Recalcitrant Betas: Intraday Variation in the Cross-Sectional Dispersion of Systematic Risk*

Torben G. Andersen† Martin Thyrsgaard‡ Viktor Todorov§

November 16, 2020

Abstract

We study the temporal behavior of the cross-sectional distribution of assets' market exposure, or betas, using a large panel of high-frequency returns. The asymptotic setup has the sampling frequency of returns increasing to infinity, while the time span of the data remains fixed, and the cross-sectional dimension of the panel is either fixed or increasing. We derive functional limit results for the cross-sectional distribution of betas evolving over time. We demonstrate, for constituents of the S&P 500 market index, that the dispersion in betas is elevated at the market open and gradually declines over the trading day. This intraday pattern varies significantly over time and reacts to information shocks such as clustered earning announcements and releases of macroeconomic news. We find that earnings news increase beta dispersion while FOMC announcements have the opposite effect on market betas.

Keywords: asset pricing, cross-sectional dispersion, functional convergence, high-frequency data, intraday variation, market beta, nonparametric inference, systematic risk.

JEL classification: C51, C52, G12.

*Research partially supported by NSF grant SES-1530748 and DFF grant 9033-00003B. We would like to thank the CoEditor (Tao Zha), three anonymour referees, participants at various conferences and seminars as well as our discussants Federico Bandi, Walter Distaso and Nancy Xu for helpful comments and suggestions.

†Kellogg School of Management, Northwestern University; NBER; CREATES.

‡Kellogg School of Management, Northwestern University; CREATES.

§Kellogg School of Management, Northwestern University.

1 Introduction

Measuring assets' exposure to systematic risk, including the sensitivity to the overall market, plays a central role in the implementation and testing of asset pricing models. Indeed, measurement error in the estimated betas, i.e., covariances of asset returns with the systematic risk factors, can have nontrivial consequences for assessing the ability of asset pricing models to explain the cross-section of asset prices, see, e.g., Shanken (1992), Jagannathan and Wang (1998), Kan and Zhang (1999), Gospodinov, Kan, and Robotti (2009) and Kleibergen (2009). The standard approach of measuring betas is based on running linear regressions using daily or lower frequency data. This requires the use of periods spanning multiple years to control the size of measurement errors in the estimated betas. However, asset betas change over time, and this can be crucial for applications, i.e., an asset pricing model may hold only conditionally, see, e.g., Hansen and Richard (1987).

One way to accommodate time variation in betas is to model them as functions of observables such as firm characteristics, macro variables and systematic risk factors, see, e.g., Shanken (1990), Jagannathan and Wang (1996), Ferson and Harvey (1999), Connor, Hagmann, and Linton (2012), Gagliardini, Ossola, and Scaillet (2016), among others. An alternative nonparametric approach, following Barndorff-Nielsen and Shephard (2004a) and Andersen, Bollerslev, Diebold, and Wu (2005a,b), is to exploit high-frequency returns. Intraday data hold the promise of attaining the identical precision, but over significantly shorter time windows. The increased liquidity in financial markets renders this approach practically feasible; indeed, Bollerslev, Li, and Todorov (2016) demonstrate the advantage of using high-frequency betas for cross-sectional asset pricing.

Prior studies using intraday data typically ignore any potential intraday variation in betas by averaging across the trading day. They either compute beta as a ratio of the daily integrated covariance of the asset with the systematic factor divided by the daily integrated variance of the systematic factor (as in Barndorff-Nielsen and Shephard (2004a)), or they aggregate betas estimated over local blocks (as in Mykland and Zhang (2009) and Li, Todorov, and Tauchen (2017)) across the day. In either case, the objective is to enhance the precision of the estimates. The premise is that intraday variation in beta is nonexistent or that it is immaterial for asset pricing. Given the extensive evidence of pronounced intraday variation in second-order return moments, it is natural to ask if this assumption is empirically justified. As an illustration, in Figure 1, we plot estimates of the market beta for two large stocks, Caterpillar (CAT) and Johnson & Johnson (JNJ), using high-frequency returns from distinct parts of the trading day. The figure suggests that market betas vary in a highly systematic manner. The betas of both stocks converge strongly towards unity–the cross-sectional population average of market betas–over the course of the trading day.

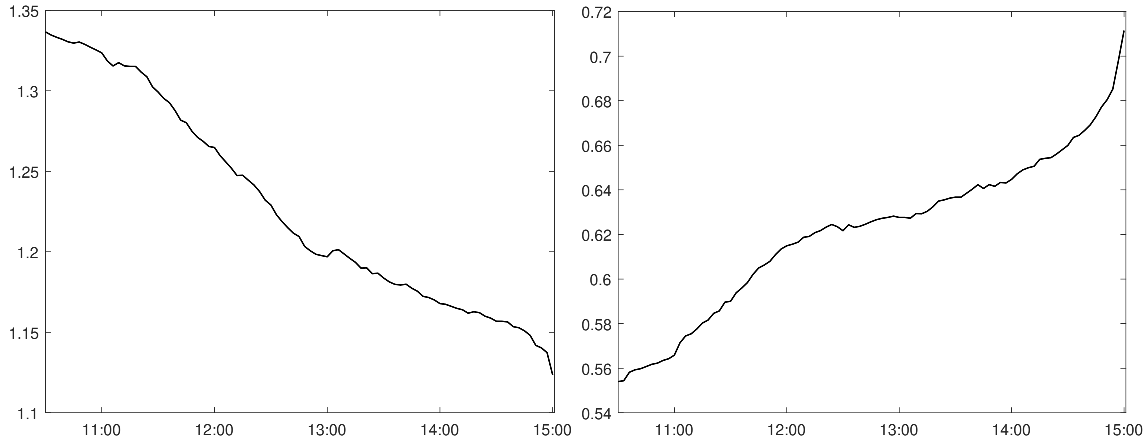

Figure 1: **Intraday Variation in Market Betas.** The betas are computed over local windows of two hours using 3-minute return data, according to equation (18), and averaged over the full sample of 2010-2018. The left panel represents Caterpillar and the right Johnson & Johnson.

More generally, is the evidence for intraday variation in market betas statistically significant in the cross-section of stocks? The goal of the current paper is to develop econometric tools for answering this type of question, characterizing the pattern of intraday variation in betas, and to provide an initial exploration of factors that may rationalize the intraday patterns in market betas.

Figure 1 suggests that, if present, the time variation in market beta takes a particular form, with the cross-sectional dispersion declining across the trading day. Formally, we define the cross-sectional beta dispersion at each point in time as,

$$D_{t,\kappa}^N = \frac{1}{N}\sum_{j=1}^{N}\left(\beta_{t-1+\kappa}^{(j)} - 1\right)^2, \quad t \in \mathbb{N}_+, \quad \kappa \in [0,1]. \tag{1}$$

Here, t indicates the trading day, κ denotes the timing within the trading day (our time unit is one day), and N is the number of assets in the cross-section. We construct estimates of $D_{t,\kappa}^N$ and develop the feasible limit theory needed for formal econometric inference of such objects. Our measures are constructed from a panel of high-frequency returns on a large cross-section of assets. The asymptotic setting is one in which the mesh of the observation grid shrinks to zero while the time span of the data remains fixed. The size of the cross-section may remain fixed or increase along with the sampling frequency. In the latter case, our inference is for the cross-sectional limit of $D_{t,\kappa}^N$ which, in general, is a random quantity.

We form our measure of the cross-sectional dispersion in market beta from the ratio of a local quadratic covariation estimate for the asset and the market return divided by a local estimate for the quadratic return variation of the market. If the size of the cross-section grows asymptotically, then the associated limit distribution, evaluated at a fixed and finite set of points during the trading day (as indicated by the values of κ) is determined solely by the systematic risk factors in the asset returns, where we allow for an arbitrary fixed number (unknown to the econometrician) of latent systematic risk factors beyond the market. In contrast, if the size of the cross-section remains fixed, then the idiosyncratic risks will also impact the limit distribution of our statistic. Importantly, however, our feasible inference procedures are valid both for a fixed or increasing N. Moreover, the inference is conducted in a way that does not require knowledge of the number of systematic factors in the returns or the corresponding factor loadings.

This limit result enables one to compare $D^N_{t,\kappa}$ across a fixed set of distinct values for κ. If the dispersion changes during the day, however, it is natural to view $D^N_{t,\kappa}$ as a function of κ, and ask whether this function varies over time, or on days including prescheduled macroeconomic announcements, when the information flow may differ from that on regular trading days. To address such questions, we develop functional limit results for $D^N_{t,\kappa}$, with $D^N_{t,\kappa}$ viewed as a function of κ. This is challenging, as estimates of $D^N_{t,\kappa}$, as a function of κ, is complicated due to the associated convergence not holding uniformly in κ. Instead, we take advantage of the one-to-one mapping between a function and its Fourier transform. This is convenient, as the Fourier transform of estimates for $D^N_{t,\kappa}$ consists of weighted sums of block estimates for the dispersion over the day, and their limit distribution is mixed Gaussian. Furthermore, we show that the convergence of these estimated Fourier transforms is uniform in their argument, thus delivering a functional convergence result enabling inference about $D^N_{t,\kappa}$ as a function of κ.

We extend the above analysis by developing inference for the full cross-sectional distribution of betas at a given point in time. Specifically, we derive a functional limit result for an estimate of the characteristic function of the cross-sectional market beta distribution at fixed points in time. This limit result can be used to test if there are other changes in the cross-sectional distribution of betas, not fully captured by the cross-sectional beta dispersion. In addition, through Fourier inversion, we may exploit this result to recover the density of the cross-sectional beta distribution at any given point in time nonparametrically (assuming of course that one exists).

We assess the finite sample performance of the new econometric tools through simulations from a model that mimics key features of the data used in our empirical study. The Monte Carlo analysis confirms that our limit theory provides a satisfactory basis for finite sample inference. Our empirical application is based on the constituents of the S&P 500 index over the sample period, 2010-2018. The estimated $D^N_{t,\kappa}$ for the full sample as well as select subsamples strongly reject the hypothesis of a constant cross-sectional dispersion of market betas across the trading day. Consistent with Figure 1, we find the highest beta dispersion at the market open, followed by a gradual decay during the trading day. That is, high/low beta stocks, with betas above/below unity, tend to have downward/upward sloping market beta trajectories across the trading day. This intraday pattern holds throughout the sample, with the decline of the cross-sectional beta dispersion during the trading hours turning even more pronounced towards the latter part of the sample.

The sharp discrepancy in the beta dispersion at the market open and market close points to heterogeneity in the type of information the market is processing across the trading day. Consistent with an information-based explanation, we find that the behavior of $D_{t,\kappa}^{N}$, as a function of κ, changes significantly during active earnings announcement weeks, around prescheduled Federal Open Market Committee (FOMC) meeting announcements, and when the markets are hit by sudden spikes in market-based uncertainty measures. In particular, clusters of company earnings releases enhance the market beta dispersion, reflecting heightened cross-sectional heterogeneity, while macroeconomic and uncertainty shocks have the opposite impact, inducing a contraction in the beta dispersion, suggesting the reaction to such news is more homogenous.

Finally, to conclude, we discuss a potential economic rationalization for the observed intraday beta dispersion pattern. It is inspired by the heterogeneous response to shifts in the relative intensity of different types of information flow. If periods of intense earnings releases induce significant systematic risk due to the anticipated arrival of new information regarding the recent economic performance of distinct sectors, regions, products, and commodities, then the impact is likely to differ in the cross section depending on the exposure of the stocks to the fortunes of different business activities. On the other hand, macroeconomic announcements and news updates speak more directly to the general level of economic activity, and may be associated with a more homogenous response, inducing the observed compression in the beta dispersion through an effect operating primarily through the discount factor. These observations suggest that market risk reflects different underlying factors with distinct cross-sectional implications, i.e., the market beta may contain separate components whose relative strength varies with the economic environment.

The rest of the paper is organized as follows. Section 2 introduces the setup and notation. We provide our measures of cross-sectional dispersion in market betas and develop the associated feasible limit theory in Section 3. In Section 4, we develop tests for changes in the beta dispersion measures both within and across trading days. Section 5 extends the theory by developing inference for the entire cross-sectional beta distribution. We assess the finite-sample properties of the econometric tools via simulations in Section 6. Section 7 contains the empirical analysis and studies the impact of distinct informational shocks. Section 8 concludes and discusses potential rationales for our findings. Assumptions, proofs and additional evidence are collected in Section 9.

2 Setup and Notation

We first introduce the basic setup. We consider a set of stocks, indexed by $j = 1, ..., N$, whose prices are defined on some filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$. The market portfolio is assigned the index $j = 0$. The evolution of these processes is given by,

$$X_t^{(0)} = X_0^{(0)} + \int_0^t \alpha_s^{(0)} ds + \int_0^t \sigma_s^{(0)} dW_s^{(0)} + \sum_{s\leq t} \Delta X_s^{(0)}, \tag{2}$$

$$\begin{aligned} X_t^{(j)} = X_0^{(j)} &+ \int_0^t \alpha_s^{(j)} ds + \int_0^t \beta_s^{(j)} \sigma_s^{(0)} dW_s^{(0)} + \int_0^t \gamma_s^{(j)} dB_s \\ &+ \int_0^t \widetilde{\sigma}_s^{(j)} d\widetilde{W}_s^{(j)} + \sum_{s\leq t} \Delta X_s^{(j)}, \quad j = 1, .., N, \end{aligned} \tag{3}$$

where $W_t^{(0)}$, $(\widetilde{W}_t^{(j)})_{j=1,...,N}$ and B_t are independent Brownian motions, the dimension of B_t is $r \times 1$ and the rest of the Brownian motions are univariate; $\{\alpha_t^{(j)}\}_{j=0,1,...,N}$, $\sigma_t^{(0)}$, $\{\beta_t^{(j)}\}_{j=1,...,N}$, $\{\widetilde{\sigma}_t^{(j)}\}_{j=1,...,N}$ and $\{\gamma_t^{(j)}\}_{j=1,...,N}$ are processes with càdlàg paths, with $\gamma_t^{(j)}$ being $1 \times r$ dimensional, and the remainder are scalar-valued functions of time. Finally, for any process Y, $\Delta Y_t = Y_t - Y_{t-}$ denotes the size of a jump at time t.

The technical assumptions regarding the processes appearing in equations (2)-(3) are provided in the appendix. We briefly discuss on them here. There are two main sets of assumptions. The first (Assumption A) concerns the dynamics of the stochastic processes involved in equations (2)-(3). It is largely left unrestricted, but we impose a "smoothness in expectation type condition that is satisfied by most continuous-time models used in prior work. In particular, it is satisfied, when these processes are Itô semimartingales. We also assume that the price jumps are of finite activity.[1] Our second set of assumptions (Assumptions B and C) concerns the convergence in probability of cross-sectional averages of various stochastic processes, evaluated over a fixed time interval. These assumptions are trivially satisfied, when the cross-sectional dimension of the panel is bounded. If the cross-section is asymptotically increasing, they can be verified by appealing to a law of large numbers, provided there is only weak conditional cross-sectional dependence between the relevant summands. Unconditionally, however, we do allow these quantities to exhibit strong dependence.

The specification (2)-(3) is an exact factor model for returns, albeit with the cross-sectional jump dependence left unrestricted, and it nests many existing cross-sectional asset pricing models. In particular, we allow for an arbitrary number of systematic factors and factor loadings that may be time-varying. Apart for the market index, the systematic factors are latent and their number r is

[1] This is mainly for convenience. We suspect that our results can be extended to allow for infinite variation jumps as well.

unknown to the econometrician. Similarly, except for regularity type conditions, the time-varying factor loadings and the time-varying stochastic volatilities are left unrestricted. Idiosyncratic diffusive risk in asset prices is captured by the independent Brownian motions $(\widetilde{W}_t^{(j)})_{j=1,...,N}$. Likewise, the dependence structure of the jumps in the cross-section is also unconstrained. If asset price jumps have the identical beta with respect to market jumps as the diffusive beta, $\beta_t^{(j)}$, they may be included in the inference. We do not impose this restriction, so we eliminate returns containing (identified) jumps, when constructing our statistics. Nonetheless, in Section 9.2, we document that including jumps in the inference does not alter any of our qualitative empirical findings.

We focus on the evolution of the cross-sectional distribution of market betas over time. Formally, the spot market beta is defined as a ratio of the spot diffusive covariance between the asset and the market and the spot diffusive market variance. Denoting the continuous part of the (predictable) quadratic variation for two semimartingales X and Y by $\langle X, Y\rangle^c$ (see, e.g., Section I.4 in Jacod and Shiryaev (2003)), the market betas in the model (2)-(3) are given by,

$$\beta_t^{(j)} = \frac{d\langle X^{(j)}, X^{(0)}\rangle_t^c}{d\langle X^{(0)}, X^{(0)}\rangle_t^c}, \quad t \in \mathbb{R}_+, \quad j = 1, ..., N. \tag{4}$$

Inference for spot volatility from high-frequency data in various settings is considered by Foster and Nelson (1996), Bandi and Phillips (2003), Fan and Wang (2008), Kristensen (2010) and Liu, Liu, and Liu (2018), while inference for the covariance (and beta) is studied in Ang and Kristensen (2012), Bibinger and Reiß (2014) and Bibinger, Hautsch, Malec, and Reiß (2019) among others. We build on this work and study the cross-sectional distribution of beta and its behavior as a function of time. Of course, the analysis that follows can be trivially extended to study the cross-sectional distribution of factor loadings with respect to other (observable) systematic risk factors.

3 Inference for the Cross-Sectional Dispersion of Market Betas

This section introduces our cross-sectional dispersion measures for market betas and derives an associated feasible limit theory. The inference is based on discrete observations of $\{X^{(j)}\}_{j=0,1,...,N}$ at equidistant times $0, \frac{1}{n}, \frac{2}{n}, ..., T$, where T refers to the time span of our data, which is fixed throughout, the integer n denotes the number of times we sample within a unit interval, $\Delta_n = 1/n$ signifies the length of the sampling interval, and the high-frequency increment of $X^{(j)}$ is given by,

$$\Delta_{t,i}^n X^{(j)} = X_{(t-1)+i/n}^{(j)} - X_{(t-1)+(i-1)/n}^{(j)}, \quad t \in \mathbb{N}_+, \quad i = 1, ..., n, \ j = 0, 1, ..., N. \tag{5}$$

As noted in the introduction, we measure time in units of one day. The notation above, therefore, implicitly assumes that we sample at high frequency throughout the full day. In practice, of course, the day consists of an active trading session and an overnight period with no, or only limited, trading. We may accommodate this feature by splitting the unit interval $[t-1,t]$ into a trading part $[t-1,t-1+\overline{\kappa}]$ and an overnight period $[t-1+\overline{\kappa},t]$, for some $\overline{\kappa} \in (0,1]$. We may then assume that we utilize high-frequency observations only during the interval $[t-1,t-1+\overline{\kappa}]$, i.e., that we sample at $t-1, t-1+\frac{1}{n}, ..., t-1+\frac{\lfloor \overline{\kappa} n \rfloor}{n}$, for $t = 1, ..., \lfloor T \rfloor$. However, to simplify notation, henceforth, we set $\overline{\kappa} = 1$, so we exclude the overnight periods. It is evident that our results can be trivially extended to allow for $\overline{\kappa} < 1$.

3.1 Estimates of the Market Beta Dispersion

We start by forming estimates for the market beta dispersion at any given point in time. In constructing our statistics, we rely on a standard truncation approach (see, e.g., Mancini (2001, 2009), Jacod and Protter (2012)) to eliminate the jumps in the (partially) observed sample path of the assets. More specifically, we truncate the increments of $X^{(k)}$ via,

$$\nu_{t,n}^{(j)} \asymp \Delta_n^{\varpi}, \quad \varpi \in (0,1/2), \qquad t \in \mathbb{N}_+, \qquad j = 0,1,...,N. \tag{6}$$

In the Monte Carlo study and the empirical analysis we provide additional detail on the particular choice of $\nu_{t,n}^{(j)}$. Exploiting the above notation, we define the sets,

$$\mathcal{A}_{t,i}^{(j)} = \{|\Delta_{t,i}^n X^{(j)}| \leq \nu_{t,n}^{(j)}\}, \quad \mathcal{A}_{t,i}^{(j,l)} = \{|\Delta_{t,i}^n X^{(j)}| \leq \nu_{t,n}^{(j)} \ \cap \ |\Delta_{t,i}^n X^{(l)}| \leq \nu_{t,n}^{(l)}\}, \tag{7}$$

for $j,l = 0,1,...,N$. Our statistics will be computed on these sets.

To construct an estimate for the cross-sectional dispersion of betas at a given point in time, $D_{t,\kappa}^N$, we must account for the fact that both the market beta and the asset volatilities are stochastic and change over time. As a result, we construct our measures on an interval whose length shrinks as we sample more frequently. Specifically, we use a local window of k_n high-frequency increments, for a sequence k_n satisfying,

$$k_n \asymp n^{\varrho}, \quad \varrho \in (0,1). \tag{8}$$

The sets of indices for the price increments over which we compute our measures on a given trading day is then denoted,

$$\mathcal{I}_\kappa^n \ = \ \{\lfloor \kappa n \rfloor - k_n + 1, ..., \lfloor \kappa n \rfloor\}, \quad \widetilde{\mathcal{I}}_\kappa^n \ = \ \{\lfloor \kappa n \rfloor - k_n + 2, ..., \lfloor \kappa n \rfloor\}, \quad \kappa \in [0,1]. \tag{9}$$

To compute market betas and their cross-sectional dispersion, we require estimates for the quadratic (co-)variation. The continuous part of the quadratic variation for an asset is estimated by,

$$\widehat{V}_{t,\kappa}^{(j,l)} \ = \ \frac{n}{|\mathcal{I}_\kappa^n|} \sum_{i \in \mathcal{I}_\kappa^n} (\Delta_{t,i}^n X^{(j)})^2 \, 1_{\{\mathcal{A}_{t,i}^{(j,l)}\}}, \quad j,l = 0,1,...,N, \quad t \in \mathbb{N}_+, \kappa \in [0,1]. \tag{10}$$

If the objective is to measure the continuous quadratic variation of asset j, it is natural to set $l = j$ in $\widehat{V}_{t,\kappa}^{(j,l)}$. However, for measuring beta, it makes more sense to compute the quadratic variation and covariation used in its construction on the identical set of increments. Of course, the jump times of a semimartingale on a finite interval are of Lebesgue measure zero, and hence whether we set $l = j$ or $l \neq j$ makes no difference for the asymptotic analysis that follows.

We next define a measure for the difference between the asset covariation with the market and the market variation as follows,

$$\begin{aligned}\widehat{C}_{t,\kappa}^{(j)}(2) \;=\; \frac{1}{2}\,\frac{n}{|\widetilde{\mathcal{I}}_\kappa^n|} \sum_{i\in\widetilde{\mathcal{I}}_\kappa^n} &\big[(\Delta_{t,i-1}^n X^{(j)} + \Delta_{t,i}^n X^{(j)} - \Delta_{t,i-1}^n X^{(0)} - \Delta_{t,i}^n X^{(0)}) \\ &\times\; (\Delta_{t,i-1}^n X^{(0)} + \Delta_{t,i}^n X^{(0)})\, 1_{\{\mathcal{A}_{t,i-1}^{(j,0)} \,\cap\, \mathcal{A}_{t,i}^{(j,0)}\}}\big], \quad t \in \mathbb{N}_+, \quad \kappa \in [0,1],\end{aligned} \tag{11}$$

where $j = 1, ..., N$. Note that we use a coarser frequency in the construction of $\widehat{C}_{t,\kappa}^{(j)}(2)$, as we double the length of the high-frequency increments used in $\widehat{C}_{t,\kappa}^{(j)}(2)$ compared to those for $\widehat{V}_{t,\kappa}^{(j,l)}$ (the use of an even coarser grid for the covariance measure is readily accommodated). The reason is twofold. First, from a theoretical perspective, the coarser frequency avoids a potential "degeneracy." If we were to use the identical frequency for $\widehat{C}_{t,\kappa}^{(j)}(2)$ and $\widehat{V}_{t,\kappa}^{(j,l)}$ then, in the absence of other systematic risk factors beyond the market return, the leading term for the error of our estimate of the dispersion measure will stem from the time variation in betas and stochastic volatilities over the local window. The behavior of this error is difficult to characterize in our general nonparametric setup. Given existing empirical evidence for multiple systematic risk factors in asset returns, the degenerate scenario is not likely to be of practical relevance. Nonetheless, we define our statistics such that this issue is avoided, with the cost being a slight loss of efficiency in our estimator if the market is the sole systematic risk factor. Second, from a practical perspective, the use of a coarser frequency in $\widehat{C}_{t,\kappa}^{(j)}(2)$ helps minimize the impact of potential asynchroneity effects on the statistics due to the lower liquidity of individual assets relative to the market index.[2]

Since $\widehat{C}_{t,\kappa}^{(j)}(2)$ is an estimate of $(\beta_{t+\kappa}^{(j)} - 1)(\sigma_{t+\kappa}^{(0)})^2$ and $\widehat{V}_{t,\kappa}^{(0,j)}$ is an estimate of $(\sigma_{t+\kappa}^{(0)})^2$, our local measure of cross-sectional market beta dispersion, $\widehat{D}_{t,\kappa}^N$, is simply given by,

$$\widehat{D}_{t,\kappa}^N \;=\; \frac{1}{N}\sum_{j=1}^N \left(\left(\frac{\widehat{C}_{t,\kappa}^{(j)}(2)}{\widehat{V}_{t,\kappa}^{(0,j)}} \right)^2 1_{\{\widehat{V}_{t,\kappa}^{(0,j)} > \alpha_n\}} \right), \quad \kappa \in [0,1], \tag{12}$$

[2] Asynchronicity in trading times induces a downward bias in the estimated covariances, known as the Epps effect (Epps, 1979). A number of solutions have been proposed to deal with it, see, e.g., Hayashi and Yoshida (2005), Christensen, Kinnebrock, and Podolskij (2010), Zhang (2011), Barndorff-Nielsen, Hansen, Lunde, and Shephard (2011), Bibinger (2011), Bibinger (2012), Varneskov (2016), Koike (2016), and recently Bandi, Pirino, and Renó (2017). As we show in Section 9.2, however, for the frequency used in our empirical analysis, the effect of asynchronicity in trading times for the stocks and market index is negligible, and it has no impact on our empirical findings.

where

$$\alpha_n \asymp 1/\log(n). \tag{13}$$

Because the estimate of the beta dispersion involves division by $\widehat{V}_{t,\kappa}^{(0,j)}$, which is not bounded away from zero in finite samples, we impose a lower bound for the local estimate of market variance. This bound does not have an asymptotic effect on the statistic or its limit distribution.

It is easy to show (and it also follows from our convergence results in the next section) that as $n \to \infty$ and $N \to \overline{N}$, for some $\overline{N} \in (0, \infty]$, we have the following convergence in probability under Assumption A, given in the appendix,

$$\widehat{D}_{t,\kappa}^{N} - D_{t,\kappa}^{N} \xrightarrow{\mathbb{P}} 0. \tag{14}$$

We note that we allow for the size of the cross-section either to remain fixed or to increase as we sample more frequently. In the latter case, $D_{t,\kappa}^{N}$ typically converges, under suitable assumptions, to a cross-sectional limit, which may still be a random number. Allowing for $\overline{N}$ to be finite or infinite is convenient from an empirical point of view, as one need not take a stand on whether the cross-sectional dimension of the return panel is sufficiently large relative to the sampling frequency to justify one finite-sample approximation versus another.

