^{(A),j}_p)^q\big) \leq C_q\Delta_n$ for any $q \geq 2$. Thus $\mathbb{E}\big(\Psi^{n,N}_{1,p}\big) \leq C\Delta_n$, implying $\sqrt{n}\,\Psi^{n,N}_{1,p} \xrightarrow{\mathbb{P}} 0$. Since $|\delta^{N,j}_p| \leq C\chi^{N,j}$ with $\chi^{N,j} = \chi^j + \overline{\chi}^{(1)}_N$, and $\sup_{N,j}\,\mathbb{E}((\chi^{N,j})^q) < \infty$ by (36), we obtain by Hölder's inequality:

$$\mathbb{E}(\Psi^{n,N}_{3,p}) \leq \frac{C}{N^2}\sum_{j,k=1}^N \big(\mathbb{E}(\chi^{N,j})^4)\big)^{\frac{1}{4}}\,\big(\mathbb{E}(|\widehat{X}^{(A),j}_p|^{8/3}\big)^{\frac{3}{4}} \leq C\Delta_n^{\frac{3}{4}}.$$

This implies $\sqrt{n}\,\Psi^{n,N}_{3,p} \xrightarrow{\mathbb{P}} 0$. For the case $m = 2$ we use the fact that the variables $\xi^{j,k} = g(\delta^{N,j}_p)g'(\delta^{N,k}_p)$ are $\mathcal{J}'$-measurable and

$$(\Psi^{n,N}_{2,p})^2 = \frac{1}{N^4}\sum_{j,j',k,k'=1}^N \xi^{j,k}\xi^{j',k'}\,\widehat{X}^{(I),k}_p\,\widehat{X}^{(I),k'}_p,$$

implying

$$\mathbb{E}\big((\Psi^{n,N}_{2,p})^2\big) \leq \frac{1}{N^4}\sum_{j,j',k,k'=1}^N \rho_{j,j',k,k'}, \qquad \rho_{j,j',k,k'} = \mathbb{E}\Big(|\xi^{j,k}_p\xi^{j',k'}_p|\,\big|\mathbb{E}'^n_{i_{n,p}-1}(\widehat{X}^{(I),k}_p\,\widehat{X}^{(I),k'}_p)\big|\Big).$$

Since $|\xi^{j,k}_p| \leq C(\chi^{N,j})^2\chi^{N,k}$ we deduce that $\rho_{j,j',k,k'}$ is smaller than $C\Delta_n$ if $k = k'$ from (40) and than $C\Delta_n^{3/2}$ if $k \neq k'$ from (43). As a consequence, $\mathbb{E}((\Psi^{n,N}_{2,p})^2) \leq C\big(1/n^{3/2} + 1/nN\big)$, and thus $\sqrt{n}\,\Psi^{n,N}_{2,p} \xrightarrow{\mathbb{P}} 0$. □

Proof of Lemma 12. Throughout we fix ω_0 and argue under $\mathbb{P}_{\overline{\mathcal{J}},\omega_0}$, under which the $(\delta^j_p)_{p\geq 1}$ are i.i.d. as j varies, with finite moments of all order. In particular, the sets $\mathcal{P}', \mathcal{P}'$ depend on ω_0 only, so under $\mathbb{P}_{\overline{\mathcal{J}},\omega_0}$ they are non random. Since $|g(x+y) - g(x) - g'(x)y| \leq Cy^2$ and recalling the notation (3), we see that for $p \in \mathcal{P}'$:

$$\big|g(\delta^{N,j}_p) - g(\delta^j_p) + \psi g'(\delta^j_p)\overline{\delta}^N_{\mathbb{I},p}\big| \leq C(\overline{\delta}^N_{\mathbb{I},p})^2, \qquad \Big|\frac{1}{N}\sum_{j=1}^N g(\delta^{N,j}_p) - \overline{\delta}^N_{g,p} + \psi\overline{\delta}^N_{g',p}\overline{\delta}^N_{\mathbb{I},p}\Big| \leq C(\overline{\delta}^N_{\mathbb{I},p})^2. \tag{58}$$

Then with the notation $U^N_p = \sqrt{N}\,\overline{\delta}^N_{\mathbb{I},p}$ and $\overline{U}^N_p = \sqrt{N}(\overline{\delta}^N_{g,p} - \overline{\delta}_{g,p})$ and $\overline{U}'^N_p = \sqrt{N}(\overline{\delta}^N_{g',p} - \overline{\delta}_{g',p})$, a calculation (simple for Y''^N below because $\delta^{N,j}_p = \delta^j_p$ when $p \in \mathcal{P}''$, more involved for Y'^N) yields

$$Y''^N = \overline{Y}''^N + \widetilde{Y}''^N, \qquad \overline{Y}''^N = 2\sum_{p\in\mathcal{P}''}\overline{\delta}_{g,p}\,\overline{U}^N_p, \qquad \widetilde{Y}''^N = \tfrac{1}{\sqrt{N}}\sum_{p\in\mathcal{P}''}(\overline{U}^N_p)^2$$

$$Y'^N = \overline{Y}'^N + \widetilde{Y}'^N, \qquad \overline{Y}'^N = 2\sum_{p\in\mathcal{P}'}\overline{\delta}_{g,p}\big(\overline{U}^N_p - \psi\overline{\delta}_{g',p}U^n_p\big),$$

$$|\widetilde{Y}'^N| \leq C_{\omega_0}\,\tfrac{1}{\sqrt{N}}\sum_{p\in\mathcal{P}'}\big(|U^N_p|^2 + |\overline{U}^N_p|^2 + |\overline{U}'^N_p|^2 + |U^N_p|^4 + |\overline{U}^N_p|^4 + |\overline{U}'^N_p|^4\big),$$