We next introduce a few measures that are used in the construction of a bias-correction term for $\widehat{D}_{t,\kappa}^{N}$ as well as an estimate for its asymptotic variance. Towards this end, we split the set $\mathcal{I}_\kappa^n$ into the following two "even" and "odd" subsets,

$$\mathcal{E}_\kappa^n = \left\{ \lfloor \kappa n \rfloor - 2\lfloor \frac{k_n - 1}{2} \rfloor, \lfloor \kappa n \rfloor - 2\lfloor \frac{k_n - 3}{2} \rfloor, \ldots, \lfloor \kappa n \rfloor - 2, \lfloor \kappa n \rfloor \right\}, \quad \mathcal{O}_\kappa^n = \mathcal{I}_\kappa^n \setminus \mathcal{E}_\kappa^n, \tag{15}$$

and define the following two measures for $j = 1, ..., N$,

$$\breve{C}_{t,\kappa}^{(j)} = \frac{n}{|\mathcal{I}_\kappa^n|} \sum_{i \in \mathcal{I}_\kappa^n} \left[(\Delta_{t,i}^n X^{(j)} - \Delta_{t,i}^n X^{(0)}) \, \Delta_{t,i}^n X^{(0)} \, 1_{\{\mathcal{A}_{t,i}^{(j,0)}\}} \left(1_{\{i \in \mathcal{O}_\kappa^n\}} - 1_{\{i \in \mathcal{E}_\kappa^n\}}\right) \right], \tag{16}$$

$$\breve{V}_{t,\kappa}^{(0,j)} = \frac{n}{|\mathcal{I}_\kappa^n|} \sum_{i \in \mathcal{I}_\kappa^n} \left[(\Delta_{t,i}^n X^{(0)})^2 \, 1_{\{\mathcal{A}_{t,i}^{(j,0)}\}} \left(1_{\{i \in \mathcal{O}_\kappa^n\}} - 1_{\{i \in \mathcal{E}_\kappa^n\}}\right) \right]. \tag{17}$$

The motivation behind these definitions of the two statistics is as follows. By subtracting consecutive summands in $\breve{C}_{t,\kappa}^{(j)}$ and $\breve{V}_{t,\kappa}^{(0,j)}$, we cancel their (conditional) mean which, although random, is locally constant (i.e., approximately constant over the short time window over which $\widehat{C}_{t,\kappa}^{(j)}(2)$ and $\widehat{V}_{t,\kappa}^{(0,j)}$ are computed). After this centering, the consecutive summands are approximately independent and possess a symmetric distribution. Therefore, $\breve{C}_{t,\kappa}^{(j)}$ and $\breve{V}_{t,\kappa}^{(0,j)}$ can be used to measure the variability in $\widehat{C}_{t,\kappa}^{(j)}(2)$ and $\widehat{V}_{t,\kappa}^{(0,j)}$. This is very convenient for our setup, in which the size of the cross-section N can increase to infinity asymptotically, as we avoid estimation of the additional systematic factors driving the asset returns as well as the loadings on them. Similarly, we do not need to make an assumption or impose a bound on the number of systematic factors driving the returns, which is typically the case when making inference for factor models.

This approach of computing the asymptotic variance is reminiscent of the so-called observed asymptotic variance, proposed recently by Mykland and Zhang (2017), although there are a couple of nontrivial differences. First, in our setting N may be asymptotically increasing, which complicates matters, because of the (unknown) factor structure in the returns. Second, we use statistics from "even" and "odd" increments in generating the "observed asymptotic variance," unlike Mykland and Zhang (2017), who use successive blocks. Our procedure, therefore, generates less of a bias in situations where the estimand may vary across the local block (which will be the case for our dispersion measures).

Finally, in the construction of the bias-correction term as well as the asymptotic variance, we need a local estimate of the market betas. We use the following,

$$\widehat{\beta}_{t,\kappa}^{(j)} = \frac{\sum_{i \in \mathcal{I}_\kappa^n} \Delta_{t,i}^n X^{(j)} \Delta_{t,i}^n X^{(0)} 1_{\{\mathcal{A}_{t,i}^{(j,0)}\}}}{\sum_{i \in \mathcal{I}_\kappa^n} (\Delta_{t,i}^n X^{(0)})^2 1_{\{\mathcal{A}_{t,i}^{(j,0)}\}}}, \quad j = 1, ..., N, \tag{18}$$

and note that, as in the construction of $\widehat{D}_{t,\kappa}^N$, we could use a coarser frequency for computing the covariation in the numerator. An estimate for the sample variance of $\widehat{\beta}_{t,\kappa}^{(j)}$ is given by,

$$\widehat{V}_{\beta,t,\kappa}^{(j)} = \frac{\sum_{i \in \mathcal{I}_\kappa^n} \left[(\Delta_{t,i}^n X^{(0)})^2 (\Delta_{t,i}^n X^{(j)} - \widehat{\beta}_{t,\kappa}^{(j)} \Delta_{t,i}^n X^{(0)})^2 1_{\{\mathcal{A}_{t,i}^{(j,0)}\}} \right]}{\left(\sum_{i \in \mathcal{I}_\kappa^n} (\Delta_{t,i}^n X^{(0)})^2 1_{\{\mathcal{A}_{t,i}^{(j,0)}\}} \right)^2}, \quad j = 1, ..., N. \tag{19}$$

The nonlinear transformation of $\widehat{C}_{t,\kappa}^{(j)}(2)$ and $\widehat{V}_{t,\kappa}^{(0,j)}$ in the construction of the dispersion measure $\widehat{D}_{t,\kappa}^N$ introduces an (upward) bias of asymptotic order $O_p(1/k_n)$. It may be estimated via,

$$\widehat{B}_{t,\kappa}^N = \frac{1}{N} \sum_{j=1}^N \left(\left[-\frac{(\widehat{\beta}_{t,\kappa}^{(j)} - 1)^2 - \widehat{V}_{\beta,t,\kappa}^{(j)}}{(\widehat{V}_{t,\kappa}^{(0,j)})^2} (\breve{V}_{t,\kappa}^{(0,j)})^2 + \frac{3}{2(\widehat{V}_{t,\kappa}^{(0,j)})^2} (\breve{C}_{t,\kappa}^{(j)})^2 \right] 1_{\{\widehat{V}_{t,\kappa}^{(0,j)} > \alpha_n\}} \right). \tag{20}$$

$\widehat{B}_{t,\kappa}^N$ provides an estimate of the second-order terms in the Taylor expansion of $\widehat{D}_{t,\kappa}^N$, viewed as a function of $\widehat{C}_{t,\kappa}^{(j)}(2)$ and $\widehat{V}_{t,\kappa}^{(0,j)}$. These terms, in turn, depend on the variance of $\widehat{C}_{t,\kappa}^{(j)}(2)$ and $\widehat{V}_{t,\kappa}^{(0,j)}$, which we estimate using $\breve{V}_{t,\kappa}^{(0,j)}$ and $\breve{C}_{t,\kappa}^{(j)}$. Because of the different frequency used in computing $\widehat{C}_{t,\kappa}^{(j)}(2)$ and $\widehat{V}_{t,\kappa}^{(0,j)}$, we need to weight $\breve{V}_{t,\kappa}^{(0,j)}$ and $\breve{C}_{t,\kappa}^{(j)}$ appropriately to account for the different contribution to the bias term of, on the other hand, $W^{(0)}$, and, on the one hand, B and $\{\widetilde{W}^{(j)}\}_{j=1,...,N}$, relative to the case where we rely on the identical frequency in constructing the measures.

Finally, our estimator of the asymptotic variance of $\widehat{D}_{t,\kappa}^{N}$ is given by

$$
\begin{aligned}
\widehat{\text{Avar}}(\widehat{D}_{t,\kappa}^{N}) \;=\; & \frac{3}{2}\left(\frac{1}{N}\sum_{j=1}^{N}\left[\frac{2(\widehat{\beta}_{t,\kappa}^{(j)}-1)}{\widehat{V}_{t,\kappa}^{(0,j)}}\left(\breve{C}_{t,\kappa}^{(j)}-(\widehat{\beta}_{t,\kappa}^{(j)}-1)\breve{V}_{t,\kappa}^{(0,j)}\right)1_{\{\widehat{V}_{t,\kappa}^{(0,j)}>\alpha_n\}}\right]\right)^2 \\
& +\;\frac{1}{2}\left(\frac{1}{N}\sum_{j=1}^{N}\left[\frac{2\left(\widehat{\beta}_{t,\kappa}^{(j)}-1\right)^2-2\widehat{V}_{\beta,t,\kappa}^{(j)}}{\widehat{V}_{t,\kappa}^{(0,j)}}\breve{V}_{t,\kappa}^{(0,j)}\,1_{\{\widehat{V}_{t,\kappa}^{(0,j)}>\alpha_n\}}\right]\right)^2.
\end{aligned}
\tag{21}
$$

The discussion regarding the weights assigned to $\breve{V}_{t,\kappa}^{(0,j)}$ and $\breve{C}_{t,\kappa}^{(j)}$ in the construction of the bias terms above apply here for the construction of $\widehat{\text{Avar}}(\widehat{D}_{t,\kappa}^{N})$ as well.

3.2 Feasible Limit Theory

This section presents our limit results for the market beta dispersion statistic.

3.2.1 Finite Dimensional Convergence

We start with a feasible Central Limit Theorem (CLT) for $\widehat{D}_{t,\kappa}^{N}$ across different fixed points in time. In the theorems below, $\xrightarrow{\mathcal{L}-s}$ denotes stable convergence in law, implying, importantly, that the convergence holds jointly with any $\mathcal{F}$-measurable random variable, see, e.g., Jacod and Shiryaev (2003).

Theorem 1. *Suppose Assumptions A and B hold. Let $n \to \infty$ and $N \to \overline{N}$, for $\overline{N} \in (0,\infty]$, with $\varpi \in (3/8, 1/2)$, $\varrho \in (0, 1/2)$ and $\varrho > 2 - 4\varpi$. For $\mathcal{K}$ an arbitrary finite set of points in $[0,1]$, and $\mathcal{T}$ a finite set of positive integers in $[0,T]$, we have,*

$$
\left\{\frac{\widehat{D}_{t,\kappa}^{N}-\widehat{B}_{t,\kappa}^{N}-D_{t,\kappa}^{N}}{\sqrt{\widehat{Avar}(\widehat{D}_{t,\kappa}^{N})}}\right\}_{t\in\mathcal{T},\,\kappa\in\mathcal{K}} \quad \xrightarrow{\mathcal{L}-s} \quad \{Z_{t,\kappa}\}_{t\in\mathcal{T},\,\kappa\in\mathcal{K}}\,, \tag{22}
$$

where $\{Z_{t,\kappa}\}_{t\in\mathcal{T},\,\kappa\in\mathcal{K}}$ is a sequence of independent standard normal random variables defined on an extension of the original probability space and independent of $\mathcal{F}$.

The limiting result above holds both when N is fixed and when it increases to infinity asymptotically. There is no restriction on the relative growth of the sampling frequency and the size of the cross-section in Theorem 1. The asymptotic distribution of $\widehat{D}_{t,\kappa}^{N}$ is determined solely by the systematic factors in the asset returns when $N \to \infty$. This is because systematic risk, unlike idiosyncratic risk, is not averaged out in a cross-sectional aggregation. On the other hand, when N is fixed, both systematic and idiosyncratic risks determine the limit distribution of $\widehat{D}_{t,\kappa}^{N}$. Importantly, the econometrician does not need to take a stand on which of the two asymptotic setups (N fixed or asymptotically increasing) is at work. Similarly, there is no need to estimate or impose a known upper bound on the number of systematic risk factors in the inference.

The asymptotic variance estimator, $\widehat{\mathrm{Avar}}(\widehat{D}^N_{t,\kappa})$, is of order $O_p(1/k_n)$, which implies that the rate of convergence of $\widehat{D}^N_{t,\kappa}$ is $\sqrt{k_n}$. In addition, the limit distribution of $\widehat{D}^N_{t,\kappa}$ is mixed Gaussian, as the probability limit of $k_n\,\widehat{\mathrm{Avar}}(\widehat{D}^N_{t,\kappa})$, in general, will be a random variable. Finally, the dispersion measures computed at distinct points in time are asymptotically $\mathcal{F}$-conditionally independent.

We remind the reader that the bias-correction term $\widehat{B}^n_{t,\kappa}$ is of order $O_p(1/k_n)$ and since the rate of convergence in Theorem 1 is $\sqrt{k_n}$, this CLT result will continue to hold even if we do not bias-correct. That said, from a practical point of view, the bias-correction term is important, particularly in applications in which k_n is relatively small.

As with other applications involving truncation-based estimates of volatility functionals, it is optimal to pick the truncation parameter ϖ near $1/2$. When we do this, our restriction for the block size is very weak, suggesting a feasible CLT result will apply for a wide range of k_n. Of course, the bigger the (asymptotic) block size, the faster the rate of convergence for the dispersion measure $\widehat{D}^N_{t,\kappa}$.

We conclude this section with a comment regarding a possible extension of the above result. Our model setup allows for latent systematic factors that are orthogonal – in a martingale sense – to $X^{(0)}$. Thus, $\beta^{(j)}_t$ is the local sensitivity at time t of $X^{(j)}$ with respect to the diffusive shocks in $X^{(0)}$. We can also consider inference for local sensitivities of $X^{(j)}$ towards the diffusive shocks in $X^{(0)}$ *after* controlling for exposure to a vector of observable factors. More specifically, suppose that there is an additional vector Y of observable systematic factors, which are orthogonal to the shocks in the latent vector of Brownian motions B, but may interact with $X^{(0)}$. In this case, instead of the quantity $\left(d\langle X^{(0)}, X^{(0)}\rangle^c_t\right)^{-1} d\langle X^{(j)}, X^{(0)}\rangle^c_t$, which we focus on here, one may also be interested in the first element of the vector $(d\langle Z, Z\rangle^c_t)^{-1}\, d\langle Z, X^{(j)}\rangle^c_t$, where $Z_t = (X^{(0)}_t, Y^\top_t)^\top$. This quantity can be estimated at time $t+\kappa$ using exactly the same procedure as above, but with the increments $\Delta^n_{t,i}X^{(0)}$ replaced by $\Delta^n_{t,i}\widetilde{X}^{(0)} = \Delta^n_{t,i}X^{(0)} - \eta^\top_{t+\kappa}\Delta^n_{t,i}Y$, where $\eta_t = (\Sigma^Y_t)^{-1}\Sigma^{(Y,0)}_t$, Σ^Y_t is the spot diffusive variance matrix of Y and $\Sigma^{(Y,0)}_t$ is the spot diffusive covariance between Y and $X^{(0)}$. The CLT of Theorem 1 will obviously still apply for this estimator. Of course, such an estimator is infeasible, because $\eta_{t+\kappa}$ is unobservable. However, this quantity is easy to estimate via linear regression on the basis of the truncated increments:

$$\widehat{\eta}^n_{t,\kappa} = \left(\sum_{i \in \mathcal{I}^n_\kappa} \Delta^n_{t,i} Y \Delta^n_{t,i} Y^\top 1_{\{||\Delta^n_{t,i} Y|| \le \alpha \Delta^\varpi_n\}} \right)^{-1} \times \sum_{i \in \mathcal{I}^n_\kappa} \Delta^n_{t,i} Y \Delta^n_{t,i} X^{(0)} 1_{\{||\Delta^n_{t,i} Y|| \le \alpha \Delta^\varpi_n \,\cap\, |\Delta^n_{t,i} X^{(0)}| \le \alpha \Delta^\varpi_n\}}. \tag{23}$$

To derive the CLT for the feasible estimator of the market beta dispersion in this modified setting, one may utilize the Delta method and a joint CLT for the infeasible dispersion, defined on the basis of $\Delta^n_{t,i} \widetilde{X}^{(0)}$ and $\widehat{\eta}^n_{t,\kappa} - \eta^n_{t+\kappa}$. This is relatively easy to do, replicating the exact steps in the proof of Theorem 1. We defer from a detailed discussion here, as it requires introducing a fair amount of additional notation. Moreover, the market factor plays a pivotal role in theory while, empirically, standard observable factors tend to be only weakly correlated with the market portfolio.

3.2.2 Functional Convergence

Theorem 1 enables us to formally compare the cross-sectional market beta dispersion across different days as well as different times within the trading day. However, if the dispersion changes during the day, it is natural to study the behavior of $\widehat{D}^N_{t,\kappa}$ as a function of its time-of-day argument, κ. For this purpose, we now develop a functional convergence result for $\widehat{D}^N_{t,\kappa}$.

Using the one-to-one mapping between a function (in L_2) and its Fourier transform, we can characterize the behavior of the dispersion function during the trading day through its Fourier transform. Therefore, we introduce a family of functions to weight the dispersion measure across the different times of the trading day and analyze the associated functional convergence of the weighted dispersions within a family of weight functions. More specifically, suppose for every $u \in \mathcal{U}$, where $\mathcal{U}$ is a compact subset of $\mathbb{R}$, that we have a weight function $\omega_u : [0,1] \to \mathbb{C}$. We then define our weighted (complex-valued) total dispersion measures via,

$$\widehat{TD}^N_t(u) \;=\; \frac{1}{n - k_n + 1} \sum_{i=k_n}^{n} \omega_u(i\Delta_n) \, (\widehat{D}^N_{t,i\Delta_n} - \widehat{B}^N_{t,i\Delta_n}), \quad t \in \mathbb{N}_+, \quad u \in \mathcal{U}. \tag{24}$$

The measure $\widehat{TD}^N_t(u)$ is a consistent estimator of,

$$TD^N_t(u) \;=\; \int_0^1 \omega_u(\kappa) \, D^N_{t,\kappa} d\kappa, \quad t \in \mathbb{N}_+, \quad u \in \mathcal{U}, \tag{25}$$

i.e., $\widehat{TD}^N_t(u) - TD^N_t(u) \;=\; o_p(1)$ uniformly in $u \in \mathcal{U}$. The following theorem states the CLT associated with this convergence in probability.

Theorem 2. *Suppose Assumptions A and B hold and, in addition, that the family of complex-valued functions $(\omega_u(z))_{u\in\mathcal{U}}$ satisfies $|\omega_u(z)-\omega_v(z)| \le K|u-v|$ and $|\omega_u(z)-\omega_u(w)| \le K|z-w|$ for all $u,v\in\mathcal{U}$, where $\mathcal{U}\in\mathbb{R}$ is a compact set, $z,w\in[0,1]$, and $K>0$ is a constant. Let $n\to\infty$ and $N\to\overline{N}$ for $\overline{N}\in(0,\infty]$, with $\varpi\in(3/8,1/2)$, $\varrho\in(1/3,1/2)$ and $\varrho>2-4\varpi$. For $t\in\mathbb{N}_+\cap[0,T]$, we have,*

$$\sqrt{n-k_n+1}\ \left(\widehat{TD}_t^N(u)-TD_t^N(u)\right)\ \xrightarrow{\mathcal{L}-s}\ Z_t(u), \tag{26}$$

where the above convergence is for u-indexed processes under the uniform metric, and further $Z_t(u)$ is a complex-valued process, defined on an extension of the original probability space, which is $\mathcal{F}$-conditionally a centered Gaussian process on $\mathcal{U}$ with covariance and relation functions given by,

$$\mathbb{E}(Z_t(u)Z_t(v)) \ \equiv\ \Sigma_t(u,v)\ =\ \underset{N\to\overline{N},\ n\to\infty}{plim}\ \widehat{\Sigma}_t(u,v), \qquad \text{for } u,v\in\mathcal{U}, \tag{27}$$

$$\mathbb{E}(Z_t(u)\overline{Z_t(v)}) \ \equiv\ \Xi_t(u,v)\ =\ \underset{N\to\overline{N},\ n\to\infty}{plim}\ \widehat{\Xi}_t(u,v), \qquad \text{for } u,v\in\mathcal{U}, \tag{28}$$

where

$$\widehat{\Sigma}_t(u,v)\ =\ \frac{k_n}{n-k_n+1}\sum_{i=1}^{n}\overline{\omega}_u^n(i)\ \overline{\omega}_v^n(i)\ \frac{\left(\widehat{Avar}(\widehat{D}^N_{t,(i\vee k_n)\Delta_n})+\widehat{Avar}(\widehat{D}^N_{t,((i+k_n)\wedge n)\Delta_n})\right)}{2}, \tag{29}$$

$$\widehat{\Xi}_t(u,v)\ =\ \frac{k_n}{n-k_n+1}\sum_{i=1}^{n}\overline{\omega}_u^n(i)\ \overline{\overline{\omega}_v^n(i)}\ \frac{\left(\widehat{Avar}(\widehat{D}^N_{t,(i\vee k_n)\Delta_n})+\widehat{Avar}(\widehat{D}^N_{t,((i+k_n)\wedge n)\Delta_n})\right)}{2}, \tag{30}$$

with

$$\overline{\omega}_u^n(i)=\begin{cases}\frac{1}{k_n}\sum_{j=0}^{i-1}\omega_u((i+j)\Delta_n), & i=1,...,k_n,\\ \frac{1}{k_n}\sum_{j=0}^{k_n-1}\omega_u((i+j)\Delta_n), & i=k_n+1,...,n,\end{cases} \tag{31}$$

and, in the above, we set $\omega_u(s)=0$ for $s>1$. We further have,

$$\sup_{u,v\in\mathcal{U}}|\widehat{\Sigma}_t(u,v)-\Sigma_t(u,v)|\ \xrightarrow{\mathbb{P}}\ 0, \qquad \sup_{u,v\in\mathcal{U}}|\widehat{\Xi}_t(u,v)-\Xi_t(u,v)|\ \xrightarrow{\mathbb{P}}\ 0. \tag{32}$$

Moreover, for $\mathcal{T}$ a finite set of positive integers in $[0,T]$, the above convergence in law holds jointly for $t\in\mathcal{T}$, with $Z_t(u)$ and $Z_s(u)$ being $\mathcal{F}$-conditionally independent for $s,t\in\mathcal{T}$ with $s\neq t$.

This convergence result is in the space of continuous functions on a compact interval, equipped with the uniform topology. Alternatively, we could have stated a functional convergence result for functions taking values in a weighted L_2 space. However, since the functions of interest (mainly, $D^N_{t,\kappa}$ as functions of κ) are defined on the bounded interval $[0,1]$, it is enough to look at their Fourier transforms only on a compact interval including zero for their analysis.

We further note that the convergence rate of $\widehat{TD}_t^N(u)$ is faster than that of $\widehat{D}_{t,\kappa}^N$, namely $\sqrt{n}$ versus $\sqrt{k_n}$. This is because, in the case of $\widehat{TD}_t^N(u)$, the errors in measuring dispersion on the entire unit interval are averaged out, which enhances the convergence rate. As a consequence, the requirement on the local window k_n, i.e., the restriction on ϱ, is now much stricter relative to the one needed for Theorem 1. Nevertheless, ϱ can still take values in a range without impacting the CLT result in Theorem 2, which is a desirable feature for practical applications. We note also here that, unlike the case of Theorem 1, the bias-correction is unavoidable, because the bias term of the statistic is of order $O_p(1/k_n)$, while the rate of convergence in equation (26) is $\sqrt{n}$ (and recall that $k_n/\sqrt{n} \to 0$ because $\varrho < 1/2$).

4 Tests for Cross-Sectional Dispersion in Market Betas

We now exploit the limit results in the previous section to design tests for hypotheses regarding the beta dispersion. We assume that the limit of $D_{t,\kappa}^N$ exists, as the size of the cross-section N converges to $\overline{N}$, for $\overline{N}$ either a finite number or infinity. We denote this limit by,

$$D_{t,\kappa} \;=\; \operatorname*{plim}_{N\to\overline{N}} D_{t,\kappa}^N, \quad t \in \mathbb{N}_+, \quad \kappa \in [0,1], \tag{33}$$

and further define,

$$TD_t(u) \;=\; \int_0^1 \omega_u(\kappa)\, D_{t,\kappa}\, d\kappa, \quad t \in \mathbb{N}_+, \quad u \in \mathcal{U}. \tag{34}$$

We will apply our theorems by averaging the dispersion measures across trading days. Towards this end, we introduce the notation,

$$\widehat{D}_{\mathcal{T},\kappa}^N = \sum_{t\in\mathcal{T}} \widehat{D}_{t,\kappa}^N\,, \quad D_{\mathcal{T},\kappa}^N = \sum_{t\in\mathcal{T}} D_{t,\kappa}^N\,, \quad D_{\mathcal{T},\kappa} = \sum_{t\in\mathcal{T}} D_{t,\kappa}\,, \tag{35}$$

for $\mathcal{T}$ being a finite set of integers in $[0,T]$. We define similarly $\widehat{TD}_{\mathcal{T}}^N(u)$, $TD_{\mathcal{T}}^N(u)$, and $TD_{\mathcal{T}}(u)$.

4.1 Tests for Intraday Variation in Dispersion

We start by designing tests for determining whether the market beta dispersion varies across the trading day. Specifically, we introduce the set,

$$\Omega_{\mathcal{T}}(\kappa,\kappa') = \left\{\omega : D_{\mathcal{T},\kappa} = D_{\mathcal{T},\kappa'}\right\}, \;\kappa,\kappa' \in [0,1],\; \kappa \neq \kappa'. \tag{36}$$

We then seek to test whether the sample path belongs to $\Omega_{\mathcal{T}}(\kappa,\kappa')$ or its compliment. For that purpose, we propose a test statistic with the following critical region,

$$\mathcal{C}_n \;=\; \left\{ \frac{|\widehat{D}_{\mathcal{T},\kappa}^N - \widehat{B}_{\mathcal{T},\kappa}^N - \widehat{D}_{\mathcal{T},\kappa'}^N + \widehat{B}_{\mathcal{T},\kappa'}^N|}{\sqrt{\sum_{t\in\mathcal{T}}\left(\widehat{\mathrm{Avar}}(\widehat{D}_{t,\kappa}^N) + \widehat{\mathrm{Avar}}(\widehat{D}_{t,\kappa'}^N)\right)}} \;>\; z_{1-\alpha/2} \right\}, \quad \alpha \in (0,1), \tag{37}$$

where z_α is the α-quantile of the standard normal distribution. Then, from Theorem 1, we have,

$$\mathbb{P}(\mathcal{C}_n|\Omega_{\mathcal{T}}(\kappa,\kappa')) \to \alpha, \quad \mathbb{P}(\mathcal{C}_n|\Omega_{\mathcal{T}}(\kappa,\kappa')^c) \to 1, \tag{38}$$

provided

$$\begin{cases} (D^N_{\mathcal{T},\kappa} - D_{\mathcal{T},\kappa} - D^N_{\mathcal{T},\kappa'} + D_{\mathcal{T},\kappa'})\, 1(\Omega_{\mathcal{T}}(\kappa,\kappa')) \;=\; o_p\big(1/\sqrt{k_n}\big), \\ (D^N_{\mathcal{T},\kappa} - D_{\mathcal{T},\kappa} - D^N_{\mathcal{T},\kappa'} + D_{\mathcal{T},\kappa'})\, 1(\Omega_{\mathcal{T}}(\kappa,\kappa')^c) \;=\; o_p(1). \end{cases} \tag{39}$$

The above condition obviously holds, when $\overline{N}$ is finite. When $\overline{N}$ is infinite, the second part of equation (39) typically follows by invoking a Law of Large Numbers for $N \to \infty$. For the first part of equation (39) with an asymptotically increasing N, we require the given rate of convergence for the above-mentioned cross-sectional Law of Large Numbers, and we need N to grow sufficiently fast relative to k_n. Alternatively, this condition automatically holds without restrictions on the relative size of the two dimensions of the return panel, if the market betas are assumed constant across the trading day (recall these assumptions concern only the null hypothesis).

4.2 Tests for Functional Variation in Dispersion

We now go on to derive a test for variation in $D_{t,\kappa}$ (as a function of κ) across different trading days. Specifically, we are interested in deciding whether the sample path belongs to the following subset of the sample space,

$$\Omega(\mathcal{T}_1,\mathcal{T}_2) \;=\; \{\omega : D_{\mathcal{T}_1,\kappa} = D_{\mathcal{T}_2,\kappa}, \quad \kappa - a.e.\}, \qquad \mathcal{T}_1 \cap \mathcal{T}_2 \;=\; \emptyset, \tag{40}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are two disjoint sets of integers in $[0,T]$.

As for the previous test, we need a condition for the asymptotic size of the difference $D^N_{t,\kappa} - D_{t,\kappa}$ which, in the current setting, takes the form,

$$\begin{cases} \sup_{\kappa\in[0,1]} |D^N_{\mathcal{T}_1,\kappa} - D_{\mathcal{T}_1,\kappa} - D^N_{\mathcal{T}_2,\kappa} + D_{\mathcal{T}_2,\kappa}|\; 1(\Omega(\mathcal{T}_1,\mathcal{T}_2)) \;=\; o_p(\sqrt{\Delta_n}), \\ \sup_{\kappa\in[0,1]} |D^N_{\mathcal{T}_1,\kappa} - D_{\mathcal{T}_1,\kappa} - D^N_{\mathcal{T}_2,\kappa} + D_{\mathcal{T}_2,\kappa}|\; 1(\Omega(\mathcal{T}_1,\mathcal{T}_2)^c) \;=\; o_p(1). \end{cases} \tag{41}$$

Similar comments to the ones following equation (39) apply here as well. In particular, the first of the above conditions hold, whenever a functional CLT for $D^N_{t,\kappa}$, as a function of κ, with $N \to \infty$ applies, and N is sufficiently large relative to n (depending on the rate of convergence of this cross-sectional CLT). Alternatively, this condition holds without any restriction on N, if $\beta^{(j)}_{t+\kappa} = \beta^{(j)}_{s+\kappa}$, for $s,t \in \mathcal{T}$, $\kappa \in [0,1]$ and $j = 1,...,N$.

For constructing the critical region of the test, we exploit the following corollary to Theorem 2.

Corollary 1. *Assume the conditions of Theorem 2 hold for $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint sets of integers in $[0, T]$. Let $\omega_u(z) = e^{iuz}$, for $u \in \mathcal{U} = [0, \overline{\mathcal{U}}]$, and $\overline{\mathcal{U}}$ is a positive constant.*
(a) If $D_{\mathcal{T}_1,\kappa} = D_{\mathcal{T}_2,\kappa}$ a.e. for $\kappa \in [0,1]$, and condition (41) holds, then,

$$\sqrt{n - k_n + 1}\ \sup_{u\in\mathcal{U}} \big| \widehat{TD}^N_{\mathcal{T}_1}(u) - \widehat{TD}^N_{\mathcal{T}_2}(u) \big| \ \xrightarrow{\mathcal{L}-s}\ \sup_{u\in\mathcal{U}} \big| \sum_{t\in\mathcal{T}_1} Z_t(u) - \sum_{t\in\mathcal{T}_2} Z_t(u) \big|, \tag{42}$$

where the limiting process $Z_t(u)$ is defined in Theorem 2.
(b) If $D_{\mathcal{T}_1,\kappa} \neq D_{\mathcal{T}_2,\kappa}$ for κ in a set of positive Lebesgue measure, then,

$$\sup_{u\in\mathcal{U}} \big| TD_{\mathcal{T}_1}(u) - TD_{\mathcal{T}_2}(u) \big| \ >\ 0. \tag{43}$$

Part (a) of the above corollary characterizes the asymptotic behavior of $\widehat{TD}^N_{\mathcal{T}_1}(u) - \widehat{TD}^N_{\mathcal{T}_2}(u)$ under the null hypothesis. Part (b) shows that for determining whether $D_{\mathcal{T}_1,\kappa}$ differs from $D_{\mathcal{T}_2,\kappa}$ (as a function of κ), we only need to look at the difference between their Fourier transforms for u on a compact subset of $\mathbb{R}$ containing zero. This result follows from Bierens (1982); see also Bierens and Ploberger (1997). Given Corollary 1, we propose a test with the following critical region,

$$\mathcal{C}_n \ =\ \left\{ \sqrt{n - k_n + 1}\ \sup_{u\in\mathcal{U}} \left| \widehat{TD}^N_{\mathcal{T}_1}(u) - \widehat{TD}^N_{\mathcal{T}_2}(u) \right| \ >\ \widehat{z}_{1-\alpha/2} \right\}, \qquad \alpha \in (0,1), \tag{44}$$

where $\widehat{z}_\alpha$ is the α-quantile of a centered Gaussian process on $\mathcal{U}$ with covariance and relation functions given by $\sum_{t\in\mathcal{T}_1 \cup \mathcal{T}_2} \widehat{\Sigma}_t(u,v)$ and $\sum_{t\in\mathcal{T}_1 \cup \mathcal{T}_2} \widehat{\Xi}_t(u,v)$, respectively. We note that $\widehat{z}_\alpha$ may be computed easily via simulation. From Corollary 1, it follows, provided condition (41) holds, that,

$$\mathbb{P}(\mathcal{C}_n | \Omega(\mathcal{T}_1, \mathcal{T}_2)) \ \to\ \alpha, \qquad \mathbb{P}(\mathcal{C}_n | \Omega(\mathcal{T}_1, \mathcal{T}_2)^c) \ \to\ 1. \tag{45}$$

Finally, we observe that a number of suitable substitutions within the above test enables us to test the hypothesis that the market beta dispersion, as a function of the time within the trading day, has a particular parametric representation. Specifically, we must replace $D_{\mathcal{T}_2,\kappa}$ by some function $\overline{D}_\kappa$ on $[0,1]$, substitute $\int_0^1 e^{iu\kappa}\overline{D}_\kappa d\kappa$ for $\widehat{TD}^N_{\mathcal{T}_2}(u)$, and set $\mathcal{T}_2 = \emptyset$, thus obtaining a test for the hypothesis $D_{\mathcal{T}_1,\kappa} = \overline{D}_\kappa$ for $\kappa \in [0,1]$.

5 Inference for the Cross-Sectional Distribution of Market Beta

We finish the theoretical analysis by proposing an estimator for the cross-sectional distribution of market betas and providing the associated limit theory. Hitherto, we have focused on the cross-sectional dispersion of market betas. However, as we now show, we can extend this analysis to study the entire distribution. We will do this by estimating the characteristic function of the cross-sectional beta distribution at a given point in time. The associated convergence result, presented below, is functional and takes place in the complex-valued Hilbert space $\mathcal{L}^2(w)$,

$$\mathcal{L}^2(w) \;=\; \left\{ f : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |f(u)|^2 w(u) du < \infty \right\}, \tag{46}$$

where w is some positive-valued and continuous weight function with exponential tail decay. The inner product on $\mathcal{L}^2(w)$ is induced from the inner products of its real and imaginary parts, i.e., for f and g two elements of $\mathcal{L}^2(w)$, we set,

$$\langle f, g\rangle \;=\; \int_{\mathbb{R}} f(z)\, \overline{g(z)}\, w(z)\, dz. \tag{47}$$

Next, for a random complex function X taking values in $\mathcal{L}^2(w)$, we introduce the covariance operator $Kh = \mathbb{E}[(X-\mathbb{E}(X))\langle h, X-\mathbb{E}(X)\rangle]$, and the relation operator $Ch = \mathbb{E}[(X-\mathbb{E}(X))\langle h, \overline{X-\mathbb{E}(X)}\rangle]$, where $h \in \mathcal{L}^2(w)$. We recall that a Gaussian law on $\mathcal{L}^2(w)$ is uniquely identified by the mean, covariance and relation operators. We refer to this law as $\mathcal{CN}(\mu, K, C)$, for μ being the mean, K being the covariance, and C being the relation operator.

Our estimator of the characteristic function of the cross-sectional distribution of market beta can be easily constructed from $\widehat{C}^{(j)}_{t,\kappa}(2)$ and $\widehat{V}^{(0,j)}_{t,\kappa}$ in the following manner,

$$\widehat{\mathcal{L}}^{\beta}_{t,\kappa}(u) \;=\; \frac{1}{N}\sum_{j=1}^{N} \exp\left(iu\, \frac{\widehat{C}^{(j)}_{t,\kappa}(2)}{\widehat{V}^{(0,j)}_{t,\kappa}} + iu \right), \qquad u \in \mathbb{R}, \tag{48}$$

where we remind the reader that $\widehat{C}^{(j)}_{t,\kappa}(2)$ is an estimate of $(\beta^{(j)}_{t+\kappa} - 1)(\sigma^{(0)}_{t+\kappa})^2$. The next theorem provides the CLT for $\widehat{\mathcal{L}}^{\beta}_{t,\kappa}(u)$.[3]

Theorem 3. *Suppose Assumptions A and C hold. Let $n \to \infty$ and $N \to \overline{N}$, for $\overline{N} \in (0, \infty]$, with $\varpi \in (3/8, 1/2)$, $\varrho \in (0, 1/2)$ and $\varrho > 2 - 4\varpi$. Then, for $t \in \mathcal{N}_+ \cap [0, T]$ and $\kappa \in [0, 1]$, we have,*

$$\sqrt{k_n}\left(\widehat{\mathcal{L}}^{\beta}_{t,\kappa}(u) - \frac{1}{N}\sum_{j=1}^{N} \exp\left(iu\beta^{(j)}_{t+\kappa} \right) \right) \;\xrightarrow{\mathcal{L}-s}\; \mathcal{Z}_{t,\kappa}, \tag{49}$$

where $\mathcal{Z}_{t,\kappa}$ is $\mathcal{F}$-conditionally $\mathcal{CN}(0, K_{t,\kappa}, C_{t,\kappa})$, and the operators $K_{t,\kappa}$ and $C_{t,\kappa}$ are given by,

$$K_{t,\kappa}h(z) = \int_{\mathbb{R}} k_{t,\kappa}(z,u) h(u) w(u)\, du, \quad C_{t,\kappa}h(z) = \int_{\mathbb{R}} c_{t,\kappa}(z,u) h(u) w(u)\, du, \quad \forall h \in \mathcal{L}^2(w), \tag{50}$$

with the functions $k_{t,\kappa}(z,u)$ and $c_{t,\kappa}(z,u)$ given in Assumption C.

[3]Similar to our dispersion measure, $\widehat{D}^{N}_{t,\kappa}$, we can also bias-correct $\widehat{\mathcal{L}}^{\beta}_{t,\kappa}(u)$. However, such bias-correction is not needed for the limit result in Theorem 3 and in order to keep the analysis simple, we do not do this here.

The above result is most useful in the case when $N \to \infty$, and there is a continuum of assets that we draw from randomly, as in Gagliardini, Ossola, and Scaillet (2016) (see also the references therein).[4] In this case, $\frac{1}{N}\sum_{j=1}^{N} \exp\left(iu\beta_{t+\kappa}^{(j)}\right)$ is an estimate of the characteristic function of the market beta distribution at time $t+\kappa$. Thus, using a feasible version of Theorem 3, we can conduct formal inference for the cross-sectional distribution of market betas at a given point (or fixed points) in time. In particular, we can test for whether the entire cross-sectional beta distribution changes during a trading day or across trading days. In addition, we can estimate the density of the cross-sectional beta distribution at time $t-1+\kappa$, denoted by $f_{t,\kappa}^{\beta}(x)$ (assuming it exists), via Fourier inversion,

$$\widehat{f}_{t,\kappa}^{\beta}(x) = \frac{1}{2\pi}\int_{-u_n}^{u_n} e^{-iux}\widehat{\mathcal{L}}_{t,\kappa}^{\beta}(u)du, \tag{51}$$

for some positive sequence $u_n \to \infty$ as $n \to \infty$.

Finally, as for the analysis of the dispersion, we can construct cross-sectional moments of temporally aggregated estimates for betas. Specifically, for $\mathcal{T}$ a set of integers in $[0,T]$, we can define,

$$\widehat{\mathcal{L}}_{\mathcal{T},\kappa}^{\beta}(u) = \frac{1}{N}\sum_{j=1}^{N}\exp\left(iu\frac{1}{|\mathcal{T}|}\sum_{t\in\mathcal{T}}\frac{\widehat{C}_{t,\kappa}^{(j)}(2)}{\widehat{V}_{t,\kappa}^{(0,j)}} + iu\right),\ \widehat{f}_{\mathcal{T},\kappa}^{\beta}(x) = \frac{1}{2\pi}\int_{-u_n}^{u_n} e^{-iux}\widehat{\mathcal{L}}_{\mathcal{T},\kappa}^{\beta}(u)du. \tag{52}$$

A functional CLT result for $\widehat{\mathcal{L}}_{\mathcal{T},\kappa}^{\beta}(u)$, analogous to the one in Theorem 3, may be obtained here as well. For brevity, we do not state it explicitly.

6 Simulation Study

We now assess the finite sample properties of the new inference procedures. To this end, we conduct a Monte Carlo study in which we vary the sampling frequency, window length, and number of assets in the cross section. The analysis is based on simulation from the following affine jump-diffusion model for the return on the market and each of the N assets,

$$\begin{aligned} dX_t^{(0)} &= \sqrt{V_t}dW_t^{(0)} + Z_t dN_t, \qquad V_t = V_t^{(1)} + V_t^{(2)}, \\ dV_t^{(i)} &= \kappa_i\big(\theta - V_t^{(i)}\big)dt + \xi_i\sqrt{V_t^{(i)}}dB_t^{(i)}, \quad i = 1,2, \\ dX_t^{(j)} &= \beta_t^{(j)}\sqrt{V_t}dW_t^{(0)} + \sqrt{V_t}d\widetilde{W}_t^{(j)} + \beta_t^{(j)}Z_t dN_t, \quad j = 1,...,N, \end{aligned} \tag{53}$$

where $B^{(1)}, B^{(2)}, W^{(0)}, \widetilde{W}^{(1)}, ..., \widetilde{W}^{(N)}$ are independent standard Brownian motions, N_t is a Poisson process with intensity per unit of time of λ_J capturing the arrival of market jumps with size given

[4] If market betas depend on observable firm characteristics, one can also estimate beta distribution conditional on such observables.

by an i.i.d. sequence $(Z_s)_{s\geq 1}$, with $Z_s \sim N\left(0, \sigma_J^2\right)$. The parameters are identical to those in Bollerslev and Todorov (2011),

$$\left(\kappa_1, \kappa_2, \theta, \xi_1, \xi_2, \lambda_J, \sigma_J^2\right) \;=\; (\,0.0128, 0.6930, 0.4068, 0.0954, 0.7023, 0.2, 0.19, 0.932\,)\,.$$

Given our objectives, we calibrate the specification for the intraday variation of beta carefully. The market betas obey the following dynamics,

$$\beta_t^{(j)} \;=\; \psi_{\lfloor t\rfloor,\, t-\lfloor t\rfloor}\left(\overline{\beta}^{(j)}\right), \quad j = 1, ..., N, \quad t \in \mathbb{R}_+, \tag{54}$$

where $\overline{\beta}^{(j)}$ is drawn from,

$$\overline{\beta}^{(j)} \overset{\text{i.i.d.}}{\sim} \text{uniform}([0.5, 1.5]), \quad j = 1, ..., N, \tag{55}$$

and the function ψ is given by,

$$\psi_{t,\,\kappa}(x) \;=\; x + \overline{\psi}_{t,\,\kappa}(x-1), \quad t \in \mathbb{N}_+, \quad \kappa \in [0,1], \quad x \in \mathbb{R}, \tag{56}$$

for some smooth function of t and κ, $\overline{\psi}_{t,\,\kappa}$: $\mathbb{N}_+ \times [0,1] \to \mathbb{R}$. We use three functions, $\{\overline{\psi}_\kappa^{(i)}\}_{i=1,2,3}$, which depend only on $\kappa \in [0,1]$, for $\overline{\psi}_{t,\,\kappa}$. The first, $\overline{\psi}_\kappa^{(1)}$, is a constant, chosen so that if $\overline{\psi}_{t,\,\kappa} = \overline{\psi}_\kappa^{(1)}$, then $\mathbb{E}(\beta_t^{(j)} - 1)^2$ equals the average daily estimate for the dispersion of the market betas observed in our data. Next, we set $\overline{\psi}_\kappa^{(2)}$, so that $\overline{\psi}_{t,\,\kappa} = \overline{\psi}_\kappa^{(2)}$ implies a value for $\mathbb{E}(\beta_t^{(j)} - 1)^2$ that matches the average dispersion of the market betas as a function of time-of-day in our data. Finally, $\overline{\psi}_\kappa^{(3)}$ is calibrated in an analogous way to $\overline{\psi}_\kappa^{(2)}$, with the difference being that the implied $\mathbb{E}(\beta_t^{(j)} - 1)^2$ now matches the average dispersion of the market betas as a function of time-of-day observed in our data only on days with low volatility. For the latter, we find that the beta dispersion as a function of the time-of-day appears to differ substantially from that computed on the other days in the sample. The three functions $\{\overline{\psi}_\kappa^{(i)}\}_{i=1,2,3}$ are displayed in Figure 2.

We fix the various tuning parameters for the statistics with a view towards our empirical application. Throughout, we set $|\mathcal{T}| = 65$ or $|\mathcal{T}| = 250$ (this applies also to the size of the sets $\mathcal{T}_1$ and $\mathcal{T}_2$, when performing tests across trading days), corresponding to averaging across a period of one quarter or one year. The truncation level is set at $\nu_{t,n}^{(j)} = 4\sqrt{BV_{t,n}^{(j)}}\Delta_n^{0.49}$, $j = 0, ..., N$, where $BV_{t,n}^{(j)}$ is the so-called bipower variation of asset j, given by $BV_{t,n}^{(j)} = \frac{\pi}{2}\sum_{i=2}^{n} |\Delta_{t,i}^n X^{(j)}||\Delta_{t,i-1}^n X^{(j)}|$, which is a nonparametric estimate of daily integrated volatility, see Barndorff-Nielsen and Shephard (2004b). Finally, we set $\alpha_n = Q_{0.1}(BV)/\log(n)$ (recall equations (12)-(13)), where $Q_{0.1}(BV)$ is the 10-th quantile of the empirical distribution of $BV^{(0)}$.

We begin by studying the finite-sample properties of the test for equal cross-sectional dispersion in betas during parts of the trading day, given in Section 4.1. The empirical rejection rates of the

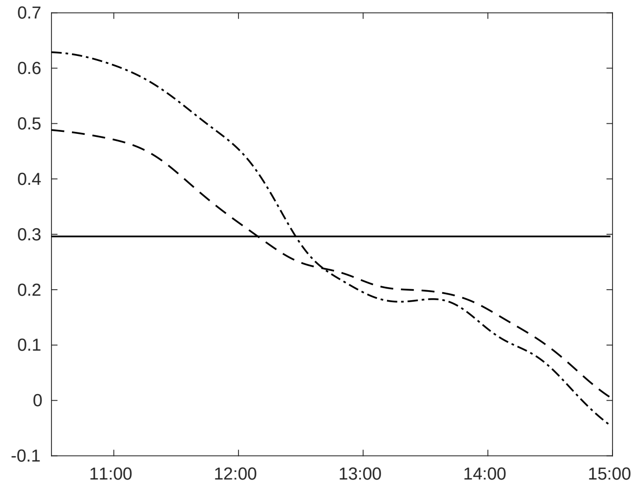

Figure 2: **The functions $\{\overline{\psi}_\kappa^{(i)}\}_{i=1,2,3}$ used in the Monte Carlo.** The figure plots $\overline{\psi}_\kappa^{(1)}$ (solid line), $\overline{\psi}_\kappa^{(2)}$ (dashed line) and $\overline{\psi}_\kappa^{(3)}$ (dashed-dotted line) used in computing the different beta functions according to equations (55)-(56) in the various Monte Carlo setups detailed in the text.

test under the null hypothesis are provided in Table 1, while those under alternative hypotheses are reported in Table 2. The results in Table 1 point to a satisfactory behavior of the test under the null hypothesis, with the empirical rejection rates being very close to its nominal size. This holds true for all the different values of n, $|\mathcal{T}|$, and N that we consider.

Turning to our ability to detect intraday variation in market beta, Table 2 shows that our test has excellent power properties against the given alternative (we consider equation (56) with $\overline{\psi}_{t,\kappa} = \overline{\psi}_\kappa^{(2)}$ as our alternative). This holds true even for the scenario in which we average the dispersion statistic over the smallest of our choices for $|\mathcal{T}|$, 65 trading days. As expected, the power of the test is somewhat lower when comparing the cross-sectional dispersion not at the market open, but rather at lunch, versus the market close. This is because the discrepancy in $D_{t,\kappa}$ between the two points within the trading day now is decidedly smaller, see Figure 2.

Next, we explore the properties of the functional test for variation in dispersion, developed in Section 4.2. We implement it with $\overline{\mathcal{U}} = 2\pi$ and discretize the domain using increments of $\pi/3$ (i.e., $u \in \{0, \pi/3, 2\pi/3, ..., 2\pi\}$). The critical test values are obtained from the procedure outlined in Section 4.2 with $10,000$ simulations. Table 3 provides results under the null hypothesis, $\overline{\psi}_{t,\kappa} = \overline{\psi}_\kappa^{(2)}$. We notice a slight over-rejection for $T = 250$ and low values of n. Overall, however, the test has empirical rejections rates under the null hypothesis close to the corresponding nominal size.

To examine the power of the test, we set $\overline{\psi}_{t,\kappa} = \overline{\psi}_\kappa^{(2)}$ for $t \in \mathcal{T}_1$ and let $\overline{\psi}_{t,\kappa} = \overline{\psi}_\kappa^{(3)}$ for $t \in \mathcal{T}_2$. Results for this simulation scenario are given in Table 4 and reveal good power. Not surprisingly, the power improves as the sampling frequency increases. We further note that the rejection rates

n \ N	$\alpha = 0.05$			$\alpha = 0.1$		
	100	300	500	100	300	500
			$\lvert\mathcal{T}\rvert$ = *65 days*			
390	0.054	0.073	0.065	0.106	0.121	0.117
120	0.068	0.056	0.065	0.121	0.110	0.121
78	0.073	0.056	0.069	0.122	0.111	0.125
			$\lvert\mathcal{T}\rvert$ = *250 days*			
390	0.058	0.062	0.053	0.115	0.115	0.107
120	0.053	0.070	0.061	0.104	0.141	0.114
78	0.075	0.065	0.062	0.135	0.119	0.120

Table 1: **Monte Carlo Results: Test for constant beta dispersion across the trading day under the null hypothesis (54)-(56) with $\overline{\psi}_{t,\kappa} = \overline{\psi}_{\kappa}^{(1)}$.** The table reports empirical rejection rates of the test in Section 4.1 for nominal size 0.05 and 0.10, using 1,000 simulations. The time windows for the test are the first and last 2 hours of the trading day.

n \ N	65 days			250 days		
	100	300	500	100	300	500
			open versus close			
390	1.000	1.000	1.000	1.000	1.000	1.000
120	1.000	1.000	1.000	1.000	1.000	1.000
78	1.000	1.000	1.000	1.000	1.000	1.000
			lunch versus close			
390	1.000	1.000	1.000	1.000	1.000	1.000
120	0.990	0.991	0.993	1.000	1.000	1.000
78	0.839	0.854	0.845	1.000	1.000	1.000

Table 2: **Monte Carlo Results: Test for constant beta dispersion across the trading day under the alternative hypothesis (54)-(56) with $\overline{\psi}_{t,\kappa} = \overline{\psi}_{\kappa}^{(2)}$.** The table reports empirical rejection rates of the test in Section 4.1 for nominal size 0.05 using 1,000 simulations. The time windows in the test are: open (first two hours), lunch (11am-1pm) and close (last two hours).

n \ N	$\alpha = 0.05$			$\alpha = 0.1$		
	100	300	500	100	300	500
			$\overline{\psi}_{t,\kappa} = \overline{\psi}^{(2)}_{t,\kappa}$, $T = 65$			
390	0.048	0.055	0.055	0.103	0.112	0.104
120	0.051	0.056	0.055	0.104	0.102	0.100
78	0.044	0.060	0.068	0.100	0.111	0.109
			$\overline{\psi}_{t,\kappa} = \overline{\psi}^{(2)}_{t,\kappa}$, $T = 250$			
390	0.066	0.063	0.056	0.116	0.117	0.111
120	0.065	0.082	0.070	0.122	0.147	0.132
78	0.074	0.070	0.055	0.127	0.131	0.118

Table 3: **Monte Carlo Results: Test for constant dispersion of betas across time under the null hypothesis (54)-(56) with $\overline{\psi}_{t,\kappa} = \overline{\psi}^{(2)}_{t,\kappa}$.** The table reports empirical rejection rates of the test in Section 4.2 of nominal size 0.05 (first three columns) and 0.1 (last three columns) using $1,000$ simulations.

appear insensitive to the choice of N (the size of the cross-section).

n \ N	65 days			250 days		
	100	300	500	100	300	500
390	1.000	1.000	1.000	1.000	1.000	1.000
120	0.889	0.878	0.873	1.000	1.000	1.000
78	0.601	0.593	0.579	0.991	0.989	0.993

Table 4: **Monte Carlo Results: Test for constant beta dispersion over time under the alternative hypothesis.** The table reports empirical rejection rates of the test of Section 4.2 for nominal size 0.05 using $1,000$ simulations. The testing is based on two blocks with $\overline{\psi}_{t,\kappa} = \overline{\psi}^{(2)}_{\kappa}$ for $t \leq 65$ ($t \leq 250$) and $\overline{\psi}_{t,\kappa} = \overline{\psi}^{(3)}_{\kappa}$ for $t > 65$ ($t > 250$) for the first (last) three columns.

7 Empirical Evidence on the Intraday Beta Dispersion

We now use our newly developed econometric tools to explore the intraday behavior of market betas. The analysis uses high-frequency returns on the constituents of the S&P 500 market index, and our market proxy is the SPY ETF on the S&P 500 index. Each trading day, we sample the asset prices every 3 minutes. We exclude the first half hour of trading to avoid any potential issues associated with market opening. This leaves us with 120 returns per day for each asset. We remove days with partial trading. Overall, the sample contains 2243 full trading days over 2010-2018.