where C_{ω_0} is a "constant" depending on the fixed ω_0 through $\overline{\delta}_{g,p}(\omega_0)$ and $\overline{\delta}_{g',p}(\omega_0)$. Observe that

$$\overline{Y}'^N = \tfrac{1}{\sqrt{N}}\sum_{j=1}^N \widetilde{\delta}'^j_p, \qquad \widetilde{\delta}'^j_g = 2\sum_{p\in\mathcal{P}'}\overline{\delta}_{g,p}\big(g(\delta^j_p) - \overline{\delta}_{g,p} - \psi\overline{\delta}_{g',p}\delta^j_p\big)$$

$$\overline{Y}''^N = \tfrac{1}{\sqrt{N}}\sum_{j=1}^N \widetilde{\delta}''^j_p, \qquad \widetilde{\delta}''^j_g = 2\sum_{p\in\mathcal{P}''}\overline{\delta}_{g,p}\big(g(\delta^j_p) - \overline{\delta}_{g,p}\big).$$

Since $g(\delta_p^j)-\overline{\delta}_{g,p}$ and $g'(\delta_p^j)-\overline{\delta}_{g',p}$, and also δ_p^j when $p\in\mathcal{P}'$, are i.i.d. centered under $\mathbb{P}_{\overline{\mathcal{J}},\omega_0}$ with finite moments, by a classical result the variables $\overline{U}_p^N$, $\overline{U}_p'^N$, and also U_p^N when $p\in\mathcal{P}'$, have finite moments of all order, and it follows that $\widetilde{Y}'^N\stackrel{\mathbb{P}}{\longrightarrow}0$ and $\widetilde{Y}''^N\stackrel{\mathbb{P}}{\longrightarrow}0$. Finally, under $\mathbb{P}_{\overline{\mathcal{J}},\omega_0}$ the 2-dimensional variables $(\delta_g'^j,\delta_g''^j)$ are i.i.d. centered, its two components being uncorrelated and with respective variances $V_g'(\omega_0)$ and $V_g''(\omega_0)$. Then, the claim follows from the usual CLT. $\square$

Proof of Lemma 13. 1) For dealing with the two cases of $\mathcal{P}$ simultaneously, we set $\overline{\psi}_p=\psi$ if $p\in\mathcal{P}'$ and $\overline{\psi}_p=0$ if $p\in\mathcal{P}''$. With $\Delta_p^{n+}W=W_{i_{n,p}\Delta_n}-W_{\tau_p}$ and $\Delta_p^{n-}W=W_{\tau_p}-W_{(i_{n,p}-1)\Delta_n}$, and recalling (45), the desired decompositions are obtained by setting

$$\mathcal{Y}_{\mathcal{P}}^{n,N}=2\sqrt{n}\sum_{p\in\mathcal{P}}\overline{\delta}_{g,p}\overline{\delta}_{g',p}(1-\overline{\psi}_p)\big(\overline{\lambda}_{\tau_p-}^{(1)\top}\Delta_p^{n-}W+\overline{\lambda}_{\tau_p}^{(1)\top}\Delta_p^{n+}W\big),\qquad \widetilde{Y}_{\mathcal{P}}^{n,N}=2\sqrt{n}\sum_{p\in\mathcal{P}}\sum_{m=1}^{4}y_{p,m}^{n,N}$$

$$\begin{aligned}y_{p,1}^{n,N}&=\overline{\delta}_{g,p}\tfrac{1}{N}\sum_{j=1}^{N}\Big(\big(g'(\delta_p^j)(\lambda_{\tau_p-}^j-\overline{\psi}_p\overline{\lambda}_{\tau_p-}^{(1)})-(1-\overline{\psi}_p)\overline{\delta}_{g',p}\overline{\lambda}_{\tau_p-}^{(1)}\big)^\top\Delta_p^{n-}W\\&\qquad+\big(g'(\delta_p^j)(\lambda_{\tau_p}^j-\overline{\psi}_p\overline{\lambda}_{\tau_p}^{(1)})-(1-\overline{\psi}_p)\overline{\delta}_{g',p}\overline{\lambda}_{\tau_p}^{(1)}\big)^\top\Delta_p^{n+}W\Big)\end{aligned}$$

$$y_{p,2}^{n,N}=\overline{\delta}_{g,p}\,\tfrac{1}{N}\sum_{j=1}^{N}g'(\delta_p^j)\big(\widehat{X}_p^{(\lambda),j}-(\lambda_{\tau_p-}^j-\overline{\psi}_p\overline{\lambda}_{\tau_p-}^{(1)})^\top\Delta_p^{n-}W-(\lambda_{\tau_p}^j-\overline{\psi}_p\overline{\lambda}_{\tau_p}^{(1)})^\top\Delta_p^{n+}W\big)$$

$$y_{p,3}^{n,N}=\overline{\delta}_{g,p}\,\tfrac{1}{N}\sum_{j=1}^{N}\big(g'(\delta_p^{N,j})-g'(\delta_p^j)\big)\,\widehat{X}_p^{(\lambda),j}$$

$$y_{p,4}^{n,N}=G_p^N\,H_p^{n,N},\qquad G_p^N=\tfrac{1}{N}\sum_{k=1}^{N}g(\delta_p^{N,k})-\overline{\delta}_{g,p},\qquad H_p^{n,N}=\tfrac{1}{N}\sum_{j=1}^{N}g'(\delta_p^{N,j})\,\widehat{X}_p^{(\lambda),j}.$$