For most of our illustrations, we retain only stocks belonging to the index for the entire sample period, which results in a cross-section of 335 stocks. This choice alleviates the concern that our comparisons are influenced by substantial shifts in the composition of the underlying stocks. However, this procedure does introduce an element of survivorship bias. Hence, for some parts of the analysis, we rely on all stocks available in the index over shorter windows. In all cases, the qualitative results are not impacted by these choices.

7.1 Unconditional Properties

First, we compute the dispersion measure averaged across the entire sample, $\frac{1}{T}\sum_{t=1}^{T}\widehat{D}_{t,\kappa}^{N}$. The function is plotted in the left panel of Figure 3. It shows that the cross-sectional dispersion of the market betas declines monotonically over the trading day. The reduction is substantial: the dispersion at the market close is less than half of its value at the open. This is consistent with the illustrative plot for the two representative stocks in Figure 1 of the introductory Section 1. We can formally test whether the cross-sectional dispersion in market betas is invariant across the trading day using the procedures developed in Section 4.1. A natural concern in such comparisons is the potential impact of confounding effects arising from excessive noise in the beta estimates due to the high idiosyncratic volatility, especially at the market open. However, our tests are explicitly designed to account for such features, and they retain power to discriminate between the scenarios involving idiosyncratic noise versus true changes in the distribution of market betas. Given the pronounced patterns observed, it is not surprising that our tests overwhelmingly reject the null hypothesis of equal intraday beta dispersion, with p-values below 0.0001, or 0.01%.

The evidence from the left panel of Figure 3 suggests that high market beta stocks (in excess of unity) tend to have declining betas throughout the trading day, while the opposite is true for low beta stocks. This conjecture is confirmed in the right panel of Figure 3. It plots the cross-sectional quantiles of the market betas across the trading day. The changes for the two extreme quantiles (the 10'th and 90'th) are most significant, while the median market beta displays little variation over the trading day. To explore whether this intraday pattern in market betas is robust, we repeated the analysis for 2-year subsamples. The finding of a declining cross-sectional dispersion in market betas over the trading day remains intact, with the evidence, if anything, strengthening in the second half of the sample. For brevity, we do not report these results here.

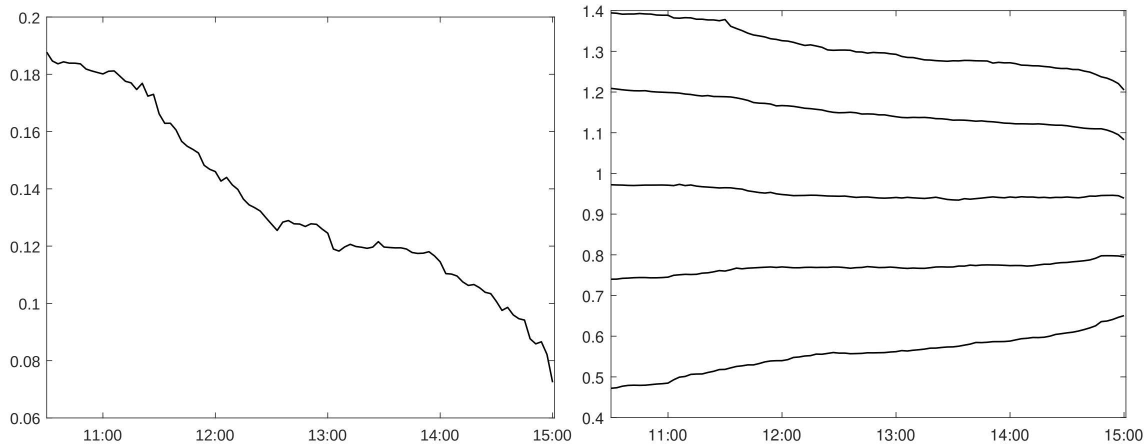

Figure 3: **Cross-Sectional Distribution of Market Betas across the Trading Day**. The left panel displays the cross-sectional dispersion in market betas and the right panel plots the corresponding quantiles. All quantities are treated as functions of the trading day and computed by averaging over the entire sample. The selected quantiles are: 10th, 25th, 50th, 75th, and 90th.

As explained in Section 5, we may also estimate the full density for the cross-sectional distribution of the market betas aggregated across time, using the Fourier inversion approach in equation (52). We apply this procedure to the averaged density estimates for the cross-sectional distribution of the market betas at the open and close of trading. Figure 4 reveals that the two distributions are very different, with the former having a significantly wider support than the latter, even if both have modes close to unity.

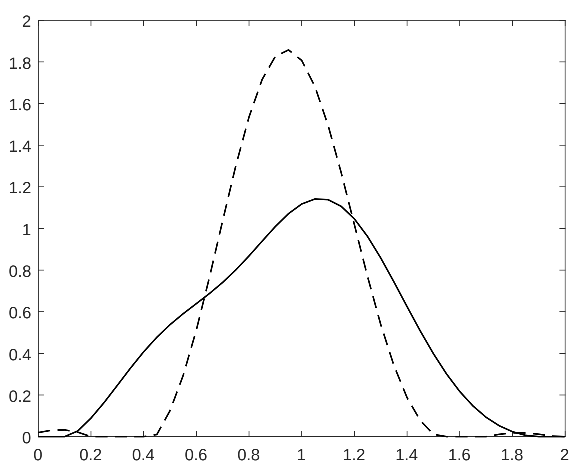

Figure 4: **Cross-Sectional Distribution of Market Betas at Open and Close of Trading.** The plot displays $\widehat{f}^{\beta}_{\mathcal{T},\kappa}(x)$ using the first two hours of trading (solid line) and the last two hours of trading (dashed line), both computed over the entire sample. The tuning parameter u_n in the Fourier inversion is set to $\inf\{0 < u : |\mathcal{L}^{\beta}_{\mathcal{T},\kappa}(u)| < 0.0005\}$.

A potential concern regarding the reported evidence is that the documented intraday pattern may be linked to imprecision or instability in the estimates of high-frequency market betas due to a lack of liquidity for the smaller stocks in our sample. This could bias our intraday pattern if the small firms are concentrated in specific beta quantiles, and liquidity shifts across the trading day. To address this concern, in Table 5, we report the relative number of firms belonging to each five-by-five quintile for market open beta and firm size, averaged across the full sample. The results in the table reveal a roughly homogeneous distribution, with only a mild tendency for the smaller firms to feature a higher beta. Moreover, in results available upon request, we document that the intraday beta pattern within each of the size quintiles is qualitatively identical. They all display a pronounced declining market beta dispersion across the trading day, implying that the systematic intraday beta pattern is robust in this respect.

	Beta				
Size	1(low)	2	3	4	5(high)
1 (small)	0.0346	0.0364	0.0391	0.0422	0.0479
2	0.0415	0.0377	0.0377	0.0396	0.0440
3	0.0457	0.0414	0.0394	0.0383	0.0361
4	0.0404	0.0409	0.0411	0.0412	0.0371
5 (big)	0.0387	0.0443	0.0430	0.0393	0.0325

Table 5: **Joint distribution of beta and size.** The table reports the joint distribution of betas at the open and the size of the firms. For each day, firms are split into 5 bins based on both their beta and size. The table reports the time-series average share of firms that belong to a given size-beta bin.

7.2 Time Series Properties

Given the overwhelming evidence for intraday variation in market betas, we now explore the evolution of this pattern over time. Figure 5 plots the time-series for the cross-sectional dispersion of betas at the market open and close. To mitigate the impact of estimation error, we report dispersion measures computed over rolling windows of 250 days. The dispersion of the market open betas fluctuates greatly, unlike that of the market close betas. In fact, following a decline towards the end of 2013, the beta dispersion at market close has been remarkably stable. In contrast, the dispersion at the market open increases initially till some time in 2012, and then gradually declines, reaching a low during 2015, when the gap between the dispersion of the market open and close betas is also the smallest in our sample. Since then, the dispersion in the market open betas increases sharply, about four-fold, and remains highly elevated for the remainder of our sample.

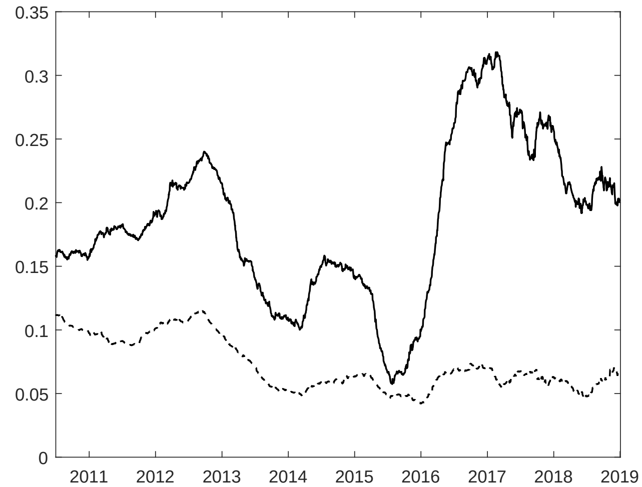

Figure 5: **Cross-Sectional Distribution of Market Betas over Time.** The figure displays the cross-sectional dispersion of market betas at open (solid line) and close (dashed line) over the full sample. Each dispersion measure is computed using a rolling window of 250 trading days.

Figure 5 strongly suggests that the intraday pattern of market betas evolves across the sample period. We may explore this hypothesis through the formal test procedures developed in Section 4.2. Specifically, we test whether the cross-sectional dispersion of betas, as a function of time-of-day, changes across the individual calendar years in our sample. Table 6 shows that, for most calendar year pairings, the null hypothesis of an equal intraday pattern for the dispersion measure is overwhelmingly rejected. Interestingly, the most stable period for adjacent intervals seems to be the last three years, when the cross-sectional dispersion of betas at market open is highly elevated.

It is outside the scope of the current paper to explore systematically, whether there is any structural relation between the evolving economic environment and the variation in the cross-sectional distribution of market betas. Nonetheless, the sharp shifts in the cross-sectional dispersion measure in 2011-2012 and, especially, towards the end of 2015, do warrant a few comments. The former episode coincides with the build up of tensions during the second wave of the European sovereign debt crises, with several countries receiving bail-out packages in the middle of 2012. Likewise, the late 2015 to early 2016 period corresponds to significant upheavals in global financial markets, as the Chinese equity market tumbled sharply between June and August 2015, triggering a devaluation of the Chinese yuan and a prolonged slide in oil prices. In addition, the beginning of 2016

	2011	2012	2013	2014	2015	2016	2017	2018
2010	0.010	0.000	0.000	0.005	0.000	0.000	0.001	0.033
2011		0.058	0.000	0.000	0.000	0.000	0.009	0.069
2012			0.000	0.000	0.000	0.002	0.063	0.004
2013				0.001	0.003	0.000	0.000	0.000
2014					0.000	0.000	0.000	0.004
2015						0.000	0.000	0.000
2016							0.540	0.002
2017								0.003

Table 6: **Tests for Changes in the Dispersion of Market Betas over Time.** The table reports p-values for tests of equal cross-sectional dispersion of market betas as functions of time-of-day. Each entry corresponds to a pairwise test, detailed in Section 4.2, involving the years in the corresponding rows and columns.

witnessed the decision to hold a vote on Brexit, with the subsequent majority coming out in favor of withdrawal in June 2016. Moreover, these events took place during a period when the uncertainty about monetary policy was elevated, as the Federal reserve was signaling a "data-dependent" end to the zero-rate policy. Action was postponed in the Fall of 2015 given the challenging international financial environment, but an interest rate hike was eventually implemented in December 2015.

Even if the evidence above is merely suggestive, it points towards the possibility that the broader economic environment has an impact on the market exposure of individual stocks, with higher uncertainty or more depressed market conditions generating a wider spread in the beta distribution. Perhaps equally noteworthy, this effect is almost exclusively present in the earlier part of the trading day. Combined with the prior evidence in Section 7.1, we conclude that the cross-sectional beta distribution is much wider and more variable at the market open than close. We conjecture this occurs because the market is processing different types of information across the trading day. For example, firm-specific news may be particularly prevalent or important in the early parts of the day due to accumulation of news and order flow for individual companies overnight. This stems both from the reduced trading activity after the market close and regulation prohibiting the release of pertinent corporate information during active trading.[5]

To illustrate this, we plot in Figure 6 the average number of firm-specific news items released

[5] See Hong and Wang (2000) for an early theoretical account of how market closures may impact the intraday return pattern through the introduction of additional overnight risk, cumulating news, and asymmetric information.

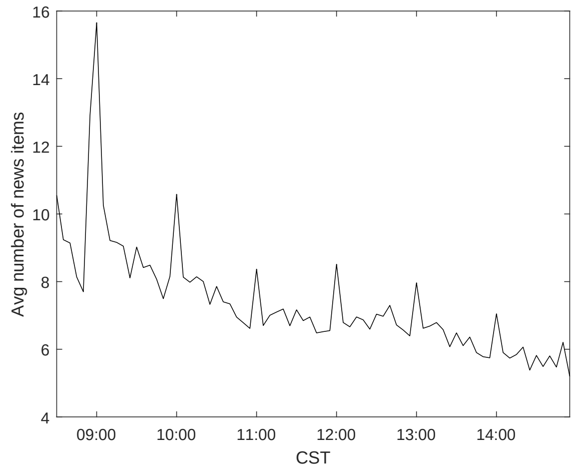

Figure 6: **Average Number of Firm-Specific News Items across the Trading Day**. The figure plots the average number of news items about firms in our sample arriving within each 5-minute interval.

every 5 minutes across the trading day for the stocks in our sample.[6] The intensity of news arrivals displays a periodic structure with pronounced spikes every hour. More importantly, beside these hourly bursts, a steady decline in intensity across the trading day is evident, with the two features interacting, but the spikes shrinking drastically, as the day progresses.

The above pattern raises a number of questions. In particular, one can explore whether the heightened intensity of firm-specific news has wider systematic risk and pricing implications. This may occur if the news regarding one company is not purely idiosyncratic, but also is expected to be indicative about the prospects of related or competing firms, either through supply chain links, industry-wide interactions, or signals regarding the underlying economic fundamentals. The implication of this is that the covariance between firm-specific and systematic market-wide cash flow news spikes during periods with an elevated rate of of firm-specific earnings relevant news; see, e.g., the extensive discussion and evidence in Da and Warachka (2009), Savor and Wilson (2016) and Ben-Rephael, Carlin, Da, and Israelsen (2020). In addition, Patton and Verardo (2012) document a pronounced day-by-day shift in betas around earnings announcement days, which they ascribed to learning across stocks. Although we cannot pursue these issues in depth within the confines of this paper, we dedicate the next section to an initial investigation of empirical hypotheses inspired by the diversity in the information flow over time, and in Section 8 we provide a few summary reflections on the potential asset pricing implications.

[6]This information is generated from the Ravenpack database. We only consider news items with a Ravenpack score of 100, ensuring that the firm "plays a key role in the news story and is considered highly relevant." Furthermore, we filter out items that merely describe the past evolution in trading or for which no news category was assigned.

7.3 Reaction to Informational Shocks

In Sections 7.1 and 7.2, we established that the cross-sectional distribution of market betas varies significantly over time and across the trading day. We also documented a significant heterogeneity in the information flow regarding firm-specific news across the trading day. In this section, we explore more directly whether different types of news shocks have a differential impact on the cross-sectional beta dispersion. For this purpose, we identify particular periods where we expect large innovations to occur for specific types of information. In particular, we focus on trading days following a large batch of corporate earnings news, days for which scheduled FOMC announcements are forthcoming, and days that involve a large shift in a risk-neutral market tail measure.

We start with the earnings announcements. Firms go through a quarterly cycle, with earnings news being released in the overnight period between two trading days. The announcements are clustered, with most firms releasing within a few weeks of each other. We define the quarterly earnings week to be the first week for which the market-weighted share of announcing firms exceeds 20%. Because the beta dispersion changes over time, the earnings week is compared to a nearby control week, namely the nearest preceding week with a low number of earnings releases, defined as less than 0.5% of stocks by market value. Thus, with the weeks being close in time and selected only by reference to the number of announcing firms, they should – all else equal – be similar.

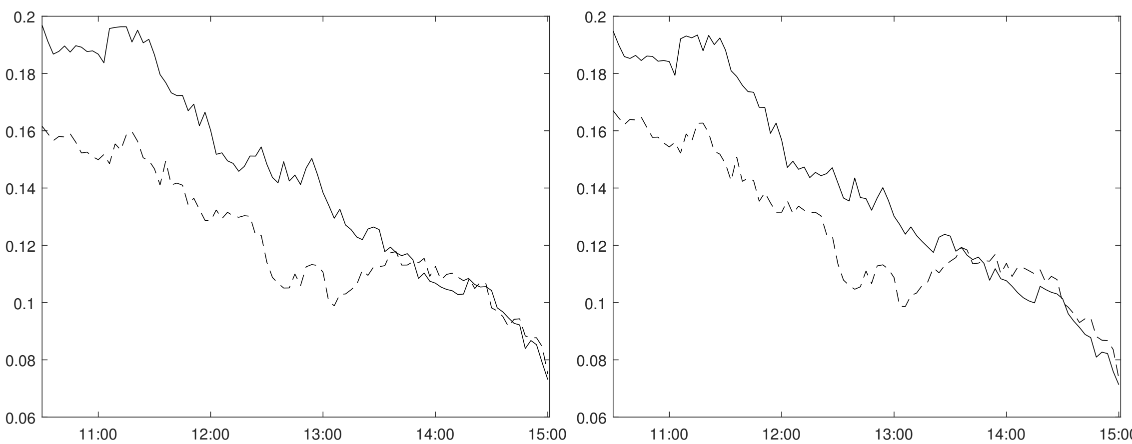

Figure 7: **The effect of earnings information**. The left panel plots the dispersion on days in earnings periods (solid line) and in the week leading up to these earnings days (dashed line). The right panel presents the same plot, but only including the stocks that do not release earnings announcements during the given week.

The left panel of Figure 7 depicts the intraday cross-sectional market beta dispersion for the quarterly earnings announcement weeks versus the non-announcement weeks preceding them. We observe a fairly constant gap, with the earnings weeks displaying an elevated dispersion relative to non-announcement weeks, ranging from the market open until about 1pm. Subsequently, the gap closes and the two curves are nearly identical for the last 90 minutes of active trading. This finding is consistent with the idea that firm-specific news has a heterogeneous impact on stocks, exacerbating the beta dispersion for a substantial part of the trading day. This does not speak directly to the systematic impact of earnings announcements, as the discrepancy between the two curves may be due exclusively to a reaction among the announcing firms. Therefore, in the right panel, we display the same curves, but only for stocks that are not releasing earnings during this week. It is evident that the qualitative impact is identical. In other words, the earnings announcements may be firm-specific, but the news shock they generate is not idiosyncratic; they carry important information regarding the future prospects of related firms, either through the information conveyed about the recent economic developments or through trends about specific industries.

One well-identified example of economic news with no firm-specific component is the 1pm CT Wednesday announcements following the regular FOMC meetings held, usually, every six weeks. The left panel of Figure 8 displays the intraday cross-sectional beta dispersion across all FOMC announcement days along with the corresponding dispersion on all other Wednesdays.[7] Due to the limited number of days with FOMC announcements in our sample (a total of 69), the estimated beta dispersion function for these days is visibly noisier. Nevertheless, the difference relative to days without an FOMC announcement is evident. The two curves are closely aligned, until there is a sharp drop in the beta dispersion right around the announcement time, and it then remains flat afterwards for the remainder of the trading day. The formal test for equal dispersion functions on days with and without FOMC announcements is in line with the above discussion. It rejects the hypothesis with p-value well below conventional significance levels. At an intuitive level, our results suggest that the stocks become more similar in terms of their exposure to systematic macro level risk in the period around and immediately following the FOMC announcement.

More broadly, enhanced market or economic policy uncertainty tend to manifest itself in an elevated compensation for exposure to market risk, especially on the downside. Such scenarios may be proxied by trading days for which the VIX volatility index rises sharply, and likely even more accurately through alternative option-implied measures capturing the compensation for return variation on the downside.[8] In the right panel of Figure 8, we depict the intraday market beta

[7] Because there is substantial variation in the distribution of firm-specific and macroeconomic announcements over the weekdays, it is likely that the intraday beta dispersion patterns are not identical either. Hence, we compare the FOMC announcements occurring on Wednesdays only to other Wednesdays, which should ensure that we control for this effect and get a direct read on the impact of the FOMC news release.

[8] Andersen, Fusari, and Todorov (2015) and Andersen, Todorov, and Ubukata (2019) find such tail measures to be more effective than the commonly adopted variance risk premium in capturing risk compensation in equity markets.

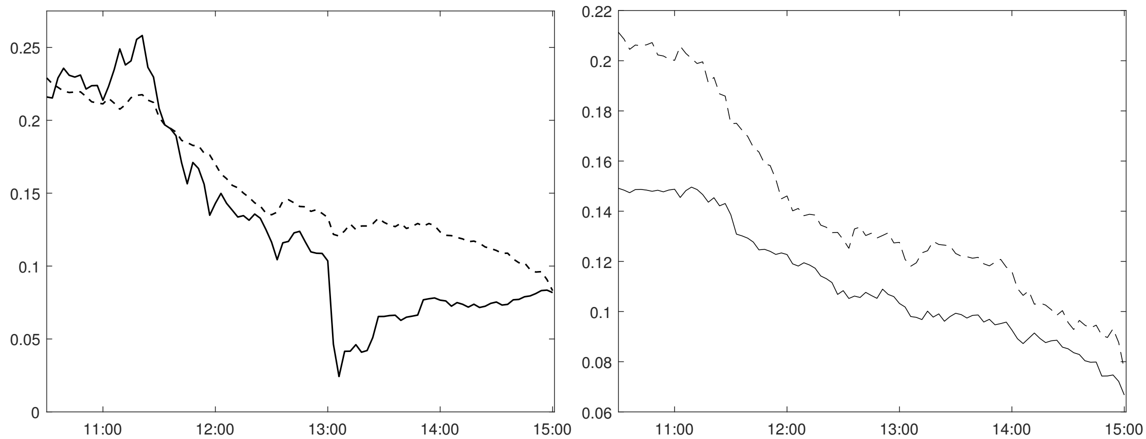

Figure 8: **Market Beta Dispersion following general Economic Shocks.** The left panel depicts the dispersion of the market betas computed on Wednesdays with (solid line) and without (dashed line) FOMC announcements. The right panel displays the cross-sectional dispersion in the market betas computed for days with the 10% largest increases in the risk-neutral tail index (solid line), and the 25% lowest increases in the tail index (dashed line).

dispersion for the trading days witnessing the 10% largest versus the 25% lowest increases in a risk-neutral downside tail measure.[9] We now find a persistently lower beta dispersion for days featuring the higher increases in the tail measure, and with a gap that declines almost monotonically across the trading day.[10] This lines up well with the left panel, where the macroeconomic release time is known, and the contraction of the beta dispersion occurs only following the news release.

The strikingly different responses of the intraday cross-sectional market beta dispersion to firm-specific (earnings) versus broader economic uncertainty shocks or announcements are telling. It provides strong evidence that the release and nature of informational shocks have a pronounced short-term impact on the cross-sectional sensitivity of stock returns to overall market movements.

8 Concluding Remarks and Directions for Future Work

This paper explores the intraday variation in market betas in a general nonparametric setting. The inference is based on a cross-section of stocks and a related proxy for the market portfolio. The approximations to the distribution of the relevant statistics are developed for a fixed time span of the return panel, while the sampling frequency increases to infinity, and the size of the cross-section is either fixed or asymptotically increasing. We form estimates of the cross-sectional market beta

[9]We use the smoothed version of the left-sided tail variation index for S&P 500 stocks from the website: tailindex.com. It is computed using end-of-day SPX option prices from Cboe. A detailed description of the index construction is provided in the white paper available on the site.

[10]Sorting stocks into high and low volatility regimes based on the end-of-day VIX measure produces qualitatively similar results.

dispersion over local time windows. We derive a feasible limit theory for the beta dispersion measures, both for a fixed number of distinct times during the trading day and in a functional sense. We further extend the analysis through a functional limit theory for estimates of the characteristic function for the cross-sectional beta distribution at given points in time. Exploiting these econometric tools, we find strong evidence for systematic variation in the cross-sectional beta dispersion, both during the trading day and across days, with the betas having the highest dispersion at market open and compressing gradually towards unity over the course of the trading day.

We conclude with a few comments on directions for future work. One may wonder if there is any plausible economic rationale for the documented intraday market beta changes and the associated dispersion pattern. In prior work, firm characteristics have been used to model market beta dynamics over lower frequencies, but the innate nature of firms does not change within the trading day. Instead, we deem a risk-based explanation more likely. We conjecture that stocks load differently (i.e., have different betas) on different types of shocks to the market portfolio, and that the volatility of these distinct shocks vary systematically within the trading day. To illustrate how this scenario can induce rapid shifts in the market exposure, suppose that the diffusive market shocks can be split into two types, generically labeled a and b. Further assuming these shocks are orthogonal with variances $(\sigma_t^{(0,a)})^2$ and $(\sigma_t^{(0,b)})^2$, so that the total diffusive market variance $(\sigma_t^{(0)})^2$ is equal to their sum. Finally, let the exposure of asset j to a market shock of type a and b, respectively, be $\beta^{(j,a)}$ and $\beta^{(j,b)}$. Then, denoting the share of the market variance stemming from shocks of type c at time t by $\omega_t^{(c)} = (\sigma_t^{(0,c)})^2/(\sigma_t^{(0)})^2$, for $c = a, b$, the market beta decomposition for asset j takes the form,

$$\beta_t^{(j)} \;=\; \omega_t^{(a)}\,\beta^{(j,a)} \;+\; \omega_t^{(b)}\,\beta^{(j,b)}\,, \qquad \text{with} \quad \omega_t^{(a)} + \omega_t^{(b)} = 1. \tag{57}$$

This decomposition shows that the market beta can vary, even if the latent exposures to market shocks of type a and b remain constant, because the composition of type a and b shocks embedded in the market variance may fluctuate across the trading day. Specifically, if the "fundamental" betas, $\beta^{(j,a)}$ and $\beta^{(j,b)}$, have different cross-sectional properties, with the former being, say, more cross-sectionally dispersed than the latter, then the observed intraday market beta behavior can be rationalized by a weakening in the relative strength of the component $\sigma_t^{(0,a)}$ in the overall market variance, that is, for $\omega_t^{(a)}$ declining over the course of the trading day.

The above discussion is purely generic. However, this type of decomposition of shocks to the market portfolio arises naturally within equilibrium or reduced-form asset pricing models, where the relative importance of distinct risk factors vary over time. For example, Campbell and Vuolteenaho (2004) decomposes shocks to the market portfolio into news about future cash flows and discount rates. In this case, even if the cash-flow and discount-rate betas are constant, but not identical, systematic variation in the volatility of cash-flow versus discount-rate news across the trading day will manifest itself in a corresponding intraday variation in market betas. Informally, our finding of a strongly declining intraday pattern in firm-specific information arrivals in Section 7.2 suggests that cash-flow news are dominant during the earlier parts of the trading day, while other factors such as general macroeconomic and trading-related shocks become relatively more important towards the market close. Moreover, the strikingly different response of the cross-sectional beta dispersion to general market uncertainty and FOMC announcements versus the firm-specific earnings releases points to the importance of allowing for cross-sectionally heterogeneous exposures and pricing implications of diverse aggregate news shocks.

To test whether a hypothesis like the one above can explain the documented intraday market beta behavior, one can use a high-frequency identification approach and study stock price behavior around a pre-scheduled announcement (e.g., FOMC) where the market shock can be plausibly identified as being of a specific type (say, a discount-rate shock for FOMC announcements). An alternative is to seek identification through heteroskedasticity in the spirit of Rigobon (2003), relying on scenarios where heightened market volatility can be associated with a certain type of fundamental shock to the market portfolio. The econometric tools developed here should be helpful in this context, but we defer formal explorations of such identification strategies to future work.

At a general level, our results illustrate the potential of high-frequency data to assist in the identification of sources of variation in risk exposure for large cross-sections of financial assets. In particular, focusing on systematic intraday variation in information flow, diverse market conditions, or distinct economic events, along with the concurrent cross-section of high-frequency returns, future work should be able to more robustly identify the sensitivity of assets to certain types of economic shocks. As documented in this paper, the variation in these high-frequency features is very large, both within the day and for the same time-of-day across different trading days. Moreover, since we can observe such variation over relatively short calendar time windows, it is plausible that fundamental risk exposures and firm characteristics remain stable over such limited horizons. This should facilitate the identification of sources of priced risk, and allow us to assess their implications for cross-sectional pricing in greater detail than is feasible from daily or lower-frequency data, for which much longer (calendar time) samples are needed to obtain sufficient variation and statistical power.

9 Appendix

9.1 Assumptions and Proofs

In the proofs, we denote with K a positive constant that does not depend on n and N, and can change from one line to another.

9.1.1 Assumptions

Assumption A. *For the processes $(X^{(j)})_{j\geq 0}$ we have:*

(a) For a sequence of stopping times, $(T_m)_{m\geq 1}$, increasing to infinity, the processes $(\alpha^{(j)})_{j\geq 0}$, $(\beta^{(j)})_{j\geq 1}$, $(\gamma^{(j)})_{j\geq 1}$, and $(\widetilde{\sigma}^{(j)})_{j\geq 1}$, are all uniformly bounded on $[0, T\wedge T_m]$.

(b) The processes $|\sigma_t^{(0)}|$ and $|\sigma_{t-}^{(0)}|$ take positive values on $[0,T]$.

(c) For a sequence of stopping times, $(T_m)_{m\geq 1}$, increasing to infinity and a sequence of constants, $(K_m)_{m\geq 1}$, we have uniformly in $j\geq 1$:

$$\mathbb{E}\Big[\sup_{s,t\in[0,T\wedge T_m]}|\sigma_t^{(0)}-\sigma_s^{(0)}|^2+\sup_{s,t\in[0,T\wedge T_m]}|\beta_t^{(j)}-\beta_s^{(j)}|^2+\sup_{s,t\in[0,T\wedge T_m]}|\widetilde{\sigma}_t^{(j)}-\widetilde{\sigma}_s^{(j)}|^2 \\ +\sup_{s,t\in[0,T\wedge T_m]}|\gamma_t^{(j)}-\gamma_s^{(j)}||\gamma_t^{(j)}-\gamma_s^{(j)}|^\top\Big]\leq K_m|t-s|, \tag{58}$$

$$|\mathbb{E}(\beta_{t\wedge T_m}^{(j)}-\beta_{s\wedge T_m}^{(j)})|+|\mathbb{E}(\sigma_{t\wedge T_m}^{(0)}-\sigma_{s\wedge T_m}^{(0)})|\leq K_m|t-s|, \tag{59}$$

$$|\mathbb{E}(\chi^{(1)}_{s\wedge T_m,t\wedge T_m}\chi^{(2)}_{s\wedge T_m,t\wedge T_m})|\leq K_m|t-s|^2, \tag{60}$$

for $\chi^{(1)}_{s,t}$ equal to $\beta_t^{(j)}-\beta_s^{(j)}$ or $\sigma_t^{(0)}-\sigma_s^{(0)}$, and $\chi^{(2)}_{s,t}$ equal to one of $(W_t^{(0)}-W_s^{(0)})^2-(t-s)$, $(W_t^{(0)}-W_s^{(0)})(B_t-B_s)$ and $(W_t^{(0)}-W_s^{(0)})(\widetilde{W}_t^{(j)}-\widetilde{W}_s^{(j)})$.

(d) We have $\sum_{s\leq t}\Delta X_s^{(j)}=\int_0^t\int_E\delta^{(j)}(s,u)\mu(ds,du)$, for $j\geq 0$, and where μ is a Poisson random measure on $\mathbb{R}_+\times E$ with compensator $ds\otimes\nu(du)$, for some σ-finite measure ν on a Polish space E. Furthermore, the jump size functions $\delta^{(j)}$ are mappings $\Omega\times\mathbb{R}_+\times E\to\mathbb{R}$ which are locally predictable. For $j=0,1,...,$ we have $\int_0^T\int_E 1(\delta^{(j)}(s,u)\neq 0)ds\nu(du)<\infty$. For a sequence of stopping times, $(T_m)_{m\geq 1}$, increasing to infinity and a sequence of nonnegative functions $(\Gamma_m(u))_{m\geq 1}$ satisfying $\int_E(1\vee\Gamma_m^2(u))\nu(du)<\infty$, we have $|\delta^{(j)}(s,u)|\leq\Gamma_m(u)$, uniformly in $j\geq 0$, for $s\in[0,T\wedge T_m]$.

Assumption B. *We have the following uniform convergence in probability as $N\to\overline{N}$ for some $\overline{N}\in(0,\infty]$:*

$$\sup_{t=1,\ldots,T,\,\kappa\in[0,1]}\left|\frac{6}{N}\sum_{j=1}^N\left[\left(\beta_{t-1+\kappa}^{(j)}-1\right)^2\frac{(\widetilde{\sigma}_{t-1+\kappa}^{(j)})^2}{(\sigma_{t-1+\kappa}^{(0)})^2}\right]-\psi_{t,\kappa}^{(a)}\right|\xrightarrow{\mathbb{P}}0, \tag{61}$$

$$\sup_{t=1,\dots,T,\,\kappa\in[0,1]} \left| 4\left(\frac{1}{N}\sum_{j=1}^{N}(\beta^{(j)}_{t-1+\kappa}-1)^2\right)^2 - \psi^{(b)}_{t,\kappa}\right| \xrightarrow{\mathbb{P}} 0, \tag{62}$$

$$\sup_{t=1,\dots,T,\,\kappa\in[0,1]} \left\| \frac{6}{N}\sum_{j=1}^{N}\left[\left(\beta^{(j)}_{t-1+\kappa}-1\right)\frac{\gamma^{(j)}_{t-1+\kappa}}{\sigma^{(0)}_{t-1+\kappa}}\right] - \Lambda_{t,\kappa}\right\| \xrightarrow{\mathbb{P}} 0, \tag{63}$$

for some $\psi^{(a)}_{t,\kappa}$, $\psi^{(b)}_{t,\kappa}$ and $\Lambda_{t,\kappa}$, which are càdlàg functions of $t-1+\kappa$, and $||\cdot||$ denoting the Frobenius norm of a matrix. We further set $\psi^{(c)}_{t,\kappa} = \Lambda_{t,\kappa}\Lambda^{\top}_{t,\kappa}$.

To state the next assumption, we introduce the following notation:

$$\begin{aligned} k^N_{t,\kappa}(z,u) = -\frac{1}{N}\sum_{j=1}^{N} e^{i(u+z)\beta^{(j)}_{t-1+\kappa}}uz\Bigg[&\left(\beta^{(j)}_{t-1+\kappa}-1\right)^4 + \frac{3}{2}\left(\beta^{(j)}_{t-1+\kappa}-1\right)^2\frac{\gamma^{(j)}_{t-1+\kappa}(\gamma^{(j)}_{t-1+\kappa})^{\top}}{(\sigma^{(0)}_{t-1+\kappa})^2} \\ &+\frac{3}{2}\left(\beta^{(j)}_{t-1+\kappa}-1\right)^2\frac{(\widetilde{\sigma}^{(j)}_{t-1+\kappa})^2}{(\sigma^{(0)}_{t-1+\kappa})^2}\Bigg], \end{aligned} \tag{64}$$

$$\begin{aligned} c^N_{t,\kappa}(z,u) = \frac{1}{N}\sum_{j=1}^{N} e^{i(z-u)\beta^{(j)}_{t-1+\kappa}}uz\Bigg[&\left(\beta^{(j)}_{t-1+\kappa}-1\right)^4 + \frac{3}{2}\left(\beta^{(j)}_{t-1+\kappa}-1\right)^2\frac{\gamma^{(j)}_{t-1+\kappa}(\gamma^{(j)}_{t-1+\kappa})^{\top}}{(\sigma^{(0)}_{t-1+\kappa})^2} \\ &+\frac{3}{2}\left(\beta^{(j)}_{t-1+\kappa}-1\right)^2\frac{(\widetilde{\sigma}^{(j)}_{t-1+\kappa})^2}{(\sigma^{(0)}_{t-1+\kappa})^2}\Bigg]. \end{aligned} \tag{65}$$

Assumption C. *We have the following convergence in probability as $N \to \overline{N}$, with some $\overline{N} \in (0,\infty]$:*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |k^N_{t,\kappa}(z,u) - k_{t,\kappa}(z,u)|^2 w(u)w(z)dudz \xrightarrow{\mathbb{P}} 0, \tag{66}$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |c^N_{t,\kappa}(z,u) - c_{t,\kappa}(z,u)|^2 w(u)w(z)dudz \xrightarrow{\mathbb{P}} 0. \tag{67}$$

for some random functions $k_{t,\kappa}(z,u)$ and $c_{t,\kappa}(z,u)$, some $t \in \mathbb{N}_+ \cap [0,T]$ and $\kappa \in [0,1]$, and with w being the weight function for the $\mathcal{L}^2(w)$ space defined in (46).

9.1.2 Localization

Assumption SA. *We have Assumption A with $T_1 = \infty$. Furthermore, the processes $(\alpha^{(j)})_{j\geq 0}$, $(\beta^{(j)})_{j\geq 1}$, $(\gamma^{(j)})_{j\geq 1}$, and $(\widetilde{\sigma}^{(j)})_{j\geq 1}$ are uniformly bounded on $[0,T]$, and $|\sigma^{(0)}_t|$ is bounded from below by a positive constant on $[0,T]$.*

We will prove the results under the stronger Assumption SA. A standard localization argument then can be used to show that they continue to hold under the weaker Assumption A.

9.1.3 Notation and Decomposition

We start with introducing some notation that will be used in the proofs. Throughout, we will use the shorthand notation $\mathbb{E}^n_{t,i}(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_{t-1+(i-1)/n})$ for $t \in \mathbb{N}_+$. For a generic process Z_t, we set

$$Z_{t,\kappa,n} = Z_{t-1+\frac{\lfloor \kappa n \rfloor - k_n}{n}}, \quad t \in \mathbb{N}_+, \ \kappa \in [0,1]. \tag{68}$$

and

$$\Delta^{n,2}_{t,i} Z = Z_{(t-1)+i/n} - Z_{(t-1)+(i-2)/n}, \quad t \in \mathbb{N}_+, \quad i = 2, ..., n. \tag{69}$$

We denote the spot variances of the asset prices by

$$V^{(0)}_t = (\sigma^{(0)}_t)^2, \quad V^{(j)}_t = \left(\beta^{(j)}_t\right)^2 V^{(0)}_t + \gamma^{(j)}_t (\gamma^{(j)}_t)^\top + \left(\widetilde{\sigma}^{(j)}_t\right)^2, \quad j = 1, ..., N, \tag{70}$$

and their continuous martingale components by

$$X^{c,(j)}_t = \int_0^t \beta^{(j)}_s \sigma^{(0)}_s dW^{(0)}_s + 1_{\{j \geq 1\}} \int_0^t \gamma^{(j)}_s dB_s + 1_{\{j \geq 1\}} \int_0^t \widetilde{\sigma}^{(j)}_s d\widetilde{W}^{(j)}_s, \quad k = 0, 1, ..., N, \tag{71}$$

where we used the normalization $\beta^{(0)}_s = 1$, for $s \in [0, T]$. We further set for $j = 1, ..., N$:

$$C^{(j)}_t = \left(\beta^{(j)}_t - 1\right) V^{(0)}_t, \quad t \in \mathbb{R}_+, \tag{72}$$

and for $t \in \mathbb{N}_+$ and $\kappa \in [0,1]$:

$$\widetilde{C}^{(j)}_{t,\kappa} = \frac{n}{|\widetilde{\mathcal{I}}^n_\kappa|} \sum_{i \in \widetilde{\mathcal{I}}^n_\kappa} \int_{t-1+(i-1)\Delta_n}^{t-1+i\Delta_n} (\beta^{(j)}_s - 1) V^{(0)}_s ds, \quad \widetilde{V}^{(j)}_{t,\kappa} = \frac{n}{|\mathcal{I}^n_\kappa|} \sum_{i \in \mathcal{I}^n_\kappa} \int_{t-1+(i-1)\Delta_n}^{t-1+i\Delta_n} V^{(j)}_s ds. \tag{73}$$

as well as

$$\overline{C}^{(j)}_{t,\kappa}(2) = \frac{n}{2|\widetilde{\mathcal{I}}^n_\kappa|} \sum_{i \in \widetilde{\mathcal{I}}^n_\kappa} \left[(\Delta^{n,2}_{t,i} X^{c,(j)} - \Delta^{n,2}_{t,i} X^{c,(0)}) \Delta^{n,2}_{t,i} X^{c,(0)} \right], \ \overline{V}^{(j)}_{t,\kappa} = \frac{n}{|\mathcal{I}^n_\kappa|} \sum_{i \in \mathcal{I}^n_\kappa} \left(\Delta^n_{t,i} X^{c,(j)} \right)^2. \tag{74}$$

We similarly define $\breve{\widetilde{C}}^{(j)}_{t,\kappa}$ and $\breve{\widetilde{V}}^{(0)}_{t,\kappa}$ as well as $\breve{\overline{C}}^{(j)}_{t,\kappa}$ and $\breve{\overline{V}}^{(0)}_{t,\kappa}$ from $\breve{C}^{(j)}_{t,\kappa}$ and $\breve{V}^{(0)}_{t,\kappa}$. Finally, we denote for $t \in \mathbb{N}_+$ and $\kappa \in [0,1]$:

$$\begin{aligned} \overline{\eta}^{n,(j)}_{t,i} &= \sigma^{(0)}_{t,\kappa,n} \gamma^{(j)}_{t,\kappa,n} n \Delta^n_{t,i} W^{(0)} \Delta^n_{t,i} B + \sigma^{(0)}_{t,\kappa,n} \widetilde{\sigma}^{(j)}_{t,\kappa,n} n \Delta^n_{t,i} W^{(0)} \Delta^n_{t,i} \widetilde{W}^{(j)} \\ &\quad + (\beta^{(j)}_{t,\kappa,n} - 1) V^{(0)}_{t,\kappa,n} (n (\Delta^n_{t,i} W^{(0)})^2 - 1), \ k = 1, ..., N, \end{aligned} \tag{75}$$

$$\begin{aligned} \overline{\eta}^{n,(j)}_{t,i}(2) &= \frac{1}{2} \sigma^{(0)}_{t,\kappa,n} \gamma^{(j)}_{t,\kappa,n} n \Delta^{n,2}_{t,i} W^{(0)} \Delta^{n,2}_{t,i} B + \frac{1}{2} \sigma^{(0)}_{t,\kappa,n} \widetilde{\sigma}^{(j)}_{t,\kappa,n} n \Delta^{n,2}_{t,i} W^{(0)} \Delta^{n,2}_{t,i} \widetilde{W}^{(j)} \\ &\quad + \frac{1}{2} (\beta^{(j)}_{t,\kappa,n} - 1) V^{(0)}_{t,\kappa,n} (n (\Delta^{n,2}_{t,i} W^{(0)})^2 - 2), \ k = 1, ..., N, \end{aligned} \tag{76}$$

$$\overline{\eta}_{t,i}^{n,(0)} = V_{t,\kappa,n}^{(0)}(n(\Delta_{t,i}^n W^{(0)})^2 - 1), \tag{77}$$

and we use them to define the following processes:

$$\mathcal{C}_{t,\kappa}^{(j)}(2) = \frac{1}{|\widetilde{\mathcal{I}}_\kappa^n|}\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n}\overline{\eta}_{t,i}^{n,(j)}(2),\ \breve{\mathcal{C}}_{t,\kappa}^{(j)} = \frac{1}{|\mathcal{I}_\kappa^n|}\sum_{i\in\mathcal{I}_\kappa^n}\overline{\eta}_{t,i}^{n,(j)}(1_{\{i\in\mathcal{O}_\kappa^n\}} - 1_{\{i\in\mathcal{E}_\kappa^n\}}), \tag{78}$$

$$\mathcal{V}_{t,\kappa}^{(0)} = \frac{1}{|\mathcal{I}_\kappa^n|}\sum_{i\in\mathcal{I}_\kappa^n}\overline{\eta}_{t,i}^{n,(0)},\ \breve{\mathcal{V}}_{t,\kappa}^{(0)} = \frac{1}{|\mathcal{I}_\kappa^n|}\sum_{i\in\mathcal{I}_\kappa^n}\overline{\eta}_{t,i}^{n,(0)}(1_{\{i\in\mathcal{O}_\kappa^n\}} - 1_{\{i\in\mathcal{E}_\kappa^n\}}). \tag{79}$$

9.1.4 Preliminary Results

We start with establishing some preliminary results about the moments of components of the differences $\widehat{C}_{t,\kappa}^{(j)}(2) - C_{t,\kappa,n}^{(j)}$ and $\widehat{V}_{t,\kappa}^{(j)} - V_{t,\kappa,n}^{(j)}$ as well as $\breve{C}_{t,\kappa}^{(j)}$ and $\breve{V}_{t,\kappa}^{(0)}$.

Lemma 1. *Assume Assumption SA holds. For $j,l = 0,1,...,N$ and $p \geq 1$, we have:*

$$\begin{aligned}&\mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^n|\widehat{C}_{t,\kappa}^{(j)}(2) - \overline{C}_{t,\kappa}^{(j)}(2)|^p + \mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^n|\widehat{V}_{t,\kappa}^{(j,l)} - \overline{V}_{t,\kappa}^{(j)}|^p + \mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^n|\breve{C}_{t,\kappa}^{(j)} - \breve{\overline{C}}_{t,\kappa}^{(j)}|^p \\ &+ \mathbb{E}_{t,\lfloor\kappa n\rfloor-k_n+1}^n|\breve{V}_{t,\kappa}^{(0,l)} - \breve{\overline{V}}_{t,\kappa}^{(0)}|^p \leq K\left(k_n^{1-p}\Delta_n^{1+p(2\varpi-1)} \vee \Delta_n^{p+p(2\varpi-1)}\right),\ t = 1,...,T,\ \kappa\in[0,1],\end{aligned} \tag{80}$$

for some positive constant K which can depend on p and ϖ but does not depend on k.

Proof of Lemma 1. We show only the bound for the terms involving $\widehat{C}_{t,\kappa}^{(j)}(2)$ and $\breve{C}_{t,\kappa}^{(j)}$, with the result for the terms involving $\widehat{V}_{t,\kappa}^{(j,l)}$ and $\breve{V}_{t,\kappa}^{(0,l)}$ established analogously. For $j,l = 0,1,...,N$, we can make the decompositions:

$$\begin{aligned}&\Delta_{t,i}^n X^{(j)}\Delta_{t,i}^n X^{(l)}1_{\{\mathcal{A}_{t,i}^{(j,l)}\}} - \Delta_{t,i}^n X^{c,(j)}\Delta_{t,i}^n X^{c,(l)} = -\Delta_{t,i}^n X^{c,(j)}\Delta_{t,i}^n X^{c,(l)}1_{\{(\mathcal{A}_{t,i}^{(j,l)})^c\}} \\ &+ \Delta_{t,i}^n X^{d,(j)}\Delta_{t,i}^n X^{d,(l)}1_{\{\mathcal{A}_{t,i}^{(j,l)}\}} + \left(\Delta_{t,i}^n X^{c,(j)}\Delta_{t,i}^n X^{d,(l)} + \Delta_{t,i}^n X^{d,(j)}\Delta_{t,i}^n X^{c,(l)}\right)1_{\{\mathcal{A}_{t,i}^{(j,l)}\}},\end{aligned} \tag{81}$$

$$\begin{aligned}&\Delta_{t,i}^{n,2} X^{(j)}\Delta_{t,i}^{n,2} X^{(l)}1_{\{\mathcal{A}_{t,i-1}^{(j,l)}\,\cap\,\mathcal{A}_{t,i}^{(j,l)}\}} - \Delta_{t,i}^{n,2} X^{c,(j)}\Delta_{t,i}^{n,2} X^{c,(l)} \\ &= -\Delta_{t,i}^{n,2} X^{c,(j)}\Delta_{t,i}^{n,2} X^{c,(l)}1_{\{(\mathcal{A}_{t,i-1}^{(j,l)})^c\,\cup\,(\mathcal{A}_{t,i}^{(j,l)})^c\}} + \Delta_{t,i}^{n,2} X^{d,(j)}\Delta_{t,i}^{n,2} X^{d,(l)}1_{\{\mathcal{A}_{t,i-1}^{(j,l)}\,\cap\,\mathcal{A}_{t,i}^{(j,l)}\}} \\ &+ \left(\Delta_{t,i}^{n,2} X^{c,(j)}\Delta_{t,i}^{n,2} X^{d,(l)} + \Delta_{t,i}^{n,2} X^{d,(j)}\Delta_{t,i}^{n,2} X^{c,(l)}\right)1_{\{\mathcal{A}_{t,i-1}^{(j,l)}\,\cap\,\mathcal{A}_{t,i}^{(j,l)}\}}.\end{aligned} \tag{82}$$

Next, using the bounds for the continuous and jump components of Itô semimartingales in Section 2.1.5 of Jacod and Protter (2012), together with Hölder's inequality, we get for $p \geq 1$ and arbitrary small $\iota > 0$:

$$\mathbb{E}_{t,i}^n\left(|\Delta_{t,i}^n X^{c,(j)}\Delta_{t,i}^n X^{c,(l)}|^p 1_{\{(\mathcal{A}_{t,i}^{(j,l)})^c\}}\right) \leq K\Delta_n^{p+1-\iota}. \tag{83}$$

$$\mathbb{E}_{t,i}^n\left(|\Delta_{t,i}^n X^{c,(j)}\Delta_{t,i}^n X^{d,(l)}|^p 1_{\{\mathcal{A}_{t,i}^{(j,l)}\}}\right) \leq K\Delta_n^{p/2+p\varpi+1-\iota}. \tag{84}$$

$$\mathbb{E}^n_{t,i}\left(|\Delta^n_{t,i}X^{d,(j)}\Delta^n_{t,i}X^{d,(l)}|^p 1_{\{\mathcal{A}^{(j,l)}_{t,i}\}}\right) \leq K\Delta_n^{1+2p\varpi}. \tag{85}$$

Combining these bounds, and using successive application of Burkholder-Gundy-Davis inequalities as well as inequality in means, the result of the lemma follows. □

Lemma 2. *Assume Assumption SA holds. For $j = 0, 1, ..., N$, $t = 1, ..., T$ and $\kappa \in [0, 1]$, we have:*

$$\begin{aligned}
&\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}|\overline{C}^{(j)}_{t,\kappa}(2) - \widetilde{C}^{(j)}_{t,\kappa}|^p + \mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}|\overline{V}^{(j)}_{t,\kappa} - \widetilde{V}^{(j)}_{t,\kappa}|^p + \mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}|\breve{\mathcal{C}}^{(j)}_{t,\kappa}|^p \\
&\quad + \mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}|\mathcal{C}^{(j)}_{t,\kappa}(2)|^p + \mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}|\mathcal{V}^{(0)}_{t,\kappa}|^p + \mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}|\breve{\mathcal{V}}^{(0)}_{t,\kappa}|^p \\
&\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}|\breve{\overline{C}}^{(j)}_{t,\kappa} - \breve{\widetilde{C}}^{(j)}_{t,\kappa}|^p + \mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}|\breve{\overline{V}}^{(0)}_{t,\kappa} - \breve{\widetilde{V}}^{(0)}_{t,\kappa}|^p \leq Kk_n^{-p/2},\ p \geq 2,
\end{aligned} \tag{86}$$

for some positive constant K which can depend on p but does not depend on j. Furthermore, we have the following decompositions

$$\overline{C}^{(j)}_{t,\kappa}(2) = \widetilde{C}^{(j)}_{t,\kappa} + \mathcal{C}^{(j)}_{t,\kappa}(2) + R^{(j)}_{t,\kappa,n},\ \overline{V}^{(0)}_{t,\kappa} = \widetilde{V}^{(0)}_{t,\kappa} + \mathcal{V}^{(0)}_{t,\kappa} + R_{t,\kappa,n}, \tag{87}$$

$$\breve{\overline{C}}^{(j)}_{t,\kappa} = \breve{\widetilde{C}}^{(j)}_{t,\kappa} + \breve{\mathcal{C}}^{(j)}_{t,\kappa} + \breve{R}^{(j)}_{t,\kappa,n},\ \breve{\overline{V}}^{(0)}_{t,\kappa} = \breve{\widetilde{V}}^{(0)}_{t,\kappa} + \breve{\mathcal{V}}^{(0)}_{t,\kappa} + \breve{R}_{t,\kappa,n}, \tag{88}$$

where for $\mathcal{R}_{t,\kappa,n} = R^{(k)}_{t,\kappa,n}$, $R_{t,\kappa,n}$, $\breve{R}^{(k)}_{t,\kappa,n}$ or $\breve{R}_{t,\kappa,n}$, we have

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}|\mathcal{R}_{t,\kappa,n}|^p \leq \frac{K}{k_n^{p-2}n},\ p \geq 2, \tag{89}$$

and for any bounded function $\omega : [0, 1] \to \mathbb{R}_+$, we also have

$$\mathbb{E}\left|\sum_{s=k_n}^{n}\omega(s\Delta_n)\mathcal{R}_{t,s\Delta_n,n}\right|^2 \leq Kk_n. \tag{90}$$

Proof of Lemma 2. The first result of the lemma follows by (successive) use of Burkholder-Davis-Gundy inequality. We now show the remaining claims for $\overline{C}^{(j)}_{t,\kappa}(2)$ only, with the corresponding result for $\breve{\overline{C}}^{(j)}_{t,\kappa}$, $\overline{V}^{(0)}_{t,\kappa}$ and $\breve{\overline{V}}^{(0)}_{t,\kappa}$ being established in an analogous way. Using Itô's lemma, we have

$$n(\Delta^{n,2}_{t,i}X^{c,(j)} - \Delta^{n,2}_{t,i}X^{c,(0)})\Delta^{n,2}_{t,i}X^{c,(0)} = n\Delta^{n,2}_{t,i}C^{(j)} + \eta^{n,(j)}_{t,i}(2), \tag{91}$$

where the term $\eta^{n,(j)}_{t,i}(2)$ satisfies

$$\mathbb{E}^n_{t,i-1}(\eta^{n,(j)}_{t,i}(2) - \overline{\eta}^{n,(k)}_{t,i}(2)) = 0,\quad \mathbb{E}^n_{t,i-1}|\eta^{n,(j)}_{t,i}(2) - \overline{\eta}^{n,(j)}_{t,i}(2)|^p \leq Kk_n/n,\ p \geq 2, \tag{92}$$

and for the second result we made use of the smoothness in expectation assumption for the processes $\sigma^{(0)}_t$, $\beta^{(j)}_t$, $\gamma^{(j)}_t$ and $\widetilde{\sigma}^{(j)}_t$ as well as Burkholder-Davis-Gundy inequality. From here the result of the lemma follows. □

Lemma 3. *Assume Assumption SA holds. For $j = 0, 1, ..., N$, $t = 1, ..., T$ and $\kappa \in [0, 1]$, we have*

$$\begin{aligned} &\mathbb{E}^n_{t,\lfloor \kappa n \rfloor - k_n + 1}(|\widetilde{C}^{(j)}_{t,\kappa} - C^{(j)}_{t,\kappa,n}|^p 1_{\{k \neq 0\}}) + \mathbb{E}^n_{t,\lfloor \kappa n \rfloor - k_n + 1}|\widetilde{V}^{(j)}_{t,\kappa} - V^{(j)}_{t,\kappa,n}|^p \\ &\quad + \mathbb{E}^n_{t,\lfloor \kappa n \rfloor - k_n + 1}(|\breve{\widetilde{C}}^{(j)}_{t,\kappa}|^p 1_{\{k \neq 0\}}) + \mathbb{E}^n_{t,\lfloor \kappa n \rfloor - k_n + 1}|\breve{\widetilde{V}}^{(j)}_{t,\kappa}|^p \leq K k_n / n, \quad p \geq 2, \end{aligned} \tag{93}$$

for some positive constant K which can depend on p but does not depend on j. In addition, for $\omega : [0, 1] \to \mathbb{R}_+$ that is Lipschitz continuous and $t = 1, ..., T$, we have

$$\frac{1}{n - k_n + 1} \sum_{s=k_n}^{n} \left[\omega(s\Delta_n) \left(\frac{\widetilde{C}^{(j)}_{t,s\Delta_n}}{\widetilde{V}^{(0)}_{t,s\Delta_n}} \right)^2 \right] = \int_{t-1}^{t} \omega(s - t + 1) \left(\beta^{(j)}_s - 1 \right)^2 ds + O_p\left(\frac{k_n}{n} \right), \tag{94}$$

with $j = 1, ..., N$.