2) Since the τ_p's are the jump times of a Poisson process independent of W, Proposition 4.4.10 in Jacod and Protter (2012) tells us that there are $\mathcal{N}(0,1)$ variables $Z_p^{-,l}$ and $Z_p^{+,l}$ and $[0,1]$-uniform variables κ_p defined on an extension of the space $(\Omega,\mathcal{F},\mathbb{P})$, all mutually independent and also independent of $\mathcal{F}$, such that

$$\frac{1}{\sqrt{\Delta_n}}\big(\Delta_p^{n-}W,\,\Delta_p^{n+}W\big)_{p\geq1}\ \stackrel{\mathcal{L}-s}{\longrightarrow}\ \big(\sqrt{\kappa_p}(Z_p^{-,l})_{1\leq l\leq K},\,\sqrt{1-\kappa_p}(Z_p^{+,l})_{1\leq l\leq K}\big)_{p\geq1},$$

and one easily checks that the variables

$$Z_p^{\pm}=\begin{cases}\frac{1}{\|\overline{\lambda}_{\tau_p\pm}^{(1)}\|}\sum_{l=1}^{K}\overline{\lambda}_{\tau_p\pm}^{(1),l}\ Z_p^{\pm,l} & \text{if }\ \overline{\lambda}_{\tau_p\pm}^{(1)}\neq0\\ Z_p^{\pm,1} & \text{otherwise,}\end{cases}$$

are again $\mathcal{N}(0,1)$, mutually independent and independent of $\mathcal{F}$ and all κ_p. The joint stable convergence $(\mathcal{Y}_{\mathcal{P}'}^{n,N},\mathcal{Y}_{\mathcal{P}''}^{n,N})\stackrel{\mathcal{L}-s}{\longrightarrow}(Z'(g)_T,Z''(g)_T)$ is then obvious, because $n\sim T/\Delta_n$.

3) The variables $\widetilde{Y}_{\mathcal{P}'}^{n,N},\widetilde{Y}_{\mathcal{P}''}^{n,N}$ being clearly $\overline{\mathcal{J}}$-measurable, it remains to prove $\sqrt{n}\,y_{p,m}^{n,N}\stackrel{\mathbb{P}}{\longrightarrow}0$ for each $p=1,\dots,P$ and $m=1,\dots,4$. First, by (38) the variables $\big\|\frac{1}{N}\sum_{j=1}^{N}g'(\delta_p^j)(\lambda_{\tau_p\pm}^j-\overline{\psi}_p\overline{\lambda}_{\tau_p\pm}^{(1)})-(1-\overline{\psi}_p)\overline{\delta}_{g',p}\overline{\lambda}_{\tau_p\pm}^{(1)}\big\|$ go to 0 in probability, and they are smaller than $C\overline{\chi}_N^{(2)}$, whose moments are bounded in N by (36). Hence, by C-S and $\mathbb{E}(\|\Delta_p^{n\pm}W\|^2)\leq\Delta_n$, we readily get $\sqrt{n}\,y_{p,1}^{N,n}\stackrel{\mathbb{P}}{\longrightarrow}0$.

Next, we have $\widehat{X}^{(\lambda),j}_p = \Delta^n_{i_{n,p}}\big(X^{(\lambda),j} - \overline{\psi}_p\overline{X}^{N,(\lambda)}\big)$. In view of the definition of $\overline{X}^{N,(\lambda)}$, plus (36) and (iii) of (A-2) (with $T_1 = \infty$), we have for some sequence $\varepsilon_{p,n}$ going to 0 as $n \to \infty$:

$$\begin{array}{l} \qquad \mathbb{E}\big(|\widehat{X}^{(\lambda),j}_p - (\lambda^j_{\tau_p-} - \overline{\psi}_p\overline{\lambda}^{(1)}_{\tau_p-})^\top\Delta^{n-}_p W - (\lambda^j_{\tau_p} - \overline{\psi}_p\overline{\lambda}^{(1)}_{\tau_p})^\top\Delta^{n+}_p W)|^2\big) \\ \leq 2|\psi|^2\mathbb{E}\Big(\int_{(i_{n,p}-1)\Delta_n}^{i_{n,p}\Delta_n}\Big|\frac{1}{N}\sum\limits_{k=1}^N\lambda^k_t - \overline{\lambda}^{(1)}_t\Big|^2 dt\Big) \\ \qquad +4\sup_k\,\mathbb{E}\Big(\int_{(\tau_p-\Delta_n)^+}^{\tau_p}\|\lambda^k_t - \lambda^k_{\tau_p-}\|^2\,dt + \int_{\tau_p}^{\tau_p+\Delta_n}\|\lambda^k_t - \lambda^k_{\tau_p}\|^2\,dt\Big) \leq C\Delta_n\varepsilon_{n,p}. \end{array}$$

Since $\overline{\delta}_{g,p}$ and $g'(\delta^j_p)$ have fourth moments bounded in j, we deduce $\sqrt{n}\,y^{n,N}_{p,2} \xrightarrow{\mathbb{P}} 0$.

Finally, since $\delta^{N,j} = \delta^j_p - \overline{\psi}_p\overline{\delta}^N_{\mathbb{I},p}$ and also $\overline{\psi}_p\overline{\delta}_{\mathbb{I},p} = 0$, we have $\overline{\delta}^{N,j}_p \xrightarrow{\mathbb{P}} \delta^j_p$ uniformly. Therefore $\sup_j\,|g'(\delta^{N,j}) - g'(\delta^j_p)| \xrightarrow{\text{a.s.}} 0$ and $G^N_p \xrightarrow{\text{a.s.}} 0$ (use (3) for this). Since $E(|H^{n,N}_p|) + \mathbb{E}(|\widehat{X}^{(\lambda),j}_p|) \leq C\sqrt{\Delta_n}$ by (41) and C-S, we deduce $\sqrt{n}\,y^{N,n}_{p,m} \xrightarrow{\mathbb{P}} 0$ for $m = 3, 4$. $\square$

Proof of Lemma 14. For shorter notation we write $U_i = h(\Delta^n_i X^{(A),N,j})$ and $V_i = \widetilde{a}(g)^{n,N}_i$, and observe that since $g, h \in \mathcal{C}$ (40) implies that both $\mathbb{E}'^n_{i-1}(U_i)$ and $\mathbb{E}'^n_{i-1}(V_i)$ are smaller than $C\Delta_n\,\mathbb{E}'^n_{i-1}((\chi^j)^2 + \overline{\chi}^{(2)}_N)$, for any $\mathcal{J}'$-measurable index i. Then successive conditioning and Hölder's inequality yield $\mathbb{E}(U_iV_{i+1}) + \mathbb{E}(U_{i+1}V_i) \leq C\Delta_n^{7/4}$, and of course $\mathbb{E}(U_i + V_i) \leq C\Delta_n$. This and Lemma 4 imply that

$$\begin{array}{c} \sqrt{n}\,U_i + \sqrt{n}\,V_i + \sqrt{n}U_iV_{i+1} + \sqrt{n}\,U_{i+1}V_i \xrightarrow{\mathbb{P}} 0, \qquad \sqrt{n \wedge N}\,\widehat{Y} \xrightarrow{\mathbb{P}} 0, \\ \text{where} \qquad \widehat{Y} = \sum\limits_{i=1}^n U_iV_i + \sum\limits_{i=2}^n(U_{i-1}V_i + U_iV_{i-1}). \end{array} \tag{59}$$

As in the proof of Lemma 10, in restriction to the set $\overline{\Omega}'_n$ we have

$$\begin{array}{c} |\widehat{SJ}'_g - \widehat{SJ}'^*_g| + |\widehat{SJ}'_g - \widehat{SJ}''^*_g| \leq \sum\limits_{p=1}^P(z_p + z'_p), \quad |\widehat{SJ}^\flat_g - \widehat{SJ}^{\flat\flat}_g| \leq \widehat{Y} + \sum\limits_{p=1}^P z'_p, \quad |\widehat{SJ}^{\flat\flat}_g - \widehat{SJ}^*_g| \leq \widehat{Y} \\ z_p = \big(U_{i_{n,p}-1} + U_{i_{n,p}+1}\big)\big(V_{i_{n,p}-1} + V_{i_{n,p}+1}\big) \\ z'_p = \big(U_{i_{n,p}-1} + U_{i_{n,p}+1}\big)\widehat{a}(g)_{i_{n,p}} + h(\widehat{r}^j_{i_{n,p}})\big(V_{i_{n,p}-1} + V_{i_{n,p}+1}\big). \end{array}$$

Observing that the variables $\widehat{a}(g)_{i_{n,p}}$ and $h(\widehat{r}^j_{i_{n,p}})$ are bounded in probability, and upon using (59), we deduce all claims in exactly the same way as in Lemma 10. $\square$

Proof of Lemma 15. The proof is exactly the same as for Lemma 11, up to replacing the weighted sums $\frac{1}{N^2}\sum_{j,k=1}^N \ldots$ by $\frac{1}{N}\sum_{k=1}^N \ldots$. $\square$

Proof of Lemma 16. We follow the proof of Lemma 12, arguing now under $\mathbb{P}_{\overline{\mathcal{J}}_j,\omega_0}$ for some fixed ω_0. With $B_N = \{1, \ldots, N\}\backslash\{j\}$, we replace $U^N, \overline{U}^N_p, \widehat{U}^N_p$ of that lemma by

$$U^N_p = \frac{1}{\sqrt{N}}\sum_{k\in B_N}\delta^k_p, \qquad \overline{U}^N_p = \frac{1}{\sqrt{N}}\sum_{k\in B_N}\big(g(\delta^k_p) - \overline{\delta}_{g,p}\big), \qquad \widetilde{U}^N_p = \frac{1}{\sqrt{N}}\sum_{k\in B_N}\big(g'(\delta^k_p) - \overline{\delta}_{g',p}\big).$$

Then (58) yields $Y''^{N,j} = \overline{Y}''^{N,j} + \widetilde{Y}''^{N,j}$, where

$$\overline{Y}''^{N,j} = \sum_{p\in\mathcal{P}''} h(\delta_p^j)\overline{U}_p^N, \qquad \widetilde{Y}''^{N,j} = \frac{1}{\sqrt{N}} \sum_{p\in\mathcal{P}''} h(\delta_p^j)\big(g(\delta_p^j) - \overline{\delta}_{g,p}\big),$$

and, after a more complicated calculation, $Y'^{N,j} = \overline{Y}'^{N,j} + \widetilde{Y}'^{N,j}$, where

$$\overline{Y}'^{N,j} = \sum_{p\in\mathcal{P}'} \big(h(\delta_p^j)\overline{U}_p^N - \psi\big(h(\delta_p^j)\overline{\delta}_{g',p} + h'(\delta_p^j)\overline{\delta}_{g,p}\big)U_p^N\big) = \tfrac{1}{\sqrt{N}} \sum_{k\in B_N} \widetilde{\delta}'^{j,k}_{g,h}$$

$$|\widetilde{Y}'^{N,j}| \le C_{\omega_0} \tfrac{1}{\sqrt{N}} \sum_{p\in\mathcal{P}'} \big(|\delta_p^j|^2 + |\delta_p^j|^4 + |U_p^N|^2 + |\overline{U}_p^N|^2 + |\overline{U}_p'^N|^2 + |U_p^N|^4 + |\overline{U}_p^N|^4 + |\overline{U}_p'^N|^4\big).$$