Proof of Lemma 3. Using Assumption SA regarding the smoothness in expectation of the processes $\beta^{(j)}_t$ and $\sigma^{(0)}_t$ as well as the boundedness of these processes, we can write

$$\left| \mathbb{E}^n_{t,\lfloor \kappa n \rfloor - k_n + 1} \left(\widetilde{C}^{(j)}_{t,\kappa} - C^{(j)}_{t,\kappa,n} \right) \right| + \left| \mathbb{E}^n_{t,\lfloor \kappa n \rfloor - k_n + 1} \left(\widetilde{V}^{(j)}_{t,\kappa} - V^{(j)}_{t,\kappa,n} \right) \right| \leq K k_n / n, \tag{95}$$

$$\mathbb{E}^n_{t,\lfloor \kappa n \rfloor - k_n + 1} \left| \widetilde{C}^{(j)}_{t,\kappa} - C^{(j)}_{t,\kappa,n} \right|^p + \mathbb{E}^n_{t,\lfloor \kappa n \rfloor - k_n + 1} \left| \widetilde{V}^{(j)}_{t,\kappa} - V^{(j)}_{t,\kappa,n} \right|^p \leq K k_n / n, \quad p \geq 2, \tag{96}$$

where the constant K does not depend on κ. From here the results of the lemma follow directly by taking into account that ω is Lipschitz continuous. $\square$

Lemma 4. *Assume Assumption SA holds and $k_n \Delta_n^{2-4\varpi} \to \infty$. For $j, l = 0, 1, ..., N$ and $p \geq 1$, we have:*

$$\mathbb{E}^n_{t,\lfloor \kappa n \rfloor - k_n + 1} \left| \frac{1}{|\mathcal{I}^n_\kappa|} \sum_{i \in \mathcal{I}^n_\kappa} \left(\Delta^n_{t,i} X^{(0)} \Delta^n_{t,i} X^{(j)} 1_{\{\mathcal{A}^{(0,l)}_{t,i}\}} \right)^2 \right|^p \leq K, \ \ t = 1, ..., T, \ \ \kappa \in [0, 1]. \tag{97}$$

Proof of Lemma 4. The result of the lemma follows by application of the bounds in (83)-(85) together with an application of Burkholder-Davis-Gundy inequalities, and further taking into account that $k_n \Delta_n^{2-4\varpi} \to \infty$. $\square$

9.1.5 Proof of Theorem 1

Second-order Taylor expansion yields for $j = 1, ..., N$:

$$\begin{aligned} &\left(\frac{\widehat{C}^{(j)}_{t,\kappa}(2)}{\widehat{V}^{(0,j)}_{t,\kappa}} \right)^2 1_{\{\widehat{V}^{(0,j)}_{t,\kappa} > \alpha_n\}} - \left(\frac{\widetilde{C}^{(j)}_{t,\kappa}}{\widetilde{V}^{(0)}_{t,\kappa}} \right)^2 = Z^{(j)}_{t,\kappa,n} + B^{(j)}_{t,\kappa,n} \\ &\quad - \left(\left(\frac{\widetilde{C}^{(j)}_{t,\kappa}}{\widetilde{V}^{(0)}_{t,\kappa}} \right)^2 + Z^{(j)}_{t,\kappa,n} + B^{(j)}_{t,\kappa,n} \right) 1_{\{\widehat{V}^{(0,j)}_{t,\kappa} \leq \alpha_n\}} + \overline{R}^{(j)}_{t,\kappa,n}, \end{aligned} \tag{98}$$

where

$$Z_{t,\kappa,n}^{(j)} = 2\frac{\widetilde{C}_{t,\kappa}^{(j)}}{\widetilde{V}_{t,\kappa}^{(0)}}\frac{1}{\widetilde{V}_{t,\kappa}^{(0)}}\left(\widehat{C}_{t,\kappa}^{(j)}(2) - \widetilde{C}_{t,\kappa}^{(j)} - \frac{\widetilde{C}_{t,\kappa}^{(j)}}{\widetilde{V}_{t,\kappa}^{(0)}}(\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)})\right), \tag{99}$$

$$\begin{aligned} B_{t,\kappa,n}^{(j)} = & 3\frac{\left(\widetilde{C}_{t,\kappa}^{(j)}\right)^2}{\left(\widetilde{V}_{t,\kappa}^{(0)}\right)^4}\left(\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}\right)^2 + \frac{1}{\left(\widetilde{V}_{t,\kappa}^{(0)}\right)^2}\left(\widehat{C}_{t,\kappa}^{(j)}(2) - \widetilde{C}_{t,\kappa}^{(j)}\right)^2 \\ & - 4\frac{\widetilde{C}_{t,\kappa}^{(j)}}{\left(\widetilde{V}_{t,\kappa}^{(0)}\right)^3}\left(\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}\right)\left(\widehat{C}_{t,\kappa}^{(j)}(2) - \widetilde{C}_{t,\kappa}^{(j)}\right), \end{aligned} \tag{100}$$

and the residual term $\overline{R}_{t,\kappa,n}^{(j)}$ satisfies

$$\begin{aligned} |\overline{R}_{t,\kappa,n}^{(j)}| \leq & K\alpha_n^{-4}(|\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}|^3 \vee |\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}|^6) \\ & + K\alpha_n^{-4}|\widehat{C}_{t,\kappa}^{(j)}(2) - \widetilde{C}_{t,\kappa}^{(j)}|(|\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}|^2 \vee |\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}|^4) \\ & + K\alpha_n^{-4}|\widehat{C}_{t,\kappa}^{(j)}(2) - \widetilde{C}_{t,\kappa}^{(j)}|^2(|\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}| \vee |\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}|^4), \end{aligned} \tag{101}$$

where the constant K does not depend on j. Finally, in what follows we use the following notation

$$\begin{aligned} \overline{Z}_{t,\kappa,n}^{(j)} = & 2\big(\beta_{t,\kappa,n}^{(j)} - 1\big)^2\frac{n}{|\widetilde{\mathcal{I}}_\kappa^n|}\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n}\left[\Delta_{t,i-1}^n W^{(0)}\Delta_{t,i}^n W^{(0)}\right] \\ & + \big(\beta_{t,\kappa,n}^{(j)} - 1\big)\frac{n}{|\widetilde{\mathcal{I}}_\kappa^n|}\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n}\left[\frac{\gamma_{t,\kappa,n}^{(j)}}{\sigma_{t,\kappa,n}^{(0)}}\Delta_{t,i}^{n,2}W^{(0)}\Delta_{t,i}^{n,2}B + \frac{\widetilde{\sigma}_{t,\kappa,n}^{(j)}}{\sigma_{t,\kappa,n}^{(0)}}\Delta_{t,i}^{n,2}W^{(0)}\Delta_{t,i}^{n,2}\widetilde{W}^{(j)}\right]. \end{aligned} \tag{102}$$

The proof of the theorem consists of a sequence of lemmas.

Lemma 5. *Under Assumption SA, and provided $k_n\Delta_n \to 0$ and $k_n\Delta_n^{1-2\varpi} \to \infty$, we have for $t = 1, ..., T$ and $\kappa \in [0,1]$:*

$$\frac{1}{N}\sum_{j=1}^N \overline{R}_{t,\kappa,n}^{(j)} + \frac{1}{n-k_n+1}\sum_{s=k_n}^n \frac{1}{N}\sum_{j=1}^N |\overline{R}_{t,s\Delta_n,n}^{(j)}| = O_p\left(\frac{1}{\alpha_n^4}\left(k_n^{-2}\Delta_n^{1+3(2\varpi-1)}\bigvee\frac{1}{k_n^{3/2}}\right)\right). \tag{103}$$

Proof of Lemma 5. The result of the lemma follows by an application of Hölder's inequality and making use of the bounds of Lemmas 1-2 as well as the inequality in (101). □

Lemma 6. *Under Assumption SA, and provided $\varpi \in (1/4, 1/2)$, $\varrho \in (0, 1/2)$ and $k_n\Delta_n^{2-4\varpi} \to \infty$, we have for $t = 1, ..., T$, $\kappa \in [0,1]$ and some arbitrary small $\iota > 0$:*

$$\widehat{B}_{t,\kappa}^N - \frac{1}{N}\sum_{j=1}^N B_{t,\kappa,n}^{(j)} = O_p\left(\frac{1}{\alpha_n^4}\left(\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n}\bigvee\left(\frac{k_n}{n}\right)^{1-\iota}\bigvee\frac{1}{k_n^{3/2}}\right)\right), \tag{104}$$

and further for any bounded function $\omega : [0,1] \to \mathbb{R}$:

$$\begin{aligned}\frac{1}{n-k_n+1}\sum_{s=k_n}^{n}\left[\omega(s\Delta_n)\left(\widehat{B}^N_{t,s\Delta_n} - \frac{1}{N}\sum_{j=1}^{N} B^{(j)}_{t,s\Delta_n,n}\right)\right] \\ = O_p\left(\frac{1}{\alpha_n^4}\left(\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n}\bigvee\left(\frac{k_n}{n}\right)^{1-\iota}\bigvee\frac{1}{k_n^{3/2}}\right)\right).\end{aligned} \tag{105}$$

Proof of Lemma 6. We start with defining the processes for $j = 1, ..., N$ (recall the notation in equations (75)-(79)):

$$\mathcal{B}^{1,(j)}_{t,\kappa,n} = -\frac{(\beta^{(j)}_{t,\kappa,n}-1)^2}{(V^{(0)}_{t,\kappa,n})^2}\left(\breve{\mathcal{V}}^{(0)}_{t,\kappa}\right)^2 + \frac{3}{2(V^{(0)}_{t,\kappa,n})^2}\left(\breve{\mathcal{C}}^{(j)}_{t,\kappa}\right)^2, \tag{106}$$

$$\mathcal{B}^{2,(j)}_{t,\kappa,n} = 3\frac{(\beta^{(j)}_{t,\kappa,n}-1)^2}{(V^{(0)}_{t,\kappa,n})^2}(\mathcal{V}^{(0)}_{t,\kappa})^2 + \frac{1}{(V^{(0)}_{t,\kappa,n})^2}(\mathcal{C}^{(j)}_{t,\kappa}(2))^2 - 4\frac{(\beta^{(j)}_{t,\kappa,n}-1)}{(V^{(0)}_{t,\kappa,n})^2}\mathcal{V}^{(0)}_{t,\kappa}\mathcal{C}^{(j)}_{t,\kappa}(2). \tag{107}$$

By direct calculation:

$$\begin{aligned}\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}(\breve{\mathcal{C}}^{(j)}_{t,\kappa})^2 = \frac{1}{|\mathcal{I}^n_\kappa|}V^{(0)}_{t,\kappa,n}\gamma^{(j)}_{t,\kappa,n}(\gamma^{(j)}_{t,\kappa,n})^\top + \frac{1}{|\mathcal{I}^n_\kappa|}V^{(0)}_{t,\kappa,n}(\widetilde{\sigma}^{(j)}_{t,\kappa,n})^2 \\ + \frac{1}{|\mathcal{I}^n_\kappa|}(V^{(0)}_{t,\kappa,n})^2\left(\beta^{(j)}_{t,\kappa,n}-1\right)^2,\end{aligned} \tag{108}$$

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}(\mathcal{C}^{(j)}_{t,\kappa}(2))^2 = \frac{3}{2}\frac{|\mathcal{I}^n_\kappa|}{|\widetilde{\mathcal{I}}^n_\kappa|}\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}(\breve{\mathcal{C}}^{(j)}_{t,\kappa})^2, \tag{109}$$

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}(\mathcal{V}^{(0)}_{t,\kappa})^2 = \mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}(\breve{\mathcal{V}}^{(0)}_{t,\kappa})^2 = \frac{2}{|\mathcal{I}^n_\kappa|}(V^{(0)}_{t,\kappa,n})^2, \tag{110}$$

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}(\mathcal{V}^{(0)}_{t,\kappa}\mathcal{C}^{(j)}_{t,\kappa}(2)) = (\beta^{(j)}_{t,\kappa,n}-1)\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}(\mathcal{V}^{(0)}_{t,\kappa})^2. \tag{111}$$

From here, using the first bound in Lemma 2, we have

$$\left|\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}\left(\mathcal{B}^{1,(j)}_{t,\kappa,n} - \mathcal{B}^{2,(j)}_{t,\kappa,n}\right)\right| \le \frac{K}{k_n^2},\ \mathbb{E}[(\mathcal{B}^{1,(j)}_{t,\kappa,n})^2 + (\mathcal{B}^{2,(j)}_{t,\kappa,n})^2] \le \frac{K}{k_n^2}, \tag{112}$$

where the constant K does not depend on t, κ and j. Next, using Lemmas 1-2, Cauchy-Schwarz and Burkholder-Davis-Gundy inequalities and taking into account that $\varpi > 1/4$ and $\varrho \in (0, 1/2)$, we have

$$\mathbb{E}|B^{(j)}_{t,\kappa,n} - \mathcal{B}^{2,(j)}_{t,\kappa,n}| \le K\left(\frac{\Delta_n^{1+2(2\varpi-1)}}{k_n}\bigvee\frac{1}{\sqrt{k_n n}}\right). \tag{113}$$

We proceed with analyzing the difference $\widehat{B}^N_{t,\kappa,n} - \frac{1}{N}\sum_{j=1}^N \mathcal{B}^{1,(j)}_{t,\kappa,n}$. First, using Lemma 1-4, Hölder's inequality as well as the restriction $k_n\Delta_n^{2-4\varpi} \to \infty$ of the lemma, we have

$$\mathbb{E}\left|\widehat{V}^{(j)}_{\beta,t,\kappa}(\breve{V}^{(0,j)}_{t,\kappa})^2 1_{\{\breve{V}^{(0,j)}_{t,\kappa}>\alpha_n\}}\right| \le \frac{K}{\alpha_n^2}\frac{1}{k_n}\left[\frac{1}{k_n}\bigvee\Delta_n^{4\varpi-\iota}\right], \tag{114}$$

for some arbitrary small $\iota > 0$. We continue with introducing the following notation

$$\xi_{t,\kappa}^{(j,1)} = \frac{\widehat{\beta}_{t,\kappa}^{(j)} - 1}{\widehat{V}_{t,\kappa}^{(0,j)}} \breve{V}_{t,\kappa}^{(0)} - \frac{\beta_{t,\kappa,n}^{(j)} - 1}{V_{t,\kappa,n}^{(0)}} \breve{\mathcal{V}}_{t,\kappa}^{(0)}, \tag{115}$$

$$\xi_{t,\kappa}^{(j,2)} = \frac{1}{\widehat{V}_{t,\kappa}^{(0,j)}} \breve{C}_{t,\kappa}^{(j)} - \frac{1}{V_{t,\kappa,n}^{(0)}} \breve{\mathcal{C}}_{t,\kappa}^{(j)}. \tag{116}$$

Then, we have

$$\begin{aligned} |\xi_{t,\kappa}^{(j,1)}| 1_{\{\widehat{V}_{t,\kappa}^{(0,j)} > \alpha_n\}} \leq & \frac{K}{\alpha_n^2} (|\widehat{C}_{t,\kappa}^{(j,0)} - C_{t,\kappa,n}^{(j,0)}| \vee |\widehat{V}_{t,\kappa}^{(0,j)} - V_{t,\kappa,n}^{(0)}| \vee 1) |\breve{V}_{t,\kappa}^{(0)} - \breve{\mathcal{V}}_{t,\kappa}^{(0)}| \\ & + \frac{K}{\alpha_n^2} |\breve{\mathcal{V}}_{t,\kappa}^{(0)}| (|\widehat{C}_{t,\kappa}^{(j,0)} - C_{t,\kappa,n}^{(j,0)}| \vee |\widehat{V}_{t,\kappa}^{(0,j)} - V_{t,\kappa,n}^{(0)}|^2 \vee |\widehat{V}_{t,\kappa}^{(0,j)} - V_{t,\kappa,n}^{(0)}|), \end{aligned} \tag{117}$$

$$|\xi_{t,\kappa}^{(j,2)}| 1_{\{\widehat{V}_{t,\kappa}^{(0,j)} > \alpha_n\}} \leq \frac{K}{\alpha_n} |\breve{C}_{t,\kappa}^{(j)} - \breve{\mathcal{C}}_{t,\kappa}^{(j)}| + \frac{K}{\alpha_n} |\breve{\mathcal{C}}_{t,\kappa}^{(j)}| |\widehat{V}_{t,\kappa}^{(0,j)} - V_{t,\kappa,n}^{(0)}|, \tag{118}$$

where we denote

$$\widehat{C}_{t,\kappa}^{(j,0)} = \frac{n}{|\mathcal{I}_\kappa^n|} \sum_{i \in \mathcal{I}_\kappa^n} \Delta_{t,i}^n X^{(j)} \Delta_{t,i}^n X^{(0)} 1_{\{\mathcal{A}_{t,i}^{(j,0)}\}}, \quad C_t^{(j,0)} = \int_0^t \beta_s^{(j)} V_s^{(0)} ds, \ j = 1, ..., N. \tag{119}$$

From here, using Lemmas 1-3, we have

$$\mathbb{E}_{t,\lfloor \kappa n \rfloor - k_n + 1}^n [(|\xi_{t,\kappa}^{(j,1)}|^2 + |\xi_{t,\kappa}^{(j,2)}|^2) 1_{\{\widehat{V}_{t,\kappa}^{(0,j)} > \alpha_n\}}] \leq \frac{K}{\alpha_n^4} \left(\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n} \bigvee \frac{1}{k_n^2} \bigvee \left(\frac{k_n}{n}\right)^{1-\iota} \right), \tag{120}$$

for some arbitrary small $\iota > 0$. Similarly, we have

$$\mathbb{E}_{t,\lfloor \kappa n \rfloor - k_n + 1}^n [(|\xi_{t,\kappa}^{(j,1)}| |\breve{\mathcal{V}}_{t,\kappa}^{(0)}| + |\xi_{t,\kappa}^{(j,2)}| |\breve{\mathcal{C}}_{t,\kappa}^{(j)}|) 1_{\{\widehat{V}_{t,\kappa}^{(0,j)} > \alpha_n\}}] \leq \frac{K}{\alpha_n^2} \left(\frac{1}{k_n^{3/2}} \vee \left(\frac{k_n}{n}\right)^{1-\iota} \frac{\Delta_n^{2\varpi-1}}{k_n^2} \right). \tag{121}$$

Furthermore, given the lower bound restriction on $V_t^{(0)}$ in Assumption SA, we have

$$1_{\{\widehat{V}_{t,\kappa}^{(0,j)} \leq \alpha_n\}} \leq K_p |\widehat{V}_{t,\kappa}^{(0)} - \widetilde{V}_{t,\kappa}^{(0)}|^p, \quad \forall p \geq 1, \tag{122}$$

where the constant K_p depends on p. Therefore, by making use again of Lemmas 1-3 and taking into account the restriction on ϖ and ϱ of the lemma, we have altogether:

$$\mathbb{E}|\widehat{B}_{t,\kappa,n}^N - \frac{1}{N} \sum_{j=1}^N \mathcal{B}_{t,\kappa,n}^{1,(j)}| \leq \frac{K}{\alpha_n^4} \left(\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n} \bigvee \frac{1}{k_n^{3/2}} \bigvee \left(\frac{k_n}{n}\right)^{1-\iota} \right), \tag{123}$$

where again the constant K does not depend on t, κ and j. Combining the above bounds, we get the first bound of the lemma in (104). For the second bound in (105) we make in addition use of the following

$$\frac{1}{n^2} \sum_{s=k_n}^n \left[\omega^2(s\Delta_n) \left(\mathcal{B}_{t,s\Delta_n,n}^{1,(j)} - \mathcal{B}_{t,s\Delta_n,n}^{2,(j)} - \mathbb{E}_{t,s-k_n+1}^n (\mathcal{B}_{t,s\Delta_n,n}^{1,(j)} - \mathcal{B}_{t,s\Delta_n,n}^{2,(j)}) \right)^2 \right] \leq \frac{K}{k_n n}, \tag{124}$$

which in turn follows by application of Cauchy-Schwarz inequality and the second bound in (112).□

Lemma 7. *Under Assumption SA, and provided $\varrho \in (0, 1/2)$ we have for $t = 1, ..., T$ and $\kappa \in [0, 1]$ as well as some arbitrary small $\iota > 0$:*

$$\left|\mathbb{E}^n_{t,\lfloor\kappa n\rfloor - k_n+1}\left(Z^{(j)}_{t,\kappa,n} - \overline{Z}^{(j)}_{t,\kappa,n}\right)\right| \leq K\left(\Delta_n^{2\varpi} \bigvee \frac{k_n}{n} \bigvee \left(\frac{k_n}{n}\right)^{1-\iota} \frac{1}{\sqrt{k_n}}\right), \tag{125}$$

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor - k_n+1}\left(Z^{(j)}_{t,\kappa,n} - \overline{Z}^{(j)}_{t,\kappa,n}\right)^2 \leq K\left(\Delta_n^{2\varpi} \bigvee \left(\frac{k_n}{n}\right)^{1-\iota} \frac{1}{k_n} \bigvee \frac{1}{k_n^2}\right), \tag{126}$$

where the constant K does not depend on t, κ and j, and $Z^{(j)}_{t,\kappa,n}$ and $\overline{Z}^{(j)}_{t,\kappa,n}$ are defined in (99) and (102), respectively.

Proof of Lemma 7. We denote with $\mathcal{Z}^{(j)}_{t,\kappa,n}$, the counterpart of $Z^{(j)}_{t,\kappa,n}$ in which $\widehat{C}^{(j)}_{t,\kappa}(2)$ and $\widehat{V}^{(0,j)}_{t,\kappa}$ are replaced with $\overline{C}^{(j)}_{t,\kappa}(2)$ and $\overline{V}^{(0)}_{t,\kappa}$, respectively. Then, using Lemma 1 and taking into account that $\varrho \in (0, 1/2)$, we have

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor - k_n+1}|Z^{(j)}_{t,\kappa,n} - \mathcal{Z}^{(j)}_{t,\kappa,n}| + \mathbb{E}^n_{t,\lfloor\kappa n\rfloor - k_n+1}|Z^{(j)}_{t,\kappa,n} - \mathcal{Z}^{(j)}_{t,\kappa,n}|^2 \leq K\Delta_n^{2\varpi}, \tag{127}$$

where the constant K does not depend on t, κ and j. Using successive conditioning and Assumption SA, we have

$$|\mathbb{E}^n_{t,\lfloor\kappa n\rfloor - k_n+1}(\chi_{1,n}\chi_{2,n})| \leq K\frac{k_n}{n}, \tag{128}$$

for

$$\chi_{1,n} = \widetilde{C}^{(j)}_{t,\kappa} - C^{(j)}_{t,\kappa,n} \text{ or } \widetilde{V}^{(0)}_{t,\kappa} - V^{(0)}_{t,\kappa,n},\ \chi_{2,n} = \overline{C}^{(j)}_{t,\kappa,n}(2) - \widetilde{C}^{(j)}_{t,\kappa,n} \text{ or } \overline{V}^{(0)}_{t,\kappa} - \widetilde{V}^{(0)}_{t,\kappa,n}. \tag{129}$$

From here, using Taylor expansion and Lemmas 2-3, we have

$$|\mathbb{E}^n_{t,\lfloor\kappa n\rfloor - k_n+1}(\mathcal{Z}^{(j)}_{t,\kappa,n} - \overline{Z}^{(j)}_{t,\kappa,n})| \leq K\left(\frac{k_n}{n} \bigvee \left(\frac{k_n}{n}\right)^{1-\iota} \frac{1}{\sqrt{k_n}}\right). \tag{130}$$

Similar analysis leads to

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor - k_n+1}(\mathcal{Z}^{(j)}_{t,\kappa,n} - \overline{Z}^{(j)}_{t,\kappa,n})^2 \leq K\left(\left(\frac{k_n}{n}\right)^{1-\iota} \frac{1}{k_n} \bigvee \frac{1}{k_n^2}\right). \tag{131}$$

From here the results of the lemma follow. $\square$

Lemma 8. *Under Assumption SA, and provided $k_n\Delta_n \to 0$ and $k_n\Delta_n^{1-2\varpi} \to \infty$, we have for $t = 1, ..., T$, $\kappa \in [0, 1]$ and $j = 1, ..., N$:*

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor - k_n+1}[(1 + |Z^{(j)}_{t,\kappa,n}| + |B^{(j)}_{t,\kappa,n}|)1_{\{\widehat{V}^{(0,j)}_{t,\kappa} \leq \alpha_n\}}] \leq K\left(\frac{k_n}{n} \bigvee \frac{1}{k_n^{p/2}} \bigvee k_n^{1-p-\iota}\Delta_n^{1+p(2\varpi-1)-\iota}\right), \tag{132}$$

for some arbitrary big $p > 2$ and some arbitrary small $\iota > 0$, and where the constant K depends on p but not on j, t and κ.