Furthermore, we can write

$$\overline{Y}'^{N,j} = \tfrac{1}{\sqrt{N}} \sum_{k\in B_N} \widetilde{\delta}'^{j,k}_{g,h}, \qquad \widetilde{\delta}'^{j,k}_{g,h} = \sum_{p\in\mathcal{P}'} \big(h(\delta_p^j)\big(g(\delta_p^k) - \overline{\delta}_{g,p}\big) - \psi h(\delta_p^j)\overline{\delta}_{g',p}\delta_p^k - \psi h'(\delta_p^j)\overline{\delta}_{g,p}\delta_p^k\big)$$

$$\overline{Y}''^{N,j} = \tfrac{1}{\sqrt{N}} \sum_{k\in B_N} \widetilde{\delta}''^{j,k}_{g,h}, \qquad \widetilde{\delta}''^{j,k}_{g,h} = \sum_{p\in\mathcal{P}''} h(\delta_p^j)\big(g(\delta_p^k) - \overline{\delta}_{g,p}\big).$$

The same arguments as in Lemma 12 imply $\widetilde{Y}'^{N,j} \xrightarrow{\mathbb{P}} 0$ and $\widetilde{Y}''^{N,j} \xrightarrow{\mathbb{P}} 0$. Moreover, under $\mathbb{P}_{\overline{\mathcal{J}}_j,\omega_0}$ the 2-dimensional variables $(\widetilde{\delta}'^{j,k}_{g,h}, \widetilde{\delta}''^{j,k}_{g,h})$ are i.i.d. centered as k varies in B_N, and their two components are uncorrelated and with respective variances $V'^j_{g,h}(\omega_0)$ and $V''^j_{g,h}(\omega_0)$, so the claim follows from the usual CLT. $\square$

Proof of Lemma 17. 1) We use the notation $\overline{\psi}_p$ and $\Delta_p^{n\pm}W$ of the proof of Lemma 13, and define similarly the increments $\Delta_p^{n\pm}\widetilde{W}^j$ for the Brownian motion $\widetilde{W}^j$. The desired decompositions are obtained by setting the desired decompositions are obtained by setting

$$\mathcal{Y}_{\mathcal{P}}^{n,N,j} = \sqrt{n} \sum_{p\in\mathcal{P}} \Big(\big((1-\overline{\psi}_p)h(\delta_p^j)\overline{\delta}_{g',p} - \overline{\psi}_p h'(\delta_p^j)\overline{\delta}_{g,p}\big)\big(\overline{\lambda}_{\tau_p-}^{(1)\top}\Delta_p^{n-}W + \overline{\lambda}_{\tau_p}^{(1)\top}\Delta_p^{n+}W\big)$$
$$+h'(\delta_p^j)\overline{\delta}_{g,p}\big(\lambda_{\tau_p-}^{j,\top}\Delta_p^{n-}W + \lambda_{\tau_p}^{j,\top}\Delta_p^{n+}W + \sigma_{\tau_p-}^j\Delta_p^{n-}\widetilde{W}^j + \sigma_{\tau_p}^j\Delta_p^{n+}\widetilde{W}^j\big)\Big)$$

$$\widetilde{Y}_{\mathcal{P}}^{n,N,j} = \sqrt{n} \sum_{p\in\mathcal{P}} \sum_{m=1}^{7} y_{p,m}^{n,N,j}, \qquad \text{where}$$

$$y_{p,1}^{n,N,j} = h(\delta_p^j)\, \tfrac{1}{N} \sum_{k=1}^{N} \Big(\big(g'(\delta_p^k)(\lambda_{\tau_p-}^k - \overline{\psi}_p\overline{\lambda}_{\tau_p-}^{(1)}) - (1-\overline{\psi}_p)\overline{\delta}_{g',p}\overline{\lambda}_{\tau_p-}^{(1)}\big)^\top\Delta_p^{n-}W$$
$$+\big(g'(\delta_p^k)(\lambda_{\tau_p}^k - \overline{\psi}_p\overline{\lambda}_{\tau_p}^{(1)}) - (1-\overline{\psi}_p)\overline{\delta}_{g',p}\overline{\lambda}_{\tau_p}^{(1)}\big)^\top\Delta_p^{n+}W\Big)$$

$$y_{p,2}^{n,N,j} = h'(\delta_p^j)\,\overline{\delta}_{g,p}\big(\Delta_{i_{n,p}}^n X^{(\sigma),j} - \sigma_{\tau_p-}^j\Delta_p^{n-}\widetilde{W}^j - \sigma_{\tau_p}^j\Delta_p^{n+}\widetilde{W}^j\big)$$

$$y_{p,3}^{n,N,j} = h'(\delta_p^j)\,\overline{\delta}_{g,p}\big(\widehat{X}_p^{(\lambda),j} - (\lambda_{\tau_p-}^j - \overline{\psi}_p\overline{\lambda}_{\tau_p-}^{(1)})^\top\Delta_p^{n-}W - (\lambda_{\tau_p}^j - \overline{\psi}_p\overline{\lambda}_{\tau_p}^{(1)})^\top\Delta_p^{n+}W\big)$$

$$y_{p,4}^{n,N,j} = h(\delta_p^j)\,\frac{1}{N}\sum_{k=1}^{N} g'(\delta_p^k)\big(\widehat{X}_p^{(\lambda),k} - (\lambda_{\tau_p-}^k - \overline{\psi}_p\overline{\lambda}_{\tau_p-}^{(1)})^\top\Delta_p^{n-}W - (\lambda_{\tau_p}^k - \overline{\psi}_p\overline{\lambda}_{\tau_p}^{(1)})^\top\Delta_p^{n+}W\big)$$

$$y_{p,5}^{n,N,j} = h'(\delta_p^j)\overline{\delta}_{g,p}\big(\widehat{X}_p^{(A),j} - \widehat{X}_p^{(\lambda),j} - \Delta_{i_{n,p}}^n X^{(\sigma),j}\big)$$

$$y_{p,6}^{n,N,j} = h(\delta_p^j)\,\frac{1}{N}\sum_{j=1}^{N}\big(g'(\delta_p^{N,k}) - g'(\delta_p^k)\big)\,\widehat{X}_p^{(\lambda),k} + h'(\delta_p^j)\,\widehat{X}_p^{(A),j}\,\Big(\frac{1}{\sum}{}_{j=1}^{N} g(\delta_p^{N,k}) - \overline{\delta}_{g,p}\Big)$$

$$y_{p,7}^{n,N,j} = \big(h(\delta_p^{N,j}) - h(\delta_p^j)\big)\frac{1}{N}\sum_{k=1}^N g'(\delta_p^{N,k})\widehat{X}_p^{(\lambda),k} + \big(h'(\delta_p^{N,j}) - h'(\delta_p^j)\big)\widehat{X}_p^{(A),j}\,\frac{1}{N}\sum_{k=1}^N g(\delta_p^{N,k}).$$

2) As in Lemma 13, there are $\mathcal{N}(0,1)$ variables $Z_p^{-,l}$ and $Z_p^{+,l}$ (for $l=1,\dots,K+1$ here) and $[0,1]$-uniform variables κ_p defined on an extension of the space $(\Omega,\mathcal{F},\mathbb{P})$, all mutually independent and also independent of $\mathcal{F}$, such that with the convention $\Delta_p^{n\pm}W^{K+1} = \Delta^{n\pm}\widetilde{W}^j$, we have

$$\begin{aligned}&\tfrac{1}{\sqrt{\Delta_n}}\,\big((\Delta_p^{n-}W^l)_{1\le l\le K+1},(\Delta_p^{n+}W^l)_{1\le l\le K+1}\big)_{p\ge1}\\ &\qquad\qquad\stackrel{\mathcal{L}-s}{\longrightarrow}\ \big(\sqrt{\kappa_p}(Z_p^{-,l})_{1\le l\le K+1},\ \sqrt{1-\kappa_p}(Z_p^{+,l})_{1\le l\le K+1}\big)_{p\ge1}.\end{aligned}$$

It easily follows that the pair $(\mathcal{Y}_{\mathcal{P}'}^{n,N,j},\mathcal{Y}_{\mathcal{P}''}^{n,N,j})$ converges stably in law to $(\sqrt{T}\,\overline{Z}',\sqrt{T}\,\overline{Z}'')$, where

$$\begin{aligned}&\overline{Z}' = \sum_{p\in\mathcal{P}'}\big(\sqrt{\kappa_p}\,\overline{Z}_p'^{-} + \sqrt{1-\kappa_p}\,\overline{Z}_p'^{+}\big), \qquad\qquad \overline{Z}'' = \sum_{p\in\mathcal{P}''}\big(\sqrt{\kappa_p}\,\overline{Z}_p''^{-} + \sqrt{1-\kappa_p}\,\overline{Z}_p''^{+}\big)\\ &\overline{Z}_p'^{\pm} = h'(\delta_p^j)\overline{\delta}_{g,p}\sigma_{\tau_p\pm}^j Z_p^{\pm,K+1} + \sum_{l=1}^K\Big(h'(\delta_p^j)\overline{\delta}_{g,p}\lambda_{\tau_p\pm}^{j,l} + \big((1-\psi)h(\delta_p^j)\overline{\delta}_{g',p} - \psi h'(\delta_p^j)\overline{\delta}_{g,p}\big)\overline{\lambda}_{\tau_p\pm}^{(1),l}\big)Z_p^{\pm,l}\Big)\\ &\overline{Z}_p''^{\pm} = h'(\delta_p^j)\overline{\delta}_{g,p}\sigma_{\tau_p\pm}^j Z_p^{\pm,K+1} + \sum_{l=1}^K\Big(h'(\delta_p^j)\overline{\delta}_{g,p}\lambda_{\tau_p\pm}^{j,l} + h(\delta_p^j)\overline{\delta}_{g',p}\overline{\lambda}_{\tau_p\pm}^{(1),l}\big)Z_p^{\pm,l}\Big).\end{aligned}$$

As in Lemma 13 again, one can write $\overline{Z}_p'^{\pm}$ and $\overline{Z}_p''^{\pm}$ as $\Gamma_{g,h,p}'^{\pm,j}Z_p^{\pm}$ and $\Gamma_{g,h,p}''^{\pm,j}Z_p^{\pm}$, where all Z_p^+,Z_p^- are standard normal variables mutually independent and also independent of $\mathcal{F}$, Z, Z' and all κ_p. We then deduce the claimed stable convergence in law of $(\mathcal{Y}_{\mathcal{P}'}^{n,N,j},\mathcal{Y}_{\mathcal{P}''}^{n,N,j})$.

3) The variables $\widetilde{Y}_{\mathcal{P}'}^{n,N,j},\widetilde{Y}_{\mathcal{P}''}^{n,N,j}$ are clearly $\overline{\mathcal{J}}$-measurable, so it remains to prove $\sqrt{n}\,y_{p,m}^{n,N,j}\stackrel{\mathbb{P}}{\longrightarrow}0$ for each $p=1,\dots,P$ and $m=1,\dots,7$. By exactly the same arguments as in Lemma 13 (even simpler in some cases), this holds for $m=1,2,3,4,6,7$.