Proof of Lemma 8. Using the inequality $1_{\{\widehat{V}^{(0,j)}_{t,\kappa}\leq\alpha_n\}} \leq K_p|\widehat{V}^{(0,j)}_{t,\kappa} - \widetilde{V}^{(0)}_{t,\kappa}|^p$, for arbitrary $p \geq 1$, we have

$$
\begin{aligned}
&(1 + |Z^{(j)}_{t,\kappa,n}| + |B^{(j)}_{t,\kappa,n}|)1_{\{\widehat{V}^{(0,j)}_{t,\kappa}\leq\alpha_n\}} \\
&\leq K|\widehat{V}^{(0,j)}_{t,\kappa} - \widetilde{V}^{(0)}_{t,\kappa}|^p\big(1 + |\widehat{C}^{(j)}_{t,\kappa}(2) - \widetilde{C}^{(j)}_{t,\kappa}| + |\widehat{C}^{(j)}_{t,\kappa}(2) - \widetilde{C}^{(j)}_{t,\kappa}|^2\big),
\end{aligned} \tag{133}
$$

for some arbitrary $p > 2$ and with the constant K not depending on j, t and κ. From here, the result of the lemma follows by application of Lemmas 1-3 as well as the assumed relative growth conditions for k_n. □

For the statement of the next lemma, we need some additional notation. We denote with $\widetilde{\mathrm{Avar}}(\widehat{D}^N_{t,\kappa})$ the counterpart of $\widehat{\mathrm{Avar}}(\widehat{D}^N_{t,\kappa})$ in which we replace $\widehat{\beta}^{(j)}_{t,\kappa}$ with $\beta^{(j)}_{t-1+\kappa}$, $\widehat{V}^{(0)}_{t,\kappa}$ with $V^{(0)}_{t-1+\kappa}$, $\breve{C}^{(j)}_{t,\kappa}$ with $\breve{\mathcal{C}}^{(j)}_{t,\kappa}$, and $\breve{V}^{(0)}_{t,\kappa}$ with $\breve{\mathcal{V}}^{(0)}_{t,\kappa}$. In addition, we set

$$
\begin{aligned}
\mathrm{Avar}(\widehat{D}^N_{t,\kappa}) = {}& \frac{6}{N^2|\widetilde{\mathcal{I}}^n_\kappa|}\sum_{j=1}^N\left[\left(\beta^{(j)}_{t-1+\kappa} - 1\right)^2\frac{(\widetilde{\sigma}^{(j)}_{t-1+\kappa})^2}{(\sigma^{(0)}_{t-1+\kappa})^2}\right] + \frac{4}{|\widetilde{\mathcal{I}}^n_\kappa|}\left(\frac{1}{N}\sum_{j=1}^N\left(\beta^{(j)}_{t-1+\kappa} - 1\right)^2\right)^2 \\
&+ \frac{6}{|\widetilde{\mathcal{I}}^n_\kappa|}\left(\frac{1}{N}\sum_{j=1}^N\frac{(\beta^{(j)}_{t-1+\kappa} - 1)\gamma^{(j)}_{t-1+\kappa}}{\sigma^{(0)}_{t-1+\kappa}}\right)\left(\frac{1}{N}\sum_{j=1}^N\frac{(\beta^{(j)}_{t-1+\kappa} - 1)(\gamma^{(j)}_{t-1+\kappa})^\top}{\sigma^{(0)}_{t-1+\kappa}}\right).
\end{aligned} \tag{134}
$$

Lemma 9. *Under Assumption SA, and provided $\varpi \in (0, 1/2)$, $\varrho \in (0, 1/2)$ and $k_n\Delta_n^{2-4\varpi} \to \infty$, we have for $t = 1, ..., T$ and $\kappa \in [0, 1]$ and some arbitrary small $\iota > 0$:*

$$
\begin{aligned}
&\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}|\widehat{Avar}(\widehat{D}^N_{t,\kappa}) - Avar(\widehat{D}^N_{t,\kappa})| \\
&\qquad \leq \frac{K}{\alpha_n^6}\left(\frac{1}{\sqrt{n}}\bigvee\frac{1}{k_n^{3/2}}\bigvee\frac{\Delta_n^{2\varpi-\iota}}{k_n}\bigvee\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n}\right),
\end{aligned} \tag{135}
$$

$$
\begin{aligned}
&\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}(\widetilde{Avar}(\widehat{D}^N_{t,\kappa}) - Avar(\widehat{D}^N_{t,\kappa})) = 0, \\
&\mathbb{E}^n_{t,\lfloor\kappa n\rfloor-k_n+1}((\widetilde{Avar}(\widehat{D}^N_{t,\kappa})) - Avar(\widehat{D}^N_{t,\kappa}))^2 \leq \frac{K}{k_n^3}.
\end{aligned} \tag{136}
$$

Proof of Lemma 9. We start with the first bound. If we denote

$$
\xi^{(j,3)}_{t,\kappa} = \frac{\widehat{\beta}^{(j)}_{t,\kappa} - 1}{\widehat{V}^{(0,j)}_{t,\kappa}}\breve{C}^{(j)}_{t,\kappa} - \frac{\beta^{(j)}_{t,\kappa,n} - 1}{V^{(0)}_{t,\kappa,n}}\breve{\mathcal{C}}^{(j)}_{t,\kappa}, \tag{137}
$$

$$
\xi^{(j,4)}_{t,\kappa} = \frac{(\widehat{\beta}^{(j)}_{t,\kappa} - 1)^2}{\widehat{V}^{(0,j)}_{t,\kappa}}\breve{V}^{(0)}_{t,\kappa} - \frac{(\beta^{(j)}_{t,\kappa,n} - 1)^2}{V^{(0)}_{t,\kappa,n}}\breve{\mathcal{V}}^{(0)}_{t,\kappa}, \tag{138}
$$

then direct calculation shows

$$
\begin{aligned}
|\xi^{(j,3)}_{t,\kappa}|1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}} \leq {}& \frac{K}{\alpha_n^2}(|\widehat{C}^{(j,0)}_{t,\kappa} - C^{(j,0)}_{t,\kappa,n}| \vee |\widehat{V}^{(0,j)}_{t,\kappa} - V^{(0)}_{t,\kappa,n}| \vee 1)|\breve{C}^{(j)}_{t,\kappa} - \breve{\mathcal{C}}^{(j)}_{t,\kappa}| \\
&+ \frac{K}{\alpha_n^2}|\breve{\mathcal{C}}^{(j)}_{t,\kappa}|(|\widehat{C}^{(j,0)}_{t,\kappa} - C^{(j,0)}_{t,\kappa,n}| \vee |\widehat{V}^{(0,j)}_{t,\kappa} - V^{(0)}_{t,\kappa,n}|^2 \vee |\widehat{V}^{(0,j)}_{t,\kappa} - V^{(0)}_{t,\kappa,n}|),
\end{aligned} \tag{139}
$$

$$
\begin{aligned}
|\xi^{(j,4)}_{t,\kappa}|1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}} \leq \frac{K}{\alpha_n^3}(|\widehat{C}^{(j,0)}_{t,\kappa} - C^{(j,0)}_{t,\kappa,n}|^2 \vee |\widehat{V}^{(0,j)}_{t,\kappa} - V^{(0)}_{t,\kappa,n}|^2 \vee 1)|\breve{V}^{(0)}_{t,\kappa} - \breve{\mathcal{V}}^{(0)}_{t,\kappa}| \\
+ \frac{K}{\alpha_n^3}|\breve{\mathcal{V}}^{(0)}_{t,\kappa}|(|\widehat{V}^{(0,j)}_{t,\kappa} - V^{(0)}_{t,\kappa,n}|^3 \vee |\widehat{V}^{(0,j)}_{t,\kappa} - V^{(0)}_{t,\kappa,n}| \vee |\widehat{C}^{(j,0)}_{t,\kappa} - C^{(j,0)}_{t,\kappa,n}|^2 \vee |\widehat{C}^{(j,0)}_{t,\kappa} - C^{(j,0)}_{t,\kappa,n}|),
\end{aligned} \tag{140}
$$

where we use the notation $\widehat{C}^{(j,0)}_{t,\kappa}$ and $C^{(j,0)}_{t}$ as defined in the proof of Lemma 6 (see equation (119)). From here, using Lemmas 1-3, we have

$$
\mathbb{E}^n_{t,\lfloor \kappa n\rfloor - k_n+1}[(|\xi^{(j,3)}_{t,\kappa}|^2 + |\xi^{(j,4)}_{t,\kappa}|^2)1_{\{\widehat{V}^{(0,j)}_{t,\kappa}>\alpha_n\}}] \leq \frac{K}{\alpha_n^6}\left(\frac{\Delta_n^{1+2(2\varpi-1)-\iota}}{k_n} \bigvee \frac{1}{k_n^2} \bigvee \left(\frac{k_n}{n}\right)^{1-\iota}\right), \tag{141}
$$

for some arbitrary small $\iota > 0$. Similarly, we have

$$
\mathbb{E}^n_{t,\lfloor \kappa n\rfloor - k_n+1}[(|\xi^{(j,3)}_{t,\kappa}| + |\xi^{(j,4)}_{t,\kappa}|)(|\breve{\mathcal{C}}^{(j)}_{t,\kappa}| + |\breve{\mathcal{V}}^{(0)}_{t,\kappa}|)1_{\{\widehat{V}^{(0)}_{t,\kappa}>\alpha_n\}}] \leq \frac{K}{\alpha_n^3}\left(\frac{1}{k_n^{3/2}} \vee \left(\frac{k_n}{n}\right)^{1-\iota} \frac{\Delta_n^{2\varpi-1}}{k_n^2}\right). \tag{142}
$$

Finally, using similar proof as that of the bound in (114), we can show

$$
\mathbb{E}\left|(\widehat{V}^{(j)}_{\beta,t,\kappa})^2(\breve{V}^{(0,j)}_{t,\kappa})^2 1_{\{\breve{V}^{(0,j)}_{t,\kappa}>\alpha_n\}}\right| \leq \frac{K}{\alpha_n^4}\frac{1}{k_n^2}\left[\frac{1}{k_n}\bigvee \Delta_n^{4\varpi-\iota}\right], \tag{143}
$$

for arbitrary small $\iota > 0$. From here, using again Lemmas 1-3, we get the first bound of the lemma. The second and third results of the lemma follow by direct calculation. $\square$

For the statement of the next lemma, we introduce some additional notation. We denote

$$
\widehat{Z}^{(a)}_{t,\kappa,n} = \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa} \chi^{(a)}_{i,t,\kappa,n},\ \widehat{Z}^{(b)}_{t,\kappa,n} = \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa} \chi^{(b)}_{i,t,\kappa,n},\ \widehat{Z}^{(c)}_{t,\kappa,n} = \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa} \chi^{(c)}_{i,t,\kappa,n},\ t = 1, ..., T,\ \kappa \in [0,1], \tag{144}
$$

where

$$
\chi^{(a)}_{i,t,\kappa,n} = \frac{1}{\sqrt{N}}\frac{n}{\sqrt{|\widetilde{\mathcal{I}}^n_\kappa|}}\sum_{j=1}^N \left[(\beta^{(j)}_{t,\kappa,n} - 1)\frac{\widetilde{\sigma}^{(j)}_{t,\kappa,n}}{\sigma^{(0)}_{t,\kappa,n}}\Delta^{n,2}_{t,i}W^{(0)}\Delta^{n,2}_{t,i}\widetilde{W}^{(j)}\right], \tag{145}
$$

$$
\chi^{(b)}_{i,t,\kappa,n} = 2\left(\frac{1}{N}\sum_{j=1}^N(\beta^{(j)}_{t,\kappa,n} - 1)^2\right)\frac{n}{\sqrt{|\widetilde{\mathcal{I}}^n_\kappa|}}\Delta^n_{t,i-1}W^{(0)}\Delta^n_{t,i}W^{(0)}, \tag{146}
$$

$$
\chi^{(c)}_{i,t,\kappa,n} = \frac{1}{N}\frac{n}{\sqrt{|\widetilde{\mathcal{I}}^n_\kappa|}}\sum_{j=1}^N \left[(\beta^{(j)}_{t,\kappa,n} - 1)\frac{\gamma^{(j)}_{t,\kappa,n}}{\sigma^{(0)}_{t,\kappa,n}}\Delta^{n,2}_{t,i}W^{(0)}\Delta^{n,2}_{t,i}B\right], \tag{147}
$$

and we note that we have

$$
\frac{1}{N}\sum_{j=1}^N \overline{Z}^{(j)}_{t,\kappa,n} = \frac{1}{\sqrt{|\widetilde{\mathcal{I}}^n_\kappa|}\sqrt{N}}\widehat{Z}^{(a)}_{t,\kappa,n} + \frac{1}{\sqrt{|\widetilde{\mathcal{I}}^n_\kappa|}}\left(\widehat{Z}^{(b)}_{t,\kappa,n} + \widehat{Z}^{(c)}_{t,\kappa,n}\right). \tag{148}
$$

Lemma 10. *Assume Assumptions SA and B hold. For $n \to \infty$, $k_n \to \infty$ and $N \to \overline{N}$, with $\overline{N} \in (0, \infty]$, we have*

$$\{\widehat{Z}^{(a)}_{t,\kappa,n}, \widehat{Z}^{(b)}_{t,\kappa,n}, \widehat{Z}^{(c)}_{t,\kappa,n}\}_{t\in\mathcal{T},\, \kappa\in\mathcal{K}} \xrightarrow{\mathcal{L}-s} \left\{\sqrt{\psi^{(a)}_{t,\kappa}} Z^{(a)}_{t,\kappa}, \sqrt{\psi^{(b)}_{t,\kappa}} Z^{(b)}_{t,\kappa}, \sqrt{\psi^{(c)}_{t,\kappa}} Z^{(c)}_{t,\kappa}\right\}_{t\in\mathcal{T},\, \kappa\in\mathcal{K}}, \tag{149}$$

for $t \in \mathcal{T}$, $\kappa \in \mathcal{K}$, with $\mathcal{K}$ being an arbitrary finite set of distinct points in $(0, 1)$, and $\{Z^{(a)}_{t,\kappa}\}_{t\in\mathcal{T},\, \kappa\in\mathcal{K}}$, $\{Z^{(b)}_{t,\kappa}\}_{t\in\mathcal{T},\, \kappa\in\mathcal{K}}$ and $\{Z^{(c)}_{t,\kappa}\}_{t\in\mathcal{T},\, \kappa\in\mathcal{K}}$ being three sequences of i.i.d. standard normal random variables defined on an extension of the original probability space and independent of $\mathcal{F}$ and each other.

Proof of Lemma 10. We denote

$$\widetilde{\chi}^{(k)}_{i,t,\kappa,n} = \chi^{(k)}_{i,t,\kappa,n} - \mathbb{E}^n_{t,i}(\chi^{(k)}_{i,t,\kappa,n}) + \mathbb{E}^n_{t,i+1}(\chi^{(k)}_{i+1,t,\kappa,n}) - \mathbb{E}^n_{t,i}(\chi^{(k)}_{i+1,t,\kappa,n}),\ k = a, b, c. \tag{150}$$

Then, for $k = a, b, c$, we have

$$\mathbb{E}^n_{t,\lfloor\kappa n\rfloor - k_n}\left(\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n} - \chi^{(k)}_{i,t,\kappa,n})\right) = 0,\ \mathbb{E}^n_{t,\lfloor\kappa n\rfloor - k_n}\left(\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n} - \chi^{(k)}_{i,t,\kappa,n})\right)^2 \le \frac{K}{k_n}, \tag{151}$$

and from here

$$\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n} - \chi^{(k)}_{i,t,\kappa,n}) = o_p(1),\ k = a, b, c. \tag{152}$$

Therefore, it suffices to prove the convergence result with $\chi^{(k)}_{i,t,\kappa,n}$ replaced by $\widetilde{\chi}^{(k)}_{i,t,\kappa,n}$ in $\widehat{Z}^{(k)}_{t,\kappa,n}$. Such a convergence result will follow by an application of Theorem IX.7.3 of Jacod and Shiryaev (2003) (by noting that $\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}) = 0$) if we show the following convergence results:

$$\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(a)}_{i,t,\kappa,n})^2 \xrightarrow{\mathbb{P}} \psi^{(a)}_{t,\kappa},\ \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(b)}_{i,t,\kappa,n})^2 \xrightarrow{\mathbb{P}} \psi^{(b)}_{t,\kappa},\ \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(c)}_{i,t,\kappa,n})^2 \xrightarrow{\mathbb{P}} \psi^{(c)}_{t,\kappa}, \tag{153}$$

$$\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}\widetilde{\chi}^{(l)}_{i,t,\kappa,n}) \xrightarrow{\mathbb{P}} 0,\ k, l = a, b, c \text{ with } k \neq l, \tag{154}$$

$$\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}\Delta^n_{t,i}M) \xrightarrow{\mathbb{P}} 0,\ k = a, b, c, \tag{155}$$

for M being $W^{(0)}$, a component of B, or a bounded martingale orthogonal to them (in a martingale sense). The first two convergence results in (153) and (154) follow directly by taking into account that the volatility processes and the beta process all have càdlàg paths. The last convergence result in (155) when M is $W^{(0)}$ or a component of B holds trivially because due to the symmetry of the standard normal distribution, $\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}\Delta^n_{t,i}M) = 0$ in this case.

Suppose now that M in (155) is equal to a bounded martingale orthogonal to $W^{(0)}$ and B. First, $\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}\Delta^n_{t,i}M) = 0$ for $k = b, c$ because M is orthogonal to $W^{(0)}$ and B. Second, if M is a discontinuous martingale, we again trivially have $\mathbb{E}^n_{t,i}(\widetilde{\chi}^{(k)}_{i,t,\kappa,n}\Delta^n_{t,i}M) = 0$ for $k = a, b, c$. Thus, we are left with showing (155) with $k = a$ and M being a continuous bounded martingale that is orthogonal to $W^{(0)}$ and B. In this case, we can write

$$\mathbb{E}^n_{t,i}(\Delta^n_{t,i-1}W^{(0)}\Delta^n_{t,i}\widetilde{W}^{(j)}\Delta^n_{t,i}M) = 0, \tag{156}$$

and

$$\mathbb{E}^n_{t,i}(\Delta^n_{t,i}W^{(0)}\Delta^n_{t,i}\widetilde{W}^{(j)}\Delta^n_{t,i}M) = \mathbb{E}^n_{t,i}(\Delta^n_{t,i}W^{(0)}\Delta^n_{t,i}\widetilde{W}^{(j)}Z^N_{t-1+i/n}), \tag{157}$$

where $Z^N_s = \mathbb{E}(\Delta^n_{t,i}M|\widetilde{\mathcal{F}}^{(N)}_s)$ for $s \in [t-1+(i-1)/n, t-1+i/n]$ and for $\{\widetilde{\mathcal{F}}^{(N)}_t\}_{t\geq 0}$ being the filtration generated by the Brownian motions $W^{(0)}$, B and $\{\widetilde{W}^{(j)}\}_{j=1,\dots,N}$. Note that Z^N_s is a $\widetilde{\mathcal{F}}^{(N)}$-martingale for $s \in [t-1+(i-1)/n, t-1+i/n]$. Therefore, by a martingale representation theorem, the orthogonality of M to $W^{(0)}$ and B, the fact that $W^{(0)}$ and B are independent from each other, and using integration by parts, we have

$$\begin{aligned}\mathbb{E}^n_{t,i}(\Delta^n_{t,i}W^{(0)}\Delta^n_{t,i}\widetilde{W}^{(j)}Z^N_{t-1+i/n}) &= \mathbb{E}^n_{t,i}\left(\Delta^n_{t,i}W^{(0)}\Delta^n_{t,i}\widetilde{W}^{(j)}\sum_{j=1}^{N}\int_{t-1+(i-1)/n}^{t-1+i/n}\eta^{(j)}_s d\widetilde{W}^{(j)}_s\right)\\ &= \mathbb{E}^n_{t,i}\left(\sum_{j=1}^{N}\int_{t-1+(i-1)/n}^{t-1+i/n}(W^{(0)}_s - W^{(0)}_{t-1+(i-1)/n})\eta^{(j)}_s ds\right),\end{aligned} \tag{158}$$

where $\{\eta^{(j)}_s\}_{j=1,\dots,N}$ are some $\mathcal{F}_s$-adapted processes. From here, using the shorthand notation $\zeta^{(j)}_{t,\kappa,n} = (\beta^{(j)}_{t,\kappa,n} - 1)\widetilde{\sigma}^{(j)}_{t,\kappa,n}$, we can write

$$\begin{aligned}&\sum_{j=1}^{N}\left(\zeta^{(j)}_{t,\kappa,n}\sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}(\Delta^n_{t,i}W^{(0)}\Delta^n_{t,i}\widetilde{W}^{(j)}\Delta^n_{t,i}M)\right)\\ &\quad= \sum_{i\in\widetilde{\mathcal{I}}^n_\kappa}\mathbb{E}^n_{t,i}\left(\sum_{j=1}^{N}\zeta^{(j)}_{t,\kappa,n}\int_{t-1+(i-1)/n}^{t-1+i/n}(W^{(0)}_s - W^{(0)}_{t-1+(i-1)/n})\eta^{(j)}_s ds\right).\end{aligned} \tag{159}$$

From here, by applying Cauchy-Schwarz inequality, we have

$$\left|\mathbb{E}^n_{t,i}\left(\int_{t-1+(i-1)/n}^{t-1+i/n}(W^{(0)}_s - W^{(0)}_{t-1+(i-1)/n})\eta^{(j)}_s ds\right)\right| \leq \frac{K}{\sqrt{n}}\int_{t-1+(i-1)/n}^{t-1+i/n}\sqrt{\mathbb{E}^n_{t,i}\left(\eta^{(j)}_s\right)^2}ds. \tag{160}$$

Using inequality in means, we have

$$\sum_{j=1}^{N}\sqrt{\mathbb{E}^n_{t,i}\left(\eta^{(j)}_s\right)^2} \leq K\sqrt{N}\sqrt{\mathbb{E}^n_{t,i}\left(\sum_{j=1}^{N}\left(\eta^{(j)}_s\right)^2\right)}, \tag{161}$$

and therefore by another application of equality in means, we have

$$\begin{aligned}\sum_{j=1}^{N}\int_{t-1+(i-1)/n}^{t-1+i/n}\sqrt{\mathbb{E}_{t,i}^{n}\left(\eta_s^{(j)}\right)^2}ds &\leq K\sqrt{\frac{N}{n}}\sqrt{\mathbb{E}_{t,i}^{n}\left(\int_{t-1+(i-1)/n}^{t-1+i/n}\sum_{j=1}^{N}\left(\eta_s^{(j)}\right)^2 ds\right)} \\ &\leq K\sqrt{\frac{N}{n}}\sqrt{\mathbb{E}_{t,i}^{n}\left(\int_{t-1+(i-1)/n}^{t-1+i/n} d\langle Z^N, Z^N\rangle_s\right)}.\end{aligned} \tag{162}$$

Applying again inequality in means, we can finally write

$$\mathbb{E}\left(\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n}\sum_{j=1}^{N}\int_{t-1+(i-1)/n}^{t-1+i/n}\sqrt{\mathbb{E}_{t,i}^{n}\left(\eta_s^{(j)}\right)^2}ds\right) \leq K\sqrt{k_n}\sqrt{\frac{N}{n}}\sqrt{\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n}\mathbb{E}(\Delta_{t,i}^n\langle Z^N, Z^N\rangle)}. \tag{163}$$

Now, using the definition of the martingale Z^N, successive conditioning and Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}(\Delta_{t,i}^n\langle Z^N, Z^N\rangle) &= \mathbb{E}\left(\mathbb{E}\left(\Delta_{t,i}^n M|\widetilde{\mathcal{F}}_{t-1+i/n}^{(N)}\right)\right)^2 - \mathbb{E}\left(\mathbb{E}\left(\Delta_{t,i}^n M|\widetilde{\mathcal{F}}_{t-1+(i-1)/n}^{(N)}\right)\right)^2 \\ &\leq 2\mathbb{E}\left(\Delta_{t,i}^n M\right)^2.\end{aligned} \tag{164}$$

Because of the boundedness of the martingale M and its continuity, we therefore have

$$\sum_{i\in\widetilde{\mathcal{I}}_\kappa^n}\mathbb{E}(\Delta_{t,i}^n\langle Z^N, Z^N\rangle) \leq 2\mathbb{E}(\langle M, M\rangle_{t-1+\frac{\lfloor \kappa n\rfloor}{n}} - \langle M, M\rangle_{t-1+\frac{\lfloor \kappa n\rfloor - k_n+1}{n}}) \downarrow 0, \tag{165}$$

as $k_n/n \to 0$. $\square$

Combining Lemmas 3, 5-8, we have for $\varpi \in (1/4, 1/2)$, $\varrho \in (0, 1/2)$ with $\varrho > 2 - 4\varpi$:

$$\widehat{D}_{t,\kappa} - \widehat{B}_{t,\kappa} - D_{t,\kappa}^N = \frac{1}{N}\sum_{j=1}^{N}\overline{Z}_{t,\kappa,n}^{(j)} + O_p\left(\frac{1}{\alpha_n^4}\left(\frac{1}{k_n}\bigvee\sqrt{\frac{k_n}{n}}\bigvee\Delta_n^{\varpi}\right)\right). \tag{166}$$

Moreover, from Lemma 9 and Assumption B, under the same conditions for ϖ and ϱ as above,

$$\frac{\widehat{\text{Avar}}(\widehat{D}_{t,\kappa}^N)}{\text{Avar}(\widehat{D}_{t,\kappa}^N)} \xrightarrow{\mathbb{P}} 1, \quad \frac{|\widetilde{\mathcal{I}}_\kappa^n|\text{Avar}(\widehat{D}_{t,\kappa}^N)}{\psi_{t,\kappa}^{(a)}/N + \psi_{t,\kappa}^{(b)} + \psi_{t,\kappa}^{(c)}} \xrightarrow{\mathbb{P}} 1. \tag{167}$$

Combining these results with Lemma 10, we get the result of the theorem.

9.1.6 Proof of Theorem 2

We start with showing the counterpart of Lemma 10 in the current context. The result of Lemma 11 below is slightly more restrictive than what we showed in Lemma 10 when $\overline{N} = \infty$. Nevertheless, it suffices for the purposes of proving Theorem 2.

Lemma 11. *Assume Assumptions SA and B hold and let $\{\omega_l\}_{l\in I}$ be Lipschitz real-valued continuous functions on $[0,1]$, where I is a countable set. For $n\to\infty$, $k_n\to\infty$ and $N\to\overline{N}$, with $\overline{N}\in(0,\infty)$, we have:*

$$\sqrt{\frac{k_n}{n}}\left\{\begin{array}{c}\sum_{s=k_n}^{n}\omega_l\big(s\Delta_n\big)\widehat{Z}^{(a)}_{t,s\Delta_n,n}\\ \sum_{s=1}^{n}\omega_l\big(s\Delta_n\big)\widehat{Z}^{(b)}_{t,s\Delta_n,n}\\ \sum_{s=k_n}^{n}\omega_l\big(s\Delta_n\big)\widehat{Z}^{(c)}_{t,s\Delta_n,n}\end{array}\right\}_{t\in\mathcal{T},i\in I}\stackrel{\mathcal{L}-s}{\longrightarrow}\left\{\begin{array}{c}\int_{t-1}^{t}\omega_i(s-\lfloor s\rfloor)\sqrt{\psi^{(a)}_{t,s-\lfloor s\rfloor}}dZ^{(a)}_s\\ \int_{t-1}^{t}\omega_i(s-\lfloor s\rfloor)\sqrt{\psi^{(b)}_{t,s-\lfloor s\rfloor}}dZ^{(b)}_s\\ \int_{t-1}^{t}\omega_i(s-\lfloor s\rfloor)\sqrt{\psi^{(c)}_{t,s-\lfloor s\rfloor}}dZ^{(c)}_s\end{array}\right\}_{t\in\mathcal{T},i\in I}, \tag{168}$$

where $Z^{(a)}_s$, $Z^{(b)}_s$ and $Z^{(c)}_s$ are three independent Brownian motions sequences defined on an extension of the original probability space and independent of $\mathcal{F}$. If in the above setting $\overline{N}=\infty$, then the convergence result in (168) for the sums involving $\widehat{Z}^{(b)}_{t,s\Delta_n,n}$ and $\widehat{Z}^{(c)}_{t,s\Delta_n,n}$ continues to hold.

Proof of Lemma 11. Using the notation and the bounds derived in Lemma 10, we have

$$\sqrt{\frac{k_n}{n}}\sum_{s=k_n}^{n}\sum_{i\in\widetilde{\mathcal{I}}^n_{s\Delta_n}}\omega_l(s\Delta_n)(\widetilde{\chi}^{(k)}_{t,i,s\Delta_n,n}-\chi^{(k)}_{t,i,s\Delta_n,n})=O_p(1/\sqrt{k_n}),\ k=a,b,c. \tag{169}$$

From here the proof of the lemma follows exactly the same steps as corresponding ones in the proof of Lemma 10. □

Combining Lemmas 3, 5-8, we have for $\varpi\in(1/4,1/2)$, $\varrho\in(0,1/2)$ with $\varrho>2-4\varpi$ and for ω as in Lemma 11 above:

$$\begin{aligned}&\frac{1}{n-k_n+1}\sum_{s=k_n}^{n}[\omega(s\Delta_n)(\widehat{D}^N_{t,s\Delta_n}-\widehat{B}^N_{t,s\Delta_n})]-\int_{t-1}^{t}\omega(s-t+1)D^N_{t,s-\lfloor s\rfloor}ds\\ &=\frac{1}{n-k_n+1}\sum_{s=k_n}^{n}\omega(s\Delta_n)\frac{1}{N}\sum_{j=1}^{N}\overline{Z}^{(j)}_{t,s\Delta_n,n}+O_p\left(\frac{1}{\alpha_n^4}\left(\frac{1}{k_n^{3/2}}\bigvee\left(\frac{k_n}{n}\right)^{1-\iota}\bigvee\Delta_n^{2\varpi}\right)\right).\end{aligned} \tag{170}$$

Moreover, from Lemma 9 and Assumption B, under the same conditions for ϖ and ϱ as above, we have for arbitrary ω,ω' that are Lipschitz real-valued continuous functions on $[0,1]$:

$$\frac{\sum_{s=1}^{n}[\overline{\omega}^n(s)\overline{\omega}'^n(s)(\widehat{\mathrm{Avar}}(\widehat{D}^N_{t,(s\vee k_n)\Delta_n})+\widehat{\mathrm{Avar}}(\widehat{D}^N_{t,(s+k_n\wedge n)\Delta_n}))]}{\sum_{s=1}^{n}[\overline{\omega}^n(s)\overline{\omega}'^n(s)(\mathrm{Avar}(\widehat{D}^N_{t,(s\vee k_n)\Delta_n})+\mathrm{Avar}(\widehat{D}^N_{t,(s+k_n\wedge n)\Delta_n}))]}\stackrel{\mathbb{P}}{\longrightarrow}1, \tag{171}$$

and

$$\frac{1}{2}\frac{k_n}{n}\frac{\sum_{s=1}^{n}[\overline{\omega}^n(s)\overline{\omega}'^n(s)(\mathrm{Avar}(\widehat{D}^N_{t,(s\vee k_n)\Delta_n})+\mathrm{Avar}(\widehat{D}^N_{t,(s+k_n\wedge n)\Delta_n}))]}{\int_{t-1}^{t}[\omega(s-t+1)\omega'(s-t+1)(\psi^{(a)}_{t,s-\lfloor s\rfloor}/N+\psi^{(b)}_{t,s-\lfloor s\rfloor}+\psi^{(c)}_{t,s-\lfloor s\rfloor})]ds}\stackrel{\mathbb{P}}{\longrightarrow}1, \tag{172}$$

where we define $\overline{\omega}^n(i)$ and $\overline{\omega}'^n(i)$, for $i=1,...,n$, from ω and ω' exactly as in (31).

Furthermore, the above two convergences hold uniformly for ω, ω' belonging to the set of weighting functions of the theorem. Overall, the above results, together with Lemma 11 imply the convergence result of the theorem holds finite-dimensionally, i.e., for any finite set of points in $\mathcal{U}$. Therefore, we are left with showing tightness of the sequence in the space of continuous functions on $\mathcal{U}$ equipped with the uniform topology. For this, we make use of Theorem 12.3 of Billingsley (2013) and Lemmas 3, 5-8 as well as the smoothness of $\omega_u(z)$ in u assumed in the statement of the theorem.

9.1.7 Proof of Corollary 1

Part (a) follows from Theorem 2 while part (b) follows from Theorem 1 of Bierens (1982).

9.1.8 Proof of Thoerem 3

Throughout this proof, $||f|| = \langle f, f \rangle$ is the norm of $f \in \mathcal{L}^2(w)$. We start the proof with denoting the function

$$g(u, x, y) = e^{iu\frac{y}{x} + iu}, \quad \text{for} \quad u,\, x,\, y \in \mathbb{R}. \tag{173}$$

Using Taylor expansion, we then have the decomposition

$$g(u, \widehat{V}_{t,\kappa}^{(0,j)}, \widehat{C}_{t,\kappa}^{(j)}(2)) - g(u, \widetilde{V}_{t,\kappa}^{(0)}, \widetilde{C}_{t,\kappa}^{(j)}) = Z_{t,\kappa,n}^{(j)}(u) + R_{t,\kappa}^{(j)}(u), \tag{174}$$

where

$$Z_{t,\kappa,n}^{(j)}(u) = \nabla_x g(u, \widetilde{V}_{t,\kappa}^{(0)}, \widetilde{C}_{t,\kappa}^{(j)})(\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)}) + \nabla_y g(u, \widetilde{V}_{t,\kappa}^{(0)}, \widetilde{C}_{t,\kappa}^{(j)})(\widehat{C}_{t,\kappa}^{(j)}(2) - \widetilde{C}_{t,\kappa}^{(j)}), \tag{175}$$

and the residual term $R_{t,\kappa}^{(j)}(u)$ satisfies

$$|R_{t,\kappa}^{(j)}(u)| \leq K(|u| \vee 1)1_{\{|\widehat{V}_{t,\kappa}^{(0,j)}| \leq \alpha_n\}} + K(|u|^2 \vee 1)((\widehat{V}_{t,\kappa}^{(0,j)} - \widetilde{V}_{t,\kappa}^{(0)})^2 + (\widehat{C}_{t,\kappa}^{(j)}(2) - \widetilde{C}_{t,\kappa}^{(j)})^2), \tag{176}$$

with a constant K that does not depend on n, j and u. We further denote

$$\begin{aligned}
\overline{Z}_{t,\kappa,n}^{(j)}(u) &= e^{iu\beta_{t,\kappa,n}^{(j)}} iu(\beta_{t,\kappa,n}^{(j)} - 1)^2 \frac{n}{|\widetilde{\mathcal{I}}_\kappa^n|} \sum_{i \in \widetilde{\mathcal{I}}_\kappa^n} \Delta_{t,i-1}^n W^{(0)} \Delta_{t,i}^n W^{(0)} \\
&+ \frac{1}{2} e^{iu\beta_{t,\kappa,n}^{(j)}} iu(\beta_{t,\kappa,n}^{(j)} - 1) \left(\frac{n}{|\widetilde{\mathcal{I}}_\kappa^n|} \sum_{i \in \widetilde{\mathcal{I}}_\kappa^n} \frac{\gamma_{t,\kappa,n}^{(j)}}{\sigma_{t,\kappa,n}^{(0)}} \Delta_{t,i}^{n,2} W^{(0)} \Delta_{t,i}^{n,2} B + \frac{n}{|\widetilde{\mathcal{I}}_\kappa^n|} \sum_{i \in \widetilde{\mathcal{I}}_\kappa^n} \frac{\widetilde{\sigma}_{t,\kappa,n}^{(0)}}{\sigma_{t,\kappa,n}^{(0)}} \Delta_{t,i}^{n,2} W^{(0)} \Delta_{t,i}^{n,2} \widetilde{W}^{(j)} \right).
\end{aligned} \tag{177}$$

Using Lemmas 1-3 and Lemma 8 and the exponential tail decay of the weighting function w, we have

$$\mathbb{E} \left\| \frac{1}{N} \sum_{j=1}^N R_{t,\kappa}^{(j)} \right\| \leq \frac{K}{\alpha_n^2} \left(\frac{1}{k_n} \bigvee \frac{k_n}{n} \right), \tag{178}$$

where we have made use of the fact that $\varpi > 1/4$. Similarly, using Lemmas 1-3, we have that

$$\mathbb{E}\left|\left|\frac{1}{N}\sum_{j=1}^{N}\left(Z_{t,\kappa,n}^{(j)} - \overline{Z}_{t,\kappa,n}^{(j)}\right)\right|\right| \leq \frac{K}{\alpha_n^2}\left(\frac{1}{k_n}\bigvee\frac{k_n}{n}\right) \leq K\left(\Delta_n^{2\varpi}\bigvee\sqrt{\frac{k_n}{n}}\bigvee\frac{1}{k_n}\right). \tag{179}$$

Given the rate condition on the sequence k_n, we are left with showing

$$\sqrt{k_n}\frac{1}{N}\sum_{j=1}^{N}\overline{Z}_{t,\kappa,n}^{(j)} \xrightarrow{\mathcal{L}-s} \mathcal{Z}_{t,\kappa}, \tag{180}$$

with $\mathcal{Z}_{t,\kappa}$ being the limit in the statement of the theorem. Using Bessel's inequality and dominated convergence we have

$$\mathbb{E}\left(\sum_{i>I}\left\langle\frac{k_n}{N}\sum_{j=1}^{N}\overline{Z}_{t,\kappa,n}^{(j)}, e_i\right\rangle^2\right) \to 0, \quad \text{as } I \to \infty, \tag{181}$$

where $\{e_i\}_{i\geq 1}$ denotes an orthonormal basis in $\mathcal{L}^2(w)$. This means that the sequence is asymptotically finite-dimensional, see 1.8 in Vaart and Wellner (1996). Therefore, the limit result of the theorem will follow from Theorem 1.8.4 in Vaart and Wellner (1996) if we can establish

$$\left\langle\frac{\sqrt{k_n}}{N}\sum_{j=1}^{N}\overline{Z}_{t,\kappa,n}^{(j)}, h\right\rangle \xrightarrow{\mathcal{L}-s} \langle\mathcal{Z}_{t,\kappa}, h\rangle, \tag{182}$$

for $\mathcal{Z}_{t,\kappa}$ denoting the limit of Theorem 1 and h an arbitrary element in $\mathcal{L}^2(w)$. This convergence follows by an application of Lemma 11.

9.2 Additional Evidence

9.2.1 Cross-sectional Dispersion of Betas and Jumps

In the paper, we eliminate all jumps in the individual assets as well as the market. To verify that the documented contraction of betas over the trading day is not due to this truncation procedure, we reproduce the quantile plot of Figure 3 using the following standard estimator of beta which includes returns with jumps:

$$\widetilde{\beta}_{t,\kappa}^{(j)} = \frac{\sum_{i\in\mathcal{I}_\kappa^n}\Delta_{t,i}^n X^{(j)}\Delta_{t,i}^n X^{(0)}}{\sum_{i\in\mathcal{I}_\kappa^n}(\Delta_{t,i}^n X^{(0)})^2}, \quad j = 1, ..., N. \tag{183}$$

The results are displayed in Figure 9. As seen from the figure, the impact of truncation on the cross-sectional distribution of betas is minimal. Furthermore, the strong contraction of betas towards unity is preserved even when jumps are included.

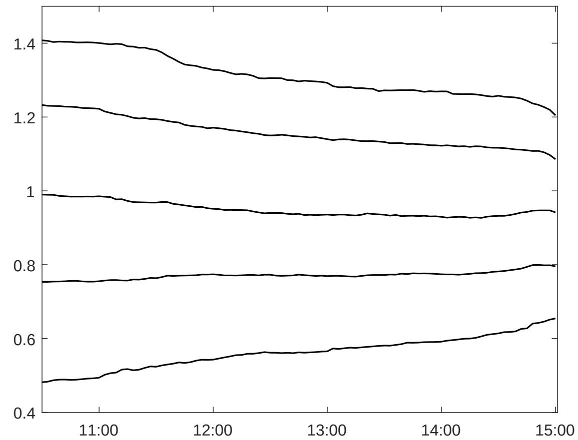

Figure 9: **Cross-Sectional Distribution of Total Market Betas across the Trading Day**. The figure plots the cross-sectional quantiles of the un-truncated betas defined in equation (183). All quantities are treated as functions of the trading day and computed by averaging over the entire sample. The selected quantiles are: 10th, 25th, 50th, 75th, and 90th.

9.2.2 Cross-sectional Dispersion of Betas and Microstructure Effects

While stocks can be traded continuously throughout the trading day, only a finite number of trades of a given stock are made over any fixed period of time. These trade times are not necessarily the same across stocks. This asynchronicity will cause the estimated covariances to be downward biased (Epps effect). Note that the sign of this bias is independent of the stocks beta, thus if staleness is to explain our findings it would need to be high (low) in the morning and low (high) near the close for low (high) beta stocks.

To assess whether this is the case, we compute for each stock the average staleness at any time of the day. In the left panel of Figure 10 we plot the cross-sectional average staleness over the trading day. Several things are worth highlighting. First, staleness is below 12 seconds across the trading day, which is low when compared to the 6 minute frequency used in calculating the covariances. Thus, staleness is not going to have a major effect on our inference procedures. Secondly, staleness gradually increases during the first part of the trading day, and then decreases over the second half of the trading day, ending at an average staleness of less than 1 second.

However, this does not address the question of whether staleness of high and low beta stocks evolves differently. To access whether this is the case, we sort the stocks based on their beta at the open into 5 groups. For each group of stocks, we compute the average staleness. The resulting series are plotted in the right panel of Figure 10. The key takeaway is that high and low beta stocks exhibit the same intraday pattern in staleness, thus staleness cannot explain our findings.

As a secondary robustness check for potential adverse effects on our results from the presence

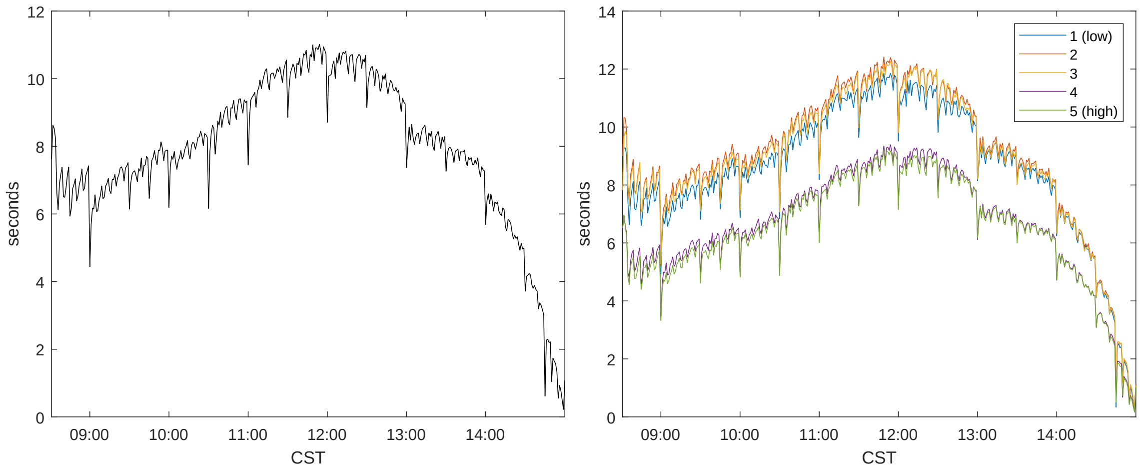

Figure 10: **Staleness of Prices**. The left figure plots the cross-sectionally averaged average staleness. The right figure plots the average staleness for of prices of stocks sorted according to their beta at market open.

of market microstructure noise, we compute the cross-sectional dispersion of market betas using 30 minute returns (thus relying on the time-series dimension rather than infill one). We then compute the cross-sectional deviation from 1, i.e. $\frac{1}{N}\sum_{j=1}^{N}\left(\bar{\beta}_{j,k}-1\right)^2$, where $\bar{\beta}_{j,k}$ is the OLS estimate of market beta in the following time-series regression:

$$r_{t,k}^{(j)} = \alpha_j + \beta_j r_{t,k}^{(0)} + \epsilon_{t,k}^{(j)}, \quad t = 1, ..., T, \tag{184}$$

where $r_{t,k}^{(j)}$ is the return of the asset j over the k'th, $k = 1, ..., 13$, half hour interval of the trading day on day t. The result is plotted in Figure 11.

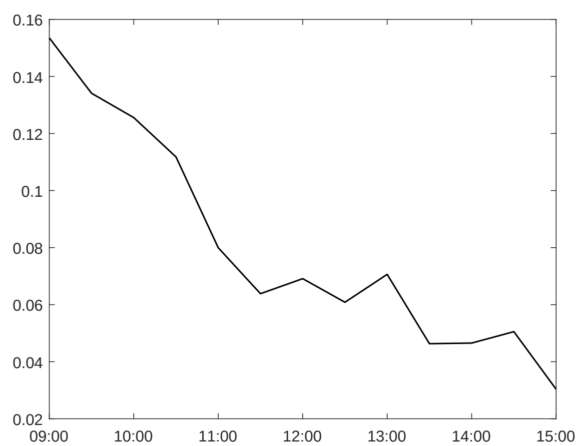

Figure 11: **Cross-Sectional Dispersion of Low-frequency Betas**. The figure plots the cross-sectional dispersion of betas estimated using 30 minute returns according to equation (184).

Note that this approach differs in one important way from the one taken in the paper. Here we are computing the dispersion of time-series averaged betas, whereas in the main text we compute the time-series average of the cross-sectional dispersion of betas. The dispersion presented below will be lower by construction. However, the finding of monotonically decreasing cross-sectional dispersion of market betas during the day can be clearly seen even when using lower frequency returns.

References

Andersen, T. G., T. Bollerslev, F. X. Diebold, and J. G. Wu (2005a): "A framework for exploring the macroeconomic determinants of systematic risk," *American Economic Review*, 95, 398–404.

——— (2005b): "Realized beta: persistence and predictability," in *Advances in Econometrics: Econometric Analysis of Economic and Financial Time Series*, ed. by T. Fomby, vol. B, pp. 1–40.

Andersen, T. G., N. Fusari, and V. Todorov (2015): "The risk premia embedded in index options," *Journal of Financial Economics*, 117(3), 558–584.

Andersen, T. G., V. Todorov, and M. Ubukata (2019): "Tail risk and return predictability for the Japanese equity market," *Journal of Econometrics, forthcoming.*

Ang, A., and D. Kristensen (2012): "Testing conditional factor models," *Journal of Financial Economics*, 106(1), 132–156.

Bandi, F. M., and P. C. Phillips (2003): "Fully nonparametric estimation of scalar diffusion models," *Econometrica*, 71, 241–283.

Bandi, F. M., D. Pirino, and R. Renó (2017): "EXcess Idle Time," *Econometrica*, 85, 1793–1846.

Barndorff-Nielsen, O. E., P. R. Hansen, A. Lunde, and N. Shephard (2011): "Multivariate realised kernels: Consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading," *Journal of Econometrics*, 162(2), 149–169.

Barndorff-Nielsen, O. E., and N. Shephard (2004a): "Econometric analysis of realized covariation: High frequency based covariance, regression, and correlation in financial economics," *Econometrica*, 72(3), 885–925.

Barndorff-Nielsen, O. E., and N. Shephard (2004b): "Power and bipower variation with stochastic volatility and jumps," *Journal of Financial Econometrics*, 2, 1–37.

Ben-Rephael, A., B. I. Carlin, Z. Da, and R. D. Israelsen (2020): "Information consumption and asset pricing," *Journal of Finance, forthcoming.*

Bibinger, M. (2011): "Efficient covariance estimation for asynchronous noisy high-frequency data," *Scandinavian Journal of Statistics*, 38, 2345.

——— (2012): "An estimator for the quadratic covariation of asynchronously observed Itô processes with noise: Asymptotic distribution theory," *Stochastic Processes and their Applications*, 122, 24112453.

Bibinger, M., N. Hautsch, P. Malec, and M. Reiss (2019): "Estimating the spot covariation of asset prices - statistical theory and empirical evidence," *Journal of Business & Economic Statistics*, 37(3), 419–435.

Bibinger, M., and M. Reiss (2014): "Spectral estimation of covolatility from noisy observations using local weights," *Scandinavian Journal of Statistics*, 41(1), 23–50.

Bierens, H. J. (1982): "Consistent model specification tests," *Journal of Econometrics*, 20(1), 105–134.

Bierens, H. J., and W. Ploberger (1997): "Asymptotic theory of integrated conditional moment tests," *Econometrica*, 65(5), 1129–1151.

Billingsley, P. (2013): *Convergence of probability measures.* John Wiley & Sons.

Bollerslev, T., S. Z. Li, and V. Todorov (2016): "Roughing up beta: Continuous versus discontinuous betas and the cross section of expected stock returns," *Journal of Financial Economics*, 120(3), 464–490.

Bollerslev, T., and V. Todorov (2011): "Estimation of jump tails," *Econometrica*, 79(6), 1727–1783.

Campbell, J. Y., and T. Vuolteenaho (2004): "Bad beta, good beta," *American Economic Review*, 94(5), 1249–1275.

Christensen, K., S. Kinnebrock, and M. Podolskij (2010): "Pre-averaging estimators of the ex-post covariance matrix in noisy diffusion models with non-synchronous data," *Journal of Econometrics*, 159, 116–133.

Connor, G., M. Hagmann, and O. Linton (2012): "Efficient semiparametric estimation of the Fama–French model and extensions," *Econometrica*, 80(2), 713–754.

Da, Z., and M. C. Warachka (2009): "Cash flow risk, systematic earnings revisions, and the cross-section of stock returns," *Journal of Financial Economics*, 94, 448–468.

Epps, T. W. (1979): "Comovements in stock prices in the very short run," *Journal of the American Statistical Association*, 74(366), 291–298.

Fan, J., and Y. Wang (2008): "Spot volatility estimation for high-frequency data," *Statistics and its Interface*, 1(2), 279–288.

Ferson, W. E., and C. R. Harvey (1999): "Conditioning variables and the cross section of stock returns," *Journal of Finance*, 54(4), 1325–1360.

Foster, D., and D. Nelson (1996): "Continuous record asymptotics for rolling sample variance estimators," *Econometrica*, 64, 139–174.

Gagliardini, P., E. Ossola, and O. Scaillet (2016): "Time-varying risk premium in large cross-sectional equity data sets," *Econometrica*, 84(3), 985–1046.

Gospodinov, N., R. Kan, and C. Robotti (2009): "Misspecification-robust inference in linear asset pricing models with irrelevant risk factors," *Review of Financial Studies*, 27, 2139–2170.

Hansen, L. P., and S. F. Richard (1987): "The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models," *Econometrica*, 55(3), 587–613.

Hayashi, T., and N. Yoshida (2005): "On covariance estimation of non-synchronously observed diffusion processes," *Bernoulli*, 11(2), 359–379.

Hong, H., and J. Wang (2000): "Trading and returns under periodic market closure," *Journal of Finance*, 55(1), 297–354.

Jacod, J., and P. Protter (2012): *Discretization of processes*. Springer-Verlag, Berlin.

Jacod, J., and A. Shiryaev (2003): *Limit theorems for stochastic processes*. Springer-Verlag, Berlin, 2nd edn.

Jagannathan, R., and Z. Wang (1996): "The conditional CAPM and the cross-section of expected returns," *Journal of Finance*, 51(1), 3–53.

Jagannathan, R., and Z. Wang (1998): "An asymptotic theory for estimating beta-pricing models using cross-sectional regression," *Journal of Finance*, 53, 1285–1309.

Kan, R., and C. Zhang (1999): "Two-pass tests of asset pricing models with useless factors," *Journal of Finance*, 54, 204–235.

Kleibergen, F. (2009): "Tests of risk premia in linear factor models," *Journal of Econometrics*, 149, 149–173.

Koike, Y. (2016): "Estimation of integrated covariances in the simultaneous presence of nonsynchronicity, microstructure noise and jumps," *Econometric Theory*, 32(3), 533–611.

Kristensen, D. (2010): "Nonparametric filtering of the realized spot volatility: A kernel-based approach," *Econometric Theory*, 26(1), 60–93.

Li, J., V. Todorov, and G. Tauchen (2017): "Adaptive estimation of continuous-time regression models using high-frequency data," *Journal of Econometrics*, 200(1), 36–47.

Liu, Q., Y. Liu, and Z. Liu (2018): "Estimating spot volatility in the presence of infinite variation jumps," *Stochastic Processes and their Applications*, 128(6), 1958–1987.

Mancini, C. (2001): "Disentangling the jumps of the diffusion in a geometric jumping Brownian motion," *Giornale dell'Istituto Italiano degli Attuari*, 64(19-47), 44.

——— (2009): "Non-parametric threshold estimation for models with stochastic diffusion coefficient and jumps," *Scandinavian Journal of Statistics*, 36(2), 270–296.

Mykland, P., and L. Zhang (2009): "Inference for continuous semimartingales observed at high frequency," *Econometrica*, 77, 1403–1445.

Mykland, P. A., and L. Zhang (2017): "Assessment of uncertainty in high frequency data: The observed asymptotic variance," *Econometrica*, 85(1), 197–231.

Patton, A. J., and M. Verardo (2012): "Does beta move with news? Firm-specific information flows and learning about profitability," *Review of Financial Studies*, 25(9), 398–404.

Rigobon, R. (2003): "Identification through heteroskedasticity," *Review of Economics and Statistics*, 85(4), 777–792.

Savor, P., and M. Wilson (2016): "Earnings announcements and systematic risk," *Journal of Finance*, 71(1), 83–138.

Shanken, J. (1990): "Intertemporal asset pricing: An empirical investigation," *Journal of Econometrics*, 45(1-2), 99–120.

Shanken, J. (1992): "On the estimation of beta pricing models," *Review of Financial Studies*, 5, 1–33.

Vaart, A. W., and J. Wellner (1996): *Weak convergence and empirical processes.* Springer, New York.

Varneskov, R. T. (2016): "Flat-top realized kernel estimation of quadratic covariation with nonsynchronous and noisy asset prices," *Journal of Business & Economic Statistics*, 34(1), 1–22.

Zhang, L. (2011): "Estimating covariation: Epps effect, microstructure noise," *Journal of Econometrics*, 160(1), 33–47.