Finally, for the case of $m=5$ we use (41), which yields $\mathbb{E}\big(|\widehat{X}_p^{(A),j} - \widehat{X}_p^{(\lambda),j} - \widehat{X}_p^{(\sigma),j}|^w\big)\le C\Delta_n$. We also have $\mathbb{E}\big(|\Delta_{i_{n,p}}^n\overline{X}^{N,(\sigma)}|^2\big)\le C\Delta_n/N$ because the processes $\widetilde{W}^j$'s are independent. Recalling $w<2$, we deduce $\mathbb{E}\big(|\widehat{X}_p^{(A),j} - \widehat{X}_p^{(\lambda),j} - \Delta_{i_{n,p}}^n X^{(\sigma),j}|^w\big)\le C(\Delta_n + (\Delta_n/N)^{w/2})$. Since $h'(\delta_p^j)\overline{\delta}_{g,p}$ has finite moments of all orders and $w>1$, by Hölder's inequality we deduce $\sqrt{n}\,y_{p,5}^{N,n}\stackrel{\mathbb{P}}{\longrightarrow}0$. $\square$

Proof of (a) of (52)**.** Below we let f be either one of g, g', h, h', $\mathbb{I}$, with g,h as in (52). In restriction to the set $\overline{\Omega}_n'$ of (46) and if $p\in\{1,\dots,P\}$, we have with the notation (45):

$$\breve{f}_{i_{n,p}}^j = f(\delta_p^{N,j} + \widehat{X}_p^{(A),j}) - \frac{1}{2}f(\Delta_{i_{n,p}-1}^n X^{(A),N,j}) - \frac{1}{2}f(\Delta_{in,p+1}^n X^{(A),N,j}).$$

Since $|f(x+y)-f(x)|\le C|y|(1+|x|+|y|)$, by (41) and $\psi\overline{\delta}_{\mathbb{I},p}^N\stackrel{\text{a.s.}}{\longrightarrow}0$, plus the bound $\mathbb{E}(|\delta_p^k|^q)\le C_q$ for any $q>0$, one easily deduces that $\sup_j\mathbb{E}\big(|\breve{f}_{i_{n,p}}^j - f(\delta_p^j)|^q\big)\to0$. Using further (3) and $\mathbb{P}(\overline{\Omega}_n')\to1$, plus $\breve{a}(f)_i = \frac{1}{N}\sum_{j=1}^N\breve{f}_i^j$, it then follows that

$$\mathbb{E}\big(|\breve{a}(f)_{i_{n,p}} - \overline{\delta}_{f,p}|^q\big)\to0, \qquad \sup_{j\ge1}\mathbb{E}\big(|\breve{f}_{i_{n,p}}^j - f(\delta_p^j)|^q\big)\to0. \tag{60}$$

Thus, if

$$
\begin{aligned}
&\widehat{V}'^{*} = \tfrac{1}{N}\sum_{j=1}^{N}\widetilde{c}'_j, && \widetilde{c}'_j := 4\Big(\sum_{p\in\mathcal{P}'}\overline{\delta}_{g,p}\big(g(\delta_p^j)\big)\Big)^2-\overline{\delta}_{g,p}-\psi\overline{\delta}_{g',p}\delta_p^j\big)\\
&\widehat{V}''^{*} = \tfrac{1}{N}\sum_{j=1}^{N}\widetilde{c}''_j, && \widetilde{c}''_j := 4\Big(\sum_{p\in\mathcal{P}''}\overline{\delta}_{g,p}\big(g(\delta_p^j)-\overline{\delta}_{g,p}\big)\Big)^2\\
&\widehat{V}'^{*j} = \tfrac{1}{N-1}\sum_{k\in L_N^j}\widetilde{c}'^{j}_k, && \widetilde{c}'^{j}_k = \Big(\sum_{p\in\mathcal{P}'}\big(\breve{h}_i^j\big(g(\delta_p^k)-\overline{\delta}_{g,p}\big)-\psi\big(\breve{h}_i^k\overline{\delta}_{g',p}+\breve{h}'^{k}_i\overline{\delta}_{g,p}\big)\mathbb{I}_p^k\big)\Big)^2\\
&\widehat{V}''^{*j} = \tfrac{1}{N-1}\sum_{k\in L_N^j}\widetilde{c}''^{j}_k, && \widetilde{c}''^{j}_k = \Big(\sum_{p\in\mathcal{P}''}\big(\breve{h}_i^j\big(g(\delta_p^k)-\overline{\delta}_{g,p}\big)\Big)^2,
\end{aligned}
$$

we deduce from (60) that

$$
\mathbb{E}\big(|\widehat{V}'_g-\widehat{V}'^{*}|+|\widehat{V}''_g-\widehat{V}''^{*}|+|\widehat{V}'^{j}_{g,h}-\widehat{V}'^{*j}|+|\widehat{V}''^{j}_{g,h}-\widehat{V}''^{*j}|\big)\to 0.
$$

The variables $\widetilde{c}'_j$ and $\widetilde{c}''_j$ for $j\geq 1$, resp. $\widetilde{c}'^{j}_k$ and $\widetilde{c}''^{j}_k$ for $k\in L_N^j$, are i.i.d. under $\mathbb{P}_{\mathcal{J}}$ with means V'_g and V''_g, resp. $V'^{j}_{g,h}$ and $V''^{j}_{g,h}$. So the law of large numbers implies the claim. $\square$

Proof of (b) and (c) of (52)**.** 1) We focus on $\widehat{\Gamma}^{-}_{g,i_{n,p}}$ and $\widehat{\Gamma}'^{-,j}_{g,h,i_{n,p}}$, the other cases being similar. (60) implies $\breve{a}(f)_{i_{n,p}}\xrightarrow{\mathbb{P}}\overline{\delta}_{f,p}$ for $f=g,g'$ and $\breve{f}^j_{i_{n,p}}\xrightarrow{\mathbb{P}}f(\delta_p^j)$ for $f=h,h'$. Thus, with $U=\overline{\delta}_{g,p}\overline{\delta}_{g',p}$ and $V=0$ in the case of $\widehat{\Gamma}^{-}_{g,i_{n,p}}$ and with $U=(1-\psi)\overline{\delta}_{g',p}h(\delta_p^j)$ and $V=\overline{\delta}_{g,p}\breve{h}'^{j}_p$ in the case of $\widehat{\Gamma}'^{-,j}_{g,h,i_{n,p}}$, upon comparing (23) and (28) with (31) and using $T=n\Delta_n$ it is clearly enough to show that

$$
\frac{1}{m_n\Delta_n}\sum_{i'\in M^-_{n,i_{n,p}}}\Big(U\frac{1}{N}\sum_{k=1}^{N}\Delta_{i'}^{n,\mathcal{T}}X^k+V\,\Delta_{i'}^{n,\mathcal{T}}X^j\Big)^2\xrightarrow{\mathbb{P}}\|U\overline{\lambda}^{(1)}_{\tau_p-}+V\lambda^j_{\tau_p-}\|^2+V^2(\sigma^j_{\tau_p-})^2.
$$

In restriction to $\overline{\Omega}'_n$ and with the short-hand notation $i=i_{n,p}$, the left hand side above is

$$
\begin{aligned}
\tfrac{1}{m_n\Delta_n}\sum_{l=1}^{m_n}\Big(&U^2\,\tfrac{1}{N^2}\sum_{k,k'=1}^{N}\Delta^n_{i-l}X^{(A),\mathcal{T},k}\Delta^n_{i-l}X^{(A),\mathcal{T},k'}\\
&+2UV\,\tfrac{1}{N}\sum_{k=1}^{N}\Delta^n_{i-l}X^{(A),\mathcal{T},k}\Delta^n_{i-l}X^{(A),\mathcal{T},j}+V^2(\Delta^n_{i-l}X^{(A),\mathcal{T},j})^2\Big).
\end{aligned}
$$

Hence, since $\mathbb{P}(\overline{\Omega}'_n)\to 1$, it is enough to show that

$$
\begin{aligned}
&\tfrac{1}{m_n\Delta_n}\sum_{l=1}^{m_n}\tfrac{1}{N^2}\sum_{k,k'=1}^{N}\Delta^n_{i-l}X^{(A),\mathcal{T},k}\Delta^n_{i-l}X^{(A),\mathcal{T},k'}\xrightarrow{\mathbb{P}}\Phi_1:=\|\overline{\lambda}^{(1)}_{\tau_p-}\|^2\\
&\tfrac{1}{m_n\Delta_n}\sum_{l=1}^{m_n}\tfrac{1}{N}\sum_{k=1}^{N}\Delta^n_{i-l}X^{(A),\mathcal{T},k}\Delta^n_{i-l}X^{(A),\mathcal{T},j}\xrightarrow{\mathbb{P}}\Phi_2:=(\overline{\lambda}^{(1)}_{\tau_p-})^\top\lambda^j_{\tau_p-}\\
&\tfrac{1}{m_n\Delta_n}\sum_{l=1}^{m_n}(\Delta^n_{i-l}X^{(A),\mathcal{T},j})^2\xrightarrow{\mathbb{P}}\Phi_3:=\|\lambda^j_{\tau_p-}\|^2+(\sigma^j_{\tau_p-})^2.
\end{aligned}
$$

Then, an application of Lemma 6 shows us that this amounts to having

$$
\begin{aligned}
&\tfrac{1}{m_n\Delta_n}\sum_{l=1}^{m_n}\widehat{\Phi}_{l,m}\xrightarrow{\mathbb{P}}\Phi_m\qquad\text{for } m=1,2,3,\text{ where}\\
&\widehat{\Psi}_l^{k,k'}=(\Delta^n_{i-l}X^{(\lambda),k}+\Delta^n_{i-l}X^{(\sigma),k})\,(\Delta^n_{i-l}X^{(\lambda),k'}+\Delta^n_{i-l}X^{(\sigma),k'})\\
&\widehat{\Phi}_{l,1}=\tfrac{1}{N^2}\sum_{k,k'=1}^{N}\widehat{\Psi}_l^{k,k'},\qquad \widehat{\Phi}_{l,2}=\tfrac{1}{N}\sum_{k=1}^{N}\widehat{\Psi}_l^{j,k},\qquad \widehat{\Phi}_{l,3}=\widehat{\Psi}_l^{j,j}.
\end{aligned}
\tag{61}
$$

2) By integration by parts $\widehat{\Psi}_l^{k,k'}$ is the sum of the increment over the interval $I(n,i-l)$ of a martingale, plus the integral over $I(n,i-l)$ of $(\lambda_t^k)^\top\lambda_t^{k'}+(\sigma_t^k)^2\,1_{\{k=k'\}}$. Then (36) implies

$$\mathbb{E}\big(\big|\mathbb{E}_{i-l-1}^{\prime n}(\widehat{\Psi}_l^{k,k'})-(\lambda_{\tau_p-}^k)^\top\lambda_{\tau_p-}^{k'}\Delta_n-(\sigma_{\tau_p-}^k)^2\Delta_n\,1_{\{k=k'\}}\big|\big)\leq\Delta_n\varepsilon_n,$$

for some sequence ε_n going to 0 and independent of k,k'. In turns, this and (iii) of (A2) (with $T_1=\infty$)and (36) again yield, for another sequence ε_n' going to 0 and $m=1,2,3$:

$$\mathbb{E}\big(\big|\mathbb{E}_{i-l-1}^{\prime n}(\widehat{\Phi}_{l,m})-\Phi_m\Delta_n\big|\big)\leq\Delta_n\Big(\varepsilon_n'+\frac{C}{N}\Big).$$

On the other hand, (41) implies $\mathbb{E}\big(|\widehat{\Psi}_l^{k,k'}|^2\big)\leq C\Delta_n^2$, hence $\mathbb{E}\big(|\widehat{\Phi}_{l,m}|^2\big)\leq C\Delta_n^2$ as well. Then (61) follows from a classical result on triangular arrays of variables. $\square$

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